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On the minimal spectral radii of skew-reciprocal integer matrices

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Abstract. In this paper we determine the minimal spectral radii among all skew-reciprocal integer matrices of a fixed even dimension that are primitive or nonnegative and irreducible. In particular, except for dimension six, we show that each such class of matrices realises smaller spectral radii than the corresponding reciprocal class.

Contents

1. Introduction

Curiously, orientation-reversing integer linear dynamical systems can be simpler than orientation-preserving ones in the following sense: among all matrices $A \in \mathrm{GL}_2(\mathbb{Z})$ with $\det(A) = -1$, the smallest spectral radius > 1 is the golden ratio φ, while among matrices A with $\det(A) = 1$, the smallest spectral radius > 1 is φ^2. In this article, we generalise this comparison to reciprocal and skew-reciprocal matrices of any even dimension, under the assumption of either primitivity or nonnegativity and irreducibility.

A matrix is *nonnegative* if all its coefficients are nonnegative. Such a matrix is *primitive* if some power has strictly positive coefficients. A matrix is *irreducible* if it is not conjugate via a permutation matrix to an upper triangular block matrix. We call a matrix *reciprocal* if the set of its eigenvalues (counted with multiplicity) is invariant under the transformation $t \mapsto t^{-1}$. Finally, we call a matrix *skew-reciprocal* if the set of its eigenvalues (counted with multiplicity) is invariant under the transformation $t \mapsto -t^{-1}$. Important examples of reciprocal or skew-reciprocal matrices are symplectic or antisymplectic matrices, respectively.

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We find out that with the exception of dimension six in the primitive case, the skew-reciprocal matrices always realise smaller spectral radii > 1 than the reciprocal ones.

Theorem 1.1. *Let $g \geq 1$ and $g \neq 3$. Among primitive matrices $A \in \mathrm{GL}_{2g}(\mathbb{Z})$, the skew-reciprocal ones realise a smaller spectral radius > 1 than the reciprocal ones. For $g = 3$, the reciprocal matrices realise a smaller spectral radius than the skew-reciprocal ones.*

Theorem 1.2. *Let $g \geq 1$. Among nonnegative irreducible $A \in \mathrm{GL}_{2g}(\mathbb{Z})$, the skew-reciprocal ones realise a smaller spectral radius > 1 than the reciprocal ones.*

Naturally, the following question arises by dropping the hypotheses of primitivity or irreducibility.

Question 1.3. *Let $g \geq 1$. Do the skew-reciprocal matrices $A \in \mathrm{GL}_{2g}(\mathbb{Z})$ realise a smaller spectral radius > 1 than the reciprocal ones?*

Answering Question 1.3 positively for all g that are large enough powers of two would provide an independent proof of Dimitrov's theorem [1], also known as the conjecture of Schinzel and Zassenhaus [8]. This follows from an argument[1] previously given by the author [3].

The proofs of Theorems 1.1 and 1.2 are based on McMullen's calculation of the minimal possible spectral radii > 1 for primitive and nonnegative irreducible reciprocal matrices [6]. In fact, we carry out the same calculation for skew-reciprocal matrices in order to determine the minimal spectral radii > 1 that arise, and compare the values with McMullen's result. The following two theorems summarise our results on these minimal spectral radii.

Theorem 1.4. *Let $g \geq 1$. The minimal spectral radius > 1 among skew-reciprocal nonnegative irreducible matrices $A \in \mathrm{GL}_{2g}(\mathbb{Z})$ is realised by the by the largest root λ_{2g} of the polynomial*

$$t^{2g} - t^{g} - 1$$

in case g is odd, and of the polynomial

$$t^{2g} - t^{g+1} - t^{g-1} - 1$$

in case g is even.

Theorem 1.5. *Let $g \geq 2$. The minimal spectral radius > 1 among skew-reciprocal primitive matrices $A \in \mathrm{GL}_{2g}(\mathbb{Z})$ is realised by the by the largest root μ_{2g} of the polynomial*

$$t^{2g} - t^{g+2} - t^{g-2} - 1$$

in case g is odd, and of the polynomial

$$t^{2g} - t^{g+1} - t^{g-1} - 1$$

[1] The definitions for reciprocality and skew-reciprocality used in the article [3] is slightly different, prescribing the sign of the constant coefficient so that the polynomials arise from the action induced on the first homology by mapping classes. However, the argument presented there can be adopted directly to the definitions we use here.

in case g is even.

Given Theorems 1.5 and 1.4, Theorems 1.1 and 1.2 follow readily.

Proof of Theorems 1.1 and 1.2. The normalised sequence $(\mu_{2g})^{2g}$ converges to the square of the silver ratio, $(1+\sqrt{2})^2 = 3+2\sqrt{2}$, while the normalised sequence $(\lambda_{2g})^{2g}$ has two accumulation points: again the square of the silver ratio, for even g, but also the square of the golden ratio, $\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$, for odd g. To see this, note that $(\mu_{2g})^g$ is the largest real zero of the function

$$f(t) = t^2 - t^{1+\frac{2}{g}} - t^{1-\frac{2}{g}} - 1$$

in case g is odd, and of the function

$$f(t) = t^2 - t^{1+\frac{1}{g}} - t^{1-\frac{1}{g}} - 1$$

in case g is even. Clearly for $g \to \infty$ any real zero > 1 converges to the larger root of the polynomial $t^2 - 2t - 1$, which is $1+\sqrt{2}$. Therefore $(\mu_{2g})^{2g}$ converges to $(1+\sqrt{2})^2 = 3+2\sqrt{2}$. The argument for the sequence λ_g is analogous.

In either case, these accumulation points are all smaller than the analogous smallest possible accumulation point in the case of reciprocal matrices. Indeed, McMullen proves that the minimal accumulation point for the normalised sequence of spectral radii for reciprocal matrices is φ^4, where φ is the golden ratio [6]. This number is strictly larger than the square of the silver ratio, so asymptotically the result is given. We finish the proof Theorems 1.1 and 1.2 by using the monotonicity of the sequences of normalised spectral radii, and compare these sequences for small g. It turns out that the only case where a normalised sequence of the skew-reciprocal matrices is larger than the accumulation point φ^4 of the normalised sequence of the reciprocal matrices is in the case $g = 3$ of primitive matrices. This finishes the proof for $g \neq 3$. For $g = 3$ in the primitive case, we simply check that $(\mu_6)^6 > 8.18$, whereas the smallest normalised spectral radius in the reciprocal case is ≈ 7.57 by McMullen's result [6]. This finishes the proof also in the case $g = 3$. □

Applications to pseudo-Anosov stretch factors. McMullen's result on the minimal spectral radii for reciprocal matrices [6] is interesting in the context of minimal stretch factors of pseudo-Anosov mapping classes. For example, it is used by Hironaka and Tsang to find, for a large class of examples, the optimal lower bound for normalised pseudo-Anosov stretch factors in the fully-punctured case [2]. Naturally, we hope our Theorem 1.5 will be instrumental for an analogous result in the case of orientation-reversing pseudo-Anosov mapping classes. In fact, Theorem 1.5 also presents some of the same polynomials found by the author and Strenner [4] in the search of minimal stretch factors among orientation-reversing pseudo-Anosov mapping classes with orientable invariant foliations.

Organisation. In the next section, we minimally review the notion of the clique polynomial as well as the input we need from McMullen's work [6], before proving Theorems 1.4 and 1.5 in the third and final section.

Acknowledgments. The author thanks Chi Cheuk Tsang and an anonymous referee for their comments on a first version of this article.

2. The clique polynomial

In this section, we review parts of McMullen's technique using the curve graph and its clique polynomial in order to single out minimal spectral radii among nonnegative matrices. We try to keep the discussion as concise as possible and refer to the original article [6] and the references therein for a more complete discussion.

Let Γ be a directed graph. A *simple closed curve* in Γ is the union of directed edges describing a closed directed loop in Γ that visits every vertex at most once. The *curve graph* G of Γ is obtained as follows: there is a vertex for every simple closed curve in Γ, and two vertices are connected by an edge if and only if the corresponding simple closed curves have no vertex of Γ in common. Each vertex of G is given a weight describing the number of edges contained in the simple closed curve.

A subset K of the vertices of G is a *clique* if the subgraph induced by K is complete. The *clique polynomial* of G is defined to be

$$Q(t) = \sum_K (-1)^{|K|} t^{w(K)},$$

where we also allow $K = \emptyset$, and $w(K)$ is the sum of all weights of vertices in K.

With a nonnegative square matrix A of dimension $n \times n$, we associate a directed graph Γ_A that has n vertices and directed edges between the vertices according to the coefficients of A. Let G_A be the associated curve graph and let $Q_A(t)$ be its clique polynomial. By a well-known result in graph theory, the characteristic polynomial of A is the reciprocal of $Q_A(t)$, that is, $\chi_A(t) = t^n Q_A(t^{-1})$. In particular, the spectral radius of A equals the inverse of the smallest modulus among the roots of $Q_A(t)$.

2.1. McMullen's classification of graphs with small growth. McMullen defines a minimal growth rate $\lambda(G)$ for graphs G. We refer to McMullen's original article [6] for more details. The only statement we need for our purposes is that if A is a nonnegative matrix of dimension $n \times n$ and with spectral radius $\rho(A)$, then $\lambda(G_A)$ is a lower bound for the normalised spectral radius $\rho(A)^n$.

We are interested in irreducible matrices A. This means that the directed graph Γ_A is strongly connected, which in turn implies that the associated curve graph G_A has complement G'_A that is connected. Here, the complement G'_A is defined to be the graph with the same vertex set as G_A but the complementary edges. There are very few curve graphs G with G' connected and minimal growth rate $\lambda(G) < 8$. Our argument is based on the following classification due to McMullen [6].

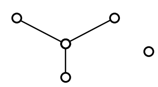
G		$\lambda(G)$
nA_1		n
A_2^*		4
A_2^{**}		$3+2\sqrt{2} \approx 5.82$
A_2^{***}		$4+2\sqrt{2} \approx 7.46$
A_3^*		$3+2\sqrt{2} \approx 5.82$
Y^*		$4+2\sqrt{2} \approx 7.46$

FIGURE 1. Some graphs and their minimal growth rates.

Theorem 2.1 (Theorem 1.6 in McMullen [6]). *The graphs G with G' connected and $1 < \lambda(G) < 8$ are given by*

$$A_2^*, A_2^{**}, A_2^{***}, A_3^*, Y^* \text{ and } nA_1,$$

For $2 \leq n \leq 7$.

This result tells us that if we want to describe all irreducible nonnegative matrices with normalised spectral radius < 8, all we have to do is check among those whose associated curve graph is among the ones shown in Figure 1. In fact, in Section 3 we split the proof of Theorems 1.4 and 1.5 into four propositions, dealing with the four distinct cases. In each case, the first thing we do is to realise the proposed minimal normalised spectral radius, which (except in one case for $g = 3$) turns out to be < 8. Then McMullen's classification result applies and we only have to check the curve graphs given in Theorem 2.1 to finish the proof.

3. Skew-reciprocity and minimal spectral radii

In this section, we prove Theorems 1.4 and 1.5. We break down the proof into four separate propositions, distinguishing between the irreducible and the primitive case, as well as the case of even and odd g.

The condition of skew-reciprocity poses slightly different constraints on the coefficients of the polynomial than reciprocity. First of all, we note that if the roots of a polynomial $f \in \mathbb{Z}[t]$ are invariant under the transformation $\lambda \mapsto -\lambda^{-1}$, then we have $f(t) = \pm t^{\deg(f)} f(-t^{-1})$. This entails the following constraints.

Lemma 3.1. *Let $f \in \mathbb{Z}[t]$ be a monic skew-reciprocal polynomial of degree $2g$. Then we have the following conditions on the coefficients of f:*

(1) *the moduli of the coefficients of t^d and t^{2g-d} agree. More precisely,*
(2) *if g is even and $f(0) = 1$, the coefficients of t^d and t^{2g-d} agree for even d and differ by a sign for odd d,*
(3) *if g is even and $f(0) = -1$, the coefficients of t^d and t^{2g-d} agree for odd d and differ by a sign for even d. In particular, the middle coefficient of f vanishes,*
(4) *if g is odd and $f(0) = 1$, the coefficients of t^d and t^{2g-d} agree for even d and differ by a sign for odd d. In particular, the middle coefficient of f vanishes,*
(5) *if g is odd and $f(0) = -1$, the coefficients of t^d and t^{2g-d} agree for odd d and differ by a sign for even d.*

Proof. Let $f(t) = a_{2g}t^{2g} + \cdots + a_0$ be a skew-reciprocal polynomial. The polynomial relation $f(t) = \pm t^{2g} f(-t^{-1})$ given by skew-reciprocity translates to the relation

$$a_d = \pm(-1)^{2g-d} a_{2g-d} = \pm(-1)^d a_{2g-d}$$

for each pair coefficients a_d and a_{2g-d}. Clearly, the coefficients are symmetric up to a possible sign that alternates between $+1$ and -1 as we change the index d of the coefficient by one. In particular, the sign is the same for all even d and it is the same for all odd d. Now recall that $f(t)$ is monic, that is, $a_{2g} = 1$. In this case, $f(0) = 1$ means that the coefficients a_d and a_{2g-d} agree for $d = 0$ and hence all even d, and they differ by a sign for odd d. Similarly, $f(0) = -1$ means that the coefficients differ by a sign for $d = 0$ and hence for all even d, and they agree for even d. Finally, the middle coefficient a_g needs to be zero if a_d and a_{2g-d} differ by a sign for all d with the same parity as g. This distinction yields the four different cases (2)–(5) described in the statement of Lemma 3.1. □

Example 3.2. To see the main proof ideas applied to the simplest nontrivial example, we now determine which spectral radii are obtained by skew-reciprocal matrices $A \in \mathrm{GL}_{2g}(\mathbb{Z})$ with curve graph $G_A = 2A_1$, which is the graph with two isolated vertices. In this case $Q(t) = 1 - t^a - t^b$, where a and b are the weights of the vertices. The polynomial $Q(t)$ is of degree $2g$, and without loss of generality we assume $b = 2g$. The only way to have the moduli of the coefficients symmetrically distributed as in (1) of Lemma 3.1 is if $a = g$. Therefore, the only clique polynomial we possibly obtain is $Q(t) = 1 - t^g - t^{2g}$. Hence, the only characteristic polynomial we possibly obtain is $t^{2g} - t^g - 1$. We make the following observations:

(1) if A is primitive, then $g = 1$. Indeed, otherwise the characteristic polynomial is a polynomial in t^g with $g > 1$. Such a polynomial cannot be the characteristic polynomial of a primitive matrix. On the other hand, for $g = 1$, the polynomial $t^2 - t - 1$ is the minimal polynomial of the golden ratio, realised as the characteristic polynomial of the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

(2) For odd g, the polynomial $t^{2g} - t^g - 1$ is the characteristic polynomial of a nonnegative irreducible matrix in $\mathrm{GL}_{2g}(\mathbb{Z})$, namely a standard companion matrix. For example, $t^6 - t^3 - 1$ is the characteristic polynomial of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

which is irreducible.

(3) For even g, we note that since the constant coefficient of the characteristic polynomial of A is negative, we are in case (3) of Lemma 3.1. In particular, the coefficient of t^g should be zero instead of -1. This means that for even g, there are no skew-reciprocal matrices $A \in \mathrm{GL}_{2g}(\mathbb{Z})$ with curve graph $2A_1$.

In summary, we obtain that the following spectral radii can be realised for matrices $A \in \mathrm{GL}_{2g}(\mathbb{Z})$ with curve graph $2A_1$:

(i) Among primitive skew-reciprocal matrices $A \in \mathrm{GL}_{2g}(\mathbb{Z})$, only the golden ratio is realised, for $g = 1$. For $g > 1$, there are no primitive skew-reciprocal matrices with curve graph $2A_1$.

(ii) Among nonnegative irreducible skew-reciprocal matrices $A \in \mathrm{GL}_{2g}(\mathbb{Z})$, the largest root of the polynomial $t^{2g} - t^g - 1$ is realised, for odd g. For even g, there are no nonnegative irreducible skew-reciprocal matrices $A \in \mathrm{GL}_{2g}(\mathbb{Z})$ with curve graph $2A_1$.

3.1. The irreducible case.

3.1.1. The case of odd g.

Proposition 3.3. *Let $g \geq 1$ odd. Among all skew-reciprocal nonnegative irreducible matrices $A \in \mathrm{GL}_{2g}(\mathbb{Z})$, the minimal spectral radius > 1 is realised by the largest root λ_{2g} of the polynomial $t^{2g} - t^g - 1$.*

Proof. Let $A \in \mathrm{GL}_{2g}(\mathbb{Z})$ be a nonnegative skew-reciprocal matrix. Then its square A^2 is a reciprocal matrix. In particular, by McMullen's result on minimal normalised spectral radii for nonnegative reciprocal matrices [6] we know that its normalised spectral radius must be at least φ^4. Therefore, the normalised spectral radius of A must be at least φ^2, which incidentally equals $(\lambda_{2g})^{2g}$. In order to finish the proof, it therefore suffices to realise the polynomial $t^{2g}-t^g-1$ as the characteristic polynomial of a nonnegative irreducible matrix in $\mathrm{GL}_{2g}(\mathbb{Z})$. This is straightforward, as it can be achieved by a standard companion matrix, see (2) in Example 3.2 for the case $g = 3$. □

Remark 3.4. It actually follows from McMullen's classification that the polynomial $t^{2g} - t^g - 1$ is the unique characteristic polynomial that can appear for a matrix that minimises the spectral radius. Indeed, only the graph $2A_1$ can appear as curve graph, with clique polynomial $1 - t^a - t^b$. For this polynomial to be skew-reciprocal we must either have $a = 2g$ and $b = g$ or $b = 2g$ and $a = g$. Both cases yield our candidate polynomial.

3.1.2. The case of even *g*. We note that the above proof does not work for even g: if g is even, then the polynomial $t^{2g} - t^g - 1$ is not skew-reciprocal, as noted in Example 3.2. We instead have the following minimisers.

Proposition 3.5. *Let $g \geq 2$ even. Among all skew-reciprocal nonnegative irreducible matrices $A \in \mathrm{GL}_{2g}(\mathbb{Z})$, the minimal spectral radius > 1 is realised by the largest root λ_{2g} of the polynomial $t^{2g} - t^{g+1} - t^{g-1} - 1$.*

Proof. The largest root λ_{2g} of the polynomial $t^{2g} - t^{g+1} - t^{g-1} - 1$ is clearly realised as the spectral radius of a nonnegative irreducible $A \in \mathrm{GL}_{2g}(\mathbb{Z})$. Indeed, again we can achieve this by a standard companion matrix. We now note that $(\lambda_{2g})^{2g}$ is a descending sequence converging to $3 + 2\sqrt{2}$ and starting at $\varphi^4 < 7$ for $g = 2$. In particular, we can finish the proof by showing that for even $g \geq 2$, λ_{2g} minimises the spectral radius among all skew-reciprocal nonnegative irreducible matrices $A \in \mathrm{GL}_{2g}(\mathbb{Z})$ that have one of the graphs in Figure 1 except A_2^{***} or Y^* as their curve graph.

(1) $G = 2A_1$. As noted in Example 3.2, there are no nonnegative irreducible skew-reciprocal matrices $A \in \mathrm{GL}_{2g}(\mathbb{Z})$ with g even and $2A_1$ as their curve graph.
(2) $G = 3A_1$. In this case $Q(t) = 1-t^a-t^b-t^c$. Without loss of generality we assume $c = 2g$. If we want the moduli of the coefficients symmetrically distributed, we are left with the options
$$1 - t^{g-d} - t^{g+d} - t^{2g}$$
for $0 \leq d \leq g$. The case $d = g$ is ruled out as the resulting polynomial is a monomial. The case $d = 0$ is ruled out by Lemma 3.1, as above. It follows that $1 < d < g$. By Proposition 3.2 in [5], we know that the largest root of the reciprocal polynomial $t^{2g} - t^{g+d} - t^{g-d} - 1$ is a strictly increasing function of d. So, the smallest spectral radius is obtained for $d = 1$, resulting in our candidate polynomial $t^{2g} - t^{g+1} - t^{g-1} - 1$.
(3) $G = 4A_1$ or $G = 6A_1$. As for $G = 2A_1$, the number of terms is odd. The only way to have the moduli of the coefficients symmetrically distributed is to have a middle coefficient, a contradiction to Lemma 3.1.

(4) $G = 5A_1$. In this case $Q(t) = 1 - t^a - t^b - t^c - t^d - t^e$. Without loss of generality $e = 2g$. As in the case $G = 3A_1$, there must be at least two paired terms of power $\neq 0, g$. We assume without loss of generality that $0 < a < g < b = 2g - a < 2g$. Since the polynomial reciprocal to $Q(t)$ is realised by a standard companion matrix, we can delete its coefficients that correspond to the terms $-t^c$ and $-t^d$ and obtain a matrix with strictly smaller spectral radius and characteristic polynomial $t^{2g} - t^b - t^a - 1$, where the a, b and g satisfy $0 < a < g < b = 2g - a < 2g$. We have shown in the case $G = 3A_1$ that the minimal spectral radius obtained by a matrix with such a characteristic polynomial is our candidate λ_{2g}.

(5) $G = A_2^*$. In this case, $Q(t) = 1 - t^a - t^b - t^c + t^{a+b}$, where c is the weight on the isolated vertex of A_2^*. Since there are three terms not paired with the constant term, we deduce there must be a nonvanishing middle coefficient. By Lemma 3.1, this means that the leading coefficient of $Q(t)$ is positive, so we have $a + b = 2g$ and $c = g$. We note that the possibilities that remain are $1 - t^{g-d} - t^g - t^{g+d} + t^{2g}$, which are reciprocal. Theorem 7.3 by McMullen [6] provides that the normalised spectral radius is $\geq \varphi^4 = \lambda_4$ and $> \lambda_{2g}$ for $g > 2$.

(6) $G = A_2^{**}$. In this case, $Q(t) = 1 - t^a - t^b - t^c - t^d + t^{a+b}$, where c and d are the weights on the isolated vertices of A_2^{**}. If the leading coefficient of $Q(t)$ is positive, then $Q(t)$ is reciprocal. By McMullen's analysis of the curve graph A_2^{**} for reciprocal weights, a normalised spectral radius in this case must be $\geq (2 + \sqrt{3})^2 > \lambda_{2g}$. It remains to consider the case where the leading coefficient of $Q(t)$ is negative. Without loss of generality, we assume $c = 2g$. We have the following conditions on the other parameters a, b, d.
   - $a + b < 2g$ and hence $a + b \leq 2g - 2$. Indeed, t^{a+b} appears with a positive sign and must be paired with a term $-t^x$ with negative sign, for $x = a, b$ or d. In particular, Lemma 3.1 implies that $a + b$ must be even.
   - either $d + a = 2g$ or $d + b = 2g$. We assume without loss of generality that $d + b = 2g$. Thus, $1 \leq a < b, d < a + b \leq 2g - 2$.

   Now consider the directed graph $\Gamma_{a,b,d}$, shown in Figure 2, where a weight w on an edge indicates $w - 1$ additional vertices placed on the edge. We note that the clique polynomial of the curve graph of $\Gamma_{a,b,d}$ is $Q(t)$. Furthermore, deleting the edge of length 1 in $\Gamma_{a,b,d}$ that forms the simple closed curve of length a strictly decreases the spectral radius of the associated adjacency matrix. Furthermore, the new curve graph is $3A_1$ and the new clique polynomial is obtained by removing the terms $-t^a$ and t^{a+b}, and hence skew-reciprocal. This is a case we have already dealt with.

(7) $G = A_3^*$. In this case, $Q(t) = 1 - t^a - t^b - t^c - t^d + t^{a+b} + t^{b+c}$, where d is the weigth of the isolated vertex and b is the weight of the vertex of

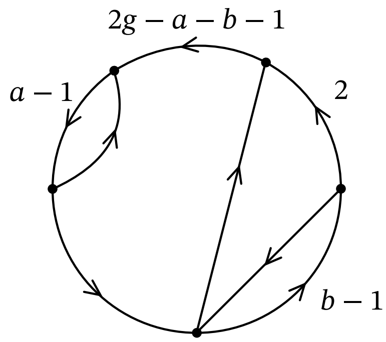


FIGURE 2. The directed graph $\Gamma_{a,b,d}$.

degree two of A_3^*. Since the number of summands is odd, the middle term must have a nonvanishing coefficient. By Lemma 3.1, the leading coefficient of $Q(t)$ is positive, and we assume without loss of generality that $b+c=2g$. Note that since the coefficients of t^b and t^c have the same sign, b and c need to be even. We now distinguish three cases: either $a+b=g$, $a=g$, or $d=g$.

- if $a+b=g$, then $Q(t)=1-t^a-t^b+t^g-t^{2g-a}-t^{2g-b}+t^{2g}$, which is reciprocal. By McMullen's analysis of the curve graph A_3^* for reciprocal weights [6], the normalised spectral radius must either be $> 12.5 > \varphi^4$, or $Q(t)$ is among the examples arising from A_2^*, a case we have dealt with already.
- if $a=g$, then $d+a+b=2g$. Since b and g are even, so must be $a+b=g+b$, and hence also d. By Lemma 3.1, the coefficients of t^d and of t^{a+b} should have the same sign, a contradiction.
- if $d=g$, we have $2a+b=2g$, and hence $a \le g-1$. Let $\Gamma'_{a,b,c}$ be defined as in Figure 3, where the edge of weight $g-a-1$ is con-

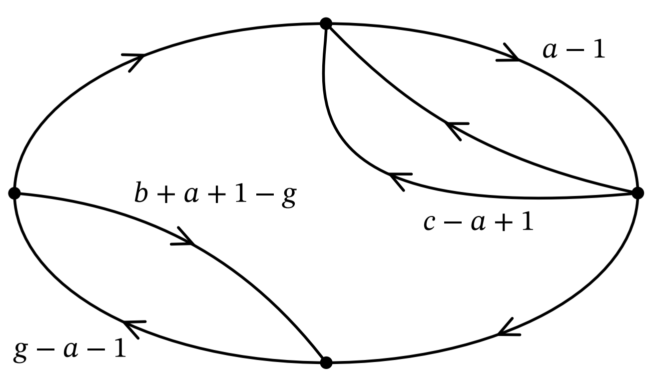


FIGURE 3. The directed graph $\Gamma'_{a,b,c}$.

tracted in case $a=g-1$. We note that the clique polynomial of the curve graph of $\Gamma'_{a,b,c}$ is exactly our $Q(t)$, where $d = g$. Deleting the edge of length one that forms the simple closed curve of length a strictly decreases the spectral radius of the associated adjacency matrix. The clique polynomial we obtain after this deletion of an edge is of the form $1 - t^b - t^g - t^{2g-b} + t^{2g}$, a case we have dealt with in our study of $G = A_2^*$.

□

Remark 3.6. Our proof of Proposition 3.5 actually shows that for $g \neq 2$, the polynomial $t^{2g} - t^{g+1} - t^{g-1} - 1$ is the unique characteristic polynomial that can appear for a matrix minimising the spectral radius. Except for $g = 2$, where we have a second possibility (appearing in (5)) for the characteristic polynomial: $t^4 - 3t^2 + 1$. In the case $g = 2$, both minimising polynomials are divisible by the minimal polynomial of the golden ratio.

Remark 3.7. In the cases (4), (6) and (7) of the proof of Proposition 3.5, we construct irreducible matrices and reduce some of their coefficients in order to obtain irreducible matrices with strictly smaller spectral radii that belong to other cases we have already dealt with. For a quicker proof of Proposition 3.5, we could use the monotonicity property for the spectral radius formulated by McMullen [6] on the level of the weighted curve graph. However, this monotonicity is not strict in general. In particular, this proof strategy seems to fail to provide the uniqueness of the minimising characteristic polynomials described in Remark 3.6.

3.2. The primitive case.

3.2.1. The case of even g.

Proposition 3.8. *Let $g \geq 2$ even. Among all skew-reciprocal primitive matrices $A \in \mathrm{GL}_{2g}(\mathbb{Z})$, the minimal spectral radius > 1 is realised by the largest root μ_{2g} of the polynomial $t^{2g} - t^{g+1} - t^{g-1} - 1$.*

Proof. By Proposition 3.5, we know that μ_{2g} is actually the minimal spectral radius among all nonnegative irreducible matrices. In order to prove the result, it is enough to show that the polynomial $t^{2g} - t^{g+1} - t^{g-1} - 1$ is the characteristic polynomial of a primitive matrix. This is the case. Indeed, we can take the standard companion matrix for the polynomial and draw its directed adjacency graph. We directly see that there are directed cycles of length $g-1, g+1$ and $2g$. In order to show that the matrix is primitive, it suffices to show that their common greatest divisor is 1. Let n be a positive integer that divides both $2g$ and $g + 1$. Since g is even, $g + 1$ is odd and so n has to to be odd as well. Now since n is odd and divides $2g$, it divides g. We have that n divides both g and $g + 1$ and therefore $n = 1$. This finishes the proof. □

3.2.2. The case of odd g.

Proposition 3.9. *Let $g \geq 3$ odd. Among all skew-reciprocal primitive matrices $A \in \mathrm{GL}_{2g}(\mathbb{Z})$, the minimal spectral radius > 1 is realised by the largest root μ_{2g} of the polynomial $t^{2g} - t^{g+2} - t^{g-2} - 1$.*

Proof for $g \geq 5$. We take a standard companion matrix to realise the largest root μ_{2g} of the polynomial $t^{2g} - t^{g+2} - t^{g-2} - 1$ as a spectral radius of a nonnegative matrix in $\mathrm{GL}_{2g}(\mathbb{Z})$. Furthermore, the associated directed graph has simple closed curves of lengths $2g, g+2, g-2$, which have greatest common divisor 1. Indeed, since g is odd so is $g+2$, so if n divides both $2g$ and $g+2$, then it must be odd itself and hence divide g. But then, since n divides both g and $g+2$, it must divide 2. But n being odd implies $n = 1$. This shows that the companion matrix we constructed is primitive.

We now note that $(\mu_{2g})^{2g}$ is a descending sequence converging to $3 + 2\sqrt{2}$, with first values $\mu_6 \approx 8.19$ and $\mu_{10} \approx 6.42$. The example in the case $g = 3$ is too large to be covered by McMullen's classification, and we give a separate argument covering this case below. For $g \geq 5$ odd, we can proceed as before, and finish the proof by showing that μ_{2g} minimises the spectral radius among all skew-reciprocal primitive matrices $A \in \mathrm{GL}_{2g}(\mathbb{Z})$ with one of the graphs in Figure 1 except A_2^{***} or Y^* as their curve graph.

(1) $G = 2A_1$. As we noted in Example 3.2, there exist no primitive skew-reciprocal matrices $A \in \mathrm{GL}_{2g}(\mathbb{Z})$ with $g > 1$ and $2A_1$ as their curve graph.

(2) $G = 3A_1$. In this case, $Q(t) = 1 - t^a - t^b - t^c$. Without loss of generality, we assume that $c = 2g$, which implies $a = 2g - b$ if we want symmetrically distributed coefficients. We have multiple possibilities for a:
   - $a = g$. In this case, $Q(t) = 1 - 2t^g - t^{2g}$, which is not primitive.
   - $a = g - 1$. In this case $Q(t) = 1 - t^{g-1} - t^{g+1} - t^{2g}$. Lemma 3.1 implies that this polynomial is not skew-reciprocal. Indeed, for it to be skew-reciprocal, the coefficients of t^{g+1} and t^{g-1} would have to differ by a sign since $g - 1$ is even.
   - $a = g - 2$. This case gives exactly our candidate polynomial with largest root μ_{2g}.
   - $a < g - 2$. By Proposition 3.2 in [5], we know that the largest root of the reciprocal polynomial $t^{2g} - t^{g+d} - t^{g-d} - 1$ is a strictly increasing function of d. In particular, the spectral radii we obtain for $a < g - 2$ are strictly larger than our candidate.

(3) $G = 4A_1$. In this case, $Q(t) = 1 - t^a - t^b - t^c - t^d$. We realise the reciprocal of $Q(t)$ as the characteristic polynomial of a standard companion matrix. Since there are five terms, there must be a middle coefficient. Deleting this middle coefficient amounts to decreasing a coefficient of the companion matrix from 1 to 0, strictly reducing the spectral radius. After this modification, the polynomial is among the examples we have already dealt with in the case $G = 3A_1$.

(4) $G = 5A_1$ or $G = 6A_1$. This case can be dealt with in the same way as the case $G = 4A_1$. We delete the coefficients of a pair of terms whose powers add to $2g$ (in the case of $G = 5A_1$) and additionally the middle coefficient (in the case of $G = 6A_1$).

(5) $G = A_2^*$. In this case, $Q(t) = 1-t^a-t^b-t^c+t^{a+b}$, where c is the weight on the isolated vertex of A_2^*. Since there are five terms, there must be a middle coefficient, which by Lemma 3.1 implies that the leading coefficient is negative. We therefore must have $c = 2g$ and we can assume without loss of generality that $b = g$ to get a polynomial of the form

$$Q(t) = 1 - t^a - t^g + t^{a+g} - t^{2g},$$

and in particular $2a + g = 2g$. But this implies that $g = 2a$ is even, a contradiction.

(6) $G = A_2^{**}$. In this case, $Q(t) = 1 - t^a - t^b - t^c - t^d + t^{a+b}$, where c and d are the weights on the isolated vertices of A_2^{**}. If the leading coefficient of $Q(t)$ is positive, then $a + b = 2g$ and the resulting polynomial is reciprocal. By McMullen's analysis of the curve graph A_2^{**} for reciprocal weights [6], a normalised spectral radius in this case must be $\geq (2+\sqrt{3})^2 > \mu_{2g}$. It remains to consider the case of a negative leading coefficient. Without loss of generality, we assume $c = 2g$. In order to have symmetrically distributed moduli of the coefficients, we must either have $2a + b = 2g$ and $b + d = 2g$ or $2b + a = 2g$ and $a + d = 2g$. Both cases imply that d is even, and hence so must be b (in the former case) or a (in the latter case). In both cases, we get a contradiction to Lemma 3.1, which states that the coefficients must differ by a sign for even powers.

(7) $G = A_3^*$. In this case, $Q(t) = 1-t^a-t^b-t^c-t^d+t^{a+b}+t^{b+c}$. Since there are seven terms, there must be a nonvanishing middle coefficient. By Lemma 3.1, this can only happen if the leading coefficient is negative. We must have $d = 2g$ and get

$$Q(t) = 1 - t^a - t^b - t^c + t^{a+b} + t^{b+c} - t^{2g}.$$

We distinguish cases depending on which term has power g.

- if one among a, b and c equals g, we have $a, b, c \leq g$ and furthermore $a + b, b + c > g$. This implies that $a + b, b + c$ and two among a, b, c are even by Lemma 3.1. But then clearly all among a, b, c are even, and hence is g, a contradiction.
- if $a + b = g$, then $a, b < g$ and hence $c, b + c > g$. Also $b + c < 2g$ so we must have $a + c = 2g = 2b + c$. In particular, $a = 2b$ is even, and hence so must be c. This contradicts Lemma 3.1, which states that coefficients must differ by a sign for terms with even powers. The argument for the case $b + c = g$ is obtained by switching a and c.

□

Proof for $g = 3$. The candidate polynomial $t^6 - t^5 - t - 1$ has maximal real root $\mu_6 \approx 1.4196 > \sqrt{2}$, so we only need to check other polynomials with roots bounded from above by this number, and bounded from below by $\sqrt{2}$. Indeed, our proof in the case $g \geq 5$ shows that there is no spectral radius $< 8^{\frac{1}{6}} = \sqrt{2}$ among skew-reciprocal primitive matrices $A \in \mathrm{GL}_6(\mathbb{Z})$. We now distinguish cases depending on the determinant of such a matrix A.

Case 1: $\det(A) = 1$. In this case, the characteristic polynomial must have a factor $(t^2 + 1)$. The reason for this is that the eigenvalues of a skew-reciprocal matrix come in groups:

- if $\lambda \notin \mathbb{R}, \lambda \neq \pm i$ is an eigenvalue, then so are $-\lambda^{-1}, \bar{\lambda}$ and $-\bar{\lambda}^{-1}$. These four roots of the characteristic polynomial contribute $+1$ to the determinant,
- if $\lambda \in \mathbb{R}, \lambda \neq 0$ is an eigenvalue, then so is $-\lambda^{-1}$. These two roots of the characteristic polynomial contribute -1 to the determinant,
- if $\lambda = \pm i$ is an eigenvalue, then so is $\bar{\lambda} = -\lambda$. These two roots contribute $+1$ to the determinant.

For $g = 3$, the only way for determinant $+1$ is if the last case appears at least once. This implies that the polynomial is divisible by $(t - i)(t + i) = t^2 + 1$. By Perron-Frobenius theory, we know that A has at least two real roots. In particular, the first case cannot occur and the spectral radius is a totally real algebraic integer with at most two Galois conjugates of modulus > 1. If it is not an integer, it is an algebraic integer of degree at least two. In particular, the Mahler measure of its minimal polynomial is at least φ^2 by Corollary 1' of Schinzel [7]. Since at most two Galois conjugates have modulus > 1, it follows that the modulus of the larger root is bounded from below by $\varphi \approx 1.61 > \mu_6$.

Case 2: $\det(A) = -1$. We first rule out the case where all eigenvalues are real. The spectral radius is an algebraic integer of degree at most six that is maximal in modulus among all its Galois conjugates. Skew-reciprocity of A and the fact that the minimal polynomial has constant coefficient ± 1 imply that at most half of the Galois conjugates of the spectral radius can have modulus > 1. Again, Schinzel's Corollary 1' in [7] implies that the spectral radius is bounded from below by $\varphi \approx 1.61 > \mu_6$.

In the remaining case, the spectral radius ρ of A is of degree six and has four non-real Galois conjugates $\lambda, -\lambda^{-1}, \bar{\lambda}$ and $-\bar{\lambda}^{-1}$. Let the characteristic polynomial of A be given by

$$\begin{aligned} P(t) &= t^6 + at^5 + bt^4 + ct^3 - bt^2 + at - 1 \\ &= (t - \rho)(t + \rho^{-1})(t - \lambda)(t + \lambda^{-1})(t - \bar{\lambda})(t + \bar{\lambda}^{-1}). \end{aligned}$$

We get the following estimates for the coefficients a, b and c.

- Since $\rho < 1.42$, we have $|\rho - \rho^{-1}| < 0.72$. For the coefficient a, we get
$$\begin{aligned}|a| &\leq |\rho - \rho^{-1}| + \left|(\lambda + \bar{\lambda}) - (\lambda^{-1} + \bar{\lambda}^{-1})\right| \\ &< 0.72 + 2|\mathrm{Re}(\lambda) - \mathrm{Re}(\lambda^{-1})| = 0.72 + 2|\mathrm{Re}(\lambda) - \frac{\mathrm{Re}(\lambda)}{|\lambda|^2}| \\ &= 0.72 + 2|\mathrm{Re}(\lambda)|\left(1 - \frac{1}{|\lambda|^2}\right) < 0.72 + 1.44 < 3,\end{aligned}$$
where we used $|\mathrm{Re}(\lambda)| \leq |\lambda| < 1.42$ in the second to last inequality. Up to replacing $P(t)$ by $P(-t)$, we may assume that $a \in \{-2, -1, 0\}$.
- Since $|\lambda| \leq \rho < 1.42$, we have $|\lambda - \lambda^{-1}| < 2.13$ and $|\bar{\lambda} - \bar{\lambda}^{-1}| < 2.13$. We calculate
$$c = -2a + (\rho^{-1} - \rho)(\lambda^{-1} - \lambda)(\bar{\lambda}^{-1} - \bar{\lambda}),$$
where
$$\left|(\rho^{-1} - \rho)(\lambda^{-1} - \lambda)(\bar{\lambda}^{-1} - \bar{\lambda})\right| < 0.72 \cdot (2.13)^2 < 3.2.$$
In particular, $c \in \{-2a - 3, \dots, -2a + 3\}$.
- We have
$$b = -3 + (\rho^{-1} - \rho)(\lambda^{-1} - \lambda + \bar{\lambda}^{-1} - \bar{\lambda}) + (\lambda^{-1} - \lambda)(\bar{\lambda}^{-1} - \bar{\lambda}),$$
where
$$\begin{aligned}&|(\rho^{-1} - \rho)(\lambda^{-1} - \lambda + \bar{\lambda}^{-1} - \bar{\lambda}) + (\lambda^{-1} - \lambda)(\bar{\lambda}^{-1} - \bar{\lambda})| \\ &\qquad\qquad < 0.72 \cdot 1.44 + (2.13)^2 < 5.58.\end{aligned}$$
In particular, $b \in \{-8, \dots, 2\}$.

There are now $3 \cdot 7 \cdot 11 = 231$ remaining polynomials to check. Listing them all as well as their roots, it is a quick check to see which ones among them have a real root with modulus between 1.41 and 1.42; only three polynomials remain. Among these three polynomials, only our candidate $t^6 - t^5 - t - 1$ remains if we insist that the real root with modulus between 1.41 and 1.42 be maximal in modulus among all the roots of the polynomial. □

Remark 3.10. Again we have shown that for $g \geq 3$, the polynomial
$$t^{2g} - t^{g+2} - t^{g-2} - 1$$
is the unique characteristic polynomial that can appear for a matrix minimising the spectral radius. The case $g = 3$ is not covered by McMullen's classification but our ad-hoc argument rules out all other possibilities for characteristic polynomials: while we gave ourselves the liberty to replace $P(t)$ by $P(-t)$ during the proof, we note that the root of $t^6 + t^5 + t - 1$ that is maximal in modulus is real and negative. Therefore, the polynomial $t^6 + t^5 + t - 1$ is not the characteristic polynomial of a primitive matrix.

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