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Kakeya-type sets for geometric maximal operators

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Abstract. We establish an estimate for arbitrary *geometric maximal operators* in the plane: we associate to any family $\mathcal{B}$ composed of rectangles and invariant by translations and central dilations a geometric quantity $\lambda_{\mathcal{B}}$ called its *analytic split* and satisfying

$$\log(\lambda_{\mathcal{B}}) \lesssim_p \|M_{\mathcal{B}}\|_p^p$$

for all $1 < p < \infty$, where $M_{\mathcal{B}}$ is the Hardy-Littlewood type maximal operator associated to the family $\mathcal{B}$.

Contents

1. Introduction

In [3], Bateman classified the behavior of directional maximal operators in the plane on the L^p scale for $1 < p < \infty$. Here, we study geometric maximal operators which are more general than directional maximal operators: in particular, their study requires to focus on the interactions between the coupling eccentricity/orientation for a given family of rectangles. Our main result is the construction of so-called Kakeya-type sets for an arbitrary geometric maximal operator which gives an *a priori* bound on their L^p norm in the same spirit than in [3].

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We work in the Euclidean plane $\mathbb{R}^2$: if U is a measurable subset we denote by $|U|$ its Lebesgue measure. We also denote by $\mathcal{R}$ the family containing all rectangles of $\mathbb{R}^2$: for $R \in \mathcal{R}$, we define its *orientation* as the angle $\omega_R \in [0,\pi)$ that its longest side makes with the x-axis and its *eccentricity* as the ratio $e_R \in (0,1]$ of its shortest side by its longest side. We will also denote by R^t the rectangle R translated in its own direction by its lentgh.

A family $\mathcal{B}$ contained in $\mathcal{R}$ is said to be geometric if it is invariant by translations and central dilations *i.e.* if for any $R \in \mathcal{R}$, any $x \in \mathbb{R}^2$ and $\lambda > 0$, we have

$$x + \lambda R \in \mathcal{B}.$$

Given any geometric family $\mathcal{B}$, we define the associated geometric maximal operator $M_{\mathcal{B}}$ as

$$M_{\mathcal{B}} f(x) := \sup_{x \in R \in \mathcal{B}} \frac{1}{|R|} \int_R |f|$$

for any $f \in L^\infty$ and $x \in \mathbb{R}^2$. We are interested in the relation between the geometry exhibited by the family $\mathcal{B}$ and the regularity of the operator $M_{\mathcal{B}}$ on the L^p space for $1 < p < \infty$.

A lot of research has been done in the case where $\mathcal{B}$ is equal to

$$\mathcal{R}_\Omega := \{R \in \mathcal{R} : \omega_R \in \Omega\}$$

where Ω is an arbitrary set of directions in $[0,\pi)$. In other words, $\mathcal{R}_\Omega$ is the set of all rectangles whose orientation belongs to Ω. We say that $\mathcal{R}_\Omega$ is a directional family and to alleviate the notation we denote

$$M_{\mathcal{R}_\Omega} := M_\Omega.$$

Naturally, the operator M_Ω is said to be a directional maximal operator. The study of those operators goes back at least to Cordoba and Fefferman's article [6] in which they use geometric techniques to show that if $\Omega = \left\{\frac{\pi}{2^k}\right\}_{k \geq 1}$ then M_Ω has weak-type $(2,2)$. A year later, using Fourier analysis techniques, Nagel, Stein and Wainger proved in [8] that M_Ω is actually bounded on $L^p(\mathbb{R}^2)$ for any $p > 1$. In [1], Alfonseca has proved that if the set of direction Ω is a lacunary set of finite order then the operator M_Ω is bounded on $L^p(\mathbb{R}^2)$ for any $p > 1$. Finally in [3], Bateman proved the converse and so characterized the L^p boundedness of directional operators in the plane.

Theorem 1.1 (Bateman). *Fix an arbitrary set of directions $\Omega \subset [0,\pi)$. We have the following alternative:*

- *if Ω is finitely lacunary, then M_Ω is bounded on L^p for any $p > 1$.*
- *if Ω is not finitely lacunary, then M_Ω is not bounded on L^p for any $p < \infty$.*

We invite the reader to look at [3] for more details and also [4] where Bateman and Katz introduced their method.

2. Results

Our main result is an *a priori* estimate in the same spirit than one of the main result of [3]. Precisely, to any family $\mathcal{B}$ contained in $\mathcal{R}$ we associate a geometric quantity

$$\lambda_{\mathcal{B}} \in \mathbb{N} \cup \{\infty\}$$

that we call *analytic split* of $\mathcal{B}$. Loosely speaking, the analytic split $\lambda_{\mathcal{B}}$ indicates if $\mathcal{B}$ contains a lot of rectangles in terms of orientation and eccentricity. We prove then the following Theorem.

Theorem 2.1. *For any geometric family $\mathcal{B}$ and any $1 < p < \infty$ we have*

$$\log(\lambda_{\mathcal{B}}) \lesssim_p \|M_{\mathcal{B}}\|_p^p.$$

An important feature of this inequality is that we do not make any assumption on the family $\mathcal{B}$. In regards of the study of geometric maximal operators, Theorem 2.1 gives a *concrete* and *a priori* lower bound on the $L^p(\mathbb{R}^2)$ norm of $M_{\mathcal{B}}$. We insist on the fact that this estimate is concrete since the analytic split is not an abstract quantity associated to $\mathcal{B}$ but has a strong *geometric interpretation*. No such results was previously known for geometric maximal operators and we give an application in order to illustrate it.

Theorem 2.2. *Fix any set of directions $\Omega \subset [0, \frac{\pi}{4})$ which is not finitely lacunary and let $\mathcal{B} \leq \mathcal{R}_\Omega$ be a geometric family satisfying for any $\omega \in \Omega$*

$$\inf_{R \in \mathcal{B}, \omega_R = \omega} e_R = 0.$$

In this case, the operator $M_{\mathcal{B}}$ is not bounded on L^p for any $p < \infty$.

Observe that since we have $\mathcal{B} \subset \mathcal{R}_\Omega$ we have the trivial pointwise estimate

$$M_{\mathcal{B}} \leq M_\Omega.$$

Hence, we have $\|M_{\mathcal{B}}\|_p < \infty$ if $\|M_\Omega\|_p < \infty$. Surprisingly, Theorem 2.2 states that the conserve is also true *i.e.* we have $\|M_{\mathcal{B}}\|_p = \infty$ if $\|M_\Omega\|_p = \infty$.

3. The family *T*

Given a geometric family $\mathcal{B} \leq \mathcal{R}$, we can always suppose, without loss of generality, that it is of the form

$$\mathcal{B} = \left\{\vec{t} + \lambda R : \vec{t} \in \mathbb{R}^2, \lambda > 0, R \in B\right\}$$

where the family B is contained in the family T defined as

$$T = \{R_n(k) : n \geq 0, 0 \leq k \leq 2^n - 1\}.$$

Here, for $n \geq 1$ and $k \leq 2^n - 1$, $R_n(k)$ is the parallelogram whose vertices are the points $(0,0), (0, \frac{1}{2^n}), (1, \frac{k-1}{2^n})$ and $(1, \frac{k}{2^n})$. The parallelogram $R_n(k)$ should be thought as a rectangle whose eccentricity and orientation are

$$\left(e_{R_n(k)}, \omega_{R_n(k)}\right) \simeq \left(\frac{1}{2^n}, \frac{k}{2^n}\frac{\pi}{4}\right).$$

In the rest of the text, we always identify a geometric family

$$\mathcal{B}, \mathcal{R}_\Omega \text{ or } \mathcal{F} \leq \mathcal{R}$$

with the family that generates it

$$B, T_\Omega \text{ or } F \subset T.$$

The family T has a natural structure of binary tree and we develop a vocabulary adapted to this structure: for any $R \in T$ of scale $n \geq 1$, there exist a unique $R_f \in T$ of scale $n-1$ such that $R \subset R_f$. We say that R_f is the *parent* of R. In the same fashion, observe that there are only two elements $R_h, R_l \in T$ of scale $n+1$ such that $R_h, R_l \subset R$. We say that R_h and R_l are the *children* of R. Observe that $R \in T$ is the child of $R' \in T$ if and only if $R \subset R'$ and $2|R| = |R'|$: we will often use those two conditions. We say that a sequence (finite or infinite) $\{R_i\}_{i \in \mathbb{N}} \subset T$ is a *path* if it satisfies $R_{i+1} \subset R_i$ and $2|R_{i+1}| = |R_i|$ for any i *i.e.* if R_i is the parent of R_{i+1} for any i. Different situations can occur. A finite path P has a first element R and a last element R' (defined in a obvious fashion) and we will write $P_{R,R'} := P$. On the other hand, an infinite path P has no endpoint. For any family B contained in T, there is a unique parallelogram $R \in T$ such that any $R' \in B$ is included in R and $|R|$ is minimal. We say that this element $R_B := R$ is the *root* of B and we define the set $[B]$ as

$$[B] := \left\{R \in T : \exists R' \in B, R' \subset R \subset R_B\right\}.$$

A subset of T of the form $[B]$ is called a *tree* generated by B. We define the set L_B as

$$L_B = \left\{R \in B : \forall R' \in B, R' \subset R \Rightarrow R' = R\right\}.$$

An element of L_B is called a *leaf* of B. Observe that for any B in T we have $[B] = [L_B]$ and also $L_B = L_{[B]}$. The first identity says that the leaves of a tree $[B]$ can be seen as the minimal set that generates $[B]$. The second identity states that $[B]$ is not bigger than B in the sense that it does not have more leaves. If P is an infinite path, we have by definition $L_P = \emptyset$.

4. Analytic split

We associate to any family B included in T a natural number $\lambda_{[B]} \in \mathbb{N} \cup \{\infty\}$ that we call *analytic split*. For any tree $[B]$, we define its *boundary* $\partial[B]$ as the set of path in $[B]$ that are maximal for the inclusion *i.e.* $P \in \partial[B]$ if and only if P is a path included in $[B]$ such that if $P' \subset [B]$ is a path that contains P then $P = P'$. For any tree $[B]$ and path $P \in \partial[B]$ we define the *splitting number of P relatively to* $[B]$ as

$$s_{P,[B]} := \#\left\{R \in [B] \setminus P : \exists R' \in P, R \subset R', 2|R| = |R'|\right\}.$$

We say that a tree $[F]$ is a *fig tree of scale n and height h* when

- $[F]$ is finite and $\#\partial[F] = 2^n$
- for any $P \in \partial[F]$ we have $s_{P,[F]} = n$ and $\#P = h$.

Observe that by construction we always have $h \geq n$. We define the *analytic split* $\lambda_{[B]}$ *of a tree* $[B]$ as the integer n such that $[B]$ contains a fig tree $[F]$ of scale n and do not contains any fig tree of scale $n+1$. In the case where $[B]$ contains fig trees of arbitrary high scale, we set $\lambda_{[B]} = \infty$. More generally for any family B contained in T (*i.e.* when B is not necessarily a tree), we define its analytic split as

$$\lambda_B := \lambda_{[B]}.$$

Hence by definition, the analytic split of a family B is the same as the analytic split of the tree $[B]$. Observe that thanks to Theorem 2.1 this definition is pertinent.

5. Bateman's construction and Kakeya-type set

In [3], Bateman proves the following Theorem.

Theorem 5.1 (Bateman's construction [3])**.** *Suppose that* $[F]$ *is a fig tree of scale* n *and height* h*: there exists a finite family* $\{R_i : i \in I\}$ *included in the geometric family* $\mathcal{F}$ *defined as*

$$\mathcal{F} = \left\{\vec{t} + \lambda R : \vec{t} \in \mathbb{R}^2, \lambda > 0, R \in [F]\right\}$$

such that

$$\log(n)\left|\bigcup_{i\in I} R_i\right| \lesssim \left|\bigcup_{i\in I} R_i^t\right|.$$

If R is a rectangle, we denote by R^t the parallelogram R but shifted of one unit length on the right along its orientation. We fix a 2^h mutually independent random variables

$$R_i : (\Omega, \mathbb{P}) \to L_{[F]}$$

who are uniformly distributed in the set $L_{[F]}$. We consider also the deterministic vectors

$$\left\{\vec{t}_i = (0, \frac{i-1}{2^h}) : i \leq 2^h\right\}$$

is a deterministic vector. Bateman's main result in [3] reads as follow

Theorem 5.2. *We have*

$$\mathbb{P}\left(\log(n)\left|\bigcup_{i\in I} t_i + R_i\right| \lesssim \left|\bigcup_{i\in I} T(t_i + R_i)\right|\right) > 0.$$

The proof of this Theorem involves fine geometric estimates, percolation theory and the use of the so-called notion of *stickiness* of thin tubes of the euclidean plane, see[3] and [4]. Those kind of geometric estimate leads, more generally, to lower bound on maximal operators.

Lemma 5.3. *Fix $N > 0$ such that there exists a finite family $\{R_i : i \in I\}$ included in a geometric family $\mathcal{B}$ such that*

$$N \left| \bigcup_{i \in I} R_i \right| \lesssim \left| \bigcup_{i \in I} R_i^t \right|.$$

In this case, for any $p \in (1, \infty)$, we have

$$N \lesssim_p \|M_{\mathcal{B}}\|_p^p.$$

6. Geometric estimates

We need different geometric estimates in order to prove Theorem 2.1. We start with geometric estimates on $\mathbb{R}$ which will help us to prove geometric estimates on $\mathbb{R}^2$. Finally we prove a geometric estimate on $\mathbb{R}^2$ involving geometric maximal operators that is crucial.

If I is a bounded interval on $\mathbb{R}$ and $\tau > 0$ we denote by τI the interval that has the same center as I and τ times its length *i.e.* $|\tau I| = \tau |I|$. The following lemma can be found in [2].

Lemma 6.1 (Austin's covering lemma)**.** *Let $\{I_\alpha\}_{\alpha \in A}$ a finite family of bounded intervals on $\mathbb{R}$. There is a disjoint subfamily*

$$\{I_{\alpha_k}\}_{k \leq N}$$

such that

$$\bigcup_{\alpha \in A} I_\alpha \subset \bigcup_{k \leq N} 3I_{\alpha_k}$$

We apply Austin's covering lemma to prove two geometric estimates on intervals of the real line. The first one concerns union of dilated intervals.

Lemma 6.2. *Fix $\tau > 0$ and let $\{I_\alpha\}_{\alpha \in A}$ a finite family of bounded intervals on $\mathbb{R}$. We have*

$$\left| \bigcup_{\alpha \in A} I_\alpha \right| \simeq_\tau \left| \bigcup_{\alpha \in A} \tau I_\alpha \right|.$$

Proof. Suppose that $\tau > 1$. We just need to prove that

$$\left| \bigcup_{\alpha \in A} \tau I_\alpha \right| \leq \tau \left| \bigcup_{\alpha \in A} I_\alpha \right|.$$

Simply observe that we have

$$\bigcup_{\alpha \in A} \tau I_\alpha \subset \left\{ M \mathbb{1}_{\cup_{\alpha \in A} I_\alpha} > \frac{1}{\tau} \right\}$$

and apply the one dimensional maximal Theorem. □

Now that we have dealt with union of dilated intervals we consider union of translated intervals.

Lemma 6.3. *Let $\mu > 0$ be a positive constant. For any finite family of intervals $\{I_\alpha\}_{\alpha\in A}$ on $\mathbb{R}$ and any finite family of scalars $\{t_\alpha\}_{\alpha\in A} \subset \mathbb{R}$ such that, for all $\alpha \in A$*

$$|t_\alpha| < \mu \times |I_\alpha|$$

we have

$$\left|\bigcup_{\alpha\in A} I_\alpha\right| \simeq_\mu \left|\bigcup_{\alpha\in A} (t_\alpha + I_\alpha)\right|.$$

Proof. We apply Austin's covering lemma to the family $\{I_\alpha\}_{\alpha\in A}$ which gives a disjoint subfamily $\left\{I_{\alpha_k}\right\}_{k\leq N}$ such that

$$\bigcup_{\alpha\in A} I_\alpha \subset \bigcup_{k\leq N} 3I_{\alpha_k}.$$

In particular we have

$$\left|\bigsqcup_{k\leq N} I_{\alpha_k}\right| \simeq \left|\bigcup_{\alpha\in A} I_\alpha\right|.$$

We consider now the family

$$\left\{(1+\mu)I_{\alpha_k}\right\}_{k\leq N}$$

which is *a priori* not disjoint. We apply again Austin's covering lemma which gives a disjoint subfamily that we will denote $\left\{(1+\mu)I_{\alpha_{k_l}}\right\}_{l\leq M}$ who satisfies

$$\bigcup_{k\leq N}(1+\mu)I_{\alpha_k} \subset \bigcup_{l\leq M} 3(1+\mu)I_{\alpha_{k_l}}.$$

In particular we have

$$\left|\bigsqcup_{l\leq M}(1+\mu)I_{\alpha_{k_l}}\right| \simeq \left|\bigcup_{k\leq N}(1+\mu)I_{\alpha_k}\right|.$$

To conclude, it suffices to observe that for any $\alpha \in A$ we have

$$t_\alpha + I_\alpha \subset (1+\mu)I_\alpha$$

because $|t_\alpha| \leq \mu \times |I_\alpha|$. Hence the family

$$\{t_{\alpha_{k_l}} + I_{\alpha_{k_l}}\}_{l\leq M}$$

is disjoint and so finally

$$\left|\bigsqcup_{l\leq M}\left(t_{\alpha_{k_l}} + I_{\alpha_{k_l}}\right)\right| = \sum_{l\leq M}\left|I_{\alpha_{k_l}}\right| \geq \frac{1}{3(1+\mu)}\left|\bigcup_{l\leq M} 3(1+\mu)I_{\alpha_{k_l}}\right| \simeq_\mu \left|\bigcup_{\alpha\in A} I_\alpha\right|$$

where we have used lemma 6.2 in the last step. □

We denote by $\mathcal{P}$ the family containing all parallelograms $R \subset \mathbb{R}^2$ whose vertices are of the form $(p,a),(p,b),(q,c)$ and (q,d) where $p-q>0$ and $b-a=d-c>0$. We say that $L_R := p-q$ is the *length* of R and that $W_R := b-a$ is the *width* of R. For $R \in \mathcal{P}$ and and a positive ratio $0<\tau<1$ we denote by $\mathcal{P}_{R,\tau}$ the family defined as

$$\mathcal{P}_{R,\tau} := \{S \in \mathcal{P} : S \subset R, L_S = L_R, |S| \geq \tau |R|\}.$$

For $R \in \mathcal{P}$ define the parallelogram $\hat{R} \in \mathcal{P}$ as the parallelogram who has same length, orientation and center than R but is 5 times wider *i.e.* $W_{\hat{R}} = 5W_R$.

Proposition 6.4. *Fix $0<\tau<1$ and any finite family of parallelograms $\{R_i\}_{i\in I} \subset \mathcal{P}$. For each $i \in I$, select an element $S_i \in \mathcal{P}_{\hat{R}_i,\tau}$. The following estimate holds*

$$\left|\bigcup_{i\in I} S_i\right| \geq \frac{\tau}{54}\left|\bigcup_{i\in I} R_i\right|.$$

Proof. Fix $x \in \mathbb{R}$ and for $i \in I$, denote by R_i^x and S_i^x the segments $R_i \cap \{x \times \mathbb{R}\}$ and $S_i \cap \{x \times \mathbb{R}\}$. For any $i \in I$, observe that there is a scalar t_i satisfying $|t_i| \leq \mu \times |R_i|$ with

$$\mu = 5$$

such that

$$t_i + \tau R_i^x \subset S_i^x.$$

Applying lemma 6.3, we then have (since $9 \times (1+\mu) = 54$)

$$\left|\bigcup_{i\in I} S_i^x\right| \geq \left|\bigcup_{i\in I} \left(t_i + \tau R_i^x\right)\right| \geq \frac{1}{54}\left|\bigcup_{i\in I} \tau R_i^x\right|.$$

We conclude using lemma 6.2

$$\frac{1}{54}\left|\bigcup_{i\in I} \tau R_i^x\right| \geq \frac{\tau}{54}\left|\bigcup_{i\in I} R_i^x\right|$$

and integrating on x. □

We state a last geometric estimate involving maximal operator: we fix an arbitrary element $R \in P$ and an element $V \in \mathcal{P}$ included in R such that $L_V = L_u$ and $|V| \leq \frac{1}{2}|R|$. Recall that we denote by R^t the parallelogram R translated in its direction by its length.

Proposition 6.5. *There is a parallelogram $S \in \mathcal{P}_{\hat{R},\frac{1}{4}}$ depending on V such that the following inclusion holds*

$$S \subset \left\{M_V \mathbb{1}_{R^t} > \frac{1}{16}\right\}.$$

Proof. Without loss of generality, we can consider that we have

$$R := [0,1]^2.$$

and that the lower left corner of V is O. The upper left corner of V is the point $(0, W_V)$ and we denote by $(d, 1)$ and $(d + W_V, 1)$ its lower right and upper right corners. Since $V \subset R$ we have

$$d + W_V \leq 1.$$

The upper right corner of $\frac{1}{2}V$ is the point $(\frac{1}{2}(d + W_V), \frac{1}{2})$ and so for any $0 \leq y \leq 1 - \frac{1}{2}(d + W_V)$ we have

$$(0, y) + \frac{1}{2}V \subset R.$$

This yields our inclusion as follow. Let $\vec{t} \in \mathbb{R}^2$ be a vector such that the center of the parallelogram $\tilde{V} = \vec{t} + 2V$ is the point $(1, 0)$. By construction we directly have

$$|\tilde{V} \cap R^t| \geq \frac{1}{16}$$

but moreover for any $0 \leq y \leq \frac{1}{2}$ we have

$$\left|\{(0, y) + \tilde{V}\} \cap R^t\right| \geq \frac{1}{16}$$

since the upper right quarter of $\tilde{V}$ is relatively to R^t in the same position than V relatively to R. Finally, denoting by V^* the parallelogram $\tilde{V} \cap [0,1] \times \mathbb{R}$, the parallelogram S defined as

$$S := \bigcup_{0 \leq y \leq \frac{1}{2}} ((0, y) + V^*)$$

satisfies the condition claimed. This concludes the proof. □

7. Proof of Theorem 2.1

We fix an arbitrary family B contained in T and we prove the following Theorem: combined with Lemma 5.3 it yields Theorem 2.1.

Theorem 7.1. *There exists a finite family $\{R_i : i \in I\}$ included in the geometric family $\mathcal{B}$ defined as*

$$\mathcal{B} = \left\{\vec{t} + \lambda R : \vec{t} \in \mathbb{R}^2, \lambda > 0, R \in B\right\}$$

which satisfies

$$\log(n) \left|\bigcup_{i \in I} R_i\right| \lesssim \left|\bigcup_{i \in I} R_i^t\right|$$

where $n = \lambda_B$.

The family B generates a tree $[B]$: we fix a fig tree $[F] \subset [\mathcal{B}]$ of scale λ_B and we denote by $h \in \mathbb{N}$ its height. We apply Bateman's Theorem to obtain a finite family $\{t_i + R_i : i \leq 2^h\}$ included in

$$\mathcal{F} = \{\vec{t} + \lambda R : \vec{t} \in \mathbb{R}^2, \lambda > 0, R \in [F]\}$$

which satisfies

$$\log(n) \left| \bigcup_{i \in I} R_i \right| \lesssim \left| \bigcup_{i \in I} R_i^t \right|.$$

We take advantage of those elements but this time using elements of B and not elements of $[F]$. Let us define A_1 as

$$A_1 := \bigcup_{i \in I} R_i$$

and similarly let us define A_2 as

$$A_2 := \bigcup_{i \in I} R_i^t$$

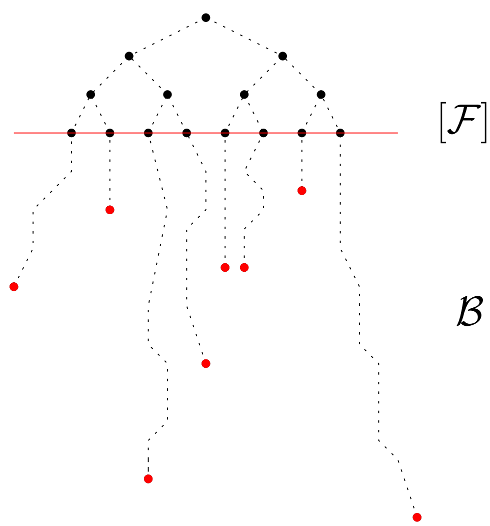

FIGURE 1. Theorem 2.1 shows that we can virtually use the tree $[F]$ for the operator $M_{\mathcal{B}}$ even if B has no structure. On the illustration, B is composed of the red dots which represent rectangles who have very different scale and yet they *interact* at the level of $[F]$.

We apply apply Proposition 6.5: for any $U \in L_{[F]}$ we fix an element V_U of B such that $V_U \subset U$. To each pair (U, V_U) we apply Proposition 6.5 and this gives a parallelogram $S_U \in \mathcal{S}_{\hat{U}, \frac{1}{4}}$ such that

$$S_U \subset \left\{ M_{V_U} \mathbb{1}_{TU} > \frac{1}{16} \right\}.$$

We define then the set B_2 as

$$B_2 := \bigcup_{i\leq 2^h} \vec{t}_i + TS_{R_i}$$

Because $V_U \in B$, we obviously have

$$M_{V_U} \leq M_{\mathcal{B}}$$

and so $S_U \subset \left\{M_{\mathcal{B}}\mathbb{1}_{TU} > \frac{1}{16}\right\}$. We take the union over $i \leq 2^h$ and we obtain

$$B_2 := \bigcup_{i\leq 2^h} \vec{t}_i + TS_{R_i} \subset \left\{M_{\mathcal{B}}\mathbb{1}_{A_1} > \frac{1}{16}\right\}$$

and so finally $|B_2| \leq \left|\left\{M_{\mathcal{B}}\mathbb{1}_{A_1} > \frac{1}{16}\right\}\right|$.

Let us compute $|B_2|$: to do so, we observe that we can use Proposition 6.4 with the families $\{\vec{t}_i + R_i^t : i \leq 2^h\}$ and $\{\vec{t}_i + TS_{R_i} : i \leq 2^h\}$. This yields

$$|B_2| \geq \frac{1}{21\times 4}|A_2|$$

and so we finally have

$$|A_1| \lesssim \frac{1}{\log(n)}\left|\left\{M_{\mathcal{B}}\mathbb{1}_{A_1} > \frac{1}{16}\right\}\right|.$$

This inequality concludes the proof of Theorem 2.1.

8. Proof of Theorem 2.2

Let Ω be a directions in $[0, \frac{\pi}{4})$ which is not finitely lacunary and let $\mathcal{B}$ be a geometric family such that we have $\mathcal{B} \subset \mathcal{R}_\Omega$ and also

$$\sup_{\omega\in\Omega} \inf_{R\in\mathcal{B},\omega_R=\omega} e_R = 0.$$

Let us denote by T_Ω the family included in T such that

$$\mathcal{R}_\Omega = \{\vec{t} + \lambda R : \vec{t} \in \mathbb{R}^2, \lambda > 0, R \in T_\Omega\}.$$

Denote also by B the family included in T that generates $\mathcal{B}$ and observe that our hypothesis implies that we have

$$[B] = T_\Omega$$

and so in particular we have

$$\lambda_B = \lambda_{T_\Omega}.$$

The following claim will concludes the proof.

Claim. *The set of direction Ω is not finitely lacunary if and only if $\lambda_{T_\Omega} = \infty$.*

Applying Theorem 2.1, we obtain for any $1 < p < \infty$

$$\infty = \lambda_{T_\Omega} = \lambda_{[B]} \lesssim \|M_{\mathcal{B}}\|_p^p.$$

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