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A note on Hardy spaces on quadratic CR manifolds

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ABSTRACT. Given a quadratic CR manifold $\mathcal{M}$ embedded in a complex space, and a holomorphic function f on a tubular neighbourhood of $\mathcal{M}$, we show that the L^p-norms of the restrictions of f to the translates of $\mathcal{M}$ is decreasing for the ordering induced by the closed convex envelope of the image of the Levi form of $\mathcal{M}$.

CONTENTS

1. Introduction

Let f be a holomorphic function on the upper half-plane $\mathbb{C}_+ = \mathbb{R} + i\mathbb{R}_+^*$. If f belongs to the Hardy space $H^p(\mathbb{C}_+)$, that is, if $\sup_{y>0} \|f_y\|_{L^p(\mathbb{R})}$ is finite, where $f_y : x \mapsto f(x+iy)$, then it is well known that the function $y \mapsto \|f_y\|_{L^p(\mathbb{R})}$ is decreasing on $\mathbb{R}_+^*$, for every $p \in]0,\infty]$. Nontheless, if f is simply holomorphic, then the lower semicontinuous function $y \mapsto \|f_y\|_{L^p(\mathbb{R})}$ need not be decreasing. Actually, the set where it is finite may be any interval in $\mathbb{R}_+^*$, or even a disconnected set.

Now, replace the upper half-plane $\mathbb{C}_+$ with a Siegel upper half-space

$$D := \{ (\zeta, z) \in \mathbb{C}^n \times \mathbb{C} \colon \operatorname{Im} z - |\zeta|^2 > 0 \},$$

and define

$$f_h \colon \mathbb{C}^n \times \mathbb{R} \ni (\zeta, x) \mapsto f(\zeta, x + i|\zeta|^2 + h)$$

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for every $h > 0$ and for every function f on D. This definition is motivated by the fact that

$$bD := \left\{ (\zeta, x + i|\zeta|^2) : (\zeta, x) \in \mathbb{C}^n \times \mathbb{R} \right\}$$

is the boundary of D, and the sets $bD + (0, ih)$, for $h > 0$, foliate D as the sets $\mathbb{R} + iy$, for $y > 0$, foliate $\mathbb{C}_+$. If f is holomorphic on D, then the mapping $h \mapsto \|f_h\|_{L^p(\mathbb{C}^n \times \mathbb{R})}$ is always decreasing (though not necessarily finite), in contrast to the preceding case (cf. Theorem 3.1). This fact is closely related with the fact that evey holomorphic function defined in a neighbourhood of bD automatically extends to D. More precisely, if one observes that bD has the structure of a CR submanifold of $\mathbb{C}^n \times \mathbb{C}$, one may actually prove that every CR function (of class C^1) is the boundary values of a unique holomorphic function on D (cf. [2, Theorem 1 of Section 15.3]).

In this note, we show that an analogous property holds when bD is replaced by a general quadratic, or quadric, CR submanifold of a complex space, and then discuss some examples of Šilov boundaries of (homogeneous) Siegel domains.

2. Preliminaries

We fix a complex hilbertian space E of dimension n, a real hilbertian space F of dimension m, and a hermitian map $\Phi : E \times E \to F_{\mathbb{C}}$. Define

$$\mathcal{M} := \{ (\zeta, x + i\Phi(\zeta)) : \zeta \in E, x \in F \} = \{ (\zeta, z) \in E \times F_{\mathbb{C}} : \operatorname{Im} z - \Phi(\zeta) = 0 \},$$

where $F_{\mathbb{C}}$ denotes the complexification of F, while $\Phi(\zeta) := \Phi(\zeta, \zeta)$ for every $\zeta \in E$. We define

$$\rho : E \times F_{\mathbb{C}} \ni (\zeta, z) \mapsto \operatorname{Im} z - \Phi(\zeta) \in F.$$

We endow $E \times F_{\mathbb{C}}$ with the product

$$(\zeta, z)(\zeta', z') := (\zeta + \zeta', z + z' + 2i\Phi(\zeta', \zeta))$$

for every $(\zeta, z), (\zeta', z') \in E \times F_{\mathbb{C}}$, so that $E \times F_{\mathbb{C}}$ becomes a 2-step nilpotent Lie group, and $\mathcal{M}$ a closed subgroup of $E \times F_{\mathbb{C}}$. In particular, the identity of $E \times F_{\mathbb{C}}$ is $(0, 0)$ and $(\zeta, z)^{-1} = (-\zeta, -z + 2i\Phi(\zeta))$ for every $(\zeta, z) \in E \times F_{\mathbb{C}}$. It will be convenient to identify $\mathcal{M}$ with the 2-step nilpotent Lie group $\mathcal{N} := E \times F$, endowed with the product

$$(\zeta, x)(\zeta', x') := (\zeta + \zeta', x + x' + 2\operatorname{Im} \Phi(\zeta, \zeta'))$$

for every $(\zeta, x), (\zeta', x') \in \mathcal{N}$, by means of the isomorphism

$$\iota : \mathcal{N} \ni (\zeta, x) \mapsto (\zeta, x + i\Phi(\zeta)) \in E \times F_{\mathbb{C}}.$$

In particular, the identity of $\mathcal{N}$ is $(0, 0)$ and $(\zeta, x)^{-1} = (-\zeta, -x)$ for every $(\zeta, x) \in \mathcal{N}$. Notice that, in this way, $\mathcal{N}$ acts holomorphically (on the left) on $E \times F_{\mathbb{C}}$. Given a function f on $E \times F_{\mathbb{C}}$, we shall define

$$f_h : \mathcal{N} \ni (\zeta, x) \mapsto f(\zeta, x + i\Phi(\zeta) + ih) \in \mathbb{C}$$

for every $h \in F$.

Observe that the preceding group structures show that, if we define the complex tangent space of $\mathcal{M}$ at (ζ, z) as

$$H_{(\zeta,z)}\mathcal{M} := T_{(\zeta,z)}\mathcal{M} \cap (iT_{(\zeta,z)}\mathcal{M})$$

for every $(\zeta, z) \in \mathcal{M}$, where $T_{(\zeta,z)}\mathcal{M}$ denotes the real tangent space to $\mathcal{M}$ at (ζ, z), identified with a subspace of $E \times F_{\mathbb{C}}$, then

$$H_{(\zeta,z)}\mathcal{M} = \mathrm{d}L_{(\zeta,z)} H_{(0,0)}\mathcal{M},$$

where $L_{(\zeta,z)}$ denotes the left translation by (ζ, z) (in $E \times F_{\mathbb{C}}$), and $\mathrm{d}L_{(\zeta,z)}$ its differential at $(0, 0)$. Therefore, $\dim_{\mathbb{C}} H_{(\zeta,z)} = n$ for every $(\zeta, z) \in \mathcal{M}$, so that $\mathcal{M}$ is a CR submanifold of $E \times F_{\mathbb{C}}$ (cf. [2, Chapter 7]), called a quadratic or quadric CR manifold (cf. [2, Section 7.3] and [10, 11]).

We observe explicitly that $\mathcal{M}$ is generic (that is, $\dim_{\mathbb{R}} \mathcal{M} - \dim_{\mathbb{R}} H_{(0,0)}\mathcal{M} = \dim_{\mathbb{R}} E \times F_{\mathbb{C}} - \dim_{\mathbb{R}} \mathcal{M}$, cf. [2, Definition 5 and Lemma 4 of Section 7.1]) and that its Levi form may be canonically identified with Φ (cf. [2, Chapter 10] and [11]).

3. A property of Hardy spaces

We denote by C the convex envelope of $\Phi(E)$.

Theorem 3.1. *Let Ω be an open subset of F such that $\Omega = \Omega + \overline{C}$, and set $D := \rho^{-1}(\Omega)$. Then, for every $f \in \mathrm{Hol}(D)$, for every $p \in]0, \infty]$, for every $h \in \Omega$ and for every $h' \in \overline{C}$,*

$$\|f_{h+h'}\|_{L^p(\mathcal{N})} \leqslant \|f_h\|_{L^p(\mathcal{N})}.$$

The proof is based on the 'anaytic disc technique' presented in [2, Section 15.3].

Observe that the assumption that $\Omega = \Omega + \overline{C}$ is not restrictive. Indeed, if Ω is connected and C has a non-empty interior $\mathrm{Int}\, C$, then every function which is holomorphic on $\rho^{-1}(\Omega)$ extends (uniquely) to a holomorphic function on $\rho^{-1}(\Omega + (\mathrm{Int}\, C \cup \{0\}))$ by [2, Theorem 1 of Section 15.3], and $\Omega + (\mathrm{Int}\, C \cup \{0\}) = \Omega + \overline{C}$ since Ω is open and $\overline{C} = \overline{\mathrm{Int}\, C}$ by convexity. The case in which $\mathrm{Int}\, C = \emptyset$ may be treated directly using similar techniques.

We also mention that, if $p < \infty$ and either Φ is degenerate or the polar of $\Phi(E)$ has an empty interior (that is, the closed convex envelope of $\Phi(E)$ contains a non-trivial vector subspace), then either $f_h = 0$ or $f_h \notin L^p(\mathcal{N})$ (at least for $p \geqslant 1$ when Φ is non-degenerate). Cf. [6] for more details in a similar case.

Proof. For every $\mathbf{v} = (v_j) \in E^m$, consider

$$A_{\mathbf{v}} \colon \mathbb{C} \ni w \mapsto \left(\sum_{j=1}^{m} v_j w^j, i \sum_{j=1}^{m} \Phi(v_j) + 2i \sum_{k<j} \Phi(v_j, v_k) w^{j-k} \right) \in E \times F_{\mathbb{C}},$$

and

$$\Psi(\mathbf{v}) := \sum_{j=1}^{m} \Phi(v_j) \in C,$$

and observe that the following hold:

- $A_{\mathbf{v}}(0) = (0, i\Psi(\mathbf{v}))$;
- $\Psi(E^m)$ is the convex envelope C of $\Phi(E)$, thanks to [12, Corollary 17.1.2];
- $\rho(A_{\mathbf{v}}(w)) = 0$ for every $w \in \mathbb{T}$;
- the mapping $A \colon E^m \ni \mathbf{v} \mapsto A_{\mathbf{v}} \in \mathrm{Hol}(\mathbb{C}; E \times F_{\mathbb{C}})$ is continuous (actually, polynomial).

Now, take $h \in \Omega$. By continuity, there is $\varepsilon > 0$ such that $A_{\mathbf{v}}(\overline{U}) + ih \subseteq D$ for every $\mathbf{v} \in B_{E^m}(0, \varepsilon)$, where U denotes the unit disc in $\mathbb{C}$, and $\overline{U}$ its closure. Then, $A_{\mathbf{v}}(\overline{U}) + ih' \subseteq D$ for every $\mathbf{v} \in B_{E^m}(0, \varepsilon)$ and for every $h' \in h + \overline{C}$. For every $h' \in \Psi(B_{E^m}(0, \varepsilon))$, denote by $\nu_{h'}$ the image of the normalized Haar measure on $\mathbb{T}$ under the mapping $\pi \circ A_{\mathbf{v}}$, for some $\mathbf{v} \in B_{E^m}(0, \varepsilon) \cap \Psi^{-1}(h')$, where $\pi \colon E \times F_{\mathbb{C}} \ni (\zeta, z) \mapsto (\zeta, \mathrm{Re}\, z) \in \mathcal{N}$. Observe that, for every $(\zeta, x) \in \mathcal{N}$ and for every $h'' \in h + \overline{C}$, the mapping

$$\overline{U} \ni w \mapsto f((\zeta, x + i\Phi(\zeta)) \cdot [A_{\mathbf{v}}(w) + (0, ih'')]) \in \mathbb{C}$$

is continuous and holomorphic on U, so that, by subharmonicity (cf., e.g., [13, Theorem 15.19]),

$$\begin{aligned}
&|f(\zeta, x + i\Phi(\zeta) + i(h' + h''))|^{\min(1,p)} \\
&\quad \leqslant \int_{\mathbb{T}} |f((\zeta, x + i\Phi(\zeta)) \cdot [A_{\mathbf{v}}(w) + (0, ih'')])|^{\min(1,p)} \, \mathrm{d}w \\
&\quad = \int_{\mathcal{N}} |f_{h''}((\zeta, x)(\zeta', x'))|^{\min(1,p)} \, \mathrm{d}\nu_{h'}(\zeta', x') \\
&\quad = (|f_{h''}|^{\min(1,p)} * \check{\nu}_{h'})(\zeta, x),
\end{aligned}$$

where $\check{\nu}_{h'}$ denotes the reflection of $\nu_{h'}$, while $\mathbf{v}$ is a suitable element of $B_{E^m}(0, \varepsilon) \cap \Psi^{-1}(h')$. Since $\nu_{h'}$ is a probability measure, by Young's inequality (cf., e.g., [4, Chapter III, § 4, No. 4]) we then infer that

$$\begin{aligned}
\|f_{h'+h''}\|_{L^p(\mathcal{N})} &= \||f_{h'+h''}|^{\min(1,p)}\|_{L^{\max(1,p)}}^{1/\min(1,p)} \\
&\leqslant \||f_{h''}|^{\min(1,p)}\|_{L^{\max(1,p)}(\mathcal{N})}^{1/\min(1,p)} = \|f_{h''}\|_{L^p(\mathcal{N})}
\end{aligned}$$

for every $h' \in \Psi(B_{E^m}(0, \varepsilon))$ and for every $h'' \in h + \overline{C}$. Since every element of C may we written as a finite sum of elements of $\Psi(B_{E^m}(0, \varepsilon))$, the arbitrariness of h'' shows that

$$\|f_{h+h'}\|_{L^p(\mathcal{N})} \leqslant \|f_h\|_{L^p(\mathcal{N})}$$

for every $h' \in C$, hence for every $h' \in \overline{C}$ by lower semi-continuity. The proof is complete. □

Corollary 3.2. *Assume that C has a non-empty interior Ω, and set $D := \rho^{-1}(\Omega)$. Then, for every $p \in]0, \infty]$ and $f \in \mathrm{Hol}(D)$,*

$$\sup_{h \in \Omega} \|f_h\|_{h \in L^p(\mathcal{N})} = \liminf_{h \to 0, h \in \Omega} \|f_h\|_{L^p(\mathcal{N})}.$$

In particular, if we define the Hardy space $H^p(D)$ as the set of $f \in \mathrm{Hol}(D)$ such that $\sup_{h\in\Omega}\|f_h\|_{h\in L^p(\mathcal{N})}$ is finite, the preceding result states that $H^p(D)$ may be equivalently defined as the set of $f \in \mathrm{Hol}(D)$ such that $\liminf\limits_{h\to 0, h\in\Omega}\|f_h\|_{L^p(\mathcal{N})}$ is finite. This result should be compared with [3], where the boundary values of the elements of $H^p(D)$ are characterized as the CR elements of $L^p(\mathcal{N})$, for $p \in [1,\infty]$. In particular, Corollary 3.2 could be deduced from the results of [3], when $p \in [1,\infty]$, though at the expense of some further technicalities.

This result extends [7, Corollary 1.43].

4. Examples

We shall now present some exmples of homogeneous Siegel domains $D = \rho^{-1}(\Omega)$ for which $\overline{\Omega}$ is the closed convex envelope of $\Phi(E)$, so that Corollary 3.2 applies.

We recall that D is said to be a Siegel domain if Ω is an open convex cone not containing affine lines, Φ is non-degenerate, and $\Phi(E) \subseteq \overline{\Omega}$. In addition, D is said to be homogeneous if the group of its biholomorphisms acts transitively on D. It is known (cf., e.g., [5, Proposition 1]) that D is homogeneous if and only if there is a triangular Lie subgroup T_+ of $GL(F)$ which acts simply transitively on Ω, and for every $t \in T_+$ there is $g \in GL(E)$ such that $t\Phi = \Phi(g \times g)$.

If T'_+ is another Lie subgroup of $GL(F)$ with the same properties as T_+, then T_+ and T'_+ are conjugated by an automorphism of F preserving Ω. Thanks to this fact, we may use the results of [7] even if a different T_+ is chosen. In particular, there is a surjective (open and) continuous homomorphism of Lie groups

$$\Delta\colon T_+ \to (\mathbb{R}_+^*)^r$$

for some $r \in \mathbb{N}$, called the rank of Ω, so that

$$\Delta^{\mathbf{s}} = \Delta_1^{s_1}\cdots\Delta_r^{s_r},$$

$\mathbf{s} \in \mathbb{C}^r$, are the characters of T_+. Once a base point $e_\Omega \in \Omega$ has been fixed, $\Delta^{\mathbf{s}}$ induces a function $\Delta_\Omega^{\mathbf{s}}$ on Ω, setting $\Delta_\Omega^{\mathbf{s}}(t(e_\Omega)) = \Delta^{\mathbf{s}}(t)$ for every $t \in T_+$.

Up to modifying Δ, we may then assume that the functions $\Delta_\Omega^{\mathbf{s}}$ are bounded on the bounded subsets of Ω if and only if $\mathrm{Re}\,\mathbf{s} \in \mathbb{R}_+^r$ (cf. [7, Lemma 2.34]). In particular, there is $\mathbf{b} \in \mathbb{R}_-^r$ such that $\Delta^{-\mathbf{b}}(t) = |\det_{\mathbb{C}} g|^2$ for every $t \in T_+$ and for every $g \in GL(E)$ such that $t\Phi = \Phi(g \times g)$ (cf. [7, Lemma 2.9]), and one may prove that $\mathbf{b} \in (\mathbb{R}_-^*)^r$ if and only if $\Phi(E)$ generates F as a vector space, in which case Ω is the interior of the convex envelope of $\Phi(E)$ (cf. [7, Proposition 2.57 and its proof, and Corollary 2.58]). Therefore, we are interested in finding examples of homogeneous Siegel domains for which $\mathbf{b} \in (\mathbb{R}_-^*)^r$.

Notice, in addition, that if $\mathbf{b} \notin (\mathbb{R}_-^*)^r$, then $\Phi(E)$ is contained in a hyperplane, so that the interior of its convex envelope is empty.

The Siegel domain D is said to be symmetric if it is homogeneous and admits an involutive biholomorphism with a unique fixed point (equivalently, if for every $(\zeta, z) \in D$ there is an involutive biholomorphism of D for which (ζ, z) is an isolated (or the unique) fixed point). The domain D is said to be irreducible if it is not biholomorphic to the product of two non-trivial Siegel domains.

It is well known that every symmetric Siegel domain is biholomorphic to a product of irreducible ones, and that the irreducible symmetric Siegel domains can be classified in four infinite families plus two exceptional domains (cf., e.g., [1, §§ 1, 2]). In particular, for an irreducible symmetric Siegel domain, either $\mathbf{b} = \mathbf{0}$ (that is, $E = \{ 0 \}$, in which case D is 'of tube type'), or $\mathbf{b} \in (\mathbb{R}_-^*)^r$ (cf., e.g., [7, Example 2.11]). Hence, when D is a symmetric Siegel domain, $\overline{\Omega}$ is the closed convex envelope of $\Phi(E)$ if and only if none of the irreducible components of D is of tube type. Note that these domains can be also characterized as those which do not admit any non-constant *rational* inner functions, thanks to [8].

We now present some examples of (homogeneous) Siegel domains.

Example 4.1. Let $\mathbb{K}$ be either $\mathbb{C}$ or the division ring of the quaternions. In addition, fix $r, k, p \in \mathbb{N}$ with $p \leqslant r$, and define

- E as the space of $k \times r$ matrices over $\mathbb{K}$ whose j-th columns have zero entries for $j = p + 1, \dots, r$;
- F as the space of self-adjoint $r \times r$ matrices over $\mathbb{K}$;
- Ω as the cone of non-degenerate positive self-adjoint $r \times r$ matrices over $\mathbb{K}$;
- $$\Phi\colon E \times E \ni (\zeta, \zeta') \mapsto \frac{1}{2}[(\zeta'^*\zeta + \zeta^*\zeta') + i(\zeta^* i\zeta' - \zeta'^* i\zeta)] \in F_{\mathbb{C}};$$
- T_+ as the group of upper triangular $r \times r$-matrices over $\mathbb{K}$ with strictly positive diagonal entries, acting on Ω (and F) by the formula $t \cdot h := tht^*$;
- $\Delta\colon T_+ \ni t \mapsto (t_{1,1}, \dots, t_{r,r}) \in (\mathbb{R}_+^*)^r$.

Then, Ω is an irreducible symmetric cone[1] of rank r on which T_+ acts simply transitively by [7, Example 2.6]. In addition, Φ is well defined, since $\zeta'^*\zeta + \zeta^*\zeta', \zeta^* i\zeta' - \zeta'^* i\zeta \in F$ for every $\zeta, \zeta' \in E$, and clearly $\Phi(\zeta) \in \overline{\Omega}$ and

$$t \cdot \Phi(\zeta) = t \cdot (\zeta^*\zeta) = (\zeta t^*)^*(\zeta t^*) = \Phi(\zeta t^*)$$

for every $t \in T_+$ and for every $\zeta \in E$ (with $\zeta t^* \in E$), so that D is homogeneous. Then, $\mathbf{b} = (b_j)$, with $b_j = -k \dim_{\mathbb{C}} \mathbb{K}$ for $j = 1, \dots, p$ and $b_j = 0$ for $j = p + 1, \dots, r$. Consequently, $\overline{\Omega}$ is the closed convex envelope of $\Phi(E)$ if and only if $p = r$ and $k > 0$.

Notice that D is irreducible since Ω is irreducible (cf. [9, Corollary 4.8]), and that D is symmetric if $kp = 0$ or if $p = r$ and $\mathbb{K} = \mathbb{C}$ (cf. [7, Examples 2.14 and 2.15]). If $kp(r - p) > 0$, or if $\mathbb{K} \neq \mathbb{C}$, $r \geqslant 3$, and $k \geqslant 2$, then D cannot be symmetric.

[1] A cone is said to be homogeneous if the group of its linear automorphisms acts transitively on it. It is said to be symmetric if, in addition, it is self-dual for some scalar product. A convex cone is said to be irreducible if it is not isomorphic to a product of non-trivial convex cones.

Example 4.2. Take $k, p, q \in \mathbb{N}$, $p \leqslant 2$. Define:

- E as the space of formal $k \times 2$ matrices whose entries of the first column belong to $\mathbb{C}$ (and are 0 if $p = 0$), and whose entires of the second column belong to $\mathbb{C}^q$ (and are 0 if $p \leqslant 1$);
- F as the space of formally self-adjoint 2×2 matrices whose diagonal entries belong to $\mathbb{R}$, and whose non-diagonal entries belong to $\mathbb{C}^q$;
- Ω as the cone of $\left(\begin{smallmatrix} a & b \\ \overline{b} & c \end{smallmatrix}\right) \in F$ with $a, c > 0$, $b \in \mathbb{C}^q$, and $ac - |b|^2 > 0$;
- Φ so that
$$\Phi\begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_k & b_k \end{pmatrix} = \begin{pmatrix} \sum_j |a_j|^2 & \sum_j \overline{a_j} b_j \\ \sum_j a_j \overline{b_j} & \sum_j |b_j|^2 \end{pmatrix}$$
for every $\left(\begin{smallmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_k & b_k \end{smallmatrix}\right) \in E$;
- T_+ as the group of formal 2×2 upper triangular matrices with diagonal entries in $\mathbb{R}_+^*$ and non-diagonal entries in $\mathbb{C}^q$, with the action[2]
$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ \overline{b}' & c' \end{pmatrix} := \begin{pmatrix} a'a^2 + c'|b|^2 + 2a\mathrm{Re}\,\langle b, b' \rangle & acb' + cc'b \\ ac\overline{b'} + cc'\overline{b} & c^2 c' \end{pmatrix};$$
- $\Delta \colon T_+ \ni t \mapsto (t_{1,1}, t_{2,2})$.

Then, Ω is an irreducible symmetric cone of rank 2 on which T_+ acts simply transitively (cf. [7, Example 2.7]). In addition, $\Phi(\zeta) \in \overline{\Omega}$ for every $\zeta \in E$, and
$$t \cdot \Phi(\zeta) = \Phi(\zeta t^*)$$
for every $t \in T_+$ and $\zeta \in E$ (with $\zeta t^* \in E$), provided that $p \leqslant 1$. Then, D is an irreducible Siegel domain, and it is homogeneous if $p \leqslant 1$ (it is symmetric if $p = 0$). In addition, $\mathbf{b} = \mathbf{0}$ if $p = 0$, while $\mathbf{b} = (k, 0)$ if $p = 1$. Further, if $p = 2$, then $\Phi(E)$ contains the boundary of Ω, since $\left(\begin{smallmatrix} a & b \\ \overline{b} & c \end{smallmatrix}\right) = \Phi\begin{pmatrix} a^{1/2} & a^{-1/2} b \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$, for every $a > 0$, for every $c \geqslant 0$ and for every $b \in \mathbb{C}^q$ such that $|b|^2 = ac$ (the case $a = 0$, $b = 0$, $c \geqslant 0$ is treated similarly). Then, $\overline{\Omega}$ is the closed convex envelope of $\Phi(E)$ if and only if $p = 2$.

References

[1] Arazy, Jonathan. A survey of invariant Hilbert spaces of analytic functions on bounded symmetric domains. *Multivariable operator theory* (Seattle, WA, 1993), 7–65. Contemp. Math., 185, *Amer. Math. Soc., Providence, RI*, 1995. MR1332053, Zbl 0831.46014, doi: 10.1090/conm/185. 1503

[2] Boggess, Albert. CR manifolds and the tangential Cauchy–Riemann complex. Studies in Advanced Mathematics. *CRC Press, Boca Raton, FL*, 1991. xvii+364 pp. ISBN: 0-8493-7152-X. MR1211412, Zbl 0760.32001, doi: 10.1201/9781315140445. 1499, 1500

[3] Boggess, Albert. CR extension for L^p CR functions on a quadric submanifold of $\mathbb{C}^n$. *Pacific J. Math.* **201** (2001), no. 1, 1–18. MR1867889, Zbl 1090.32020, doi: 10.2140/pjm.2001.201.1. 1502

[2]Formally, $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} a' & b' \\ \overline{b}' & c' \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)\left(\begin{smallmatrix} a' & b' \\ \overline{b}' & c' \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)^*$.

[4] Bourbaki, Nicolas. Integration. II. Chapters 7–9. Elements of Mathematics (Berlin). *Springer-Verlag, Berlin*, 2004. vii+326 pp. ISBN: 3-540-20585-3. MR2098271, Zbl 1095.28002, doi: 10.1007/978-3-662-07931-7. 1501

[5] Calzi, Mattia. Besov spaces of analytic type: interpolation, convolution, Fourier multipliers, inclusions. Preprint, 2021. arXiv:2109.09402. 1502

[6] Calzi, Mattia. Paley–Wiener–Schwartz theorems on quadratic CR manifolds. Preprint, 2021. arXiv:2112.07991. 1500

[7] Calzi, Mattia; Peloso, Marco M. Holomorphic function spaces on homogeneous Siegel domains. *Dissertationes Math.* **563** (2021), 168 pp. MR4300902, Zbl 07413827, arXiv:2009.11083, doi: 10.4064/dm833-3-2021. 1502, 1503, 1504

[8] Korányi, Adam; Vági, Stephen. Rational inner functions on bounded symmetric domains. *Trans. Amer. Math. Soc.* **254** (1979), 179–193. MR0539914, Zbl 0439.32006, doi: 10.2307/1998265. 1503

[9] Nakajima, Kazufumi. Some studies on Siegel domains. *J. Math. Soc. Japan* **27** (1975), 54–75. MR0367317, Zbl 0293.32030, doi: 10.2969/jmsj/02710054. 1503

[10] Peloso, Marco M.; Ricci, Fulvio. Tangential Cauchy–Riemann equations on quadratic CR manifolds. Harmonic analysis on complex homogeneous domains and Lie groups (Rome, 2001). *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **13** (2002), no. 3-4, 285–294. MR1984107, Zbl 1225.32037, doi: 10.4171/RLM. 1500

[11] Peloso, Marco M.; Ricci, Fulvio. Analysis of the Kohn Laplacian on quadratic CR manifolds. *J. Funct. Anal.* **203** (2003), no. 2, 321–355. MR2003351, Zbl 1043.32021, doi: 10.1016/S0022-1236(03)00176-9. 1500

[12] Rockafellar, R. Tyrrel. Convex analysis. Princeton Mathematical Series, 28. *Princeton University Press, Princeton, NJ*, 1970. xviii+451 pp. MR0274683, Zbl 0193.18401, doi: 10.1515/9781400873173. 1501

[13] Rudin, Walter. Real and complex analysis. Third edition. *McGraw-Hill Book Co. New York*, 1987. xiv+416 pp. ISBN: 0-07-054234-1. MR0924157, Zbl 0925.00005. 1501

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