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Generalization of the excess area and its geometric interpretation

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ABSTRACT. The image area of the unit disk under $(z \cdot h)(z)$ exceeds the image area under the holomorphic function $h(z)$. In his book, *Hermitian Analysis*, J. D'Angelo precisely determines how this excess image area of the unit disk, Area $((z \cdot h)(\mathbb{D}))$ − Area $(h(\mathbb{D}))$, grows. In our work, we replace the multiplier z with a finite Blaschke product and observe that the excess area growth is a solution for the Dirichlet problem on the unit disk. We replace holomorphic functions with harmonic ones in the formulation and observe a new identity. Furthermore, we show that the excess area growth idea can also be implemented to some other domains conformal to the unit disk.

CONTENTS

1. Introduction

The Lusin's Area Integral formulation states that for h holomorphic and $h' \neq 0$ on $\Omega \subseteq \mathbb{C}$ one can observe $\text{Area}(h(\Omega)) = \int_\Omega \left|h'(z)\right|^2 dA(z)$. It is an essential tool in studying the boundary behaviors of complex maps, see [11, Section 5.1]. If $h : \Omega \to \mathbb{C}$ is m-to-one the above area formula still holds, [5].

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In general, the multiplicity varies from point to point for a complex valued function. However, for a complex analytic function, multiplicity is locally constant, [1]. This property provides the convenience to justify the definition of *area of* $h(\Omega)$ by breaking Ω into subsets on which h has constant multiplicity, [5].

On $\mathbb{D}$, since $|z| < 1$, $|zh(z)| = |z||h(z)| \leq |h(z)|$. However, the area of the image of $\mathbb{D}$ under zh exceeds the area of the image of $\mathbb{D}$ under h, unless h is identically zero, see [5, proofs of Proposition 4.2 and Corollary 4.2]. In fact, for certain holomorphic functions h on $\mathbb{D}$, one can explain and determine the area growth precisely by using the relationship between the L^2-norm of h and the ℓ^2-norm of the Taylor coefficients of h as $\pi \sum_{n=0}^{\infty} |h_n|^2$, or by calculating it through a geometric argument on the unit circle as $\frac{1}{2}\int_0^{2\pi} |h(e^{i\theta})|^2 d\theta$.

Let $L^2(\Omega)$ be the space of square integrable functions on Ω, with inner product $\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(z)\overline{g(z)}dA(z)$. Let $W^1(\Omega)$ (Sobolev space) be the space of $u \in L^2(\Omega)$ such that $\frac{\partial u}{\partial z} \in L^2(\Omega)$. The inner product for $W^1(\Omega)$ is defined as $\langle f, g\rangle_{W^1(\Omega)} = \langle f, g\rangle_{L^2(\Omega)} + \langle \partial f/\partial z, \partial g/\partial z\rangle_{L^2(\Omega)}$, for an elementary introduction to Sobolev spaces see [5, Section 8 and the rest of Chapter 3] or [8, Chapter 6]. Let $\mathcal{O}(\Omega)$ denotes the space of holomorphic functions on Ω, $L^2_a(\Omega)$ be the space of square integrable holomorphic functions on Ω, known as Bergman space on Ω, and $W^1_a(\Omega)$ be the space of $f \in L^2_a(\Omega)$ such that $\frac{\partial f}{\partial z} \in L^2_a(\Omega)$. The subspaces $L^2_a(\Omega)$, $W^1_a(\Omega)$, and $W^1(\Omega)$ are closed in $L^2(\Omega)$, see [4, p. 70-71] or [10, Chapter 14].

For $h : \Omega \to \mathbb{C}$ holomorphic function define $A(h) := \text{Area}(h(\Omega))$, and note that $\text{Area}(h(\Omega)) = \|h'\|^2_{L^2(\Omega)}$ for $h \in W^1_a(\Omega)$.

A twice differentiable function $u(x, y)$ is known as *harmonic* if $\Delta u(x, y) = \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$. They are continuous solutions to the Laplace equation. A twice differentiable function $u(x, y)$ is known as *subharmonic* if $\Delta u(x, y) \geq 0$. For more on *harmonic* and *subharmonic* functions see for example [10, Chapter 7].

A **Finite Blaschke product** is a special function used for manipulating zeros on the unit disk. It is of the form

$$B(z) = e^{i\theta} \prod_{j=1}^{n} (f_{a_j}(z))^{m_j},$$

where m_j is the multiplicity of the zero $a_j \in \mathbb{D}$, $e^{i\theta}$ is a point on the unit circle, and $f_{a_j}(z) = \frac{z-a_j}{1-\overline{a}_j z}$ is known as **Blaschke factor**.

Some properties of a finite Blaschke product and a Blaschke factor are that B, $f_{a_j} \in \mathcal{O}(\overline{\mathbb{D}})$, B and f_{a_j} map $b\mathbb{D}$ to $b\mathbb{D}$, f_{a_j} is a one-to-one, so B is $(\sum_{j=1}^{n} m_j)$-to-one on $\overline{\mathbb{D}}$, and $f^{-1}_{a_j} = f_{-a_j}$. For more on the interesting properties of a finite Blaschke Product and their use see the book [6] and the survey [9].

In the computations of this work we use differential forms notation, $dz = dx + idy$ and $d\overline{z} = dx - idy$. For $h \in C^1(\Omega)$, we write ∂h for $\frac{\partial h}{\partial z}dz$ and $\overline{\partial} h$ for $\frac{\partial h}{\partial \overline{z}}d\overline{z}$. If h is a holomorphic function on Ω, $\overline{\partial} h = 0$ and $dh = (\partial + \overline{\partial})h = \partial h = \frac{\partial h}{\partial z}dz = h'dz$.

The *area form* on the complex plane is

$$dx \wedge dy = \frac{(-1)}{2i}dz \wedge d\overline{z} = \frac{i}{2}dz \wedge d\overline{z}.$$

The area difference between the image of the area of the unit disk under zh and under $h \in W^1_a(\mathbb{D})$, Area $\big(zh(\mathbb{D})\big)$−Area $\big(h(\mathbb{D})\big)$, is geometrically interpreted as the average value of the module-square of h on the unit circle times π, see Theorem 1.1.

When h extends continuously to the circle, let Sh to represent its restriction to the unit circle. Then, for $h \in W^1_a(\mathbb{D})$, we have $h(z) = \sum_{n=0}^{\infty} h_n z^n$ on $\mathbb{D}$ and

$$\|Sh\|^2_{L^2(b\mathbb{D})} = \frac{1}{2\pi}\int_0^{2\pi} |h(e^{i\theta})|^2 d\theta = \frac{1}{2\pi}\sum_{n=0}^{\infty}|h_n|^2\int_0^{2\pi} |e^{in\theta}|^2 d\theta = \sum_{n=0}^{\infty}|h_n|^2.$$

The measure used on the unit circle is the normalized one: $\frac{d\theta}{2\pi}$.

Theorem 1.1 ([5]). *Let Sh be the restriction of h to the unit circle for $h \in W^1_a(\mathbb{D})$. Then Sh is square integrable on the unit circle and*

$$\left\|\frac{\partial(zh)}{\partial z}\right\|^2_{L^2(\mathbb{D})} - \left\|\frac{\partial h}{\partial z}\right\|^2_{L^2(\mathbb{D})} = \pi\|Sh\|^2_{L^2(b\mathbb{D})}, \text{ for } h \in W^1_a(\mathbb{D}). \tag{1}$$

Theorem 1.1 in [5, Chapter 4, Section 2] is proved with two different approaches. One is by using the inner points of the domain. More specifically, by utilizing the relationship between the L^2-norm of h and the ℓ^2-norm of the Taylor coefficients of h. The second approach uses Stokes' Theorem to move the calculations from the inner points of the unit disk to the unit circle. The geometric approach requires $h \in W^1_a(\mathbb{D})$ to be also one time continuously differentiable on the unit circle.

In this paper, we study the concept of excess area growth formulated in Theorem 1.1. In section 2, we present one of the main results Theorem 2.1, precisely determining how the excess area grows when the multiplier z in (1) is replaced with a function f, holomorphic on a neighborhood of the closure of the unit disk, mapping the unit circle to itself. Then, applying the modified excess area growth formulation, we observe that the excess area growth generated with the multiplier function f is comparable to that generated with the function z. Moreover, we include another result, Theorem 2.7, on the unit disk, by keeping the multiplier function as z we replace holomorphic functions $h \in W^1_a(\mathbb{D})$ with harmonic functions $u \in W^1(\mathbb{D})$ in the area difference identity (1). We observe that the excess area growth with holomorphic functions h, $\|Sh\|^2_{L^2(\mathbb{D})}$, is a part of the new identity (14). In section 3, we provide the proof of Theorem 2.1. We manage to use D'Angelo's geometric approach. We also provide an argument about how one can deduce the result to only $h \in W_a^1(\mathbb{D})$ from $h \in W_a^1(\mathbb{D})$ with the extra smoothness requirement on the unit circle, Lemma 3.1. In section 4, we present the proof of Theorem 2.7. In the proof, to show that a harmonic function $u \in W^1(\mathbb{D})$ is a real part of a holomorphic function $h \in W_a^1(\mathbb{D})$ we make use of the observation that if $g \in C^1(\mathbb{D})$ and $\nabla g \in L^2(\mathbb{D})$ then $g \in L^2(\mathbb{D})$, see Lemma 4.1. In Section 5, we move the area difference formulation (1) onto suitable domains conformal to the unit disk. In the last section, we propose a future direction in this context.

2. Main results with some observations

In Theorem 2.1, we precisely determine how the excess area grows with multiplier function f holomorphic on a neighborhood of the closed unit disk that maps the unit circle to the unit circle. Such a function f can always be represented with a Finite Blaschke Product, [6, Lemma 3.2] and [9]. This representation brings convenience into the calculations. Inspired by D'Angelo's work in [5] we use a similar geometric argument to prove our main result; we make use of Stokes' Theorem to move the calculations from $\mathbb{D}$ to $b\mathbb{D}$. However, moving the calculations from the unit disk to the unit circle requires h' to extend continuously to the circle for $h \in W_a^1(\mathbb{D})$. In the proof, we use a standard limiting argument Lemma 3.1 to infer the result for $h \in W_a^1(\mathbb{D})$ from the work when $h \in W_a^1(\mathbb{D})$ is having a continuous first derivative on the unit circle, see section 3.

Theorem 2.1. *For $h \in W_a^1(\mathbb{D})$ and f holomorphic on a neighborhood of $\overline{\mathbb{D}}$ such that $f(b\mathbb{D}) = b\mathbb{D}$*

$$\begin{aligned} A(fh) - A(h) &= \left\| \frac{\partial (fh)}{\partial z} \right\|^2_{L^2(\mathbb{D})} - \left\| \frac{\partial h}{\partial z} \right\|^2_{L^2(\mathbb{D})} \\ &= \pi \sum_{j=1}^{n} m_j \left(\frac{1}{2\pi} \int_0^{2\pi} \left| h\left(f_{-a_j}(e^{i\theta}) \right) \right|^2 d\theta \right) \\ &= \pi \sum_{j=1}^{n} m_j \left\| Sh(f_{-a_j}) \right\|^2_{L^2(b\mathbb{D})} \end{aligned} \tag{2}$$

where a_j are zeros of f in $\mathbb{D}$, m_j is the multiplicity of the zero $a_j \in \mathbb{D}$, and $f_{a_j}(z) = \frac{z-a_j}{1-\overline{a}_j z}$ is the Blaschke factor with zero at a_j.

A simple outcome from Theorem 2.1 is as follows.

Corollary 2.2. *For $h \in W^1_a(\mathbb{D})$ and $0 \leq m \leq n$*

$$A(z^n h) - A(z^m h) = \pi(n-m)\left(\frac{1}{2\pi}\int_0^{2\pi}\left|h(e^{i\theta})\right|^2 d\theta\right)$$
$$= \pi(n-m)\|Sh\|^2_{L^2(b\mathbb{D})}$$

Proof. For $h \in W^1_a(\mathbb{D})$ and $0 \leq m \leq n$ we consider $A(z^n h) - A(z^m h)$ as $A(z^{n-m}(z^m h)) - A(z^m h)$ which, by (2) in Theorem 2.1, is equal to

$$\pi(n-m)\left(\frac{1}{2\pi}\int_0^{2\pi}\left|e^{im\theta}h(e^{i\theta})\right|^2 d\theta\right) = \pi(n-m)\left(\frac{1}{2\pi}\int_0^{2\pi}\left|h(e^{i\theta})\right|^2 d\theta\right)$$
$$= \pi(n-m)\|Sh\|^2_{L^2(b\mathbb{D})}.$$

Note that we use z for $f_0(z)$, that is, z is the Blaschke factor with zero at $a = 0$, and so $e^{i\theta} = f_0(e^{i\theta})$. □

Corollary 2.2 also follows from D'Angelo's Theorem 1.1 by an induction argument.

One observation that comes from Theorem 2.1 is that the excess area growth generated with multiplier function Blaschke factor $f_w(z) = \frac{z-w}{1-\overline{w}z}$ becomes a solution of the Dirichlet problem on the unit disk.

Corollary 2.3. *If $h \in W^1_a(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ and the multiplier function is a Blaschke factor $f_w(z) = \frac{z-w}{1-\overline{w}z}$ then*

$$\phi(w) := A(f_w h) - A(h) = \begin{cases} \frac{1}{2}\int_0^{2\pi}|h(e^{i\theta})|^2\frac{1-|w|^2}{|e^{i\theta}-w|^2}d\theta & \text{if } w \in \mathbb{D} \\ \pi|h(w)|^2 & \text{if } w \in b\mathbb{D}. \end{cases}$$

Thus, $\phi(w)$ is continuous on $\overline{\mathbb{D}}$ and harmonic on $\mathbb{D}$.

Proof. By using Theorem 2.1 with a multiplier $f_w(z) = \frac{z-w}{1-\overline{w}z}$ for $w \in \mathbb{D}$ we have

$$A(f_w h) - A(h) = \pi\|Sh(f_{-w})\|^2_{L^2(b\mathbb{D})} = \pi\frac{1}{2\pi}\int_0^{2\pi}|h(f_{-w}(e^{i\theta}))|^2 d\theta. \quad (3)$$

First, alter the integral in (3) with $\zeta = e^{i\theta}$ and $d\zeta = ie^{i\theta}d\theta$ for $0 \leq \theta \leq 2\pi$ as

$$\|Sh(f_{-w})\|^2_{\mathbb{D}} = \frac{1}{2\pi}\int_0^{2\pi}|h(f_{-w}(e^{i\theta}))|^2 d\theta = \frac{1}{2\pi i}\int_0^{2\pi}\frac{|h(f_{-w}(e^{i\theta}))|^2}{e^{i\theta}}ie^{i\theta}d\theta$$
$$= \frac{1}{2\pi i}\int_{b\mathbb{D}}\frac{|h(f_{-w}(\zeta))|^2}{\zeta}d\zeta. \quad (4)$$

Note that f_w is a (one-to-one) holomorphic map on a neighborhood of $\overline{\mathbb{D}}$ and $f_w(b\mathbb{D}) = b\mathbb{D}$. We make a change of coordinates with $\zeta = f_w(\xi)$ and $d\zeta = \frac{\partial f_w(\xi)}{\partial \xi} d\xi$ in the integral at (4) to obtain

$$\frac{1}{2\pi i} \int\limits_{f_w^{-1}(b\mathbb{D})=b\mathbb{D}} \frac{|h(f_{-w}(f_w(\xi)))|^2}{f_w(\xi)} \frac{\partial f_w(\xi)}{\partial \xi} d\xi. \tag{5}$$

$\frac{\partial f_w(\xi)}{\partial \xi} = \frac{1-|w|^2}{(1-\overline{w}\xi)^2}$, so for $\xi \in b\mathbb{D}$ we have

$$\begin{aligned} \frac{1}{f_w(\xi)} \cdot \frac{\partial f_w(\xi)}{\partial \xi} &= \frac{1-\overline{w}\xi}{\xi - w} \cdot \frac{1-|w|^2}{(1-\overline{w}\xi)^2} = \frac{1}{\xi - w} \cdot \frac{1-|w|^2}{1-\overline{w}\xi} \\ &= \frac{\overline{\xi}}{1-w\overline{\xi}} \cdot \frac{1-|w|^2}{1-\overline{w}\xi} = \overline{\xi} \cdot \frac{1-|w|^2}{|1-w\overline{\xi}|^2} = \overline{\xi} \cdot \frac{1-|w|^2}{|\xi - w|^2}. \end{aligned}$$

Thus the integral at (5) becomes

$$\begin{aligned} \frac{1}{2\pi i} \int\limits_{b\mathbb{D}} |h(\xi)|^2 \overline{\xi} \cdot \frac{1-|w|^2}{|\xi - w|^2} d\xi &= \frac{1}{2\pi i} \int_0^{2\pi} |h(e^{i\theta})|^2 e^{-i\theta} \frac{1-|w|^2}{|e^{i\theta} - w|^2} i e^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} |h(e^{i\theta})|^2 \frac{1-|w|^2}{|e^{i\theta} - w|^2} d\theta. \end{aligned} \tag{6}$$

The last integral at (6) provides another representation for the excess area growth generated with multiplier function Blaschke factor f_w,

$$A(f_w h) - A(h) = \pi \|Sh(f_{-w})\|^2_{\mathbb{D}} = \pi \int_0^{2\pi} |h(e^{i\theta})|^2 \frac{1}{2\pi} \frac{1-|w|^2}{|e^{i\theta} - w|^2} d\theta. \tag{7}$$

The expression

$$\frac{1}{2\pi} \frac{1-|w|^2}{|e^{i\theta} - w|^2}$$

in (7) is known as the **Poisson kernel** for the unit disk. It is used to calculate a harmonic function on the disk from its *boundary values*, that is, from its values on the circle that bounds the disk. Then use the solution of the Dirichlet Problem for the unit disk, [10], to obtain the result. □

Remark 2.4. From Corollary 2.3 we know $\phi(w) := A(f_w h) - A(h)$ is a *harmonic function* on $\mathbb{D}$. Thus, one can use the properties of harmonic functions on the unit disk to interpret the excess area growth generated by a Balschke factor with its zero in the unit disk. For example, by the Mean Value Property for harmonic functions, for $\overline{D(a,r)} \subset \mathbb{D}$, one can say that the excess area value, $\phi(a)$, generated by a Blaschke factor f_a with its zero at the center of a disk $D(a,r)$, deduces from the extra area values, $\phi(a + re^{i\theta})$, generated by Blaschke factors $f_{a+re^{i\theta}}$ with zeros on the circle $bD(a,r)$.

Another observation is a Harnack type inequality relating the excess area values $A(f_\xi h) - A(h)$ generated by a Blaschke factor f_ξ with a zero $\xi \in \mathbb{D}$ at two points, w and 0 in $\mathbb{D}$.

Corollary 2.5. *If $h \in W_a^1(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ and the multiplier function is a Blaschke factor $f_w(z) = \frac{z-w}{1-\overline{w}z}$,*

$$\frac{1-|w|}{1+|w|} \cdot (A(zh) - A(h)) \leq A(f_w h) - A(h) \leq \frac{1+|w|}{1-|w|} \cdot (A(zh) - A(h)) \quad (8)$$

The inequality (8) also works for any two points w and w' in $\mathbb{D}$. Moreover, for f equal to a finite Blaschke product $e^{i\theta} \prod_{j=1}^{n} \left(\frac{z-a_j}{1-\overline{a}_j z} \right)^{m_j}$, where m_j is the multiplicity of the zero $a_j \in \mathbb{D}$ and $e^{i\theta}$ is a point on the unit circle, (8) can be further formulated as

$$\left(\sum_{j=1}^{n} m_j \frac{1-|a_j|}{1+|a_j|} \right) \cdot (A(zh) - A(h)) \leq A(fh) - A(h) \quad (9)$$
$$\leq \left(\sum_{j=1}^{n} m_j \frac{1+|a_j|}{1-|a_j|} \right) \cdot (A(zh) - A(h)).$$

Proof. The inequality at (8) follows from triangle inequality and reversed triangle inequality applied on the Poisson kernel's denominator in the integral at (7).

As for the second inequality (9) with

$$f(z) = e^{i\theta} \prod_{j=1}^{n} \left(f_{-a_j} \right)^{m_j} = e^{i\theta} \prod_{j=1}^{n} \left(\frac{z-a_j}{1-\overline{a}_j z} \right)^{m_j},$$

we use Theorem 2.1, so for $h \in W_a^1(\mathbb{D})$ we have

$$A(fh) - A(h) = \pi \sum_{j=1}^{n} m_j \left\| Sh(f_{-a_j}) \right\|_{L^2(b\mathbb{D})}^2. \quad (10)$$

Combining (10) with (8) will give us (9). That is, rewrite (8) as

$$\frac{1-|a_j|}{1+|a_j|} \cdot \|Sh\|_{L^2(b\mathbb{D})}^2 \leq \left\| Sh(f_{-a_j}) \right\|_{L^2(b\mathbb{D})}^2 \leq \frac{1+|a_j|}{1-|a_j|} \cdot \|Sh\|_{L^2(b\mathbb{D})}^2.$$

Multiply each side of the inequality by m_j and then sum the entire inequality from $j = 1$ to $j = n$ to obtain

$$\left(\sum_{j=1}^{n} m_j \frac{1-|a_j|}{1+|a_j|}\right) \cdot \|Sh\|^2_{L^2(b\mathbb{D})} \leq \sum_{j=1}^{n} m_j \left\|Sh(f_{-a_j})\right\|^2_{L^2(b\mathbb{D})}$$
$$\leq \left(\sum_{j=1}^{n} m_j \frac{1+|a_j|}{1-|a_j|}\right) \cdot \|Sh\|^2_{L^2(b\mathbb{D})},$$

which is,

$$\left(\sum_{j=1}^{n} m_j \frac{1-|a_j|}{1+|a_j|}\right) \cdot (A(zh) - A(h)) \leq (A(fh) - A(h))$$
$$\leq \left(\sum_{j=1}^{n} m_j \frac{1+|a_j|}{1-|a_j|}\right) \cdot (A(zh) - A(h)).$$

□

The following observation is connected to the difference between $\pi|h(\xi)|^2$ and the excess area value, generated by a Blaschke factor f_ξ, being a subharmonic function (for ξ) on the closure of the unit disk.

Corollary 2.6. *Let $h \in W^1_a(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ and $f_a(z) = \frac{z-a}{1-\bar{a}z}$ with $a \in \mathbb{D}$. Then*

$$A(f_a h) - A(h) \geq \pi |h(a)|^2. \tag{11}$$

Moreover, for f holomorphic on a neighborhood of $\overline{\mathbb{D}}$ such that $f(b\mathbb{D}) = b\mathbb{D}$, we have

$$A(fh) - A(h) \geq \pi \sum_{j=1}^{n} m_j |h(a_j)|^2. \tag{12}$$

Proof. By the equation (7) in the proof of Corolarry 2.3 for $w \in \mathbb{D}$ we have

$$A(f_w h) - A(h) = \pi \left(\int_0^{2\pi} |h(e^{i\theta})|^2 \left(\frac{1}{2\pi} \frac{1-|w|^2}{|e^{i\theta} - w|^2} \right) d\theta \right).$$

Moreover, by Corollary 2.3 we know that $\phi(w) := A(f_w h) - A(h)$ is harmonic when $w \in \mathbb{D}$ and $\phi(w) = \pi|h(w)|^2$ continuous when $w \in b\mathbb{D}$. Furthermore, $\pi|h(w)|^2$ is a *subharmonic* function on $\overline{\mathbb{D}}$.

Consider the following function $\psi(w) := \pi|h(w)|^2 - \phi(w)$ for $w \in \overline{\mathbb{D}}$. The function $\psi(w)$ is a subharmonic function on $\overline{\mathbb{D}}$: $|h(w)|^2$ is subharmonic on $\overline{\mathbb{D}}$ and $\phi(w)$ is harmonic on $\mathbb{D}$. For $w \in b\mathbb{D}$ we have $\phi(w) = \pi|h(w)|^2$, so $\psi \equiv 0$ on $b\mathbb{D}$ and

$$\Delta\psi(w) = \pi\Delta|h(w)|^2 - \Delta\phi(w) \geq 0.$$

In the next step, we use the Maximum Value Property for the subharmonic functions: *The maximum value of a subharmonic function is attained only at the boundary of the domain unless the function is constant.* This tells us that the function $\psi(w) \leq 0$ for $w \in \mathbb{D}$, that is,

$$\pi|h(w)|^2 \leq \phi(w), \quad \forall w \in \mathbb{D} \tag{13}$$

which is (11).

Moreover, if f is a holomorphic function on a neighborhood of $\overline{\mathbb{D}}$ such that $f(b\mathbb{D}) = b\mathbb{D}$, that is, f is a finite Blaschke product, see [6, Lemma 3.2] or [9], and then by Theorem 2.1 we obtain (12). □

By replacing holomorphic functions with functions harmonic on $\mathbb{D}$ in the area difference formulation (1), we lose the 'area' meaning, but another interesting identity emerges (14). In the identity (14) we observe the excess area growth with holomorphic functions $\|Sh\|^2_{L^2(\mathbb{D})}$ as a part of the harmonic functions' identity.

Theorem 2.7. *For $u \in W^1(\mathbb{D})$ and harmonic on $\mathbb{D}$, there is $h \in W^1_a(\mathbb{D})$ such that $u = \Re(h)$ and the following identity holds*

$$\begin{aligned}
&\left\|\frac{\partial(zu)}{\partial z}\right\|^2_{L^2(\mathbb{D})} - \left\|\frac{\partial u}{\partial z}\right\|^2_{L^2(\mathbb{D})} \\
&= \frac{1}{4}\left(\pi\left(\frac{1}{2\pi}\int_0^{2\pi} |h(e^{i\theta})|^2 d\theta\right) + 2\pi\Re(h^2(0)) + \|h\|^2_{L^2(\mathbb{D})}\right) \\
&= \frac{1}{4}\left(\pi\|Sh\|^2_{L^2(b\mathbb{D})} + 2\pi\Re(h^2(0)) + \|h\|^2_{L^2(\mathbb{D})}\right).
\end{aligned} \tag{14}$$

3. Proof of Theorem 2.1

The proof of Theorem 2.1 goes on the lines of the argument with a geometric approach for Theorem 1.1 in [5], using Stokes' Theorem to move the computations from the unit disk to the unit circle. The following Lemma 3.1 is a standard analysis argument that allows deducing the result in Theorem 2.1 for $h \in W^1_a(\mathbb{D})$ from calculations for W^1_a-functions h on $\mathbb{D}$ with derivative, h', continuous on the closure of the unit disk.

Lemma 3.1. *Let $h \in W^1(\mathbb{D})$, $h_j(z) := h\left(\frac{z}{1+1/j}\right)$ and S is the restriction operator to the unit circle. Then*

$$\lim_{j\to\infty}\left\|Sh_j - Sh\right\|_{L^2(b\mathbb{D})} = 0.$$

Proof of the Lemma 3.1. Let $h \in W^1_a(\mathbb{D})$, for every $\varepsilon > 0$ there is a large enough disc $D(0,r) \subset \mathbb{D}$ (with $0 < r < 1$) and $j_0 \in \mathbb{N}$ such that

$$\|h\|_{W^1_a(\mathbb{D}\setminus\overline{D(0,r)})} < \frac{\varepsilon}{(1+1/j_0)^2}, \tag{15}$$

which also gives

$$\left\|h_j\right\|_{W_a^1(\mathbb{D}\setminus\overline{D(0,r)})} < \varepsilon \text{ for all } j \geq j_0. \tag{16}$$

Let's show how (16) is obtained.

$$\left\|h_j\right\|^2_{W_a^1(\mathbb{D}\setminus\overline{D(0,r)})} = \int_{\mathbb{D}\setminus\overline{D(0,r)}} \left|h\left(\frac{z}{1+1/j}\right)\right|^2 dxdy + \int_{\mathbb{D}\setminus\overline{D(0,r)}} \left|\frac{\partial}{\partial z}h\left(\frac{z}{1+1/j}\right)\right|^2 dxdy.$$

We do change of coordinates $z = F^{-1}(\zeta) = (1+1/j)\zeta$. Then

$$dxdy = \left|\frac{\partial}{\partial \zeta}F^{-1}(\zeta)\right|^2 d\xi d\eta, \qquad \frac{\partial}{\partial z} = \frac{1}{\frac{\partial}{\partial \zeta}F^{-1}(\zeta)}\frac{\partial}{\partial \zeta},$$

so

$$\begin{aligned}
&= \int_{D\left(0,\frac{1}{1+1/j}\right)\setminus\overline{D\left(0,\frac{r}{1+1/j}\right)}} \left|h(\zeta)\right|^2 (1+1/j)^2 d\xi d\eta \\
&\qquad + \int_{D\left(0,\frac{1}{1+1/j}\right)\setminus\overline{D\left(0,\frac{r}{1+1/j}\right)}} \left|\frac{\partial}{\partial \zeta}h(\zeta)\right|^2 d\xi d\eta \\
&\leq \int_{\mathbb{D}\setminus\overline{D(0,r)}} \left|h(\zeta)\right|^2 (1+1/j)^2 d\xi d\eta + \int_{\mathbb{D}\setminus\overline{D(0,r)}} \left|\frac{\partial}{\partial \zeta}h(\zeta)\right|^2 d\xi d\eta \\
&\leq (1+1/j_0)^2 \|h\|^2_{W_a^1(\mathbb{D}\setminus\overline{D(0,r)})} < \varepsilon.
\end{aligned}$$

To prove the lemma, first, we show that h_j converges to h in W^1-norm on $\mathbb{D}$. In this step we also use (15) and (16). Then we make use of Trace theorem, [8, Theorem 6.47].

Thus, let's show that $\lim_{j\to\infty}\left\|h_j - h\right\|_{W_a^1(\mathbb{D})} = 0$: To make use of (15) and (16) conveniently we split the W^1-norm on $\mathbb{D}$ to W^1-norm on $\mathbb{D}\setminus\overline{D(0,r)}$ and on $\overline{D(0,r)}$.

$$\left\|h_j - h\right\|^2_{W_a^1(\mathbb{D})} = \left\|h_j - h\right\|^2_{W_a^1(\overline{D(0,r)})} + \left\|h_j - h\right\|^2_{W_a^1(\mathbb{D}\setminus\overline{D(0,r)})} \tag{17}$$

The first term on the right hand side of (17) can be made as small as it is needed:

For $z \in \overline{D(0,r)}$ and $j,k \in \mathbb{N}$ let $\gamma_{jk}(t) := \frac{z}{1+j}t + (1-t)\frac{z}{1+k}$ for $0 \leq t \leq 1$, then we have

$$
\begin{aligned}
\left|h_j(z) - h_k(z)\right| &= \left|h\left(\frac{z}{1+1/j}\right) - h\left(\frac{z}{1+1/k}\right)\right| \\
&= \left|h\left(\gamma_{jk}(1)\right) - h\left(\gamma_{jk}(0)\right)\right| = \left|\int_0^1 h'(\gamma_{jk}(t))\gamma'_{jk}(t)dt\right| \\
&\leq C\int_0^1 |\gamma'_{jk}(t)|dt = C\left|\frac{z}{1+1/j} - \frac{z}{1+1/k}\right| \\
&\leq rC\left|\frac{1}{1+1/j} - \frac{1}{1+1/k}\right| \\
&\leq rC\left|\frac{1}{k} - \frac{1}{j}\right| \to 0 \text{ as } k,j \to \infty.
\end{aligned}
$$

Thus, $h_j(z)$ is *uniformly Cauchy* on $\overline{D(0,r)}$ which implies that $h_j(z)$ converges to $h(z)$ uniformly on $\overline{D(0,r)}$.

Note also that, by Cauchy estimate

$$\left|\frac{\partial}{\partial z}h_j(z) - \frac{\partial}{\partial z}h_k(z)\right| \leq \frac{1}{r}\sup_{z\in\overline{D(0,r)}}\left|h_j(z) - h_k(z)\right|.$$

The right hand side goes to 0 as j and k go to ∞, since $\overline{D(0,r)} \subset \mathbb{D}$ is compact, so that h_j converges uniformly on $\overline{D(0,r)}$. Thus $\frac{\partial}{\partial z}h_j$ is also *uniformly Cauchy* (Cauchy in the uniform norm) on $\overline{D(0,r)}$, as it suffices to conclude that $\frac{\partial}{\partial z}h_j(z)$ converges to $\frac{\partial}{\partial z}h(z)$ uniformly on $\overline{D(0,r)}$. Thus, choose J so large that for all $j \geq J$ we have

$$
\begin{aligned}
&\left\|h_j - h\right\|^2_{W^1_a(\overline{D(0,r)})} \qquad (18)\\
&= \int_{\overline{D(0,r)}}\left|h_j(z) - h(z)\right|^2 + \int_{\overline{D(0,r)}}\left|\frac{\partial}{\partial z}h_j(z) - \frac{\partial}{\partial z}h(z)\right|^2 \\
&\leq \pi r^2\varepsilon^2 + \pi r\varepsilon^2
\end{aligned}
$$

The second term on the right had side of (17) can be estimated from above by using the triangle inequality,

$$\leq\left\|h_j - h\right\|^2_{W^1_a(\overline{D(0,r)})} + \left\|h_j\right\|^2_{W^1_a(\mathbb{D}\backslash\overline{D(0,r)})} + \|h\|^2_{W^1_a(\mathbb{D}\backslash\overline{D(0,r)})}. \qquad (19)$$

By the choice of $\overline{D(0,r)}$, the last two norms are dominated by ε^2 and $\frac{\varepsilon^2}{(1+j_0)^4}$, respectively, see (16) and (15). By the choice of J, the first norm is dominated by $\pi r^2\varepsilon^2 + \pi r\varepsilon^2$. Altogether

$$\left\|h_j - h\right\|^2_{W^1_a(\mathbb{D})} \leq \pi r^2\varepsilon^2 + \pi r\varepsilon^2 + \varepsilon^2 + \frac{\varepsilon^2}{(1+j_0)^4},$$

so

$$\left\|h_j - h\right\|_{W^1_a(\mathbb{D})} \to 0 \text{ as } j \to \infty. \tag{20}$$

Now, the restriction operator S maps $W^1(\mathbb{D})$ to $W^{1/2}(b\mathbb{D}) \subset L^2(b\mathbb{D})$, see [8, Theorem 6.47]. Thus, $S : W^1(\mathbb{D}) \to L^2(b\mathbb{D})$ is a bounded operator, that is,

$$\left\|Sh_j\right\|_{L^2(b\mathbb{D})} \leq \left\|h_j\right\|_{W^1(\mathbb{D})} \tag{21}$$

Thus, by (20) and (21) we obtain

$$\lim_{j\to\infty}\left\|Sh_j - Sh\right\|_{L^2(b\mathbb{D})} \leq \lim_{j\to\infty}\left\|h_j - h\right\|_{W^1(\mathbb{D})} = 0.$$

This complies the proof of Lemma 3.1. □

Note that if a function f is complex analytic on a neighborhood of $\overline{\mathbb{D}}$, mapping $b\mathbb{D}$ to $b\mathbb{D}$ and with no zeros on the inner points of the unit disk, then f is a constant. An argument for proving this is as follows: Assume f is not constant, with no zeros on the unit disk. Then, functions f and $1/f$ would be complex analytic on the unit disk. By Maximum Modulus Principle we observe $\left|\frac{1}{f(z)}\right| \leq 1$ and $|f(z)| \leq 1$ on $\mathbb{D}$, which implies that f maps the unit disk to the unit circle. This outcome contradicts the Open Mapping Theorem, so f is constant.

Proof of Theorem 2.1. If f is holomorphic on $\overline{\mathbb{D}}$ and maps $b\mathbb{D}$ to $b\mathbb{D}$, then f can be represented with a Finite Blaschke Product, $f = B$, [6, Lemma 3.2].

For any $h \in W^1_a(\mathbb{D})$ we have

$$A(Bh) - A(h) = \left\|(Bh)'\right\|^2_{L^2(\mathbb{D})} - \left\|h'\right\|^2_{L^2(\mathbb{D})} \tag{22}$$

$$= \left\langle \partial(Bh), \partial(Bh)\right\rangle_{L^2(\mathbb{D})} - \left\langle \partial h, \partial h\right\rangle_{L^2(\mathbb{D})}$$
$$= \frac{i}{2}\int_{\mathbb{D}} \partial(Bh) \wedge \overline{\partial(Bh)} - \frac{i}{2}\int_{\mathbb{D}} \partial h \wedge \overline{\partial h}$$

Note that $\partial(Bh) \wedge \overline{\partial(Bh)} = (\partial + \bar\partial)\left((Bh)\overline{\partial(Bh)}\right) = d\left((Bh)\overline{\partial(Bh)}\right)$. Similarly, $\partial h \wedge \overline{\partial h} = (\partial + \bar\partial)\left(h\overline{\partial h}\right) = d\left(h\overline{\partial h}\right)$. Then, we can write the above integrals are equal to

$$\begin{aligned}\frac{i}{2}\int_{\mathbb{D}} \partial(Bh)\wedge\overline{\partial(Bh)} - \frac{i}{2}\int_{\mathbb{D}} \partial h\wedge\overline{\partial h} =& \frac{i}{2}\int_{\mathbb{D}} d\left((Bh)\overline{\partial(Bh)}\right) - \frac{i}{2}\int_{\mathbb{D}} d\left(h\overline{\partial h}\right)\\ =& \frac{i}{2}\int_{\mathbb{D}} d\left((Bh)\overline{\partial(Bh)} - h\overline{\partial h}\right)\\ =& \frac{i}{2}\int_{\mathbb{D}} d\left(B|h|^2\overline{\partial B}\right) \\ &+ \frac{i}{2}\int_{\mathbb{D}} d((|B|^2-1)h\overline{\partial h}).\end{aligned} \tag{23}$$

Now, we want to use Stokes' Theorem on (23), but $h \in W^1_a(\mathbb{D})$ is not defined on the unit circle, so we will use a standard limiting argument. Let's dilate $h \in W^1_a(\mathbb{D})$ as $h_j(z) := h\left(\frac{z}{1+1/j}\right)$. h_j is holomorphic on a neighborhood of $\overline{\mathbb{D}}$ and so it is smooth on the boundary of the unit disk. From this point on, we will work with h_j instead of h. Then, in the end, we will take the limit of h_j in W^1-norm on $\mathbb{D}$ as j goes to infinity.

Then the two integrals in (23) are modified as

$$\frac{i}{2}\int_{\mathbb{D}} d\left(B|h_j|^2\overline{\partial B}\right) + \frac{i}{2}\int_{\mathbb{D}} d((|B|^2-1)h_j\overline{\partial h_j}). \tag{24}$$

By using Stokes' Theorem we move the integration from the disk $\mathbb{D}$ to the circle $b\mathbb{D}$, so the two integrals in (24) become

$$\frac{i}{2}\int_{b\mathbb{D}} B|h_j|^2\overline{\partial B} + \frac{i}{2}\int_{b\mathbb{D}} (|B|^2-1)h_j\overline{\partial h_j}. \tag{25}$$

The second integral in (25) is vanishing due to $|B(z)| = 1$ for $z \in \{|z|^2 = 1\}$. Thus, we are left only with the first integral

$$\frac{i}{2}\int_{b\mathbb{D}} B(z)|h_j(z)|^2\overline{\partial B(z)}. \tag{26}$$

By induction it suffices to present the case for $B(z) = f_{a_1}(z)\cdot f_{a_2}(z)$. The computation for any finite Blaschke product will be the same. Thus, $B'(z) = f'_{a_1}(z)\cdot f_{a_2}(z) + f_{a_1}(z)\cdot f'_{a_2}(z)$ and the last integral (26) becomes

$$\begin{aligned}&\frac{i}{2}\int_{b\mathbb{D}} B(z)|h_j(z)|^2\overline{B'(z)}d\overline{z}\\ &\qquad = \frac{i}{2}\int_{b\mathbb{D}} f_{a_1}(z)\cdot f_{a_2}(z)|h_j(z)|^2\left(\overline{f'_{a_1}(z)}\cdot\overline{f_{a_2}(z)} + \overline{f_{a_1}(z)}\cdot\overline{f'_{a_2}(z)}\right)d\overline{z}.\end{aligned}$$

At this point, we do a change of coordinates with $w = f_{a_1}(z)$ and obtain

$$= \frac{i}{2} \int_{f_{a_1}(b\mathbb{D})=b\mathbb{D}} f_{a_1}(f_{a_1}^{-1}(w)) \cdot f_{a_2}(f_{a_1}^{-1}(w)) \tag{27}$$

$$\cdot |h_j(f_{a_1}^{-1}(w))|^2 \overline{f'_{a_1}(f_{a_1}^{-1}(w))} \cdot \overline{f_{a_2}(f_{a_1}^{-1}(w))} \overline{\frac{\partial f_{a_1}^{-1}(w)}{\partial w}} d\overline{w}$$

$$+\frac{i}{2} \int_{f_{a_1}(b\mathbb{D})=b\mathbb{D}} f_{a_1}(f_{a_1}^{-1}(w)) \cdot f_{a_2}(f_{a_1}^{-1}(w))$$

$$\cdot |h_j(f_{a_1}^{-1}(w))|^2 \overline{f_{a_1}(f_{a_1}^{-1}(w))} \cdot \overline{f'_{a_2}(f_{a_1}^{-1}(w))} \overline{\frac{\partial f_{a_1}^{-1}(w)}{\partial w}} d\overline{w}.$$

Note that in the first integral at (27) we have the term $\overline{\frac{\partial(f_{a_1}(f_{a_1}^{-1}(w)))}{\partial z}}$ which cancels with $\overline{\frac{\partial f_{a_1}^{-1}(w)}{\partial w}}$, that is,

$$\overline{\frac{\partial(f_{a_1}(f_{a_1}^{-1}(w)))}{\partial z}} \cdot \overline{\frac{\partial f_{a_1}^{-1}(w)}{\partial w}} = \overline{\frac{\partial(f_{a_1}(f_{a_1}^{-1}(w)))}{\partial w}} = \overline{\frac{\partial w}{\partial w}} = 1.$$

After the simplifications in both integrals at (27) we arrive at

$$=\frac{i}{2}\int_{b\mathbb{D}} w \cdot |f_{a_2}(f_{a_1}^{-1}(w))|^2 |h_j(f_{a_1}^{-1}(w))|^2 d\overline{w} \tag{28}$$

$$+ \frac{i}{2}\int_{b\mathbb{D}} w\overline{w} \cdot f_{a_2}(f_{a_1}^{-1}(w)) |h_j(f_{a_1}^{-1}(w))|^2 \overline{f'_{a_2}(f_{a_1}^{-1}(w))} \cdot \overline{\frac{\partial f_{a_1}^{-1}(w)}{\partial w}} d\overline{w}.$$

For further simplification in (28) we observe $|f_{a_2}(f_{a_1}^{-1}(w))|^2 = 1$ for $w \in b\mathbb{D}$. Moreover, the last two factors in the second integral in (28), $\overline{f'_{a_2}(f_{a_1}^{-1}(w))} \cdot \overline{\frac{\partial f_{a_1}^{-1}(w)}{\partial w}}$, resulted from $\overline{\frac{\partial\left(f_{a_2}(f_{a_1}^{-1}(w))\right)}{\partial w}}$ by applying the chain rule:

$$\overline{\frac{\partial\left(f_{a_2} \circ f_{a_1}^{-1}\right)(w)}{\partial w}} = \overline{\frac{\partial\left(f_{a_2}(f_{a_1}^{-1}(w))\right)}{\partial w}} = \overline{\frac{\partial f_{a_2}(f_{a_1}^{-1}(w))}{\partial z}} \cdot \overline{\frac{\partial f_{a_1}^{-1}(w)}{\partial w}}$$

$$= \overline{f'_{a_2}(f_{a_1}^{-1}(w))} \cdot \overline{\frac{\partial f_{a_1}^{-1}(w)}{\partial w}}.$$

Thus, we rewrite (28) as

$$= \frac{i}{2}\int_{b\mathbb{D}} w|h_j(f_{a_1}^{-1}(w))|^2 d\overline{w} \tag{29}$$

$$+ \frac{i}{2}\int_{b\mathbb{D}} f_{a_2}(f_{a_1}^{-1}(w))|h_j(f_{a_1}^{-1}(w))|^2 \overline{\frac{\partial\left(f_{a_2}\circ f_{a_1}^{-1}(w)\right)}{\partial w}} d\overline{w}.$$

The first integral in (29) is equal to $\pi\left\|Sh_j(f_{a_1}^{-1})\right\|_{L^2(b\mathbb{D})}^2$, but the second integral needs more work. Thus, we make another change of coordinates $\zeta = f_{a_2}\circ f_{a_1}^{-1}(w) = f_{a_2}(f_{a_1}^{-1}(w))$ (note that $w = (f_{a_2}\circ f_{a_1}^{-1})^{-1}(\zeta) = f_{a_1}\circ f_{a_2}^{-1}(\zeta) = f_{a_1}(f_{a_2}^{-1}(\zeta)))$ on the second integral in (29) and obtain

$$\frac{i}{2}\int_{f_{a_2}(f_{a_1}^{-1}(b\mathbb{D}))=b\mathbb{D}} f_{a_2}\left(f_{a_1}^{-1}\left(f_{a_1}(f_{a_2}^{-1}(\zeta))\right)\right)\cdot\left|h(f_{a_1}^{-1}\left(f_{a_1}(f_{a_2}^{-1}(\zeta))\right))\right|^2 \tag{30}$$

$$\cdot\overline{\frac{\partial\left(f_{a_2}\left(f_{a_1}^{-1}\left(f_{a_1}(f_{a_2}^{-1}(\zeta))\right)\right)\right)}{\partial w}}\cdot\overline{\frac{\partial\left(f_{a_1}(f_{a_2}^{-1}(\zeta))\right)}{\partial\zeta}}d\overline{\zeta}.$$

Note that in the integral (30) the factor $\overline{\frac{\partial\left(f_{a_2}\left(f_{a_1}^{-1}\left(f_{a_1}(f_{a_2}^{-1}(\zeta))\right)\right)\right)}{\partial w}} = \overline{\frac{\partial\zeta}{\partial w}}$ cancels with the factor $\overline{\frac{\partial\left(f_{a_1}(f_{a_2}^{-1}(\zeta))\right)}{\partial\zeta}} = \overline{\frac{\partial w}{\partial\zeta}}$.

After the simplifications, we obtain that the above integral in (30) is equal to

$$\frac{i}{2}\int_{b\mathbb{D}}\zeta\left|h_j(f_{a_2}^{-1}(\zeta))\right|^2 d\overline{\zeta} = \pi\left\|Sh_j(f_{a_2}^{-1})\right\|_{L^2(b\mathbb{D})}^2 \tag{31}$$

Thus, (29) becomes equal to

$$\pi\left\|Sh_j(f_{a_1}^{-1})\right\|_{L^2(b\mathbb{D})}^2 + \pi\left\|Sh_j(f_{a_2}^{-1})\right\|_{L^2(b\mathbb{D})}^2. \tag{32}$$

Based on the above calculations, if $B(z) = e^{i\theta}\prod_{k=1}^{n}(f_{a_k,\varepsilon}(z))^{m_k}$ then the integral in (26) by induction is equal to

$$\pi\sum_{k=1}^{n} m_j\left\|Sh_j(f_{a_k}^{-1})\right\|_{L^2(b\mathbb{D})}^2. \tag{33}$$

That is, the area difference at (22) is equal to the finite sum at (33), with h_j's which are holomorphic on $\mathbb{D}$ and smooth up to the unit circle. However, by

Lemma 3.1 we conclude that the equality also holds for $h \in W_a^1$, that is,

$$A(Bh) - A(h) = \pi \sum_{k=1}^{n} m_j \left\| Sh(f_{a_k}^{-1}) \right\|_{L^2(b\mathbb{D})}^2 \quad \text{for} \quad h \in W_a^1(\mathbb{D}).$$

□

4. Proof of Theorem 2.7

The calculations in the proof are on the unit disk, which is a simply-connected domain. Thus, we make use of the observation that on a simply connected domain, every harmonic function is the real part of a holomorphic function, [10, Lemma 7.1.2]. Moreover, the following lemma is used in the proof of Theorem 2.7 to obtain $h \in W_a^1(\mathbb{D})$.

Lemma 4.1. *Let $f \in C^1(\mathbb{D})$ and $\nabla f \in L^2(\mathbb{D})$. Then $f \in L^2(\mathbb{D})$.*

Proof. $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

$$\begin{aligned}
\left|\frac{\partial}{\partial r} f(r,\theta)\right| &= \left|\frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\right| \\
&= \left|f_x \cdot \cos(\theta) + f_y \cdot \sin(\theta)\right| \\
&= \left|\nabla_{xy} f(r,\theta) \cdot (\cos(\theta), \sin(\theta))\right| \\
&\leq \left\|\nabla_{xy} f(r,\theta)\right\|
\end{aligned}$$

$\left\|\nabla_{xy} f(r,\theta)\right\|$ represents the *pointwise-norm* of $\nabla_{xy} f(r,\theta)$.

Now, we apply the Fundamental Theorem of Calculus on $f(r,\theta)$ only on its r variable. We have

$$f(r,\theta) = f(0,\theta) + \int_0^r \frac{\partial}{\partial s} f(s,\theta) ds$$

$$\Rightarrow \; |f(r,\theta)| \leq |f(0,\theta)| + \int_0^1 \left\|\nabla_{xy} f(s,\theta)\right\| ds.$$

Then, let's consider the squared-terms

$$\begin{aligned}
|f(r,\theta)|^2 &\leq 2|f(0,\theta)|^2 + 2\left(\int_0^1 \left\|\nabla_{xy} f(s,\theta)\right\| ds\right)^2 \qquad (34) \\
&\leq 2|f(0)|^2 + 2\int_0^1 \left\|\nabla_{xy} f(s,\theta)\right\|^2 ds
\end{aligned}$$

Note that since $f(z)$ is continuous at $z = 0$ we have $|f(0,\theta)| = |f(0)|$. We used Cauchy-Schwartz Inequality on the last integral. We take integral of both sides of the inequality at (34) with respect to r and θ.

$$\int_{\mathbb{D}} |f(r,\theta)|^2 dr d\theta \leq 4\pi |f(0)|^2 + 2\int_0^1 \int_0^1 \int_0^{2\pi} \left\|\nabla_{xy} f(s,\theta)\right\|^2 d\theta ds dr \qquad (35)$$

On the triple integral on the right-hand side, we can drop the outermost integral because the result out of the two-innermost integrals is a constant, and the third integral does not add anything to the outcome.

$$\int_{\mathbb{D}} |f(r,\theta)|^2 dr d\theta \leq 4\pi |f(0)|^2 + 2\int_0^{2\pi}\int_0^1 \left\|\nabla_{xy} f(r,\theta)\right\|^2 dr d\theta. \tag{36}$$

Let's rewrite the inequality (36) in z variable

$$\int_{\mathbb{D}} |f(z)|^2 \frac{1}{|z|} dxdy \leq 4\pi |f(0)|^2 + 2\int_{\mathbb{D}} \left\|\nabla_{xy} f(z)\right\|^2 \frac{1}{|z|} dxdy. \tag{37}$$

Note that we have $|z|$ on the denominator, and it will go under the squared-norm as $\sqrt{|z|}$ and we will have $\frac{f(z)}{\sqrt{|z|}}$ in the integrand as square-integrable. That is,

$$\|f\|^2_{L^2(\mathbb{D})} \leq \left\|\frac{f(z)}{\sqrt{|z|}}\right\|^2_{L^2(\mathbb{D})} \leq 4\pi |f(0)|^2 + 2\left\|\frac{\nabla f(z)}{\sqrt{|z|}}\right\|^2_{L^2(\mathbb{D})}. \tag{38}$$

If f is bounded near 0 and ∇f is bounded near 0 and $\nabla f \in L^2(\mathbb{D})$. For r small

$$\frac{\nabla f(z)}{\sqrt{|z|}} = \underbrace{\frac{\nabla f(z)}{\sqrt{|z|}}\chi_{D(0,r_0)}}_{f_1} + \underbrace{\frac{\nabla f(z)}{\sqrt{|z|}}\chi_{(\mathbb{D}\setminus D(0,r_0))}}_{f_2} = f_1 + f_2 \tag{39}$$

Since f is bounded

$$\begin{aligned}\|f_1\|^2_{L^2(\mathbb{D})} &= \int_{D(0,r_0)} \left|\frac{\nabla f(z)}{\sqrt{|z|}}\chi_{D(0,r_0)}\right|^2 dxdy \\ &\lesssim \int_{D(0,r_0)} \frac{1}{|z|} dxdy = \int_0^{2\pi}\int_0^{r_0} \frac{1}{r} r dr d\theta = 2\pi r_0\end{aligned} \tag{40}$$

and

$$\begin{aligned}\|f_2\|^2_{L^2(\mathbb{D})} &= \int_{\mathbb{D}\setminus D(0,r_0)} \left|\frac{\nabla f(z)}{\sqrt{|z|}}\chi_{(\mathbb{D}\setminus D(0,r_0))}\right|^2 dxdy \\ &\leq \frac{1}{r_0}\int_{\mathbb{D}} |\nabla f(z)|^2 dxdy\end{aligned} \tag{41}$$

since $\nabla f(z) \in L^2(\mathbb{D})$. □

Remark 4.2. If (34) is integrated with respect to θ between 0 and 2π

$$\int_0^{2\pi} |f(r,\theta)|^2 d\theta \leq 4\pi |f(0)|^2 + 2\int_0^{2\pi}\int_0^1 \left\|\nabla_{xy} f(s,\theta)\right\|^2 ds d\theta \tag{42}$$

by (39) (with (40) and (41)) the right hand side of (42) is just a constant. That is, $\int_0^{2\pi} |f(r,\theta)|^2 d\theta$ is uniformly bounded, or equivalently, all averages of $|f|^2$ on circles centered at 0 are uniformly bounded.

Remark 4.3. Lemma 4.1 works without requiring harmonicity due to the simplicity of its domain and dimension. The result in Lemma 4.1 is a particular case of [3, (3.1)], which essentially comes from the book by Kufner ([12]). For more general circumstances, see a version of a Poincare inequality with harmonic functions in [2, Proposition 2.1] or [7, 13].

We present two different proofs for Theorem 2.7. In the first one, we employ Lemma 4.1, Stokes' Theorem, Cauchy Integral Formula, and Cauchy's Theorem; we assume harmonic functions' first-derivative extend continuously to the boundary of the unit disk and refer to the argument in the proof of Theorem 2.1. The second proof is by working on the inner points of the unit disk with series representations. We utilize the series representation of holomorphic functions h.

Proof of Theorem 2.7 by a geometric argument. Since u is harmonic on the unit disk, a simply connected domain in $\mathbb{C}$, there is a harmonic conjugate v, [10, Lemma 7.1.2]. Let $h = u + iv$ be the corresponding holomorphic function. Using Cauchy-Riemann's equations, one can see that L^2-norm of the gradient of v is equal to L^2-norm of the gradient of u, so the gradient of v is in $L^2(\mathbb{D})$. To make the conjugate harmonic function v unique, we assume $v(0) = 0$. Then by using Lemma 4.1 we obtain that $v \in L^2(\mathbb{D})$, and so $h \in W_a^1(\mathbb{D})$.

We start by calculating the differences between the square of the L^2-norms of $\frac{\partial}{\partial z}(zu)$ and $\frac{\partial}{\partial z}u$. In the beginning, we involve a factor of 4 to get rid of the 4 coming from the denominator.

$$4\left\|\frac{\partial(zu)}{\partial z}\right\|^2_{L^2(\mathbb{D})} - 4\left\|\frac{\partial u}{\partial z}\right\|^2_{L^2(\mathbb{D})} =$$

$$= 4\left\|\frac{\partial}{\partial z}\left(z\frac{(h+\overline{h})}{2}\right)\right\|^2_{L^2(\mathbb{D})} - 4\left\|\frac{\partial}{\partial z}\left(\frac{h+\overline{h}}{2}\right)\right\|^2_{L^2(\mathbb{D})}$$

$$\begin{aligned}
&= \left\|\frac{\partial}{\partial z}\left(z(h+\overline{h})\right)\right\|^2_{L^2(\mathbb{D})} - \left\|\frac{\partial}{\partial z}\left(h+\overline{h}\right)\right\|^2_{L^2(\mathbb{D})} \\
&= \left\langle \partial(z(h+\overline{h})), \partial(z(h+\overline{h}))\right\rangle_{L^2(\mathbb{D})} - \left\langle \partial(h+\overline{h}), \partial(h+\overline{h})\right\rangle_{L^2(\mathbb{D})} \\
&= \frac{i}{2}\int_{\mathbb{D}} \partial(z(h+\overline{h})) \wedge \overline{\partial(z(h+\overline{h}))} - \frac{i}{2}\int_{\mathbb{D}} \partial(h+\overline{h}) \wedge \overline{\partial(h+\overline{h})} \qquad (43)
\end{aligned}$$

Note that the form in the first integral at (43) can be rewritten as

$$\begin{aligned}\partial(z(h+\overline{h}))\wedge\overline{\partial(z(h+\overline{h}))} &= (\partial(zh)+\partial(z\overline{h}))\wedge(\overline{\partial(zh)}+\overline{\partial}(\overline{z}h))\\ &= \partial(zh)\wedge\overline{\partial(zh)}+\partial(zh)\wedge\overline{\partial}(\overline{z}h)\\ &\quad+\partial(z\overline{h})\wedge\overline{\partial(zh)}+\partial(z\overline{h})\wedge\overline{\partial}(\overline{z}h)\\ &= (\partial+\overline{\partial})\left((zh)\overline{\partial(zh)}\right)-(\partial+\overline{\partial})\left((\overline{z}h)\partial(zh)\right)\\ &\quad+(\partial+\overline{\partial})\left((z\overline{h})\overline{\partial(zh)}\right)+|h|^2dz\wedge d\overline{z},\end{aligned}$$

so

$$= d\left((zh)\overline{\partial(zh)}\right)-d\left((\overline{z}h)\partial(zh)\right)+d\left((z\overline{h})\overline{\partial(zh)}\right)+|h|^2dz\wedge d\overline{z}.$$

Similarly, the form in the second integral at (43) can be rewritten as

$$\partial(h+\overline{h})\wedge\overline{\partial(h+\overline{h})}=\partial h\wedge\overline{\partial h}=(\partial+\overline{\partial})\left(h\overline{\partial h}\right)=d\left(h\overline{\partial h}\right).$$

Then, the two integrals at (43) can written as

$$\begin{aligned}&=\frac{i}{2}\int_{\mathbb{D}}d\left((zh)\overline{\partial(zh)}\right)-\frac{i}{2}\int_{\mathbb{D}}d\left((\overline{z}h)\partial(zh)\right)+\frac{i}{2}\int_{\mathbb{D}}d\left((z\overline{h})\overline{\partial(zh)}\right)\\ &\qquad+\frac{i}{2}\int_{\mathbb{D}}|h|^2|dz\wedge d\overline{z}-\frac{i}{2}\int_{\mathbb{D}}d\left(h\overline{\partial h}\right).\end{aligned}$$

$$\begin{aligned}=\frac{i}{2}\int_{\mathbb{D}}d\left((zh)\overline{\partial(zh)}-(\overline{z}h)\partial(zh)+(z\overline{h})\overline{\partial(zh)}-h\overline{\partial h}\right)\\ +\frac{i}{2}\int_{\mathbb{D}}|h|^2z\wedge d\overline{z}.\end{aligned}\tag{44}$$

Now, at this point of the proof as in the proof of Theorem 2.1 we want to move the first integral in (44) from the unit disk to the unit circle by Stokes' Theorem, but $h\in W_a^1(\mathbb{D})$, is not defined on the unit circle. We can use the dilation argument, used in the proof of Theorem 2.1, and then at the end of the proof again, and we can refer to Lemma 3.1 to obtain the result for $h\in W_a^1(\mathbb{D})$. Keeping in mind that the same dilation argument can also work in this proof, assuming that the function $h\in W^1(\mathbb{D})$ has its derivative extending continuously to the unit circle suffices to complete the proof.

Next, we use Stoke's Theorem on the first integral of (44); the second integral is the L^2-norm of h on $\mathbb{D}$.

$$=\frac{i}{2}\int_{b\mathbb{D}}\left((zh)\overline{\partial(zh)}-(\overline{z}h)\partial(zh)+(z\overline{h})\overline{\partial(zh)}-h\overline{\partial h}\right)+\|h\|^2_{L^2(\mathbb{D})}.\tag{45}$$

By the product rule, we ohave

$$\partial(zh) = h\partial z + z\partial h = hdz + z\partial h \tag{46}$$

$$\overline{\partial}(\overline{z}h) = h\overline{\partial}\overline{z} = hd\overline{z}. \tag{47}$$

We plug in (46) and (47) into the form in (45) and then simplify, obtaining

$$\begin{aligned}
&(zh)\overline{\partial(zh)} - (\overline{z}h)\partial(zh) + (z\overline{h})\overline{\partial(zh)} - h\overline{\partial h}\\
&= (zh)(\overline{h\partial z} + \overline{z\partial h}) - (\overline{z}h)(h\partial z + z\partial h) + (z\overline{h})(\overline{h\partial z} + \overline{z\partial h}) - h\overline{\partial h}\\
&= z|h|^2\overline{\partial}\overline{z} + |z|^2 h\overline{\partial h} - \overline{z}h^2\partial z - |z|^2 h\partial h + z\overline{h}^2\overline{\partial}\overline{z} + |z|^2\overline{h\partial h} - h\overline{\partial h}\\
&= z|h|^2\overline{\partial}\overline{z} + (|z|^2 - 1)h\overline{\partial h} - \overline{z}h^2\partial z - z\overline{z}h\partial h + z\overline{h}^2\overline{\partial}\overline{z} + |z|^2\overline{h\partial h}.
\end{aligned}$$

The integral term in (45) becomes

$$\begin{aligned}
&= \frac{i}{2}\int_{b\mathbb{D}} z|h|^2\overline{\partial}\overline{z} + \frac{i}{2}\int_{b\mathbb{D}} \overline{h}^2 z\overline{\partial}\overline{z} - \frac{i}{2}\int_{b\mathbb{D}} \overline{z}h^2\partial z - \frac{i}{2}\int_{b\mathbb{D}} z\overline{z}h\partial h + \underbrace{\frac{i}{2}\int_{b\mathbb{D}}(|z|^2-1)h\overline{\partial h}}_{=0 \text{ since on } b\mathbb{D}\ |z|^2=1}\\
&\qquad + \frac{i}{2}\int_{b\mathbb{D}} |z|^2\overline{h\partial h} + \|h\|^2_{L^2(\mathbb{D})}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\int_0^{2\pi} |h(e^{i\theta})|^2 d\theta + \frac{i}{2}\int_{b\mathbb{D}} \overline{h^2(z)}zd\overline{z} - \frac{i}{2}\int_{b\mathbb{D}} h^2(z)\overline{z}dz - \frac{i}{2}\int_{b\mathbb{D}} h\partial h\\
&\qquad + \frac{i}{2}\int_{b\mathbb{D}} \overline{h\partial h} + \|h\|^2_{L^2(\mathbb{D})}.
\end{aligned} \tag{48}$$

Next, we employ Stoke's Theorem for the fourth and fifth integrals at (48) to see that they are equal to zero, one can also use Cauchy's Theorem (since $h \in W^1_a(\mathbb{D})$ we have $h\partial h$ as holomorphic on the disk and continuous on the circle.) Moreover, we use Cauchy's Integral Formula on the second and third integrals at (48):

$$\frac{i}{2}\int_{b\mathbb{D}} \overline{h^2(z)}zd\overline{z} = \frac{i}{2}\int_{b\mathbb{D}} \frac{\overline{h^2(z)}}{\overline{z}}d\overline{z} = \frac{i}{2}\left(\overline{\int_{b\mathbb{D}} \frac{h^2(z)}{z-0}dz}\right) = \frac{i}{2}\left(\overline{2\pi i h^2(0)}\right) = \pi\overline{h^2(0)},$$

$$-\frac{i}{2}\int_{b\mathbb{D}} h^2(z)\overline{z}dz = -\frac{i}{2}\int_{b\mathbb{D}} \frac{h^2(z)}{z-0}dz = -\frac{i}{2}2\pi i h^2(0) = \pi h^2(0).$$

Thus, (48) is equal to

$$= \pi\|Sh\|^2_{b\mathbb{D}} + 2\pi\Re(h^2(0)) + \|h\|^2_{L^2(\mathbb{D})}.$$

□

Proof of Theorem 2.7 by series representation. The proof with series is an elementary calculation.

$$4\left\|\frac{\partial(zu)}{\partial z}\right\|^2_{L^2(\mathbb{D})}-4\left\|\frac{\partial u}{\partial z}\right\|^2_{L^2(\mathbb{D})}=$$
$$=4\left\|\frac{\partial}{\partial z}\left(z\frac{(h+\overline{h})}{2}\right)\right\|^2_{L^2(\mathbb{D})}-4\left\|\frac{\partial}{\partial z}\left(\frac{h+\overline{h}}{2}\right)\right\|^2_{L^2(\mathbb{D})}$$
$$=\left\|\frac{\partial}{\partial z}\left(z(h+\overline{h})\right)\right\|^2_{L^2(\mathbb{D})}-\left\|\frac{\partial}{\partial z}\left(h+\overline{h}\right)\right\|^2_{L^2(\mathbb{D})}. \tag{49}$$

Let's first observe the the following equalities between the L^2-norms of some terms involved in the calculation at (49):

$$\left\|\frac{\partial(z\overline{h})}{\partial z}\right\|^2_{L^2(\mathbb{D})}=\left\langle\sum_{n=0}^{\infty}\overline{h}_n\overline{z}^n,\sum_{n=0}^{\infty}\overline{h}_n\overline{z}^n\right\rangle_{L^2(\mathbb{D})} \tag{50}$$
$$=\sum_{n=0}^{\infty}|h_n|^2\|z^n\|^2_{L^2(\mathbb{D})}=\sum_{n=0}^{\infty}|h_n|^2\frac{\pi}{n+1}=\|h\|^2_{L^2(\mathbb{D})}.$$

$$\left\|\frac{\partial\overline{h}}{\partial z}\right\|^2_{L^2(\mathbb{D})}=0\quad\Rightarrow\quad\left\|\frac{\partial(h+\overline{h})}{\partial z}\right\|^2_{L^2(\mathbb{D})}=\left\|\frac{\partial h}{\partial z}\right\|^2_{L^2(\mathbb{D})}.$$

Let's start by calculating the first term at (49): $\left\|\frac{\partial(z(h+\overline{h}))}{\partial z}\right\|^2_{L^2(\mathbb{D})}=$

$$=\left\langle\sum_{n=0}^{\infty}h_n(n+1)z^n+\sum_{n=0}^{\infty}\overline{h}_n\overline{z}^n,\sum_{n=0}^{\infty}h_n(n+1)z^n+\sum_{n=0}^{\infty}\overline{h}_n\overline{z}^n\right\rangle_{\mathbb{D}}$$
$$=\left\langle\sum_{n=0}^{\infty}h_n(n+1)z^n,\sum_{n=0}^{\infty}h_n(n+1)z^n\right\rangle_{\mathbb{D}}+\left\langle\sum_{n=0}^{\infty}\overline{h}_n\overline{z}^n,\sum_{n=0}^{\infty}\overline{h}_n\overline{z}^n\right\rangle_{\mathbb{D}}$$
$$+\left\langle\sum_{n=0}^{\infty}h_n(n+1)z^n,\sum_{n=0}^{\infty}\overline{h}_n\overline{z}^n\right\rangle_{\mathbb{D}}+\left\langle\sum_{n=0}^{\infty}\overline{h}_n\overline{z}^n,\sum_{n=0}^{\infty}h_n(n+1)z^n\right\rangle_{\mathbb{D}}$$
$$=\left\|\frac{\partial(zh)}{\partial z}\right\|^2_{L^2(\mathbb{D})}+\left\|\frac{\partial(z\overline{h})}{\partial z}\right\|^2_{L^2(\mathbb{D})}+\pi h_0^2+\pi\overline{h_0^2}$$
$$=\left\|\frac{\partial(zh)}{\partial z}\right\|^2_{L^2(\mathbb{D})}+2\pi\Re(h_0^2)+\|h\|^2_{L^2(\mathbb{D})}.$$

On the last equality we used (50).

Thus, the difference of the L^2-norms at (49) becomes

$$\left\|\frac{\partial(z(h+\overline{h}))}{\partial z}\right\|^2_{L^2(\mathbb{D})} - \left\|\frac{\partial(h+\overline{h})}{\partial z}\right\|^2_{L^2(\mathbb{D})} =$$

$$= \left\|\frac{\partial(zh)}{\partial z}\right\|^2_{L^2(\mathbb{D})} - \left\|\frac{\partial h}{\partial z}\right\|^2_{L^2(\mathbb{D})} + 2\pi\Re(h_0^2) + \|h\|^2_{L^2(\mathbb{D})}$$

$$= \pi\sum_{n=0}^{\infty}|h_n|^2 + 2\pi\Re(h_0^2) + \|h\|^2_{L^2(\mathbb{D})}.$$

□

5. Excess area growth on some simply-connected domains

We show that the area difference formulation at (1) is invariant under specific conformal maps. In the hypothesis for $h \in W^1_a(\Omega)$, we require its derivative h' to extend continuously to $b\Omega$. However, Lemma 3.1 can easily be altered for any simply connected bounded domain on the complex plain and can be used to deduce the result of Proposition 5.1 for $h \in W^1_a(\Omega)$ without the initial regularity requirement on $b\Omega$.

Proposition 5.1. *Let $\Omega \subset \mathbb{C}$ bounded domain with C^1-smooth boundary such that there is a conformal map $F : \Omega \to \mathbb{D}$. Suppose $\frac{\partial(Fh)}{\partial z}$ is in $L^2(\Omega)$ for $h \in W^1_a(\Omega)$ such that h' extends continuously to $b\Omega$. Then $S(h\circ F^{-1})$ is square-integrable on $b\mathbb{D}$ and*

$$\left\|\frac{\partial(Fh)}{\partial z}\right\|^2_{L^2(\Omega)} - \left\|\frac{\partial h}{\partial z}\right\|^2_{L^2(\Omega)} = \pi\left\|S(h\circ F^{-1})\right\|^2_{L^2(b\mathbb{D})}.$$

To prove Proposition 5.1 we can not use the series representation approach because the domain of convergence for power series is a disk, and Ω might not be a domain of convergence. In the proof we utilize the same geometric argument that we employed to prove Theorem 2.1.

Painlevé's theorem [11, Theorem 5.2.4], allows the conformal map F to extend as a C^1-function to $b\Omega$. Thus we can move the calculations from the inner points of the domain Ω to its boundaries by the Stokes' Theorem. Then we make use of the fact that $|F(z)| = 1$ for $z \in b\Omega$ to annihilate one of the boundary integrals. Then we do a change of coordinates with the conformal map $w = F(z)$ to move the integral from $b\Omega$ to $b\mathbb{D}$ and formulate it with the operator S.

Proof of Proposition 5.1. If h is holomorphic, then $\overline{\partial}h = 0$ and we have

$$dh = (\partial + \overline{\partial})h = \partial h = h'(z)dz.$$

Thus,

$$\begin{aligned} A(Fh)-A(h) &= \|\partial(Fh)\|^2_{L^2(\Omega)} - \|\partial h\|^2_{L^2(\Omega)} \\ &= \langle \partial(Fh), \partial(Fh)\rangle_{L^2(\Omega)} - \langle \partial h, \partial h\rangle_{L^2(\Omega)} \\ &= \frac{i}{2}\int_\Omega \partial(Fh)\wedge\overline{\partial(Fh)} - \frac{i}{2}\int_\Omega \partial h\wedge\overline{\partial h} \end{aligned}$$

Now, note that $\partial(Fh)\wedge\overline{\partial(Fh)} = (\partial+\bar\partial)\left((Fh)\overline{\partial(Fh)}\right) = d\left((Fh)\overline{\partial(Fh)}\right)$. Similarly, $\partial h\wedge\overline{\partial h} = (\partial+\bar\partial)\left(h\overline{\partial h}\right) = d\left(h\overline{\partial h}\right)$. Then, we can write the above integrals are equal to

$$\begin{aligned} &= \frac{i}{2}\int_\Omega d\left((Fh)\overline{\partial(Fh)}\right) - \frac{i}{2}\int_\Omega d\left(h\overline{\partial h}\right) \\ &= \frac{i}{2}\int_\Omega d\left((Fh)\overline{\partial(Fh)} - h\overline{\partial h}\right). \end{aligned} \tag{51}$$

By the product rule, $\partial(Fh) = h\partial F + F\partial h$. Now, we plug in $h\partial F + F\partial h$ into $(Fh)\overline{\partial(Fh)} - h\overline{\partial h}$ and simplify, getting

$$(Fh)\overline{\partial(Fh)} - h\overline{\partial h} = (Fh)(\overline{h\partial F} + \overline{F\partial h}) - h\overline{\partial h} = F|h|^2\overline{\partial F} + (|F|^2-1)h\overline{\partial h}$$

The integral in (51) becomes

$$= \frac{i}{2}\int_\Omega d\left(F|h|^2\overline{\partial F} + (|F|^2-1)h\overline{\partial h}\right).$$

We use Stokes Theorem:

$$= \frac{i}{2}\int_{b\Omega}\left(F|h|^2\overline{\partial F} + (|F|^2-1)h\overline{\partial h}\right). \tag{52}$$

Then,

$$\begin{aligned} \frac{i}{2}\int_{b\Omega}\left(F|h|^2\overline{\partial F} + (|F|^2-1)h\overline{\partial h}\right) &= \frac{i}{2}\int_{b\Omega} F|h|^2\overline{\partial F} + \underbrace{\frac{i}{2}\int_{b\Omega}(|F|^2-1)h\overline{\partial h}}_{=0 \text{ since on } b\Omega\ |F|^2=1} \\ &= \frac{i}{2}\int_{b\Omega}|h(z)|^2F(z)\overline{F'(z)}d\bar z \end{aligned}$$

We do change of coordinates $z = F^{-1}(w)$ (we move the boundary integral from $b\Omega$ to the unit circle $b\mathbb{D}$). Then,

$$\frac{i}{2}\int_{b\Omega} |h(z)|^2 F(z)\overline{F'(z)}d\overline{z}$$

$$= \frac{i}{2}\int_{b\mathbb{D}=F(b\Omega)} |h(F^{-1}(w))|^2 F(F^{-1}(w))\overline{F'(F^{-1}(w))}\,\overline{\frac{\partial F^{-1}(w)}{\partial w}}dw$$

Note that

$$\frac{\partial F(F^{-1}(w))}{\partial z} = \frac{1}{\frac{\partial F^{-1}(w)}{\partial w}},$$

we use this on the last integral and obtain

$$= \frac{i}{2}\int_{b\mathbb{D}} |h(F^{-1}(w))|^2 w \frac{1}{\overline{\frac{\partial F^{-1}(w)}{\partial w}}}\overline{\frac{\partial F^{-1}(w)}{\partial w}}d\overline{w} = \frac{i}{2}\int_{b\mathbb{D}} \left|h(F^{-1}(w))\right|^2 w d\overline{w}.$$

Now, let's use $w = e^{i\theta}$ on $b\mathbb{D}$, then the last integral becomes

$$= \frac{i}{2}\int_0^{2\pi} e^{i\theta}|h(F^{-1}(e^{i\theta}))|^2(-i)e^{-i\theta}d\theta = \frac{1}{2}\int_0^{2\pi} |h(F^{-1}(e^{i\theta}))|^2 d\theta.$$

Thus we observe that

$$A_{Fh} - A_h = \pi\left\|Sh(F^{-1})\right\|^2_{L^2(b\mathbb{D})}. \tag{53}$$

□

6. A future direction

The higher dimensional analog of the excess area growth idea is developed by J. D'Angelo, see [5, Section 9 of Chapter 4] and the papers he refers to in the references section. It will be interesting to explore the idea in the weighted space of square-integrable entire holomorphic functions whose first derivative is also weighted-square integrable, $W_a^1(\mathbb{C}, e^{-|z|^2})$. The differential operator $D = \frac{\partial}{\partial z}$ and the multiplier operator $M = z$ from Theorem 1.1 play important roles in Physics, in $W_a^1(\mathbb{C}^n, e^{-|z|^2})$ space, [5, Section 12 of Chapter 4]. In the case of $W_a^1(\mathbb{C}, e^{-|z|^2})$, instead of the area of the image of the unit disk, it may be interpreted as the weighted-area of the image of the entire complex plane under a holomorphic function. The relationship between the L^2-norm of f and the ℓ^2-norm of the Taylor coefficients of f can still be useful. Integration by parts will be the main tool instead of Stokes' theorem due to the absence of boundaries in $W_a^1(\mathbb{C}^n, e^{-|z|^2})$ space. Moreover, it will be interesting to see what operator can replace S in the excess area difference.

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References

[1] Ahlfors, Lars V. Complex analysis. An introduction to the theory of analytic functions of one complex variable. Third edition. International Series in Pure and Applied Mathematics. *McGraw-Hill Book Co., New York*, 1978. xi+331 pp. ISBN: 0-07-000657-1. MR510197(90c:30001), Zbl 0395.30001. 1231

[2] Boas, Harold P.; Straube, Emil J. Sobolev norms of harmonic and analytic functions. Unpublished, (1984-1986). 1247

[3] Boas, Harold P.; Straube, Emil J. Integral inequalities of Hardy and Poincaré type. *Proc. Amer. Math. Soc.* **103** (1988), no. 1, 172–176. MR938664(89f:46068), Zbl 0663.46025, doi: 10.2307/2047547. 1247

[4] D'Angelo, John P. Inequalities from complex analysis. Carus Mathematical Monographs, 28. *Mathematical Association of America, Washington, DC*, 2002. xvi+264 pp. ISBN: 0-88385-033-8. MR1899123(2003e:32001), Zbl 0996.30001, doi: 10.5948/UPO9780883859704. 1231

[5] D'Angelo, John P. Hermitian analysis. From Fourier series to Cauchy–Riemann geometry. Second edition. Cornerstones. *Birkhäuser/Springer, Cham*, 2019. x+229 pp. ISBN: 978-3-030-16513-0; 978-3-030-16514-7. MR3931729, Zbl 1428.42001, doi: 10.1007/978-3-030-16514-7. 1230, 1231, 1232, 1233, 1238, 1253

[6] Daepp, Ulrich; Gorkin, Pamela; Shaffer, Andrew; Voss, Karl. Finding ellipses. What Blaschke products, Poncelet's theorem, and the numerical range know about each other. Carus Mathematical Monographs, 34. *MAA Press, Providence, RI*, 2018. xi+268 pp. ISBN: 978-1-4704-4383-2. MR3932079, Zbl 1419.51001, doi: 10.1090/car/034. 1231, 1233, 1238, 1241

[7] Détraz, Jacqueline. Classes de Bergman de fonctions harmoniques. *Bull. Soc. Math. France* **109** (1981), no. 2, 259–268. MR623793(83m:31006), Zbl 0473.31006, doi: 10.24033/bsmf.1941. 1247

[8] Folland, Gerald B. Introduction to partial differential equations. Second edition. *Princeton University Press, Princeton, NJ*, 1995. xii+324 pp. ISBN: 0-691-04361-2. MR1357411(96h:35001), Zbl 0841.35001. 1231, 1239, 1241

[9] Garcia, Stephan Ramon; Mashreghi, Javad; Ross, William T. Finite Blaschke products: a survey. *Harmonic analysis, function theory, operator theory, and their applications*, 133–158, Theta Ser. Adv. Math., 19. *Theta, Bucharest*, 2017. MR3753897, Zbl 1438.30176, arXiv:1512.05444. 1231, 1233, 1238

[10] Greene, Robert E.; Krantz, Steven G. Function theory of one complex variable. Third edition. Graduate Studies in Mathematics, 40. *American Mathematical Society, Providence, RI*, 2006. x+504 pp. ISBN: 0-8218-3962-4. MR2215872(2006m:30001), Zbl 1114.30001, doi: 10.1090/gsm/040. 1231, 1235, 1245, 1247

[11] KRANTZ, STEVEN G. Geometric function theory. Explorations in complex analysis. Cornerstones. *Birkhäuser Boston, Inc., Boston, MA*, 2006. xiv+314 pp. ISBN: 978-0-8176-4339-3; 0-8176-4339-7. MR2167675(2006e:30001), Zbl 1086.30001. 1230, 1251

[12] KUFNER, ALOIS. Weighted Sobolev spaces. A Wiley-Interscience Publication. *John Wiley & Sons, Inc., New York*, 1985. 116 pp. ISBN: 0-471-90367-1. MR802206(86m:46033), Zbl 0567.46009. 1247

[13] LIGOCKA, EWA. The Sobolev spaces of harmonic functions. *Studia Math.* **84**, (1986), no. 1, 79–87. MR871847(88b:46057), Zbl 0627.46033, doi: 10.4064/sm-84-1-79-87. 1247

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