New York Journal of Mathematics

New York J. Math. **28** (2022) 433–444.

Potential density of projective varieties having an int-amplified endomorphism

Jia Jia, Takahiro Shibata and De-Qi Zhang

Abstract. We consider the potential density of rational points on an algebraic variety defined over a number field K, i.e., the property that the set of rational points of X becomes Zariski dense after a finite field extension of K. For a non-uniruled projective variety with an int-amplified endomorphism, we show that it always satisfies potential density. When a rationally connected variety admits an int-amplified endomorphism, we prove that there exists some rational curve with a Zariski dense forward orbit, assuming the Zariski dense orbit conjecture in lower dimensions. As an application, we prove the potential density for projective varieties with int-amplified endomorphisms in dimension ≤ 3. We also study the existence of densely many rational points with the maximal arithmetic degree over a sufficiently large number field.

Contents

1. Introduction

Let K be a number field with a fixed algebraic closure $\overline{K}$. Given a variety X over K, we are interested in the set of K-rational points $X(K)$ of X. More specifically, we study the *potential density* of varieties over K.

Definition 1.1. A variety X defined over a number field K is said to satisfy *potential density* if there is a finite field extension $K \subseteq L$ such that $X_L(L)$ is Zariski dense in X_L, where $X_L := X \times_{\operatorname{Spec} K} \operatorname{Spec} L$.

Received October 14, 2021.

2010 *Mathematics Subject Classification.* 37P55, 14G05, 14E30, 08A35.

Key words and phrases. Potential density, Int-amplified endomorphism, Arithmetic degree, Dynamical degree.

ISSN 1076-9803/2022

The potential density of varieties over number fields has been investigated in several papers. The potential density problem is attractive because the potential density of a variety is pretty much governed by its geometry. See [Cam04] for a conjecture characterising varieties satisfying potential density. However, algebraic varieties for which the potential density is verified are very few. See [Has03] for a survey of studies on the potential density problem.

In this paper, we first study the potential density of varieties admitting int-amplified endomorphisms. For the definition of int-amplified endomorphisms, see 2.1(11). Recently, the equivariant minimal model program for varieties with int-amplified endomorphisms was established (cf. [MZ20]). It has been used to study arithmetic-dynamical problems (cf. [MY19], [MMSZ20]). It turns out that the equivariant minimal model program is also useful for the potential density problem.

Our main conjecture is the following.

Conjecture 1.2 (Potential density under int-amplified endomorphisms)**.** *Let X be a projective variety defined over a number field K. Suppose that X admits an int-amplified endomorphism. Then X satisfies potential density.*

The endomorphism being int-amplified is a crucial assumption in Conjecture 1.2 above. Indeed, consider $X = X_1 \times C$ where X_1 is any smooth projective variety and C is any smooth projective curve of genus at least 2. Such X does not satisfy potential density (cf. Remark 1.4(2)). It does not have any int-amplified endomorphisms either; this is because every surjective endomorphism f of X, after iteration, has the form $(x_1, x_2) \mapsto (g(x_1, x_2), x_2)$ for some morphism $g : X_1 \times C \longrightarrow X_1$ by [San20, Lemma 4.5], and hence descends to the identity map id_C on C via the natural projection $X \longrightarrow C$; thus, the iteration and hence f itself are not int-amplified (cf. [Men20, Lemma 3.7 and Theorem 1.1]).

One might think that Conjecture 1.2 is too strong. In fact, the following even stronger conjecture has already been long outstanding. We refer to Medvedev–Scanlon [MS14, Conjecture 7.13], and Amerik–Bogomolov–Rovinsky [ABR11] for the details.

Conjecture 1.3 (Zariski dense orbit conjecture)**.** *Let X be a variety defined over an algebraically closed field $\mathbf{k}$ of characteristic zero and $f : X \dashrightarrow X$ a dominant rational map. If the f^*-invariant function field $\mathbf{k}(X)^f$ is trivial, that is, $\mathbf{k}(X)^f = \mathbf{k}$, then there exists some $x \in X(\mathbf{k})$ whose (forward) f-orbit $O_f(x) := \{f^n(x) \mid n \geq 0\}$ is well-defined (i.e., f is defined at $f^n(x)$ for any $n \geq 0$) and Zariski dense in X.*

Note that Conjecture 1.3 with f being int-amplified implies Conjecture 1.2 (cf. Lemmas 2.2 and 2.3).

Remark 1.4. We recall some known cases of the potential density problem and Conjecture 1.3.

(1) Unirational varieties and abelian varieties over number fields satisfy potential density (cf. [Has03, Corollary 3.3 and Proposition 4.2]).

(2) Let X be a variety with a dominant rational map $X \dashrightarrow C$ to a curve of genus ≥ 2 over a number field. Then X does not satisfy potential density (cf. [Fal83] and [Has03, Proposition 3.1]).
(3) Conjecture 1.3 holds for any pair (X, f) with X being a curve (cf. [Ame11, Corollary 9]).
(4) Conjecture 1.3 holds for any pair (X, f) with X being a projective surface and f a surjective endomorphism of X (cf. [Xie19], [JXZ20]).

We first prove Conjecture 1.2 for rationally connected varieties in dimension ≤ 3.

Proposition 1.5. *Let X be a rationally connected projective variety over the number field K. Suppose that* $\dim X \leq 3$ *and X admits an int-amplified endomorphism. Then X satisfies potential density.*

Conjecture 1.2 also has a positive answer for non-uniruled varieties in any dimension:

Proposition 1.6. *Let X be a non-uniruled projective variety over the number field K. Suppose that X admits an int-amplified endomorphism. Then X satisfies potential density.*

With the help of Propositions 1.5 and 1.6, we are able to show:

Theorem 1.7. *Let X be a normal projective variety over the number field K with at worst $\mathbb{Q}$-factorial klt singularities. Suppose that* $\dim X \leq 3$ *and X admits an int-amplified endomorphism. Then X satisfies potential density.*

In the last section, we study Question 1.9 below, which is also arithmetic in nature, initiated in [KS14] and further studied in [SS20] and [SS21].

Definition 1.8 (cf. [SS20, Definition 1.4])**.** Let X be a projective variety over a number field K and $f : X \longrightarrow X$ a surjective morphism. We recall the inequality

$$\alpha_f(x) \leq d_1(f)$$

between the arithmetic degree $\alpha_f(x)$ at a point $x \in X(\overline{K})$ and the first dynamical degree $d_1(f)$ of f (cf. 2.1(12) and (13)). Let L be an intermediate field: $K \subseteq L \subseteq \overline{K}$. We say that (X, f) has *densely many L-rational points with the maximal arithmetic degree* if there is a subset $S \subseteq X(L)$ satisfying the following conditions:

(1) S is Zariski dense in X_L;
(2) the equality $\alpha_f(x) = d_1(f)$ holds for all $x \in S$; and
(3) $O_f(x_1) \cap O_f(x_2) = \emptyset$ for any pair of distinct points $x_1, x_2 \in S$.

Following [SS21], we introduce the following notation. We say that (X, f) satisfies $(DR)_L$ if (X, f) has densely many L-rational points with the maximal arithmetic degree. We say that (X, f) satisfies (DR) if there is a finite field extension $K \subseteq L$ $(\subseteq \overline{K})$ such that (X, f) satisfies $(DR)_L$.

Question 1.9. *Let X be a projective variety over K and $f : X \longrightarrow X$ a surjective endomorphism. Assume that X satisfies potential density. Does (X, f) satisfy (DR)?*

Question 1.9 has a positive answer for smooth projective surfaces when $d_1(f) > 1$ (cf. [SS21, Theorem 1.5]). We generalise it to (possibly singular) projective surfaces:

Theorem 1.10. *Let X be a normal projective surface over the number field K satisfying potential density, and $f : X \longrightarrow X$ a surjective morphism with $d_1(f) > 1$. Then (X, f) satisfies (DR).*

The following is an affirmative answer to Question 1.9 for int-amplified endomorphisms on rationally connected threefolds.

Theorem 1.11. *Let X be a rationally connected smooth projective threefold over the number field K and $f : X \longrightarrow X$ an int-amplified endomorphism. Then (X, f) satisfies (DR).*

Acknowledgements. The authors would like to thank the referee for very careful reading and the suggestions to improve the paper. The first, second and third authors are supported, from NUS, by the President's scholarship, a Research Fellowship and an ARF, respectively.

2. Preliminaries

2.1. Notation and terminology

(1) Let K be a number field. We work over K when considering the potential density. We fix an algebraic closure $\overline{K}$ of K.
(2) Let $\mathbf{k}$ be an algebraically closed field of characteristic zero. We work over $\mathbf{k}$ when considering geometric properties.
(3) A *variety* means a geometrically integral separated scheme of finite type over a field.
(4) Let X be a variety over K and $f : X \longrightarrow X$ a morphism (over K). We denote $X_{\overline{K}} := X \times_{\operatorname{Spec} \overline{K}} \operatorname{Spec} K$ and $f_{\overline{K}} : X_{\overline{K}} \longrightarrow X_{\overline{K}}$ the induced morphism (over $\overline{K}$).
(5) The symbol $\sim_{\mathbb{R}}$ denotes the $\mathbb{R}$-linear equivalence on Cartier divisors.
(6) We refer to [KM98] for definitions of $\mathbb{Q}$-factoriality and klt singularities.
(7) A variety X of dimension n is *uniruled* if there is a variety U of dimension $n-1$ and a dominant rational map $\mathbb{P}^1 \times U \dashrightarrow X$.
(8) Let X be a proper variety over a field k. We say that X is *rationally connected* if there is a family of proper algebraic curves $U \longrightarrow Y$ whose geometric fibres are irreducible rational curves with cycle morphism $U \longrightarrow X$ such that $U \times_Y U \longrightarrow X \times X$ is dominant (cf. [Kol96, Chapter IV, Definition 3.2]). When k is algebraically closed of characteristic zero, if X is rationally connected, then any two closed points of X are connected by an irreducible rational curve over k (by applying [Kol96, Chapter IV, Theorem 3.9] to a

resolution of X). The converse holds when k is also uncountable (cf. [Kol96, Chapter IV, Proposition 3.6.2]).

(9) A normal projective variety X is said to be *Q-abelian* if there is a finite surjective morphism $\pi : A \longrightarrow X$, which is étale in codimension 1, with A being an abelian variety.

(10) For a morphism $f : X \longrightarrow X$ and a point $x \in X$, the forward *f-orbit* of x is the set $O_f(x) := \{x, f(x), f^2(x), ...\}$. We denote the Zariski closure of $O_f(x)$ by $Z_f(x)$. More generally, for a closed subset $Y \subseteq X$, we denote $O_f(Y) := \bigcup_{n=0}^{\infty} f^n(Y)$ and its Zariski-closure $Z_f(Y) := \overline{O_f(Y)}$. We say that $O_f(Y)$ is Zariski dense if $Z_f(Y) = X$.

(11) A surjective morphism $f : X \longrightarrow X$ of a projective variety is called *int-amplified* if there exists an ample Cartier divisor H on X such that $f^*H - H$ is ample. In particular, polarised endomorphisms are int-amplified.

(12) Let X be a projective variety and $f : X \longrightarrow X$ a surjective morphism. The *first dynamical degree $d_1(f)$* of f is the limit

$$d_1(f) := \lim_{n\to\infty} ((f^n)^*H \cdot H^{\dim X - 1})^{1/n},$$

where H is an ample Cartier divisor on X. This limit always converges and is independent of the choice of H (cf. [DS05]). Dynamical degrees are invariant under the conjugation by generically finite maps (cf. [Zha09, Lemma 2.6]).

(13) Let X be a projective variety over K and $f : X \longrightarrow X$ a surjective morphism. Fix a (logarithmic) height function $h_H \geq 1$ associated to an ample Cartier divisor H on X. For $x \in X(\overline{K})$, the *arithmetic degree $\alpha_f(x)$* of f at x is the limit

$$\alpha_f(x) := \lim_{n\to\infty} h_H(f^n(x))^{1/n}.$$

This limit always converges and is independent of the choices of H and h_H (cf. [KS16]).

Lemma 2.2. *Let X be a projective variety over $\mathbf{k}$ and $f : X \longrightarrow X$ an int-amplified endomorphism. Then $\mathbf{k}(X)^f = \mathbf{k}$. In particular, if Conjecture 1.3 holds for (X, f), then there exists some $x \in X(\mathbf{k})$ such that $O_f(x)$ is Zariski dense in X.*

Proof. Assume to the contrary that there is a nonconstant rational function $\phi : X \dashrightarrow \mathbb{P}^1$ such that $\phi \circ f = \phi$. Let Γ be the graph of the rational map $\phi : X \dashrightarrow \mathbb{P}^1$ with projections $\pi_1 : \Gamma \longrightarrow X$ being birational and $\pi_2 : \Gamma \longrightarrow \mathbb{P}^1$ being surjective. Then f lifts to an endomorphism $f|_\Gamma$ on Γ such that $\pi_1 \circ f|_\Gamma = f \circ \pi_1$ and $\pi_2 \circ f|_\Gamma = \pi_2$. It follows from [Men20, Lemmas 3.4 and 3.5] that $\mathrm{id} : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is int-amplified, which is absurd. □

Lemma 2.3. *Let X be a projective variety over K, $f : X \longrightarrow X$ a surjective morphism, and $Z \subseteq X$ a subvariety which satisfies potential density (e.g., Z is an abelian variety or unirational; see Remark 1.4(1)). If $O_f(Z)$ is Zariski dense, then X satisfies potential density.*

Proof. Replacing K with a finite extension, we may assume that $Z(K)$ is Zariski dense in Z. Then the union $\bigcup_{n=0}^{\infty} f^n(Z(K))$ is a Zariski dense set of K-rational points of X. □

3. Rationally connected varieties: Proof of Proposition 1.5

Lemma 3.1. *Let X be a rationally connected projective variety over $\mathbf{k}$ and of dimension $d \geq 1$, and $f : X \longrightarrow X$ an int-amplified endomorphism. Assume Conjecture 1.3 in dimension $\leq d-1$. Then there is a rational curve $C \subseteq X$ such that $O_f(C)$ is Zariski dense.*

Proof. If we have a Zariski dense f-orbit $O_f(x)$, take any rational curve C passing through x. Clearly, $O_f(C)$ is Zariski dense. So we may assume that f has no Zariski dense orbit.

Replacing f by some positive power, we can take a point $x \in X(\mathbf{k})$ such that $Z_f(x)$ is irreducible with dimension $r < d$ (cf. e.g. [MMSZ20, Lemma 2.7]). By [Fak03, Theorem 5.1], the subset of $X(\mathbf{k})$ consisting of f-periodic points is Zariski dense in X. Pick an f-periodic point $y \in X(\mathbf{k}) \setminus Z_f(x)$. After iterating f, we may assume that y is an f-fixed point. Take a rational curve $C \subseteq X$ containing x and y. Set $W := Z_f(C)$. If $W = X$, we are done. So we may assume that $W \subsetneq X$. If $\dim W = r$, then W has its irreducible decomposition as $W = Z_f(x) \cup W_1 \cup \cdots \cup W_m$. There is some $n \geq 0$ such that $f^n(x) \in Z_f(x) \setminus \bigcup_{i=1}^m W_i$. Then $f^n(C) \subseteq W$ but $f^n(C) \nsubseteq \bigcup_{i=1}^m W_i$. Hence $f^n(C) \subseteq Z_f(x)$. In particular, $y = f^n(y) \in f^n(C) \subseteq Z_f(x)$, a contradiction. Thus $r < \dim W$ $(< \dim X = d)$.

Now there exists an f-periodic irreducible component $W' \subseteq W$ with $r < \dim W' < d$. Replacing f by a positive power, we may assume that W' is f-invariant. Then $f|_{W'}$ is an int-amplified endomorphism on W' (cf. [Men20, Lemma 2.2]). By assumption, Conjecture 1.3 holds for $(W', f|_{W'})$. So there exists some $w \in W'(\mathbf{k})$ such that $Z_f(w) = Z_{f|_{W'}}(w) = W'$ (cf. Lemma 2.2). In particular, $Z_f(w)$ is irreducible with $\dim Z_f(w) > r$. Continuing this process, the lemma follows. □

Corollary 3.2. *Let X be a rationally connected projective variety over $\mathbf{k}$ and of dimension ≤ 3, and $f : X \longrightarrow X$ an int-amplified endomorphism. Then there is a rational curve $C \subseteq X$ such that $O_f(C)$ is Zariski dense.*

Proof. This follows from Remark 1.4(3), (4), and Lemma 3.1. □

Proof of Proposition 1.5. By applying Corollary 3.2 to $(X_{\overline{K}}, f_{\overline{K}})$, we know that there is a rational curve $C \subseteq X_{\overline{K}}$ such that $O_{f_{\overline{K}}}(C)$ is Zariski dense. Replacing K with a finite extension, we may assume that C is defined over K. Then $O_f(C)$ is Zariski dense in X. The theorem follows from Lemma 2.3. □

4. Int-amplified endomorphisms: Proofs of Proposition 1.6 and Theorem 1.7

Lemma 4.1. *(cf.* [Men20, Theorem 1.9]*) Let X be a normal projective variety over $\mathbf{k}$ and $f : X \longrightarrow X$ an int-amplified endomorphism. Assume one of the following conditions.*

(i) *X is non-uniruled.*

(ii) *X has at worst $\mathbb{Q}$-factorial klt singularities, and K_X is pseudo-effective, i.e., K_X is in the closure of the cone of effective $\mathbb{R}$-divisors.*

Then X is a Q-abelian variety. In particular, f has a Zariski dense orbit.

Proof. The first claim is [Men20, Theorem 1.9]. Now there is a finite cover $\pi : A \longrightarrow X$ (étale in codimension 1) from an abelian variety A with f lifted to an int-amplified endomorphism g on A (cf. [NZ10, Lemma 2.12] and [Men20, Lemma 3.5]). Since Conjecture 1.3 holds for endomorphisms on abelian varieties (cf. [GS16]), g has a Zariski dense orbit $O_g(a)$ for some $a \in A(\mathbf{k})$ (cf. Lemma 2.2). Then $O_f(\pi(a))$ is a Zariski dense orbit of f. □

Lemma 4.2. *Let X be a normal projective variety over $\mathbf{k}$ and of dimension ≤ 3 with at worst $\mathbb{Q}$-factorial klt singularities. Let $f : X \longrightarrow X$ be an int-amplified endomorphism. Then there exists a rational subvariety $Z \subseteq X$ of dimension ≥ 0, such that $O_f(Z)$ is Zariski dense.*

Proof. By Remark 1.4(3), (4), and Lemma 2.2, the assertion holds when $\dim X \le 2$. Then by Corollary 3.2 and Lemma 4.1, we may assume that X is a threefold that is uniruled but not rationally connected, and K_X is not pseudo-effective.

By [MZ20], replacing f with an iteration, we can run an f-equivariant minimal model program:

$$X = X_0 \overset{\mu_0}{\dashrightarrow} X_1 \overset{\mu_1}{\dashrightarrow} \cdots \overset{\mu_{m-1}}{\dashrightarrow} X_m = X' \xrightarrow{\ \pi\ } Y,$$

where each μ_i is a birational map and π is a Mori fibre space with $\dim Y < \dim X' = 3$. If $\dim Y = 0$, then X' is klt Fano. Hence X' and X are rationally connected (cf. [Zha06, Theorem 1]), contradicting our extra assumption. Thus $\dim Y = 1, 2$. Since $\dim Y \le 2$, the int-amplified endomorphism $g := f|_Y$ has a Zariski dense orbit $O_g(y)$ by Remark 1.4(3), (4) and Lemma 2.2 (cf. [Men20, Lemmas 3.4 and 3.5]). A general fibre of π is a klt Fano variety with dimension $\dim X - \dim Y$ (cf. [KM98, Lemma 5.17(1)]). So, replacing y by $g^N(y)$ for a suitable $N \ge 0$, we may assume that $F := \pi^{-1}(y)$ is a klt Fano variety of dimension equal to $\dim X - \dim Y \in \{1, 2\}$, and hence a rational variety. Clearly, $O_f(\tilde{F})$ is Zariski dense in X by construction, where $\tilde{F} \subseteq X$ is the strict transform of $F \subseteq X'$. □

Proof of Proposition 1.6. Since being uniruled and the potential density are birational properties (cf. 2.1(7) and [Has03, Proposition 3.1]), they are invariant under the normalisation map. Also, since an int-amplified endomorphism on the variety X lifts to an int-amplified endomorphism on its normalisation (cf. [Men20, Lemma 3.5]), we may assume that X is normal. Then the proposition follows from Lemmas 2.3 and 4.1. □

Proof of Theorem 1.7. This follows from Lemmas 2.3 and 4.2. □

5. The maximal arithmetic degree: Proofs of Theorems 1.10 and 1.11

In this section, we study Question 1.9. We first prepare some results and then we prove Theorem 1.10. We begin with:

Lemma 5.1. *Let X, Y be normal projective varieties over K, and $f : X \longrightarrow X$ and $g : Y \longrightarrow Y$ surjective endomorphisms. Assume that there is a surjective morphism $\pi : X \longrightarrow Y$ such that $\pi \circ f = g \circ \pi$. Then we have:*

(1) *If π is generically finite and (X, f) satisfies (DR), then (Y, g) also satisfies (DR).*
(2) *Suppose π is birational. Then (X, f) satisfies (DR) if and only if so does (Y, g).*

Proof. Assume first that π is generically finite. Let $X \xrightarrow{\pi'} X' \xrightarrow{\varphi} Y$ be the Stein factorisation of π, where π' is a projective morphism with connected fibres (indeed, $\pi'_*\mathcal{O}_X \simeq \mathcal{O}_{X'}$) to a normal variety X', and φ is a finite morphism (cf. [Har77, Chapter III, Corollary 11.5]). Since $\pi \circ f = g \circ \pi$ and φ is finite, we see that $\pi' \circ f$ contracts every fibre of π'. By the rigidity lemma (cf. [Deb01, Lemma 1.15]), there is a morphism $f' : X' \longrightarrow X'$ such that $\pi' \circ f = f' \circ \pi'$ and $\varphi \circ f' = g \circ \pi$. By [SS21, Lemma 3.2], for (1), we only need to show that (X', f') satisfies (DR), which can be deduced from (2); for (2), we only need to show that if (X, f) satisfies (DR), then so does (Y, g).

Let $\Sigma \subsetneq Y$ be the subset consisting of points y such that $\dim \pi^{-1}(y) > 0$, and $E := \pi^{-1}(\Sigma) \subseteq X$, which is a closed proper subset. Since π has connected fibres by Zariski's Main Theorem (cf. [Har77, Chapter III, Corollary 11.4]), $\pi|_{X\setminus E} : X \setminus E \longrightarrow Y \setminus \Sigma$ is an isomorphism. Since g is finite, both Σ and $Y \setminus \Sigma$ are g^{-1}-invariant. There is an induced surjective morphism $f|_{X\setminus E} : X \setminus E \longrightarrow X \setminus E$ such that $\pi|_{X\setminus E} \circ f|_{X\setminus E} = g|_{Y\setminus\Sigma} \circ \pi|_{X\setminus E}$. Let L be a finite field extension of K such that (X, f) satisfies $(DR)_L$. Then there exists a sequence of L-rational points $S_X = \{x_i\}_{i=1}^{\infty} \subseteq X(L) \setminus E$ such that

(i) S_X is Zariski dense in X_L;
(ii) $\alpha_f(x_i) = d_1(f)$ for all i; and
(iii) $O_f(x_i) \cap O_f(x_j) = \emptyset$ for $i \neq j$.

Thus $y_i := \pi(x_i)$ is well-defined and $S_Y := \{y_i\}_{i=1}^{\infty}$ satisfies the conditions of $(DR)_L$ for (Y, g); note that $d_1(f) = d_1(g)$ and $\alpha_f(x_i) = \alpha_g(y_i)$ (cf. [Sil17, Lemma 3.2] in the smooth case, or [MMSZ20, Lemma 2.8] in general). □

We need the following from [SS20].

Lemma 5.2 (cf. [SS20, Theorem 4.1]). *Let X be a projective variety over K and $f : X \longrightarrow X$ a surjective morphism with $d_1(f) > 1$. Assume the following condition:*

(†) *There is a numerically non-zero nef $\mathbb{R}$-Cartier divisor D on X such that $f^*D \sim_{\mathbb{R}} d_1(f)D$, and for any proper closed subset $Y \subseteq X_{\overline{K}}$, there exists a morphism $g : \mathbb{P}^1_K \longrightarrow X$ such that $g(\mathbb{P}^1_K) \nsubseteq Y$ and g^*D is ample.*

Then (X, f) satisfies $(DR)_K$.

We also need the following structure theorem of endomorphisms.

Proposition 5.3 (cf. [JXZ20, Theorem 1.1]). *Let $f : X \longrightarrow X$ be a non-isomorphic surjective endomorphism of a normal projective surface over* **k**. *Then, replacing f with a positive power, one of the following holds.*

(i) *$\rho(X) = 2$; there is a $\mathbb{P}^1$-fibration $X \longrightarrow C$ to a smooth projective curve of genus ≥ 1, and f descends to an automorphism of finite order on the curve C.*
(ii) *f lifts to an endomorphism $f|_V$ on a smooth projective surface V via a generically finite surjective morphism $V \longrightarrow X$.*
(iii) *X is a rational surface.*

Proof. We use [JXZ20, Theorem 1.1]. Cases (1), (3) and (8) imply our (ii). Cases (4) ~ (7) and (9) lead to our (iii). Case (2) implies our (i), noting that f cannot be polarised since it descends to an automorphism and hence $\rho(X) = 2$ by [MZ19, Theorem 5.4]. □

Proof of Theorem 1.10. When f is an automorphism, we may take an equivariant resolution of (X, f) and assume that X is smooth (cf. Lemma 5.1). In this case, the theorem follows from [SS21, Theorem 1.5].

Now we assume that $\deg(f) \geq 2$. We apply Proposition 5.3 to $(X_{\overline{K}}, f_{\overline{K}})$ (cf. [SS21, Lemma 3.3]). In either case, we may replace K with a finite field extension so that the varieties and morphisms are defined over K.

In Case 5.3(ii), the theorem follows from Lemma 5.1 and [SS21, Theorem 1.5]. In Case 5.3(iii), the theorem is a consequence of [SS20, Theorem 1.11].

In Case 5.3(i), we may assume $g(C) = 1$; otherwise, X does not satisfy potential density (cf. Remark 1.4(2)). Note that $\pi : X \longrightarrow C$ has a section S over $\overline{K}$ (the classical Tsen's theorem). After replacing K by a finite extension, we may assume that S is defined over K. Let F be a general fibre of π, which is a rational curve over $\overline{K}$ since π is a $\mathbb{P}^1$-fibration. Replacing K by a finite extension, we may assume that F is defined over K. Then S intersects F at a K-rational point. Hence $F \simeq \mathbb{P}^1$ over K. By [MMSZZ21, Theorem 6.4], there is a numerically non-zero nef $\mathbb{R}$-Cartier divisor D on X such that $f^*D \sim_{\mathbb{R}} d_1(f)D$, after possibly replacing K with a finite field extension. The numerical equivalence class of D is not a multiple of that of the fibre F since $f^*F \sim_{\mathbb{R}} F$ and $d_1(f) > 1$. Then $(D \cdot F) > 0$, by the Hodge index theorem. Thus, (X, f) satisfies the condition (†) in Lemma 5.2 and hence satisfies (DR). □

Before proving Theorem 1.11, we need a stronger version of Corollary 3.2 in dimension 3.

Lemma 5.4. *Let X be a rationally connected smooth projective threefold over* $\mathbf{k}$ *and $f : X \longrightarrow X$ an int-amplified endomorphism. Let D be a numerically non-zero nef $\mathbb{R}$-Cartier divisor on X. Then there is a rational curve $C \subseteq X$ such that $O_f(C)$ is Zariski dense and $(D \cdot C) > 0$.*

Proof. By [Yos21, Corollary 1.4], X is of Fano type. Then there is a surjective morphism $\phi : X \longrightarrow Y$ to a projective variety Y such that $D \sim_{\mathbb{R}} \phi^* H$ for some ample $\mathbb{R}$-divisor on Y by [Bir10, Theorem 3.9.1].

If f has a Zariski dense orbit $O_f(x)$, then there is a rational curve passing through x (such a curve exists since X is rationally connected) and satisfying the claims. So we may assume that f has no Zariski dense orbit.

Since Conjecture 1.3 is known for surfaces (cf. [JXZ20, Theorem 1.9]), we can take a point $x_0 \in X$ such that $\dim Z_f(x_0) = 2$ (cf. Proof of Lemma 3.1). Replacing f by a power and x_0 by $f^N(x_0)$ for some integer $N \geq 0$, we may assume that $Z_f(x_0)$ is irreducible. We can take an f-periodic point $x_1 \in X$ such that $x_1 \notin Z_f(x_0) \cup \phi^{-1}(\phi(x_0))$ since the set of f-periodic points is Zariski dense in X (cf. [Fak03, Theorem 5.1]). Take a rational curve $C \subseteq X$ containing x_0, x_1. We see that $O_f(C)$ is Zariski dense as in the proof of Lemma 3.1. Now $\phi(C)$ is not a point by construction, so

$$(D \cdot C) = (\phi^* H \cdot C) = (H \cdot \phi_* C) > 0.$$

Thus C satisfies the claims. □

Proof of Theorem 1.11. By [MMSZZ21, Theorem 6.4], replacing K by a finite extension, there is a numerically non-zero nef $\mathbb{R}$-Cartier divisor D on X such that $f^* D \sim_{\mathbb{R}} d_1(f) D$. Lemma 5.4 implies that, replacing K with a finite extension so that the curve C there (and f) are defined over K, the pair (X, f) satisfies (†) in Lemma 5.2. Hence (X, f) satisfies (DR). □

References

[Ame11] Amerik, Ekaterina. Existence of non-preperiodic algebraic points for a rational self-map of infinite order. *Math. Res. Lett.* **18** (2011), no. 2, 251–256. MR2784670, Zbl 1241.14011, arXiv:1007.1635, doi: 10.4310/MRL.2011.v18.n2.a5. 435

[ABR11] Amerik, Ekaterima; Bogomolov, Fedor A.; Rovinsky, Marat. Remarks on endomorphisms and rational points. *Compos. Math.* **147** (2011), no. 6, 1819–1842. MR2862064, Zbl 1231.14014, arXiv:1001.1150, doi: 10.1112/S0010437X11005537. 434

[Bir10] Birkar, Caucher. On existence of log minimal models and weak Zariski decompositions. *Math. Ann.* **354** (2012), no. 2, 787–799. MR2965261, Zbl 1255.14016, arXiv:0907.5182, doi: 10.1007/s00208-011-0756-y. 442

[Cam04] Campana, Frédéric. Orbifolds, special varieties and classification theory. *Ann. Inst. Fourier (Grenoble)* **54** (2004), no. 3, 499–630. MR2097416, Zbl 1062.14014, doi: 10.5802/aif.2027. 434

[Deb01] DEBARRE, OLIVIER. Higher-dimensional algebraic geometry. Universitext. *Springer-Verlag, New York*, 2001. xiv+233 pp. ISBN: 0-387-95227-6. MR1841091, Zbl 0978.14001, doi: 10.1007/978-1-4757-5406-3. 440

[DS05] DINH, TIEN-CUONG; SIBONY, NESSIM. Une borne supérieure pour l'entropie topologique d'une application rationnelle. *Ann. of Math. (2)* **161** (2005), no. 3, 1637–1644. MR2180409, Zbl 1084.54013, doi: 10.4007/annals.2005.161.1637. 437

[Fak03] FAKHRUDDIN, NAJMUDDIN. Questions on self maps of algebraic varieties. *J. Ramanujan Math. Soc.* **18** (2003), no. 2, 109–122. MR1995861, Zbl 1053.14025. 438, 442

[Fal83] FALTINGS, GERD. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. Math.* **73** (1983), no. 3, 349–366. MR718935, Zbl 0588.14026, doi: 10.1007/BF01388432. 435

[GS16] GHIOCA, DRAGOS; SCANLON, THOMAS. Density of orbits of endomorphisms of abelian varieties. *Trans. Amer. Math. Soc.* **369** (2017), no. 1, 447–466. MR3557780, Zbl 1354.11045, arXiv:1412.2029, doi: 10.1090/tran6648. 439

[Har77] HARTSHORNE, ROBIN. Algebraic geometry. Graduate Texts in Mathematics, 52. *Springer-Verlag, New York-Heidelberg*, 1977. xvi+496 pp. ISBN: 0-387-90244-9. MR0463157, Zbl 0367.14001, doi: 10.1007/978-1-4757-3849-0. 440

[Has03] HASSETT, BRENDAN. Potential density of rational points on algebraic varieties. *Higher dimensional varieties and rational points* (Budapest, 2001), 223–282. Bolyai Soc. Math. Stud., 12. *Springer, Berlin*, 2003. MR2011748, Zbl 1092.14029, doi: 10.1007/978-3-662-05123-8_8. 434, 435, 439

[JXZ20] JIA, JIA; XIE, JUNYI; ZHANG, DE-QI. Surjective endomorphisms of projective surfaces – the existence of infinitely many dense orbits. Preprint, 2020. arXiv:2005.03628. 435, 441, 442

[Kol96] KOLLÁR, JÁNOS. Rational curves on algebraic varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge. A Series of Modern Surveys in Mathematics, 32. *Springer-Verlag, Berlin*, 1996. viii+320 pp. ISBN: 3-540-60168-6. MR1440180, Zbl 0877.14012. 436, 437

[KM98] KOLLÁR, JÁNOS; MORI, SHIGEFUMI. Birational geometry of algebraic varieties. With the collaboration of C. H. Clemens and A. Corti. Translated from the 1998 Japanese original. Cambridge Tracts in Mathematics, 134. *Cambridge University Press, Cambridge*, 1998. viii+254 pp. ISBN: 0-521-63277-3. MR1658959, Zbl 1143.14014, doi: 10.1017/CBO9780511662560. 436, 439

[KS14] KAWAGUCHI, SHU; SILVERMAN, JOSEPH H. Examples of dynamical degree equals arithmetic degree. *Michigan Math. J.* **63** (2014), no. 1, 41–63. MR3189467, Zbl 1309.37084, arXiv:1212.3015, doi: 10.1307/mmj/1395234358. 435

[KS16] KAWAGUCHI, SHU; SILVERMAN, JOSEPH H. On the dynamical and arithmetic degrees of rational self-maps of algebraic varieties. *J. Reine Angew. Math.* **713** (2016), 21–48. MR3483624, Zbl 1393.37115, arXiv:1208.0815, doi: 10.1515/crelle-2014-0020. 437

[MMSZ20] MATSUZAWA, YOHSUKE; MENG, SHENG; SHIBATA, TAKAHIRO; ZHANG, DE-QI. Non-density of points of small arithmetic degrees. Preprint, 2020. arXiv:2002.10976. 434, 438, 440

[MMSZZ21] MATSUZAWA, YOHSUKE; MENG, SHENG; SHIBATA, TAKAHIRO; ZHANG, DE-QI; ZHONG, GUOLEI. Invariant subvarieties with small dynamical degree. *International Mathematics Research Notices*, **2021**, no. 39, 2021. arXiv:2005.13368, doi: 10.1093/imrn/rnab039. 441, 442

[MY19] MATSUZAWA, YOHSUKE; YOSHIKAWA, SHOU. Kawaguchi–Silverman conjecture for endomorphisms on rationally connected varieties admitting an int-amplified endomorphism. Preprint, 2019. arXiv:1908.11537. 434

[MS14] Medvedev, Alice; Scanlon, Thomas. Invariant varieties for polynomial dynamical systems. *Ann. of Math. (2)* **179** (2014), no. 1, 81–177. MR3126567, Zbl 1347.37145, arXiv:0901.2352, doi: 10.4007/annals.2014.179.1.2. 434

[Men20] Meng, Sheng. Building blocks of amplified endomorphisms of normal projective varieties. *Math. Z.* **294** (2020), no. 3–4, 1727–1747. MR4074056, Zbl 1439.14057, arXiv:1712.08995, doi: 10.1007/s00209-019-02316-7. 434, 437, 438, 439, 440

[MZ19] Meng, Sheng; Zhang De-Qi. Kawaguchi–Silverman conjecture for surjective endomorphisms. Preprint, 2019. arXiv:1908.01605. 441

[MZ20] Meng, Sheng; Zhang De-Qi. Semi-group structure of all endomorphisms of a projective variety admitting a polarized endomorphism. *Math. Res. Lett.* **27** (2020), no. 2, 523–549. MR4117085, Zbl 1440.14077, doi: 10.4310/MRL.2020.v27.n2.a8. 434, 439

[NZ10] Nakayama, Noboru; Zhang, De-Qi. Polarized endomorphisms of complex normal varieties. *Math. Ann.* **346** (2010), no. 4, 991–1018. MR2587100, Zbl 1189.14043, arXiv:0908.1688, doi: 10.1007/s00208-009-0420-y. 439

[San20] Sano, Kaoru. Dynamical degree and arithmetic degree of endomorphisms on product varieties. *Tohoku Math. J. (2)* **72** (2020), no. 1, 1–13. MR4079421, Zbl 1464.37092, arXiv:1604.04174, doi: 10.2748/tmj/1585101618. 434

[Sil17] Silverman, Joseph H. Arithmetic and dynamical degrees on abelian varieties. *J. Théor. Nombres Bordeaux* **29** (2017), no. 1, 151–167. MR3614521, Zbl 1416.11091, arXiv:1501.04205, doi: 10.5802/jtnb.973. 440

[SS20] Sano, Kaoru, Shibata, Takahiro. Zariski density of points with maximal arithmetic degree. Preprint, 2021. arXiv:2007.15180. 435, 440, 441

[SS21] Sano, Kaoru, Shibata, Takahiro. Zariski density of points with maximal arithmetic degree for surfaces. Preprint, 2021. arXiv:2101.08417. 435, 436, 440, 441

[Xie19] Xie, Junyi. The existence of Zariski dense orbits for endomorphisms of projective surfaces (with an appendix in collaboration with Thomas Tucker). Preprint, 2019. arXiv:1905.07021. 435

[Yos21] Yoshikawa, Shou. Structure of Fano fibrations of varieties admitting an int-amplified endomorphism. *Adv. Math.* **391** (2021), Paper No. 107964, 32 pp. MR4305238, Zbl 7408032, doi: 10.1016/j.aim.2021.107964. 442

[Zha06] Zhang, Qi. Rational connectedness of log **Q**-Fano varieties. *J. Reine Angew. Math.* **590** (2006), 131–142. MR2208131, Zbl 1093.14059, doi: 10.1515/CRELLE.2006.006. 439

[Zha09] Zhang, De-Qi. Dynamics of automorphisms on projective complex manifolds. *J. Differential Geom.* **82** (2009), no. 3, 691–722. MR2534992, Zbl 1187.14061, doi: 10.4310/jdg/1251122550. 437

(Jia Jia) Department of Mathematics, National University of Singapore, Singapore 119076, Republic of Singapore
jia_jia@u.nus.edu

(Takahiro Shibata) Department of Mathematics, National University of Singapore, Singapore 119076, Republic of Singapore
mattash@nus.edu.sg

(De-Qi Zhang) Department of Mathematics, National University of Singapore, Singapore 119076, Republic of Singapore
matzdq@nus.edu.sg

This paper is available via http://nyjm.albany.edu/j/2022/28-17.html.