

Potential density of projective varieties having an int-amplified endomorphism

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ABSTRACT. We consider the potential density of rational points on an algebraic variety defined over a number field K , i.e., the property that the set of rational points of X becomes Zariski dense after a finite field extension of K . For a non-uniruled projective variety with an int-amplified endomorphism, we show that it always satisfies potential density. When a rationally connected variety admits an int-amplified endomorphism, we prove that there exists some rational curve with a Zariski dense forward orbit, assuming the Zariski dense orbit conjecture in lower dimensions. As an application, we prove the potential density for projective varieties with int-amplified endomorphisms in dimension ≤ 3 . We also study the existence of densely many rational points with the maximal arithmetic degree over a sufficiently large number field.

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1. Introduction

Let K be a number field with a fixed algebraic closure \overline{K} . Given a variety X over K , we are interested in the set of K -rational points $X(K)$ of X . More specifically, we study the *potential density* of varieties over K .

Definition 1.1. A variety X defined over a number field K is said to satisfy *potential density* if there is a finite field extension $K \subseteq L$ such that $X_L(L)$ is Zariski dense in X_L , where $X_L := X \times_{\text{Spec } K} \text{Spec } L$.

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The potential density of varieties over number fields has been investigated in several papers. The potential density problem is attractive because the potential density of a variety is pretty much governed by its geometry. See [Cam04] for a conjecture characterising varieties satisfying potential density. However, algebraic varieties for which the potential density is verified are very few. See [Has03] for a survey of studies on the potential density problem.

In this paper, we first study the potential density of varieties admitting int-amplified endomorphisms. For the definition of int-amplified endomorphisms, see 2.1(11). Recently, the equivariant minimal model program for varieties with int-amplified endomorphisms was established (cf. [MZ20]). It has been used to study arithmetic-dynamical problems (cf. [MY19], [MMSZ20]). It turns out that the equivariant minimal model program is also useful for the potential density problem.

Our main conjecture is the following.

Conjecture 1.2 (Potential density under int-amplified endomorphisms). *Let X be a projective variety defined over a number field K . Suppose that X admits an int-amplified endomorphism. Then X satisfies potential density.*

The endomorphism being int-amplified is a crucial assumption in Conjecture 1.2 above. Indeed, consider $X = X_1 \times C$ where X_1 is any smooth projective variety and C is any smooth projective curve of genus at least 2. Such X does not satisfy potential density (cf. Remark 1.4(2)). It does not have any int-amplified endomorphisms either; this is because every surjective endomorphism f of X , after iteration, has the form $(x_1, x_2) \mapsto (g(x_1, x_2), x_2)$ for some morphism $g : X_1 \times C \rightarrow X_1$ by [San20, Lemma 4.5], and hence descends to the identity map id_C on C via the natural projection $X \rightarrow C$; thus, the iteration and hence f itself are not int-amplified (cf. [Men20, Lemma 3.7 and Theorem 1.1]).

One might think that Conjecture 1.2 is too strong. In fact, the following even stronger conjecture has already been long outstanding. We refer to Medvedev–Scanlon [MS14, Conjecture 7.13], and Amerik–Bogomolov–Rovinsky [ABR11] for the details.

Conjecture 1.3 (Zariski dense orbit conjecture). *Let X be a variety defined over an algebraically closed field \mathbf{k} of characteristic zero and $f : X \dashrightarrow X$ a dominant rational map. If the f^* -invariant function field $\mathbf{k}(X)^f$ is trivial, that is, $\mathbf{k}(X)^f = \mathbf{k}$, then there exists some $x \in X(\mathbf{k})$ whose (forward) f -orbit $O_f(x) := \{f^n(x) \mid n \geq 0\}$ is well-defined (i.e., f is defined at $f^n(x)$ for any $n \geq 0$) and Zariski dense in X .*

Note that Conjecture 1.3 with f being int-amplified implies Conjecture 1.2 (cf. Lemmas 2.2 and 2.3).

Remark 1.4. We recall some known cases of the potential density problem and Conjecture 1.3.

- (1) Unirational varieties and abelian varieties over number fields satisfy potential density (cf. [Has03, Corollary 3.3 and Proposition 4.2]).

- (2) Let X be a variety with a dominant rational map $X \dashrightarrow C$ to a curve of genus ≥ 2 over a number field. Then X does not satisfy potential density (cf. [Fal83] and [Has03, Proposition 3.1]).
- (3) Conjecture 1.3 holds for any pair (X, f) with X being a curve (cf. [Ame11, Corollary 9]).
- (4) Conjecture 1.3 holds for any pair (X, f) with X being a projective surface and f a surjective endomorphism of X (cf. [Xie19], [JXZ20]).

We first prove Conjecture 1.2 for rationally connected varieties in dimension ≤ 3 .

Proposition 1.5. *Let X be a rationally connected projective variety over the number field K . Suppose that $\dim X \leq 3$ and X admits an int-amplified endomorphism. Then X satisfies potential density.*

Conjecture 1.2 also has a positive answer for non-uniruled varieties in any dimension:

Proposition 1.6. *Let X be a non-uniruled projective variety over the number field K . Suppose that X admits an int-amplified endomorphism. Then X satisfies potential density.*

With the help of Propositions 1.5 and 1.6, we are able to show:

Theorem 1.7. *Let X be a normal projective variety over the number field K with at worst \mathbb{Q} -factorial klt singularities. Suppose that $\dim X \leq 3$ and X admits an int-amplified endomorphism. Then X satisfies potential density.*

In the last section, we study Question 1.9 below, which is also arithmetic in nature, initiated in [KS14] and further studied in [SS20] and [SS21].

Definition 1.8 (cf. [SS20, Definition 1.4]). Let X be a projective variety over a number field K and $f : X \rightarrow X$ a surjective morphism. We recall the inequality

$$\alpha_f(x) \leq d_1(f)$$

between the arithmetic degree $\alpha_f(x)$ at a point $x \in X(\overline{K})$ and the first dynamical degree $d_1(f)$ of f (cf. 2.1(12) and (13)). Let L be an intermediate field: $K \subseteq L \subseteq \overline{K}$. We say that (X, f) has *densely many L -rational points with the maximal arithmetic degree* if there is a subset $S \subseteq X(L)$ satisfying the following conditions:

- (1) S is Zariski dense in X_L ;
- (2) the equality $\alpha_f(x) = d_1(f)$ holds for all $x \in S$; and
- (3) $O_f(x_1) \cap O_f(x_2) = \emptyset$ for any pair of distinct points $x_1, x_2 \in S$.

Following [SS21], we introduce the following notation. We say that (X, f) satisfies $(DR)_L$ if (X, f) has densely many L -rational points with the maximal arithmetic degree. We say that (X, f) satisfies (DR) if there is a finite field extension $K \subseteq L (\subseteq \overline{K})$ such that (X, f) satisfies $(DR)_L$.

Question 1.9. *Let X be a projective variety over K and $f : X \rightarrow X$ a surjective endomorphism. Assume that X satisfies potential density. Does (X, f) satisfy (DR)?*

Question 1.9 has a positive answer for smooth projective surfaces when $d_1(f) > 1$ (cf. [SS21, Theorem 1.5]). We generalise it to (possibly singular) projective surfaces:

Theorem 1.10. *Let X be a normal projective surface over the number field K satisfying potential density, and $f : X \rightarrow X$ a surjective morphism with $d_1(f) > 1$. Then (X, f) satisfies (DR).*

The following is an affirmative answer to Question 1.9 for int-amplified endomorphisms on rationally connected threefolds.

Theorem 1.11. *Let X be a rationally connected smooth projective threefold over the number field K and $f : X \rightarrow X$ an int-amplified endomorphism. Then (X, f) satisfies (DR).*

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2. Preliminaries

2.1. Notation and terminology

- (1) Let K be a number field. We work over K when considering the potential density. We fix an algebraic closure \bar{K} of K .
- (2) Let \mathbf{k} be an algebraically closed field of characteristic zero. We work over \mathbf{k} when considering geometric properties.
- (3) A *variety* means a geometrically integral separated scheme of finite type over a field.
- (4) Let X be a variety over K and $f : X \rightarrow X$ a morphism (over K). We denote $X_{\bar{K}} := X \times_{\text{Spec } K} \text{Spec } \bar{K}$ and $f_{\bar{K}} : X_{\bar{K}} \rightarrow X_{\bar{K}}$ the induced morphism (over \bar{K}).
- (5) The symbol $\sim_{\mathbb{R}}$ denotes the \mathbb{R} -linear equivalence on Cartier divisors.
- (6) We refer to [KM98] for definitions of \mathbb{Q} -factoriality and klt singularities.
- (7) A variety X of dimension n is *uniruled* if there is a variety U of dimension $n - 1$ and a dominant rational map $\mathbb{P}^1 \times U \dashrightarrow X$.
- (8) Let X be a proper variety over a field k . We say that X is *rationally connected* if there is a family of proper algebraic curves $U \rightarrow Y$ whose geometric fibres are irreducible rational curves with cycle morphism $U \rightarrow X$ such that $U \times_Y U \rightarrow X \times X$ is dominant (cf. [Kol96, Chapter IV, Definition 3.2]). When k is algebraically closed of characteristic zero, if X is rationally connected, then any two closed points of X are connected by an irreducible rational curve over k (by applying [Kol96, Chapter IV, Theorem 3.9] to a

resolution of X). The converse holds when k is also uncountable (cf. [Kol96, Chapter IV, Proposition 3.6.2]).

- (9) A normal projective variety X is said to be *Q-abelian* if there is a finite surjective morphism $\pi : A \rightarrow X$, which is étale in codimension 1, with A being an abelian variety.
- (10) For a morphism $f : X \rightarrow X$ and a point $x \in X$, the forward *f-orbit* of x is the set $O_f(x) := \{x, f(x), f^2(x), \dots\}$. We denote the Zariski closure of $O_f(x)$ by $Z_f(x)$. More generally, for a closed subset $Y \subseteq X$, we denote $O_f(Y) := \bigcup_{n=0}^{\infty} f^n(Y)$ and its Zariski-closure $Z_f(Y) := \overline{O_f(Y)}$. We say that $O_f(Y)$ is Zariski dense if $Z_f(Y) = X$.
- (11) A surjective morphism $f : X \rightarrow X$ of a projective variety is called *int-amplified* if there exists an ample Cartier divisor H on X such that $f^*H - H$ is ample. In particular, polarised endomorphisms are int-amplified.
- (12) Let X be a projective variety and $f : X \rightarrow X$ a surjective morphism. The *first dynamical degree* $d_1(f)$ of f is the limit

$$d_1(f) := \lim_{n \rightarrow \infty} ((f^n)^*H \cdot H^{\dim X - 1})^{1/n},$$

where H is an ample Cartier divisor on X . This limit always converges and is independent of the choice of H (cf. [DS05]). Dynamical degrees are invariant under the conjugation by generically finite maps (cf. [Zha09, Lemma 2.6]).

- (13) Let X be a projective variety over K and $f : X \rightarrow X$ a surjective morphism. Fix a (logarithmic) height function $h_H \geq 1$ associated to an ample Cartier divisor H on X . For $x \in X(\overline{K})$, the *arithmetic degree* $\alpha_f(x)$ of f at x is the limit

$$\alpha_f(x) := \lim_{n \rightarrow \infty} h_H(f^n(x))^{1/n}.$$

This limit always converges and is independent of the choices of H and h_H (cf. [KS16]).

Lemma 2.2. *Let X be a projective variety over \mathbf{k} and $f : X \rightarrow X$ an int-amplified endomorphism. Then $\mathbf{k}(X)^f = \mathbf{k}$. In particular, if Conjecture 1.3 holds for (X, f) , then there exists some $x \in X(\mathbf{k})$ such that $O_f(x)$ is Zariski dense in X .*

Proof. Assume to the contrary that there is a nonconstant rational function $\phi : X \dashrightarrow \mathbb{P}^1$ such that $\phi \circ f = \phi$. Let Γ be the graph of the rational map $\phi : X \dashrightarrow \mathbb{P}^1$ with projections $\pi_1 : \Gamma \rightarrow X$ being birational and $\pi_2 : \Gamma \rightarrow \mathbb{P}^1$ being surjective. Then f lifts to an endomorphism $f|_{\Gamma}$ on Γ such that $\pi_1 \circ f|_{\Gamma} = f \circ \pi_1$ and $\pi_2 \circ f|_{\Gamma} = \pi_2$. It follows from [Men20, Lemmas 3.4 and 3.5] that $\text{id} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is int-amplified, which is absurd. \square

Lemma 2.3. *Let X be a projective variety over K , $f : X \rightarrow X$ a surjective morphism, and $Z \subseteq X$ a subvariety which satisfies potential density (e.g., Z is an abelian variety or unirational; see Remark 1.4(1)). If $O_f(Z)$ is Zariski dense, then X satisfies potential density.*

Proof. Replacing K with a finite extension, we may assume that $Z(K)$ is Zariski dense in Z . Then the union $\bigcup_{n=0}^{\infty} f^n(Z(K))$ is a Zariski dense set of K -rational points of X . \square

3. Rationally connected varieties: Proof of Proposition 1.5

Lemma 3.1. *Let X be a rationally connected projective variety over \mathbf{k} and of dimension $d \geq 1$, and $f : X \rightarrow X$ an int-amplified endomorphism. Assume Conjecture 1.3 in dimension $\leq d - 1$. Then there is a rational curve $C \subseteq X$ such that $O_f(C)$ is Zariski dense.*

Proof. If we have a Zariski dense f -orbit $O_f(x)$, take any rational curve C passing through x . Clearly, $O_f(C)$ is Zariski dense. So we may assume that f has no Zariski dense orbit.

Replacing f by some positive power, we can take a point $x \in X(\mathbf{k})$ such that $Z_f(x)$ is irreducible with dimension $r < d$ (cf. e.g. [MMSZ20, Lemma 2.7]). By [Fak03, Theorem 5.1], the subset of $X(\mathbf{k})$ consisting of f -periodic points is Zariski dense in X . Pick an f -periodic point $y \in X(\mathbf{k}) \setminus Z_f(x)$. After iterating f , we may assume that y is an f -fixed point. Take a rational curve $C \subseteq X$ containing x and y . Set $W := Z_f(C)$. If $W = X$, we are done. So we may assume that $W \subsetneq X$. If $\dim W = r$, then W has its irreducible decomposition as $W = Z_f(x) \cup W_1 \cup \dots \cup W_m$. There is some $n \geq 0$ such that $f^n(x) \in Z_f(x) \setminus \bigcup_{i=1}^m W_i$. Then $f^n(C) \subseteq W$ but $f^n(C) \not\subseteq \bigcup_{i=1}^m W_i$. Hence $f^n(C) \subseteq Z_f(x)$. In particular, $y = f^n(y) \in f^n(C) \subseteq Z_f(x)$, a contradiction. Thus $r < \dim W (< \dim X = d)$.

Now there exists an f -periodic irreducible component $W' \subseteq W$ with $r < \dim W' < d$. Replacing f by a positive power, we may assume that W' is f -invariant. Then $f|_{W'}$ is an int-amplified endomorphism on W' (cf. [Men20, Lemma 2.2]). By assumption, Conjecture 1.3 holds for $(W', f|_{W'})$. So there exists some $w \in W'(\mathbf{k})$ such that $Z_f(w) = Z_{f|_{W'}}(w) = W'$ (cf. Lemma 2.2). In particular, $Z_f(w)$ is irreducible with $\dim Z_f(w) > r$. Continuing this process, the lemma follows. \square

Corollary 3.2. *Let X be a rationally connected projective variety over \mathbf{k} and of dimension ≤ 3 , and $f : X \rightarrow X$ an int-amplified endomorphism. Then there is a rational curve $C \subseteq X$ such that $O_f(C)$ is Zariski dense.*

Proof. This follows from Remark 1.4(3), (4), and Lemma 3.1. \square

Proof of Proposition 1.5. By applying Corollary 3.2 to $(X_{\overline{K}}, f_{\overline{K}})$, we know that there is a rational curve $C \subseteq X_{\overline{K}}$ such that $O_{f_{\overline{K}}}(C)$ is Zariski dense. Replacing K with a finite extension, we may assume that C is defined over K . Then $O_f(C)$ is Zariski dense in X . The theorem follows from Lemma 2.3. \square

4. Int-amplified endomorphisms: Proofs of Proposition 1.6 and Theorem 1.7

Lemma 4.1. (cf. [Men20, Theorem 1.9]) *Let X be a normal projective variety over \mathbf{k} and $f : X \rightarrow X$ an int-amplified endomorphism. Assume one of the following conditions.*

- (i) X is non-uniruled.
- (ii) X has at worst \mathbb{Q} -factorial klt singularities, and K_X is pseudo-effective, i.e., K_X is in the closure of the cone of effective \mathbb{R} -divisors.

Then X is a \mathbb{Q} -abelian variety. In particular, f has a Zariski dense orbit.

Proof. The first claim is [Men20, Theorem 1.9]. Now there is a finite cover $\pi : A \rightarrow X$ (étale in codimension 1) from an abelian variety A with f lifted to an int-amplified endomorphism g on A (cf. [NZ10, Lemma 2.12] and [Men20, Lemma 3.5]). Since Conjecture 1.3 holds for endomorphisms on abelian varieties (cf. [GS16]), g has a Zariski dense orbit $O_g(a)$ for some $a \in A(\mathbf{k})$ (cf. Lemma 2.2). Then $O_f(\pi(a))$ is a Zariski dense orbit of f . \square

Lemma 4.2. *Let X be a normal projective variety over \mathbf{k} and of dimension ≤ 3 with at worst \mathbb{Q} -factorial klt singularities. Let $f : X \rightarrow X$ be an int-amplified endomorphism. Then there exists a rational subvariety $Z \subseteq X$ of dimension ≥ 0 , such that $O_f(Z)$ is Zariski dense.*

Proof. By Remark 1.4(3), (4), and Lemma 2.2, the assertion holds when $\dim X \leq 2$. Then by Corollary 3.2 and Lemma 4.1, we may assume that X is a threefold that is uniruled but not rationally connected, and K_X is not pseudo-effective.

By [MZ20], replacing f with an iteration, we can run an f -equivariant minimal model program:

$$X = X_0 \xrightarrow{\mu_0} X_1 \xrightarrow{\mu_1} \dots \xrightarrow{\mu_{m-1}} X_m = X' \xrightarrow{\pi} Y,$$

where each μ_i is a birational map and π is a Mori fibre space with $\dim Y < \dim X' = 3$. If $\dim Y = 0$, then X' is klt Fano. Hence X' and X are rationally connected (cf. [Zha06, Theorem 1]), contradicting our extra assumption. Thus $\dim Y = 1, 2$. Since $\dim Y \leq 2$, the int-amplified endomorphism $g := f|_Y$ has a Zariski dense orbit $O_g(y)$ by Remark 1.4(3), (4) and Lemma 2.2 (cf. [Men20, Lemmas 3.4 and 3.5]). A general fibre of π is a klt Fano variety with dimension $\dim X - \dim Y$ (cf. [KM98, Lemma 5.17(1)]). So, replacing y by $g^N(y)$ for a suitable $N \geq 0$, we may assume that $F := \pi^{-1}(y)$ is a klt Fano variety of dimension equal to $\dim X - \dim Y \in \{1, 2\}$, and hence a rational variety. Clearly, $O_f(\tilde{F})$ is Zariski dense in X by construction, where $\tilde{F} \subseteq X$ is the strict transform of $F \subseteq X'$. \square

Proof of Proposition 1.6. Since being uniruled and the potential density are birational properties (cf. 2.1(7) and [Has03, Proposition 3.1]), they are invariant under the normalisation map. Also, since an int-amplified endomorphism on the variety X lifts to an int-amplified endomorphism on its normalisation

(cf. [Men20, Lemma 3.5]), we may assume that X is normal. Then the proposition follows from Lemmas 2.3 and 4.1. \square

Proof of Theorem 1.7. This follows from Lemmas 2.3 and 4.2. \square

5. The maximal arithmetic degree: Proofs of Theorems 1.10 and 1.11

In this section, we study Question 1.9. We first prepare some results and then we prove Theorem 1.10. We begin with:

Lemma 5.1. *Let X, Y be normal projective varieties over K , and $f : X \rightarrow X$ and $g : Y \rightarrow Y$ surjective endomorphisms. Assume that there is a surjective morphism $\pi : X \rightarrow Y$ such that $\pi \circ f = g \circ \pi$. Then we have:*

- (1) *If π is generically finite and (X, f) satisfies (DR), then (Y, g) also satisfies (DR).*
- (2) *Suppose π is birational. Then (X, f) satisfies (DR) if and only if so does (Y, g) .*

Proof. Assume first that π is generically finite. Let $X \xrightarrow{\pi'} X' \xrightarrow{\varphi} Y$ be the Stein factorisation of π , where π' is a projective morphism with connected fibres (indeed, $\pi'_* \mathcal{O}_X \simeq \mathcal{O}_{X'}$) to a normal variety X' , and φ is a finite morphism (cf. [Har77, Chapter III, Corollary 11.5]). Since $\pi \circ f = g \circ \pi$ and φ is finite, we see that $\pi' \circ f$ contracts every fibre of π' . By the rigidity lemma (cf. [Deb01, Lemma 1.15]), there is a morphism $f' : X' \rightarrow X'$ such that $\pi' \circ f = f' \circ \pi'$ and $\varphi \circ f' = g \circ \varphi$. By [SS21, Lemma 3.2], for (1), we only need to show that (X', f') satisfies (DR), which can be deduced from (2); for (2), we only need to show that if (X, f) satisfies (DR), then so does (Y, g) .

Let $\Sigma \subseteq Y$ be the subset consisting of points y such that $\dim \pi^{-1}(y) > 0$, and $E := \pi^{-1}(\Sigma) \subseteq X$, which is a closed proper subset. Since π has connected fibres by Zariski's Main Theorem (cf. [Har77, Chapter III, Corollary 11.4]), $\pi|_{X \setminus E} : X \setminus E \rightarrow Y \setminus \Sigma$ is an isomorphism. Since g is finite, both Σ and $Y \setminus \Sigma$ are g^{-1} -invariant. There is an induced surjective morphism $f|_{X \setminus E} : X \setminus E \rightarrow X \setminus E$ such that $\pi|_{X \setminus E} \circ f|_{X \setminus E} = g|_{Y \setminus \Sigma} \circ \pi|_{X \setminus E}$. Let L be a finite field extension of K such that (X, f) satisfies $(DR)_L$. Then there exists a sequence of L -rational points $S_X = \{x_i\}_{i=1}^{\infty} \subseteq X(L) \setminus E$ such that

- (i) S_X is Zariski dense in X_L ;
- (ii) $\alpha_f(x_i) = d_1(f)$ for all i ; and
- (iii) $O_f(x_i) \cap O_f(x_j) = \emptyset$ for $i \neq j$.

Thus $y_i := \pi(x_i)$ is well-defined and $S_Y := \{y_i\}_{i=1}^{\infty}$ satisfies the conditions of $(DR)_L$ for (Y, g) ; note that $d_1(f) = d_1(g)$ and $\alpha_f(x_i) = \alpha_g(y_i)$ (cf. [Sil17, Lemma 3.2] in the smooth case, or [MMSZ20, Lemma 2.8] in general). \square

We need the following from [SS20].

Lemma 5.2 (cf. [SS20, Theorem 4.1]). *Let X be a projective variety over K and $f : X \rightarrow X$ a surjective morphism with $d_1(f) > 1$. Assume the following condition:*

- (\dagger) *There is a numerically non-zero nef \mathbb{R} -Cartier divisor D on X such that $f^*D \sim_{\mathbb{R}} d_1(f)D$, and for any proper closed subset $Y \subseteq X_{\overline{K}}$, there exists a morphism $g : \mathbb{P}_K^1 \rightarrow X$ such that $g(\mathbb{P}_K^1) \not\subseteq Y$ and g^*D is ample.*

Then (X, f) satisfies $(DR)_K$.

We also need the following structure theorem of endomorphisms.

Proposition 5.3 (cf. [JXZ20, Theorem 1.1]). *Let $f : X \rightarrow X$ be a non-isomorphic surjective endomorphism of a normal projective surface over \mathbf{k} . Then, replacing f with a positive power, one of the following holds.*

- (i) $\rho(X) = 2$; *there is a \mathbb{P}^1 -fibration $X \rightarrow C$ to a smooth projective curve of genus ≥ 1 , and f descends to an automorphism of finite order on the curve C .*
- (ii) *f lifts to an endomorphism $f|_V$ on a smooth projective surface V via a generically finite surjective morphism $V \rightarrow X$.*
- (iii) *X is a rational surface.*

Proof. We use [JXZ20, Theorem 1.1]. Cases (1), (3) and (8) imply our (ii). Cases (4) \sim (7) and (9) lead to our (iii). Case (2) implies our (i), noting that f cannot be polarised since it descends to an automorphism and hence $\rho(X) = 2$ by [MZ19, Theorem 5.4]. \square

Proof of Theorem 1.10. When f is an automorphism, we may take an equivariant resolution of (X, f) and assume that X is smooth (cf. Lemma 5.1). In this case, the theorem follows from [SS21, Theorem 1.5].

Now we assume that $\deg(f) \geq 2$. We apply Proposition 5.3 to $(X_{\overline{K}}, f_{\overline{K}})$ (cf. [SS21, Lemma 3.3]). In either case, we may replace K with a finite field extension so that the varieties and morphisms are defined over K .

In Case 5.3(ii), the theorem follows from Lemma 5.1 and [SS21, Theorem 1.5]. In Case 5.3(iii), the theorem is a consequence of [SS20, Theorem 1.11].

In Case 5.3(i), we may assume $g(C) = 1$; otherwise, X does not satisfy potential density (cf. Remark 1.4(2)). Note that $\pi : X \rightarrow C$ has a section S over \overline{K} (the classical Tsen’s theorem). After replacing K by a finite extension, we may assume that S is defined over K . Let F be a general fibre of π , which is a rational curve over \overline{K} since π is a \mathbb{P}^1 -fibration. Replacing K by a finite extension, we may assume that F is defined over K . Then S intersects F at a K -rational point. Hence $F \simeq \mathbb{P}^1$ over K . By [MMSZZ21, Theorem 6.4], there is a numerically non-zero nef \mathbb{R} -Cartier divisor D on X such that $f^*D \sim_{\mathbb{R}} d_1(f)D$, after possibly replacing K with a finite field extension. The numerical equivalence class of D is not a multiple of that of the fibre F since $f^*F \sim_{\mathbb{R}} F$ and $d_1(f) > 1$. Then $(D \cdot F) > 0$, by the Hodge index theorem. Thus, (X, f) satisfies the condition (\dagger) in Lemma 5.2 and hence satisfies (DR) . \square

Before proving Theorem 1.11, we need a stronger version of Corollary 3.2 in dimension 3.

Lemma 5.4. *Let X be a rationally connected smooth projective threefold over \mathbf{k} and $f : X \rightarrow X$ an int-amplified endomorphism. Let D be a numerically non-zero nef \mathbb{R} -Cartier divisor on X . Then there is a rational curve $C \subseteq X$ such that $O_f(C)$ is Zariski dense and $(D \cdot C) > 0$.*

Proof. By [Yos21, Corollary 1.4], X is of Fano type. Then there is a surjective morphism $\phi : X \rightarrow Y$ to a projective variety Y such that $D \sim_{\mathbb{R}} \phi^*H$ for some ample \mathbb{R} -divisor on Y by [Bir10, Theorem 3.9.1].

If f has a Zariski dense orbit $O_f(x)$, then there is a rational curve passing through x (such a curve exists since X is rationally connected) and satisfying the claims. So we may assume that f has no Zariski dense orbit.

Since Conjecture 1.3 is known for surfaces (cf. [JXZ20, Theorem 1.9]), we can take a point $x_0 \in X$ such that $\dim Z_f(x_0) = 2$ (cf. Proof of Lemma 3.1). Replacing f by a power and x_0 by $f^N(x_0)$ for some integer $N \geq 0$, we may assume that $Z_f(x_0)$ is irreducible. We can take an f -periodic point $x_1 \in X$ such that $x_1 \notin Z_f(x_0) \cup \phi^{-1}(\phi(x_0))$ since the set of f -periodic points is Zariski dense in X (cf. [Fak03, Theorem 5.1]). Take a rational curve $C \subseteq X$ containing x_0, x_1 . We see that $O_f(C)$ is Zariski dense as in the proof of Lemma 3.1. Now $\phi(C)$ is not a point by construction, so

$$(D \cdot C) = (\phi^*H \cdot C) = (H \cdot \phi_*C) > 0.$$

Thus C satisfies the claims. \square

Proof of Theorem 1.11. By [MMSZZ21, Theorem 6.4], replacing K by a finite extension, there is a numerically non-zero nef \mathbb{R} -Cartier divisor D on X such that $f^*D \sim_{\mathbb{R}} d_1(f)D$. Lemma 5.4 implies that, replacing K with a finite extension so that the curve C there (and f) are defined over K , the pair (X, f) satisfies (\dagger) in Lemma 5.2. Hence (X, f) satisfies (DR) . \square

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