

Schiffer operators and calculation of a determinant line in conformal field theory

David Radnell, Eric Schippers,
Mohammad Shirazi and Wolfgang Staubach

ABSTRACT. We consider an operator associated to compact Riemann surfaces endowed with a conformal map, f , from the unit disk into the surface, which arises in conformal field theory. This operator projects holomorphic functions on the surface minus the image of the conformal map onto the set of functions h so that the Fourier series $h \circ f$ has only negative powers. We give an explicit characterization of the cokernel, kernel, and determinant line of this operator in terms of natural operators in function theory.

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1. Introduction

In conformal field theory the central charge, or conformal anomaly, plays an important role and is encoded geometrically by the determinant line bundle of a certain family of operators $\pi_{(R,f)}$ over the rigged moduli space, see Y.-Z. Huang [3] or G. Segal [15]. Here R is a Riemann surface and f is a conformal map into that surface (see below). In this paper, we give a simple and explicit description of the cokernel of $\pi_{(R,f)}$, in the case of surfaces with one boundary curve. Furthermore, we explicitly relate the operator $\pi_{(R,f)}$ to classical operators of function theory: the Faber, Grunsky, and Schiffer operators. In particular, we give an explicit formula for its inverse in terms of

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an integral operator involving a Green's function. We use this to show that the cokernel of $\pi_{(R,f)}$ can be identified with the space of anti-holomorphic one-forms on R .

We now give a heuristic description of the rigged moduli space and the determinant line, postponing analytic considerations for the main text of the paper. The rigged moduli space $\widetilde{M}(g, 1)$ is a moduli space of Riemann surfaces R with one puncture, together with a "rigging". This rigging is a conformal map f from the disk into R taking the unit disk \mathbb{D} in the complex plane into the surface R , which takes 0 to the puncture. Let $\Sigma = R \setminus \text{cl } f(\mathbb{D})$ where "cl" denotes closure. Fourier expansions can be obtained for boundary values h of holomorphic functions on Σ by considering the functions $h \circ f$ on the circle. Let P_- denote the projection onto those h such that the Fourier series of $h \circ f$ has only negative powers of $e^{i\theta}$. We define $\pi_{(R,f)}$ to be the restriction of P_- to the holomorphic functions on Σ , that is, $\pi_{(R,f)} = P_-|_{\text{Hol}(\Sigma)}$. Its determinant line is the one-dimensional vector space

$$\text{Det}(\pi_{(R,f)}) = \text{Hom} \left(\wedge^{\dim \text{Ker}(\pi_{(R,f)})} \text{Ker}(\pi_{(R,f)}), \wedge^{\dim \text{Coker}(\pi_{(R,f)})} \text{Coker}(\pi_{(R,f)}) \right).$$

This determinant line is canonically isomorphic to that of the operator $\bar{\partial} \oplus P_-$ (see Y.-Z. Huang [3, Proposition D.3.3]). These lines come with a \mathbb{Z}_2 -grading, but this plays no role in the current article and so will not be discussed.

We show that the cokernel of $\pi_{(R,f)}$ can be identified with the space of the anti-holomorphic one-forms on R . We achieve this using an explicit inverse to $\pi_{(R,f)}$, in terms of natural operators in function theory constructed from the Green's function that were studied by M. Schiffer and others. Since the kernel is trivial the determinant line is isomorphic to the top exterior power of this space of one-forms. Finally, we outline the surprising direct relation between $\pi_{(R,f)}$ and natural generalizations of the Faber operator and the Grunsky operator of classical function theory and approximation theory. This relation follows naturally from the fact that the Faber and Grunsky operators and their generalizations are closely related to the jump formula on Riemann surface, as in the genus-zero case; see Y.-Z. Huang [3] and D. Radnell [5]. We show that the pull-back of the space of holomorphic functions under the rigging is the graph of the Grunsky operator. We also generalize the Grunsky inequalities, which state that the norm of the Grunsky operator is strictly less than one, to the case of general genus surfaces with one conformal map.

The usual regularity requirement on the rigging f is that it has an analytic or smooth extension to the boundary of the disk. In the analytic rigging setting, the determinant lines and corresponding holomorphic bundle structure were rigorously constructed for genus-zero surfaces in Y.-Z. Huang [3] and in higher-genus in D. Radnell [5]. However the higher-genus result was achieved without a direct description of the cokernel of $\pi_{(R,f)}$, nor the relation to the classical infinite-dimensional Teichmüller space and the function

theoretic objects mentioned above. These new connections open the path to future progress.

Throughout the paper, we use a considerably weaker analytic condition, namely that the function has a quasiconformal extension. As a consequence, the analysis is more difficult. However, the technical difficulties were resolved in publications of E. Schippers and W. Staubach [13, 14]. We also use the Dirichlet space on Σ , which is a natural conformally invariant completion of the set of harmonic functions which extend smoothly to the boundary. Aside from their naturality from an analytic perspective, these choices pave the way for the later construction of the holomorphic bundle structure over the infinite-dimensional rigged moduli space. This construction will use the correspondence between the Teichmüller space and the rigged moduli space discovered by Radnell and Schippers [6]. We will pursue this in later publications. A discussion of the reasons for these analytic choices, and related results of the authors, can be found in Radnell, Schippers and Staubach [7].

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2. Preliminaries

2.1. Riemann surfaces and quasicircles. This section gives the basic notation and set-up for Riemann surfaces used throughout the article.

Let S be an arbitrary Riemann surface and R a compact Riemann surface of genus g . We say that a Jordan curve $\Gamma \subseteq \mathbb{C}$ is a quasicircle if there is a quasiconformal map $\psi : \mathbb{C} \rightarrow \mathbb{C}$ such that $\Gamma = \psi(\mathbb{S}^1)$ [4]. Given a Riemann surface S , we say that a curve $\Gamma \subseteq S$ is a quasicircle if there is an open neighbourhood U of Γ and a holomorphic bijection $\phi : U \rightarrow A$ where A is a doubly-connected open set in \mathbb{C} and $\phi(\Gamma)$ is a quasicircle in \mathbb{C} according to the above criterion.

Throughout the paper, we consider the situation that R is compact and Γ separates R into two connected components Σ_1 and Σ_2 . The surfaces Σ_1 and Σ_2 are bordered Riemann surfaces and we can identify the borders $\partial\Sigma_1$ and $\partial\Sigma_2$ with Γ pointwise. In particular we will exclusively use the setting where Σ_2 is simply-connected. In this case we use the notation $\Sigma = \Sigma_1$ and $\Omega = \Sigma_2$. Note that an equivalent way to view this is that Ω is the image in R of a conformal map from \mathbb{D} into R and the border of Σ (and Ω) is the image of the unit circle \mathbb{S}^1 under the conformal map.

Note also that the boundary $\partial\Sigma$ is a border [1] and that furthermore it is homeomorphic to \mathbb{S}^1 . If desired, we can also identify $\partial\Sigma$ with a simple closed analytic curve in the Riemann surface Σ^d obtained by adjoining Σ to its double.

2.2. Operators on the Dirichlet and Bergman space associated to Green's function. Let S be a Riemann surface. We define a pairing on

one-forms on S by

$$(\omega_1, \omega_2) = \frac{1}{2} \iint_S \omega_1 \wedge * \overline{\omega_2} \quad (2.1)$$

provided that this is finite. Here $*$ is the dual of the almost complex structure: that is, if $z = x + iy$ is a local holomorphic parameter then

$$*(a dx + b dy) = a dy - b dx.$$

In particular, $*\omega = -i\omega$ for holomorphic one-forms and $*\omega = i\omega$ for anti-holomorphic one-forms.

Denote by $A_{\text{harm}}(S)$ those one-forms which are harmonic on S and which are L^2 with respect to the pairing (2.1). Let $A(S)$ denote the Bergman space of L^2 holomorphic one-forms and $\overline{A(S)}$ the set of L^2 anti-holomorphic one-forms on S . Of course, $\overline{A(S)}$ consists precisely of complex conjugates of elements of $A(S)$. We have the direct sum decomposition

$$A_{\text{harm}}(S) = A(S) \oplus \overline{A(S)} \quad (2.2)$$

and it follows directly from (2.1) that the decomposition is orthogonal. Finally, the subsets of exact forms will be denoted with a subscript ‘e’, that is by $A(S)_e$, $A_{\text{harm}}(S)_e$ and $\overline{A(S)}_e$.

We will also consider the Dirichlet spaces of complex harmonic functions, that is

$$\begin{aligned} \mathcal{D}_{\text{harm}}(S) &= \{h : S \rightarrow \mathbb{C} : h \text{ is harmonic and } dh \in A_{\text{harm}}(S)\} \\ \mathcal{D}(S) &= \{h \in \mathcal{D}_{\text{harm}}(S) : dh \in A(S)\} \\ \overline{\mathcal{D}(S)} &= \{h \in \mathcal{D}_{\text{harm}}(S) : dh \in \overline{A(S)}\}. \end{aligned}$$

If we define the indefinite inner-product

$$(f, g) = (df, dg)$$

then up to constants $d : \mathcal{D}_{\text{harm}}(S) \rightarrow A_{\text{harm}}(S)_e$ is an isometry, and similarly $\mathcal{D}(S)$ and $\overline{\mathcal{D}(S)}$ are isometric to $A(S)_e$ and $\overline{A(S)}_e$ under ∂ and $\bar{\partial}$. We will thus not notationally distinguish the norm and semi-norm, and it is understood that one is applied to functions and the other to one-forms.

For a fixed point $q \in S$, we let

$$\mathcal{D}_{\text{harm}}(S)_q = \{h \in \mathcal{D}_{\text{harm}}(S) : h(q) = 0\}$$

and similarly for $\mathcal{D}(S)_q$ and $\overline{\mathcal{D}(S)}_q$. The restriction of d , ∂ and $\bar{\partial}$ become honest isometries.

Finally, observe that although $\mathcal{D}(S)$ and $\overline{\mathcal{D}(S)}$ are orthogonal, there is no decomposition of $\mathcal{D}_{\text{harm}}(S)$ similar to (2.2) because given $h \in \mathcal{D}_{\text{harm}}(S)$, ∂h and $\bar{\partial} h$ are not in general exact.

As in section 2.1 let R be a compact Riemann surface and Γ a quasicircle which separates R into two connected components Σ and Ω , where Ω is simply connected. In this case, there are orthogonal projection maps

$$P(\Omega) : \mathcal{D}_{\text{harm}}(\Omega) \longrightarrow \mathcal{D}(\Omega)$$

and

$$\overline{P}(\Omega) : \mathcal{D}_{\text{harm}}(\Omega) \longrightarrow \overline{\mathcal{D}(\Omega)}.$$

These are obviously bounded with respect to the Dirichlet semi-norm. We also define, for $q \in \Omega$,

$$P(\Omega)_q h = P(\Omega)h - (P(\Omega)h)(q) \in \mathcal{D}(\Omega)_q$$

and similarly

$$\overline{P}(\Omega)_q h = \overline{P}(\Omega)h - (\overline{P}(\Omega)h)(q) \in \overline{\mathcal{D}(\Omega)}_q.$$

We also consider an operator which is given by integration against a Cauchy-type kernel obtained from the Green’s function. When restricted to a certain subspace, this can also be thought of as a projection, in a way which is explained ahead.

We define Green’s function of R [12] to be the unique function $g(w, w_0; z, q)$ such that

- (1) g is harmonic in w on $R \setminus \{z, q\}$;
- (2) for a local coordinate ϕ on an open set U containing z , $g(w, w_0; z, q) + \log |\phi(w) - \phi(z)|$ is harmonic for $w \in U$;
- (3) for a local coordinate ϕ on an open set U containing q , $g(w, w_0; z, q) - \log |\phi(w) - \phi(q)|$ is harmonic for $w \in U$;
- (4) $g(w_0, w_0; z, q) = 0$ for all z, q, w_0 .

It can be shown that g exists, is uniquely determined by these properties, and that $\partial_w g$ is independent of w_0 , so we will henceforth drop the point w_0 in the notation for g . It also follows from symmetry properties of g that it is also harmonic in z away from the poles.

Let g_Ω be the Green’s function of Ω , in the usual sense ($g_\Omega(z, p)$ has a logarithmic singularity at p and has a continuous extension to $\partial\Omega$ which vanishes there). For fixed $p \in \Omega$ let $\Gamma_\epsilon(\Omega, p)$ denote the level curves of the Green’s function $g_\Omega(\cdot, p)$, for ϵ sufficiently close to 0. For fixed $q \notin \Gamma$, we define the integral operator

$$J_q(\Gamma) : \mathcal{D}_{\text{harm}}(\Omega) \longrightarrow \mathcal{D}_{\text{harm}}(\Omega \cup \Sigma)_q$$

$$h \longmapsto - \lim_{\epsilon \searrow 0} \frac{1}{\pi i} \int_{\Gamma_\epsilon(\Omega, p)} \partial_w g(w; z, q) h(w) \quad (2.3)$$

where by $\mathcal{D}_{\text{harm}}(\Omega \cup \Sigma)_q$ we mean the set of harmonic functions on the (disjoint) union of Ω and Σ , which have bounded Dirichlet norm, and vanish at q . The operator appears to be dependent on p , but below we see that it is in fact independent.

We then have the following results.

Theorem 2.1 ([14]). *Let R be a compact Riemann surface with Green’s function g , and let Γ be a quasicircle separating R into connected components Ω and Σ , with Ω simply-connected. Fix $q \notin \Gamma$ and $p \in \Omega$. Let $J_q(\Gamma)$ be defined by (2.3), where $\Gamma_\epsilon(\Omega, p)$ are the level curves $\{w : g_\Omega(w, p) = \epsilon\}$ of*

the Green's function g_Ω . Then $J_q(\Gamma)$ is independent of the choice of $p \in \Omega$, and bounded with respect to the Dirichlet semi-norm.

Finally, we define the Schiffer operators, which are closely related to $J_q(\Gamma)$:

$$T(\Omega, \Sigma) : \overline{A(\Omega)} \longrightarrow A(\Sigma)$$

$$\alpha \longmapsto -\frac{1}{\pi i} \iint_{\Omega} \partial_z \partial_w g(w; z, q) \wedge_w \alpha(w); \quad z \in \Sigma$$

and

$$T(\Omega, \Omega) : \overline{A(\Omega)} \longrightarrow A(\Omega)$$

$$\alpha \longmapsto -\frac{1}{\pi i} \iint_{\Omega} \partial_z \partial_w g(w; z, q) \wedge_w \alpha(w); \quad z \in \Omega$$

where in both cases \wedge_w denotes wedge product in the variable of integration w . These are bounded operators [14].

2.3. Boundary values and transmission of harmonic functions. We will use a conformally invariant notion of non-tangential boundary values which is well-adapted to the Dirichlet spaces. In the remainder of this section, we define this notion and summarize the necessary results. Proofs and details can be found in [13].

We say that U is a collar neighbourhood of $\partial\Sigma$ if it is an open set bounded by $\partial\Sigma$ and a Jordan curve $\Gamma' \subseteq \Sigma$ which is isotopic to $\partial\Sigma$ from within the closure of U .

Let g_Σ denote the Green's function of Σ , and for $p \in \Sigma$, let $\Gamma_\epsilon(\Sigma, p)$ denote the level set $\{z : g_\Sigma(z, p) = \epsilon\}$ for $\epsilon > 0$. For ϵ sufficiently small $\Gamma_\epsilon(\Sigma, p)$ is an analytic curve.

For r sufficiently close to 0, the Green's function induces a biholomorphic chart $\phi : A_r \rightarrow \mathbb{A}$ from a collar neighbourhood A_r of $\partial\Sigma$ onto an annulus

$$\mathbb{A} = \{r < |z| < 1\}$$

for some $r < 1$, in such a way that the level curves $\Gamma_\epsilon(\Sigma, p)$ map to circles of radius less than one. If we consider $\partial\Sigma$ as an analytic curve in Σ^d , then this chart has an analytic extension to an open neighbourhood of the closure of A_r , which takes $\partial\Sigma$ to $|z| = 1$. In any case, one can speak without ambiguity of the continuous extension of ϕ to $\partial\Sigma$ which takes it homeomorphically to $|z| = 1$. For p fixed, this chart is unique up to multiplication by a unit modulus constant and the choice of r . We call this chart the canonical collar chart with respect to (Σ, p) .

We say that a closed subset I of $\partial\Sigma$ is null with respect to Σ if $\phi(I) \subseteq \mathbb{S}^1$ has logarithmic capacity zero where ϕ is a canonical chart with respect to (Σ, p) . This is independent of p , so the terminology "null with respect to Σ " is well-defined.

Any function $h \in \mathcal{D}_{\text{harm}}(\Sigma)$ has boundary values in a certain non-tangential sense which we now describe. Let

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

denote the unit disk in the complex plane \mathbb{C} , and let $W(q, M)$ be a non-tangential wedge in \mathbb{D} with terminal point at q . That is,

$$W(q, M) = \{z \in \mathbb{D} : |q - z| < M(1 - |z|)\}$$

where the angular width of the wedge is determined by $M \in (1, \infty)$. Fix $p \in \Sigma$ and let $\phi : U \rightarrow \mathbb{A}$ be a canonical chart with respect to (Σ, p) . For all M , the limit

$$\lim_{\substack{z \rightarrow q \\ z \in W(q, M) \cap \mathbb{A}}} h \circ \phi^{-1}$$

exists for all $q \in \mathbb{S}^1$ except possibly on a set of logarithmic capacity zero. Since ϕ^{-1} extends continuously to \mathbb{S}^1 , the limit

$$\lim_{\substack{z \rightarrow \phi^{-1}(q) \\ z \in \phi^{-1}(W(q, M) \cap \mathbb{A})}} h(z)$$

exists except possibly on a null set in $\partial\Sigma$ with respect to Σ . The existence of the limit at q is independent of the choice of p . In this case, we say that the limit exists conformally non-tangentially quasi-everywhere on $\partial\Sigma$. We will usually drop the term “quasi-everywhere”, and abbreviate “conformally non-tangentially” by “CNT”.

Remark 2.2. It follows immediately from the definition that conformally non-tangential and non-tangential limits are the same in the case of the unit disk \mathbb{D} . From this it is not hard to deduce that they are the same for analytic curves.

If two functions in $\mathcal{D}_{\text{harm}}(\Sigma)$ have the same CNT boundary values except possibly on a null set, then they are equal. Consider thus the set of functions $h : \Gamma \rightarrow \mathbb{C}$ defined quasi-everywhere. We say that two such functions h_1 and h_2 are equivalent, $h_1 \sim h_2$, if they agree quasi-everywhere. We define $\mathcal{H}(\Gamma, \Sigma)$ to be the set of functions which are CNT boundary values of elements of $\mathcal{D}_{\text{harm}}(\Sigma)$ quasi-everywhere, modulo the equivalence relation \sim .

Now let Σ_1 and Σ_2 be biholomorphically equivalent bordered Riemann surfaces each with one border homeomorphic to \mathbb{S}^1 . Let $f : \Sigma_1 \rightarrow \Sigma_2$ be a holomorphic bijection. Then f extends continuously to the closure Σ_1 so that the restriction of f to $\partial\Sigma_1$ is a homeomorphism between the borders $\partial\Sigma_1$ and $\partial\Sigma_2$. It follows immediately from the conformal invariance of Green’s functions that a closed set $I \subset \Sigma_1$ is null with respect to Σ_1 if and only if $f(I)$ is null with respect to Σ_2 . The composition operator

$$\begin{aligned} \mathcal{C}_f : \mathcal{D}_{\text{harm}}(\Sigma_2) &\longrightarrow \mathcal{D}_{\text{harm}}(\Sigma_1) \\ h &\longmapsto h \circ f \end{aligned}$$

is a semi-norm preserving bijection. Furthermore, by conformal invariance of the notion of CNT boundary values, if u are the boundary values of

$h \circ f$ quasi-everywhere on $\partial\Sigma_1$ and v are the boundary values of h quasi-everywhere on $\partial\Sigma_2$, then $u = v \circ f$. Thus we have the well-defined composition map

$$\begin{aligned} \mathcal{C}_f : \mathcal{H}(\partial\Sigma_2) &\longrightarrow \mathcal{H}(\partial\Sigma_1) \\ h &\longmapsto h \circ f. \end{aligned}$$

We summarize some necessary facts in the following theorem, using the set-up from Section 2.1.

Theorem 2.3 ([13, Theorem 3.29]). *Let R be a compact Riemann surface, and let Γ be a quasicircle which separates R into two connected components Σ_1 and Σ_2 .*

- (1) *A set $I \subset \Gamma$ is null with respect to Σ_1 if and only if it is null with respect to Σ_2 .*
- (2) *Given $h_1 \in \mathcal{D}_{\text{harm}}(\Sigma_1)$, there is a unique $h_2 \in \mathcal{D}_{\text{harm}}(\Sigma_2)$ whose CNT boundary values agree with those of h_1 quasi-everywhere; similarly an $h_2 \in \mathcal{D}_{\text{harm}}(\Sigma_2)$ uniquely determines an $h_1 \in \mathcal{D}_{\text{harm}}(\Sigma_1)$ with the same boundary values up to a null set.*
- (3) *The maps*

$$\mathfrak{D}(\Sigma_1, \Sigma_2) : \mathcal{D}_{\text{harm}}(\Sigma_1) \longrightarrow \mathcal{D}_{\text{harm}}(\Sigma_2)$$

and

$$\mathfrak{D}(\Sigma_2, \Sigma_1) : \mathcal{D}_{\text{harm}}(\Sigma_2) \longrightarrow \mathcal{D}_{\text{harm}}(\Sigma_1)$$

defined by (2) are bounded with respect to the Dirichlet semi-norms. We refer to these operators as transmission operators.

Thus, if $\Gamma = \partial\Sigma_1 = \partial\Sigma_2$ is a quasicircle, we may unambiguously define $\mathcal{H}(\Gamma)$ without reference to Σ_1 or Σ_2 .

3. Jump decomposition and the Faber and Grunsky operators

3.1. Outline of the section. In this Section, we obtain a jump decomposition on quasicircles in Riemann surfaces, and a related isomorphism which we later show to be inverse to $\pi_{(R,f)}$. We also show how this isomorphism relates to natural generalizations of the Faber and Grunsky operators of classical function theory and approximation theory.

3.2. Jump decomposition on quasicircles in Riemann surfaces. Let R, Γ, Σ and Ω be as in Section 2.1. In this section we gather theorems on the jump decomposition on such quasicircles Γ . We show that the jump decomposition induces an isomorphism between a certain subset of $\mathcal{D}(\Omega)$ and $\mathcal{D}(\Sigma)$. This isomorphism plays a central role in the description of the determinant lines.

Let $H \in \mathcal{H}(\Gamma)$. By a jump decomposition of H , we mean a pair of functions $h_\Omega \in \mathcal{D}(\Omega)$ and $h_\Sigma \in \mathcal{D}(\Sigma)$ with boundary values u_Ω and u_Σ respectively, such that

$$H = u_\Sigma - u_\Omega \tag{3.1}$$

except possibly on a null set. The jump decomposition on Riemann surfaces is known to exist classically for more regular functions and curves, see for example [2, 11], providing that H satisfies a certain algebraic condition. Assuming for the moment that Γ and H are smooth, this algebraic condition states that H has a decomposition if and only if

$$\int_\Gamma H\alpha = 0 \tag{3.2}$$

for all holomorphic one-forms α on R .

We must phrase this condition on quasicircles, which of course can be highly non-smooth, in terms of Dirichlet spaces and $\mathcal{H}(\Gamma)$. We do this now using results of [14]. For fixed $p \in \Omega$, define

$$W = \left\{ h \in \mathcal{D}_{\text{harm}}(\Omega) : \lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon(\Omega, p)} h\alpha = 0 \quad \forall \alpha \in A(R) \right\}.$$

Note that by Stokes' theorem, this integral can be expressed as an integral over Ω of a two form (see equation (3.4) below). In particular it is independent of p .

Remark 3.1. One might also consider the set

$$W_\Sigma = \left\{ h \in \mathcal{D}_{\text{harm}}(\Sigma) : \lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon(\Sigma, p)} h\alpha = 0 \quad \forall \alpha \in A(R) \right\}$$

for fixed $p \in \Sigma$. As expected, we have that $\mathfrak{D}(\Sigma, \Omega)W_\Sigma = W$ [14].

Since every element of $\mathcal{D}(\Omega)$ has a jump decomposition (namely, set u_Σ to be zero on Σ), it suffices to focus on $\overline{\mathcal{D}(\Omega)}$. Thus we define

$$W' = W \cap \overline{\mathcal{D}(\Omega)}.$$

Fixing $p \in \Omega$, we define in the usual way $W'_p = \{h \in W' : h(p) = 0\}$. The choice of p is arbitrary. Note that we have the decomposition

$$W = W'_p \oplus \mathcal{D}(\Omega)$$

which is orthogonal with respect to the inner product on $\mathcal{D}_{\text{harm}}(\Omega)$.

Let $B(R)$ denote the set of primitives in Ω of the restrictions to Ω of elements of $A(R)$. That is

$$B(R) = \{h \in \mathcal{D}(\Omega) : \exists \alpha \in A(R) \text{ such that } \partial h = \alpha|_\Omega\}$$

and as above let $B(R)_p$ denote the set of $h \in B(R)$ which vanish at p . For a linear space U in $\mathcal{D}_{\text{harm}}(\Omega)$ let U^\perp denote the set of $h \in \mathcal{D}_{\text{harm}}(\Omega)$ such

that $(h, g) = 0$ for all $g \in U$. This is not strictly speaking an orthogonal complement, since constants are in U^\perp for any U . We claim that

$$\overline{B(R)}^\perp = W. \quad (3.3)$$

To see this, let $h \in \overline{B(R)}^\perp$. Given any $\alpha \in A(R)$, let $g \in \mathcal{D}(\Omega)$ be such that $\partial g = \alpha$ on Ω .

$$\lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon(\Omega, p)} h \alpha = \iint_{\Omega} dh \wedge \partial g = 0 \quad (3.4)$$

so $h \in W$. Conversely if $h \in W$, then any $g \in \overline{B(R)}$ is orthogonal to h by the same computation.

We then have the following orthogonal decomposition in $\mathcal{D}_{\text{harm}}(\Omega)$:

$$\mathcal{D}_{\text{harm}}(\Omega) = \overline{B(R)}_p \oplus W'_p \oplus \mathcal{D}(\Omega) = \overline{B(R)}_p \oplus W. \quad (3.5)$$

For later use, denote the orthogonal projection onto $\overline{B(R)}_p$ by

$$\overline{P}_{\overline{B(R)}_p} : \mathcal{D}_{\text{harm}}(\Omega) \longrightarrow \overline{B(R)}_p$$

Theorem 3.2. *Fix $q \in \Sigma$. The map*

$$\begin{aligned} \mathfrak{J}_q(\Gamma) : W &\longrightarrow \mathcal{D}(\Omega) \oplus \mathcal{D}_q(\Sigma) \\ h &\longmapsto (J_q(\Gamma)h|_\Omega, J_q(\Gamma)h|_\Sigma) \end{aligned}$$

is an isomorphism.

Proof. This is a special case of [14, Theorem 4.29]. □

Furthermore, if we set

$$h_\Omega = J_q(\Gamma)h|_\Omega$$

and

$$h_\Sigma = J_q(\Gamma)h|_\Sigma$$

then if $H \in \mathcal{H}(\Gamma)$ is the CNT boundary values of $h \in \mathcal{D}_{\text{harm}}(\Omega)$ then h_Ω and h_Σ are solutions to the jump problem (3.1). These are the unique solutions up to a constant [14, Corollary 4.31].

There is another related isomorphism. Let

$$\begin{aligned} K_{p,q} : W'_p &\longrightarrow \mathcal{D}(\Sigma)_q \\ h &\longmapsto J_q(\Gamma)h|_\Sigma. \end{aligned} \quad (3.6)$$

We then have the following theorem.

Theorem 3.3. *Fix $q \in \Sigma$ and $p \in \Omega$. Then $K_{p,q}$ is an isomorphism with inverse $-\overline{P}(\Omega)_p \mathfrak{D}(\Sigma, \Omega)$.*

Proof. Let $A(\Sigma)_e \subseteq A(\Sigma)$ denote the set of those one-forms in $A(\Sigma)$ which are exact. Since Ω is simply connected, $A(\Omega)_e = A(\Omega)$. Let $\bar{\delta} : \overline{A(\Omega)} \rightarrow \mathcal{D}(\Omega)_p$ be the unique inverse of $\bar{\partial} : \mathcal{D}(\Omega)_p \rightarrow \overline{A(\Omega)}$. We then have by [14, Theorem 4.2] that

$$T(\Omega, \Sigma) = \partial K_{p,q} \bar{\delta} : \overline{A(\Omega)} \longrightarrow A(\Sigma)_e.$$

Applying [14, Theorem 4.25] with $V_1 = \bar{\partial}W_1$, $\Sigma_1 = \Omega$ and $\Sigma_2 = \Sigma$, we see that $T(\Omega, \Sigma)$ is an isomorphism. Since the restriction of $\bar{\delta}$ to V_1 is an isomorphism onto W' , and $\partial : \mathcal{D}(\Sigma)_q \rightarrow A(\Sigma)_e$ is an isomorphism, this shows that $K_{p,q}$ is an isomorphism.

To prove the second claim, observe by [14, Theorem 4.16] with $\Sigma_1 = \Omega$ and $\Sigma_2 = \Sigma$ as above, we have for all $\bar{h} \in W'$ that (denoting restriction to Ω and Σ by $[\cdot]_\Omega$ and $[\cdot]_\Sigma$)

$$\bar{h} = [J_q(\Gamma)\bar{h}]_\Omega - \mathfrak{D}(\Sigma, \Omega) [J_q(\Gamma)\bar{h}]_\Sigma. \tag{3.7}$$

Applying $\overline{P}(\Omega)_p$ to both sides, using the facts that the output of $J_q(\Gamma)$ applied to functions in W' is holomorphic and that $\overline{P}(\Omega)_p \bar{h} = \bar{h}$ we see that

$$\bar{h} = -\overline{P}(\Omega)_p \mathfrak{D}(\Sigma, \Omega) [J_q(\Gamma)\bar{h}]_\Sigma = -\overline{P}(\Omega)_p \mathfrak{D}(\Sigma, \Omega) K_{p,q} \bar{h}.$$

That is, $-\overline{P}(\Omega)_p \mathfrak{D}(\Sigma, \Omega)$ is a left inverse of $K_{p,q}$ as claimed. Since we have already shown that $K_{p,q}$ is invertible, this proves that it is also a right inverse. \square

Equation (3.7) combines the jump decomposition (3.1) with transmission. By doing so, it allows us to recognize the relation of the decomposition of $\mathcal{D}_{\text{harm}}(\Omega)$ into holomorphic and anti-holomorphic functions to the jump formula. This relation reveals the isomorphism $K_{p,q}$.

3.3. Faber isomorphism and Grunsky operator. Define the following “generalized Faber operator” and “generalized Grunsky operator”. Let $f : \mathbb{D} \rightarrow \Omega$ be a conformal map. Set $f(0) = p$. Define

$$V_f = \mathcal{C}_f W'_p \subseteq \overline{\mathcal{D}(\mathbb{D})}.$$

Observe that elements of V_f vanish at 0. We now define the operator

$$\begin{aligned} I_f : V_f &\longrightarrow \mathcal{D}_q(\Sigma) \\ h &\longmapsto -K_{p,q}(\Sigma) \mathcal{C}_{f^{-1}} h. \end{aligned}$$

We then have

Theorem 3.4. *Let R be a compact Riemann surface, and Γ be a quasicircle separating R into components Ω and Σ . Assume that Ω is simply connected and $f : \mathbb{D} \rightarrow \Omega$ is conformal. Then I_f is an isomorphism, with inverse $P(\mathbb{D})_0 \mathcal{C}_f \mathfrak{D}(\Sigma, \Omega)$.*

Proof. First observe that because \mathcal{C}_f is an isometry, $\overline{P}(\mathbb{D})\mathcal{C}_f = \mathcal{C}_f\overline{P}(\Omega)$. Thus since $f(0) = p$, $\overline{P}(\mathbb{D})_0\mathcal{C}_f = \mathcal{C}_f\overline{P}(\Omega)_p$. By Theorem 3.3 we have that

$$\overline{P}(\mathbb{D})_0\mathcal{C}_f\mathfrak{D}(\Sigma, \Omega)I_f = -\mathcal{C}_f\overline{P}(\Omega)_p\mathfrak{D}(\Sigma, \Omega)K_{p,q}\mathcal{C}_{f^{-1}} = \text{Id}.$$

□

Define also

$$\begin{aligned} \text{Gr}_f : V_f &\longrightarrow \mathcal{D}(\mathbb{D})_0 \\ h &\longrightarrow P(\mathbb{D})_0\mathcal{C}_f\mathfrak{D}(\Sigma, \Omega)I_f h. \end{aligned}$$

We have a generalization of the Grunsky inequalities.

Theorem 3.5. *Let R be a compact surface, and let Γ be a quasicircle separating R into two components Ω and Σ such that Ω is simply connected. Let V_f be as above and let $f : \mathbb{D} \rightarrow \Omega$ be a conformal bijection and let Gr_f be the associated Grunsky operator on V_f . Then*

$$\left\| \text{Gr}_f|_{V_f} \right\| < 1.$$

Proof. Set $p = f(0)$ as above. Let $\overline{H} \in V_f$, so that $\mathcal{C}_{f^{-1}}\overline{H} \in W'_p$. By (3.7) we have that

$$\mathcal{C}_{f^{-1}}\overline{H} = [J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_{\Omega} - \mathfrak{D}(\Sigma, \Omega) ([J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_{\Sigma}).$$

Now applying $P(\mathbb{D})_0\mathcal{C}_f$ to both sides we obtain that (using the fact that \overline{H} vanishes at 0)

$$0 = P(\mathbb{D})_0\overline{H} = P(\mathbb{D})_0\mathcal{C}_f ([J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_{\Omega}) - P(\mathbb{D})_0\mathcal{C}_f\mathfrak{D}(\Sigma, \Omega) ([J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_{\Sigma}).$$

The projection operator in the first term on the right hand side acts as the identity since the function in brackets is holomorphic. Thus

$$\text{Gr}_f\overline{H} = \mathcal{C}_f ([J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_{\Omega}) + \text{constant}.$$

Define the operator $\hat{\mathcal{C}}_f : A_{\text{harm}}(\Omega) \rightarrow A_{\text{harm}}(\mathbb{D})$ to be the unique operator satisfying $\hat{\mathcal{C}}_f d = d\mathcal{C}_f$. Now differentiating, by [14, Theorem 4.2] we obtain

$$\begin{aligned} \partial\text{Gr}_f\overline{H} &= \partial\mathcal{C}_f ([J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_{\Omega}) \\ &= \hat{\mathcal{C}}_f\partial ([J_q(\Gamma)\mathcal{C}_{f^{-1}}\overline{H}]_{\Omega}) \\ &= \hat{\mathcal{C}}_f (\partial\mathcal{C}_{f^{-1}}\overline{H} + T(\Omega, \Omega)\bar{\partial}\mathcal{C}_{f^{-1}}\overline{H}) \\ &= \hat{\mathcal{C}}_f T(\Omega, \Omega)\bar{\partial}\mathcal{C}_{f^{-1}}\overline{H}. \end{aligned}$$

Since $\mathcal{C}_{f^{-1}}\overline{H} \in W'_p$ and $\hat{\mathcal{C}}_f$ is norm-preserving,

$$\left\| T(\Omega, \Omega)|_{W'_p} \right\| < 1$$

by [14, Theorem 4.23]. This completes the proof. □

The Grunsky matrix thus can be seen as a generalization of the period matrix to surfaces with one border. In the case of genus zero surfaces with n boundary curves this was demonstrated in [9], and holomorphicity as a function on Teichmüller space was demonstrated in [8].

4. Applications to the determinant line bundle

4.1. Outline of the section. In Section 4.2 we define and briefly discuss the rigged moduli space. In Section 4.3 we define the projections corresponding to the splitting of Fourier series into positive and negative powers, induced by the rigging. We also characterize the set of boundary values of the Dirichlet space of $R \setminus \text{cl } f(\mathbb{D})$ as the graph of the Grunsky operator. In Section 4.4, we define the operator $\pi_{(R,f)}$ and give an explicit inverse. Finally we characterize its kernel, cokernel, and determinant line.

4.2. The moduli space of rigged Riemann surfaces. A central object in conformal field theory is the rigged moduli space, which can be modelled as the set of compact Riemann surfaces with n punctures (or equivalently, distinguished points), together with n mappings from the unit disk into the surface, modulo an equivalence relation which we shortly describe. Typically, the mappings are assumed to have an analytic or smooth continuation to the unit circle. Radnell, Schippers, and Staubach in various combinations have shown that there are good analytic and geometric reasons for extending this class analytically; a thorough discussion of related results can be found in [7].

Specialized to the situation at hand, the rigged moduli space of genus g surfaces with one rigging is

$$\widetilde{\mathcal{M}}(g, 1) = \{(R, f)\} / \sim$$

where R is a compact surface of genus g with one puncture p say, $f : \mathbb{D} \rightarrow R$ is a one-to-one holomorphic map with a quasiconformal extension to an open neighbourhood of the closure of \mathbb{D} , such that $f(0) = p$. We say that $(R_1, f_1) \sim (R_2, f_2)$ if there is a biholomorphism $g : R_1 \rightarrow R_2$ such that $g \circ f_1 = f_2$. The conditions on f guarantee that $f(\mathbb{S}^1)$ is a special type of Jordan curve called a quasicircle.

The determinant line bundle is the bundle of determinant lines over the rigged moduli space of a certain operator $\pi_{(R,f)}$ (see 4.4 below). For each element of the moduli space, this operator is defined on spaces of holomorphic functions determined by a suitable representative (R, f) in the equivalence class. The bundle structure is analytically non-trivial, because the moduli space is a Banach manifold. See the introduction for further discussion.

4.3. Decompositions of boundary values and the Grunsky operator. Consider $\mathcal{H}(\mathbb{S}^1)$, which by definition consists of the set of conformally non-tangential boundary values of elements of $\mathcal{D}_{\text{harm}}(\mathbb{D})$ up to null sets. Define $\mathcal{H}_+(\mathbb{S}^1)$ to be the set of elements of $\mathcal{H}(\mathbb{S}^1)$ which are CNT boundary

values of elements of $\mathcal{D}(\mathbb{D})$, and $\mathcal{H}_-(\mathbb{S}^1)$ to be the set of CNT boundary values of elements of $\overline{\mathcal{D}(\mathbb{D})}_0$. Then the decomposition

$$\mathcal{D}_{\text{harm}}(\mathbb{D}) = \mathcal{D}(\mathbb{D}) \oplus \overline{\mathcal{D}(\mathbb{D})}_0 \quad (4.1)$$

induces a direct sum decomposition

$$\mathcal{H}(\mathbb{S}^1) = \mathcal{H}_+(\mathbb{S}^1) \oplus \mathcal{H}_-(\mathbb{S}^1). \quad (4.2)$$

An element

$$h(z) = \sum_{n=0}^{\infty} h_n z^n + \sum_{n=1}^{\infty} h_{-n} \bar{z}^n \in \mathcal{D}_{\text{harm}}(\mathbb{D})$$

has power series satisfying

$$\sum_{n=-\infty}^{\infty} n |h_n|^2 < \infty,$$

so the Fourier series

$$\sum_{n=-\infty}^{\infty} h_n e^{in\theta}$$

converges except on a set I of outer logarithmic capacity zero [17, Chapter XIII, Theorem 11.3]. Thus

$$\mathcal{H}_+(\mathbb{S}^1) = \left\{ h(z) = \sum_{n=0}^{\infty} h_n e^{in\theta} \in \mathcal{H}(\mathbb{S}^1) \right\}$$

and

$$\mathcal{H}_-(\mathbb{S}^1) = \left\{ h(z) = \sum_{n=-\infty}^{-1} h_n e^{in\theta} \in \mathcal{H}(\mathbb{S}^1) \right\}.$$

Now let (R, f) be a representative of an element of the rigged moduli space $\widetilde{\mathcal{M}}(g, 1)$. We let $f(\mathbb{D}) = \Omega \subseteq R$, $\Sigma = R \setminus \text{cl}f(\mathbb{D})$, and $\Gamma = f(\mathbb{S}^1) = \partial\Omega = \partial\Sigma$. By our assumptions, Γ is a quasicircle. We also denote the puncture of R by $p = f(0)$.

If we would like to describe the set of possible Fourier series arising as the boundary values of holomorphic function on Σ , we need to parametrize Γ . We obtain a parametrization from $f : \mathbb{D} \rightarrow \Omega$. Observe that f has a homeomorphic extension to \mathbb{S}^1 . Let $\mathcal{H}(\Gamma)$ denote the set of complex-valued functions which are the boundary values quasi-everywhere of functions in $\mathcal{D}_{\text{harm}}(\Omega)$, or equivalently of functions in $\mathcal{D}_{\text{harm}}(\Sigma)$ by Theorem 2.3.

We now define the subspaces

$$\mathcal{H}_+(\Gamma, f) = \{h \in \mathcal{H}(\Gamma) : h \circ f \in \mathcal{H}_+(\mathbb{S}^1)\}$$

and

$$\mathcal{H}_-(\Gamma, f) = \{h \in \mathcal{H}(\Gamma) : h \circ f \in \mathcal{H}_-(\mathbb{S}^1)\}$$

and the corresponding projections

$$P_{\pm}(\Gamma, f) : \mathcal{H}(\Gamma) \longrightarrow \mathcal{H}_{\pm}(\Gamma, f).$$

Thus the parametrization induces a notion of Fourier series of (sufficiently regular) functions on Γ and a decomposition into positive and negative parts. By conformal invariance of the Dirichlet spaces and the notion of CNT boundary values, we can also identify $\mathcal{H}_+(\Gamma)$ as the set of CNT boundary values of $\mathcal{D}(\Omega)$ and $\mathcal{H}_-(\Gamma)$ as the set of CNT boundary values of $\overline{\mathcal{D}(\Omega)}_0$.

The pull-back of the decomposition

$$\mathcal{H}(\Gamma) = \mathcal{H}_+(\Gamma, f) \oplus \mathcal{H}_-(\Gamma, f)$$

is the decomposition (4.2), which can be identified with the holomorphic/anti-holomorphic decomposition of the harmonic Dirichlet space of the disk (4.1). Using the characterization of $\mathcal{D}(\Sigma)$ as the image of I_f given by Theorem 3.4 we can thus interpret the pull-back of $\mathcal{D}(\Sigma)$ under \mathcal{C}_f as the graph of the Grunsky operator.

Theorem 4.1. *Assume that Γ be a quasicircle separating R into components Ω and Σ , and that Ω is simply connected. Let $f : \mathbb{D} \rightarrow \Omega$ be a conformal bijection. Then $\mathcal{C}_f \mathcal{D}_q(\Sigma)$ is the graph of Gr_f in $V_f \oplus \mathcal{D}(\mathbb{D})_0$.*

Proof. This follows directly from Theorem 3.4 and the definition of Gr_f . \square

$\mathcal{D}(\Sigma)$ was also shown to be the graph of a generalized Grunsky operator in the case that Σ is a genus-zero surface bordered by n curves homeomorphic to \mathbb{S}^1 [9]. A different approach to the Grunsky matrix for higher-genus surfaces with one boundary curve, using a formulation of Faber polynomials due to H. Tietz [16], appears in K. Reimer and E. Schippers [10].

4.4. The operator $\pi_{(R,f)}$ and its determinant line. Consider the projection operator

$$\pi_{(R,f)} : \mathcal{D}(\Sigma) \rightarrow \mathcal{H}_-(\partial\Sigma, f)$$

induced by the decomposition

$$\mathcal{H}(\partial\Sigma) = \mathcal{H}_+(\partial\Sigma, f) \oplus \mathcal{H}_-(\partial\Sigma, f). \tag{4.3}$$

As explained in the introduction, this is the operator whose determinant line we are to characterize. We will consider a slight variant which maps into the isomorphic space of holomorphic functions with bounded Dirichlet semi-norm, as explained below.

We give $\pi_{(R,f)}$ a simple description in terms of the decomposition on $\mathcal{D}_{\text{harm}}(\Omega)$, which has the advantage that it is orthogonal with respect to the L^2 -type norm. Let

$$b(\Omega) : \mathcal{D}_{\text{harm}}(\Omega) \longrightarrow \mathcal{H}(\Gamma)$$

and

$$b(\Sigma) : \mathcal{D}_{\text{harm}}(\Sigma) \longrightarrow \mathcal{H}(\Gamma)$$

be the vector space isomorphisms obtained by taking the CNT boundary values. By definition we then have the identities

$$\begin{aligned}\mathfrak{D}(\Sigma, \Omega) &= b(\Omega)^{-1}b(\Sigma) \\ \mathfrak{D}(\Omega, \Sigma) &= b(\Sigma)^{-1}b(\Omega).\end{aligned}\tag{4.4}$$

Observe also that $b(\Omega)$ takes the holomorphic/antiholomorphic (orthogonal) decomposition to the decomposition 2.2. That is,

$$b(\Omega)|_{\overline{\mathcal{D}(\Omega)}_p} : \overline{\mathcal{D}(\Omega)}_p \longrightarrow \mathcal{H}_-(\Sigma, f)$$

and

$$b(\Omega)|_{\mathcal{D}(\Omega)} : \mathcal{D}(\Omega) \longrightarrow \mathcal{H}_+(\Sigma, f)$$

are isomorphisms.

Using (4.4) and the fact that

$$P_-(\Gamma, f) = b(\Omega)\overline{P}(\Omega)_p b(\Omega)^{-1}$$

we see that

$$\begin{aligned}\pi_{(R,f)} &= P_-(\Gamma, f)b(\Sigma)|_{\mathcal{D}(\Sigma)} \\ &= b(\Omega)\overline{P}(\Omega)_p b(\Omega)^{-1}b(\Sigma)|_{\mathcal{D}(\Sigma)} \\ &= b(\Omega)\overline{P}(\Omega)_p \mathfrak{D}(\Sigma, \Omega)|_{\mathcal{D}(\Sigma)}.\end{aligned}$$

In particular, since as observed above the restriction

$$b(\Omega)|_{\overline{\mathcal{D}(\Omega)}_p} \longrightarrow \mathcal{H}_-(\partial\Sigma, f)$$

is an isomorphism, we will restrict our attention to

$$\Pi_{(R,f)} = \overline{P}(\Omega)_p \mathfrak{D}(\Sigma, \Omega)|_{\mathcal{D}(\Sigma)}.$$

This has the advantage that we are able to make better use of the L^2 structure inherent in the Dirichlet space of functions and Bergman space of one-forms. This is an advantage both analytically (so that we can make easy use of certain integral operators), and algebraically, because we can make use of Corollary 4.2.

This leads to our first result which will give a characterization of the cokernel.

Corollary 4.2. *Let $K_{p,q}$ be the operator associated to R , Ω , and Σ by (3.6), and let W'_p be the subspace of $\mathcal{D}(\Omega)$ defined above. Then $-K_{p,q}|_{W'_p}$ is inverse to $\Pi_{(R,f)} = b(\Omega)^{-1}\pi_{(R,f)}|_{\mathcal{D}(\Sigma)_q}$.*

Proof. This follows directly from Theorem 3.3. \square

Remark 4.3. The construction of the operator $\pi_{(R,f)}$ that is carried out here agrees with that in the literature [3] in the case that we restrict to analytic parametrizations. Assume that the boundary Γ of Σ in R is smooth and the map $f : \mathbb{D} \rightarrow R$ extends to a smooth map of the closure of $\partial\mathbb{D}$ taking \mathbb{S}^1

analytically onto Γ . In this case the differentiable structure induced on $\partial\Sigma$ by Σ agrees with the differentiable structure of Γ as an embedded submanifold of R . Now let $H^s(\Sigma)$ denote the Sobolev space of order s on Σ , and similarly for $H^s(\Gamma)$. Then one can show that the completion of $\text{Hol}(\Sigma)$ in $H^1(\Sigma)$ can be identified with the Dirichlet space $\mathcal{D}(\Sigma)$, and $H^{1/2}(\Gamma)$ can be identified with $\mathcal{H}(\Gamma)$ which is the set of conformally nontangential boundary values of the elements of $\mathcal{D}(\Sigma)$. Using these facts one can then realize the operator $\pi_{(R,f)}$ defined by Y.-Z. Huang in [3, Appendix D], as a Fredholm operator from $\mathcal{D}(\Sigma) \rightarrow \mathcal{H}(\Gamma)$. Further discussion can be found in [7].

Now we give a canonical characterization of the cokernel of $\pi_{(R,f)}$, or equivalently of $\Pi_{(R,f)}$. That is, we will describe

$$\text{Coker}(\Pi_{(R,f)}) = \overline{\mathcal{D}(\Omega)}_p / \text{Im}(\Pi_{(R,f)}).$$

Since $\pi_{(R,f)}$ annihilates constants, one has

$$\text{Im}(\pi_{(R,f)}) = b(\Omega) \Big|_{\overline{\mathcal{D}(\Omega)}_p} \text{Im}(\overline{P}(\Omega)_p \mathfrak{D}(\Sigma, \Omega)).$$

Moreover, by Corollary 4.2, the restriction of $K_{p,q}$ to W'_{pu} is inverse to $-\overline{P}(\Omega)_p \mathfrak{D}(\Sigma, \Omega)$, and therefore

$$\text{Im}(\overline{P}(\Omega)_p \mathfrak{D}(\Sigma, \Omega)) = -\text{Im}(\overline{P}(\Omega)_p \mathfrak{D}(\Sigma, \Omega) K_{p,q}) = -W'_p = W'_p.$$

We now define

$$P_{\overline{A(R)}} : \text{Coker}(\Pi_{(R,f)}) \longrightarrow \overline{A(R)}$$

to be the map satisfying, for $[\overline{h}] \in \text{Coker}(\Pi_{(R,f)})$,

$$P_{\overline{A(R)}}[\overline{h}] \Big|_{\Omega} = \overline{\partial}(P_{\overline{B(R)}_p} \overline{h}).$$

That is, $P_{\overline{A(R)}}[\overline{h}]$ is the unique anti-holomorphic one-form on R extending the derivative of $P_{\overline{B(R)}_p} \overline{h}$. This map is well defined, since if $\overline{h}_1 - \overline{h}_2 \in \text{Im}(\Pi_{(R,f)}) = W'_p$, then it is orthogonal to $\overline{B(R)}_p$ by definition, and hence

$$P_{\overline{B(R)}_p}(\overline{h}_1 - \overline{h}_2) = 0.$$

Theorem 4.4. *Let (R, f) be a rigged Riemann surface of type $(g, 1)$. Then*

- (1) $\text{Im}(\Pi_{(R,f)}) = W'_p$;
- (2) $P_{\overline{A(R)}} : \text{Coker}(\Pi_{(R,f)}) \rightarrow \overline{A(R)}$ is an isomorphism;
- (3) $\text{Ker}(\Pi_{(R,f)})$ consists of the constant functions and is thus isomorphic to \mathbb{C} .

Proof. The first claim follows from Corollary 4.2 and the paragraph preceding the proof. The second claim follows from the first together with the orthogonal decomposition (3.5). To prove the third, assume that $\Pi_{(R,f)}h = 0$. Then $\overline{P}(\Omega)_p \mathfrak{D}(\Sigma, \Omega)h = 0$, and therefore $\mathfrak{D}(\Sigma, \Omega)h$ is holomorphic on Ω . By Theorems 4.13 and 4.16, and Proposition 4.30 in [14] this yields that h is constant on Σ . However since $h(q) = 0$, one has that $h = 0$. \square

We may rephrase this theorem in terms of $\pi_{(R,f)}$ as follows:

$$\begin{aligned}\operatorname{Im}(\pi_{(R,f)}) &= b(\Omega)W'_p; \\ \operatorname{Ker}(\pi_{(R,f)}) &= \operatorname{Ker}(\Pi_{(R,f)}) = \mathbb{C};\end{aligned}$$

and

$$P_{\overline{A(R)}}b(\Omega)^{-1} : \operatorname{Coker}(\pi_{(R,f)}) \longrightarrow \overline{A(R)}$$

is a well-defined isomorphism, which establishes the isomorphism between the cokernel of $\pi_{(R,f)}$ with the space of anti-holomorphic one forms.

Remark 4.5. Since the dimension of $A(R)$ is equal to the genus g of the Riemann surface R , we see that the dimension of $\operatorname{Coker}(\pi_{(R,f)})$ is g . Ultimately this comes from the g -dimensional obstruction (given in equation 3.2) to solving the jump problem. The connection of the cokernel to this obstruction was observed in D. Radnell [5]. In this paper we have made this connection explicit.

Corollary 4.6. *Let g be the genus of the compact Riemann surface R . Then*

$$\operatorname{Det}(\pi_{(R,f)}) \simeq \wedge^g \overline{A(R)}$$

where \simeq means canonically isomorphic and \wedge^g is the g th exterior power.

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(D. Radnell) DEPARTMENT OF MATHEMATICS AND SYSTEMS ANALYSIS, AALTO UNIVERSITY, P.O. BOX 11100, FI-00076 AALTO, FINLAND
david.radnell@aalto.fi

(E. Schippers) MACHRAY HALL, DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MANITOBA, WINNIPEG, MB R3T 2N2, CANADA
eric.schippers@umanitoba.ca

(M. Shirazi) DEPARTMENT OF MATHEMATICS AND STATISTICS, MCGILL UNIVERSITY, MONTREAL, QC H3A 0B9, CANADA
mohammad.shirazi@mcgill.ca

(W. Staubach) DEPARTMENT OF MATHEMATICS, UPPSALA UNIVERSITY, S-751 06 UPPSALA, SWEDEN
wulf@math.uu.se

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