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On complex surfaces with definite intersection form

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Abstract. A compact complex surface with positive definite intersection lattice is either the projective plane or a fake projective plane. If the intersection lattice is trivial or negative definite, the surface is either a secondary Kodaira surface, an elliptic surface with $b_1 = 1$, or a class VII surface. If the lattice is non-trivial, it is odd and diagonalizable over the integers. There are no other cases of surfaces where the intersection lattice is definite.

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1. Introduction

The intersection form of a compact connected orientable $4n$-dimensional manifold is bilinear, symmetric and, by Poincaré duality, unimodular. As is well known (cf. [Mil58, MH73]), if such a form is indefinite, its isometry class is uniquely determined by the signature and parity of the form. Recall that a form b has *even* parity if $b(x,x)$ is even for all elements x of the lattice and it has *odd* parity otherwise. Odd indefinite unimodular forms are diagonalizable over the integers, but unimodular even forms are evidently not diagonalizable. The reader may consult [Mil58, MH73] for more precise information.

In the definite situation, the situation is dramatically different: the number of isometry classes goes up drastically with the rank. See e.g. [Ser73, Ch. V.2.3]. So one might ask whether all definite forms occur as intersection forms. This is indeed the case for topological manifolds in view of the celebrated result [Fre82] by M. Freedman implying that every form can be realized as the intersection form of a simply connected compact oriented 4-manifold. Moreover,

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its oriented homeomorphism type is uniquely determined by the intersection form.

For differentiable 4-manifolds, Donaldson [Don83] proved that in the simply connected situation a definite form is diagonalizable. A little later, in [Don87], he proved this also for the non-simply connected case. Such differentiable 4-manifolds are easily constructed: take a connected sum of projective planes or projective planes with opposite orientation. In fact, almost all of these cannot have a complex structure. Indeed, e.g. [BHPvdV04, V. Thm. 1.1] implies that the only possible simply connected complex surface with a definite intersection form is the projective plane. Indeed, such a surface has $b_1 = 0$ and hence is Kähler (cf. Proposition 2.1), so that definiteness forces $b_2 = 1$ so that the intersection form is positive definite (implying that the hypotheses of loc. cit. are verified).

The main results of this note deal with complex surfaces having definite intersection forms. The convention here is that a zero form (for surfaces with $b_2 = 0$) is not called "definite”. However, since these do occur, explicit attention is given to this case, especially since their blow-ups have negative definite intersection form (cf. Lemma 3.1).

It turns out that the basic dichotomy is between even and odd b_1. For complex geometers, this is the dichotomy between Kähler and non-Kähler surfaces (see Proposition 2.1). The result in the Kähler case reads as follows:

Theorem 1.1. *Let X be a compact Kähler surface with a definite intersection form, then X is either the projective plane or a fake projective plane, that is a surface of general type with the same Betti numbers as $\mathbb{P}^2$. In these cases the intersection form is isometric to the (trivial) odd positive rank 1 form $(x, y) \mapsto xy$.*

This result is probably known to experts, but I am not aware of any proof in the literature. The proof ultimately rests on S. T. Yau's groundbreaking work [Yau77] which, for surfaces, gives a characterization of the fake planes as quotients of the complex 2-ball and so these have large fundamental groups. The first fake plane has been constructed by D. Mumford [Mum79]. A full classification has been given by G. Prasad and S.-T. Yeung [PY07]. In view of these results, one obtains a characterization of fake planes:

Corollary 1.2. *The only non-simply connected Kähler surfaces with a definite intersection form are the fake planes.*

For non-Kähler surfaces, the intersection form can also be negative definite. In this case, a distinction has to be made between minimal and non-minimal surfaces.[1] Non-minimal surfaces are obtained from minimal surfaces by repeatedly blowing up points. Each blowing up introduces an exceptional curve. The main theorem is as follows:

[1]Recall, a surface is *minimal* if it does not contain exceptional curves, i.e., rational curves of self-intersection (-1).

Theorem 1.3. *Let X be a compact non-Kähler surface with a definite intersection form. Then either X is a non-minimal surface of class VII with minimal model having $b_2 = 0$, a surface of class VII with $b_2 > 0$, a non-minimal secondary Kodaira surface, or a blown up properly elliptic surface whose minimal model has invariants $q = b_1 = 1$ and $b_2 = c_2 = 0$. In all cases the intersection form is negative definite and diagonalizable (and hence odd).*

For the (standard) terminology concerning surfaces, see [BHPvdV04, Ch. VI].

Remark 1.4. 1. The elliptic surfaces in the above theorem have been classified: each of these are deformations of some surface obtained from the product $\mathbb{P}_1 \times E$, E an elliptic curve, by doing logarithmic transformations in (lifts of) three torsion points of E with non-zero sum. (Combine the argument in [FM94] given at the beginning of section 2.7.7, and Thm. 7.7 in section. 2.7.2).
2. Donaldson's results are not used in the proof in the Kähler case, but instead the Bogomolov–Miyaoka–Yau inequality (cf. [BHPvdV04, §VII.4]) is invoked. For the non-Kähler situation, the Donaldson results can likewise be dispensed of, provided the Kato conjecture holds, i.e. *class VII surfaces with $b_2 > 0$ have global spherical shells*.

2. Basic facts from surface theory

It is well known that the Chern numbers $c_1^2(X)$ and $c_2(X)$ are topological invariants. This is obvious for c_2 since it is the Euler number. For c_1^2 this is a consequence of a special case of the index theorem [Hir66, Thm. 8.2.2] which for surfaces takes the shape

$$\tau(X) = \text{index of } X = \frac{1}{3}(c_1^2(X) - 2c_2(X)). \tag{1}$$

Here the index is the index of the intersection form of X. Also, Noether's formula (cf. [BHPvdV04, p. 26]) is used below. It is a special case of the Riemann–Roch formula and reads:

$$1 - q(X) + p_g(X) = \frac{1}{12}(c_1^2(X) + c_2(X)), \tag{2}$$

where $q(X) = \dim H^1(X, \mathcal{O}_X)$ and $p_g = \dim H^2(X, \mathcal{O}_X)$. Furthermore, an expression for the signature of the intersection form in terms of these invariants is made use of (cf. [BHPvdV04, Ch. IV.2–3]):

Proposition 2.1. *Let X be a compact complex surface. Then*

(1) *$b_1(X)$ is even and equal to $2q(X)$ if and only if X is Kähler. Otherwise $b_1(X) = 2q(X) - 1$.*
(2) *In the Kähler case the signature of the intersection form equals $(2p_g(X) + 1, b_2(X) - 2p_g(X) - 1)$ and $(2p_g(X), b_2(X) - 2p_g(X)$ otherwise.*

As a consequence, firstly, $q(X)$ and $p_g(X)$ are topological invariants. Secondly, for a Kähler surface the intersection form S_X can only be indefinite or positive definite while for a non-Kähler surface it can a priori be indefinite, positive definite or negative definite. It is positive definite if and only if $b_2 = 2p_g \neq 0$ and negative definite if and only if $p_g = 0$ and $b_2 \neq 0$.

The proof of the main results uses the Enriques–Kodaira classification which for the present purposes can be rephrased as follows (cf. [BHPvdV04, Ch. VI]):

Theorem 2.2 (Enriques–Kodaira classification). *Every compact complex surface belongs to exactly one of the following classes according to their Kodaira dimension κ. The invariants (c_1^2, c_2) are given for their minimal models:*

κ	*Class*		b_1	c_1^2	c_2
$-\infty$	*rational surfaces*	*Kähler*	0	8 *or* 9	4 *or* 3
	ruled surfaces of genus > 0	*Kähler*	$2g$	$8(1-g)$	$4(1-g)$
	class VII surfaces	*non-Kähler*	1	$-b_2$	b_2
0	*Two-dimensional tori*	*Kähler*	4	0	0
	K3 surfaces	*Kähler*	0	0	24
	primary Kodaira surfaces	*non-Kähler*	3	0	0
	secondary Kodaira surfaces	*non-Kähler*	1	0	0
	Enriques surfaces	*Kähler*	0	0	12
	bielliptic surfaces	*Kähler*	2	0	0
1	*properly elliptic surfaces*	*Kähler*	*even*	0	≥ 0
		non-Kähler	*odd*	0	≥ 0
2	*surfaces of general type*	*Kähler*	*even*	> 0	> 0

3. Proofs of Theorems 1.1 and 1.3

Let X be a compact complex surface, S_X the intersection form on the free $\mathbb{Z}$-module $\mathsf{H}_X = H^2(X, \mathbb{Z})/\text{torsion}$. So (H_X, S_X) is the intersection lattice of X.

Recall the (standard) notation concerning lattices:

- The rank 1 unimodular positive, respectively negative definite lattices are denoted $\langle 1 \rangle$ and $\langle -1 \rangle$ respectively.
- The hyperbolic plane U is the rank 2 lattice with basis $\{e, f\}$ and form (denoted by a dot) given by $e \cdot e = f \cdot f = 0, e \cdot f = 1$

For rational and ruled surfaces, the intersection forms are well known: for $\mathbb{P}^2$ it is $\langle 1 \rangle$, for the other minimal rational or ruled surfaces it is either $\langle 1 \rangle \oplus \langle -1 \rangle$ or U. See, for example, [Bea96, Prop. II.18, Prop. V.1.]. So, only $\mathbb{P}^2$ gives a definite intersection form and the other surfaces can be discarded for the proof of Theorem 1.1.

As to minimality, observe the following result:

Lemma 3.1. *If X is not minimal, then H_X is odd. If X_0 is a minimal model of X, then H_X is the orthogonal direct sum of H_{X_0} with as many copies of $\langle -1 \rangle$ as blow-ups from X_0 are needed to obtain X. If, moreover X is Kähler, H_X is indefinite.*

The reason is that if X is not minimal, the class of an exceptional curve splits off orthogonally whereas a Kähler class has positive self-intersection. This makes the latter somewhat easier to handle.

The Kähler case. One only has to consider positive definite forms. Then, by Proposition (2.1), one has $\tau = 2p_g + 1$. The index theorem (1) combined with the Noether formula (2) then yields the following expressions for c_1^2 and c_2:

$$\begin{aligned} c_1^2 &= 10p_g - 8q + 9 \\ c_2 &= 2p_g - 4q + 3 \end{aligned}$$

so that $c_1^2 - 3c_2 = 4(p_g + q)$. The class of surfaces with Kodaira dimension $-\infty$ has already been dealt with. From the table of the classification theorem 2.2, one sees that for surfaces with Kodaira dimension 0, 1 one has $c_1^2 - 3c_2 \leq 0$. For surfaces of general type, this is the Bogomolov–Miyaoka–Yau inequality named after [Bog78, Miy77, Yau77] (cf. [BHPvdV04, §VII.4]). Consequently, $p_q = q = 0$ and then necessarily $S_X \simeq \langle 1 \rangle$. Then, one also sees that $c_1^2 = 3c_2$. So, if $X \not\simeq \mathbb{P}^2$, it is of general type and, by [Yau77], X must have the complex unit ball as its universal covering, i.e. X is a fake plane.

Non-Kähler surfaces. The intersection form can either be positive definite or negative definite. In the former case, the index equals $\tau = 2p_g$ and in the latter $\tau = -b_2$ and $p_g = 0$. From the list of Theorem 2.2, the surfaces concerned are the class VII surfaces, the Kodaira surfaces and the properly elliptic surfaces.

- Minimal class VII surfaces with $b_2 = 0$. These include the Hopf surfaces [Hop48] and the Inoue surfaces [Ino74]. Hopf surfaces by definition have $\mathbb{C}^2 - \{0\}$ as their universal covering. Primary Hopf surfaces are diffeomorphic to $S^3 \times S^1$. Quotients of primary Hopf surfaces by a freely acting finite group are called secondary Hopf surfaces. Clearly, all such surfaces have trivial intersection lattice and non-minimal surfaces have negative definite intersection lattices.
- Minimal class VII surfaces with $b_2 \neq 0$. The list shows that $\tau = \frac{1}{3}(c_1^2 - 2c_2) = -c_2 = -b_2 < 0$. Since $p_g = 0$, the intersection form is negative definite. This remains so for non-minimal surfaces (Lemma 3.1). Minimal such surfaces have been constructed by Inoue in [Ino77]. M. Kato has shown in [Kat77] that these admit a holomorphically embedded copy of $\{z \in \mathbb{C}^2 \mid 1 - \epsilon < |z| < 1 + \epsilon\}$ for some $\epsilon > 0$, and for which, moreover, the complement in the surface is connected. Conversely, any such Kato surface, by definition a compact complex surface containing such a so-called "global spherical shell" must be of class VII and is a deformation of a blown up primary Hopf surface (recall, this is a complex surface diffeomorphic to $S^3 \times S^1$). Hence, the intersection form is diagonalizable and negative definite. By Donaldson's result [Don87], this is true for any class VII surface with $b_2 > 0$.

  It is conjectured that all class VII surfaces with $b_2 > 0$ are Kato surfaces, which would prove this directly. For recent work in this direction, consult [Tel17, DT20]

- By [BHPvdV04, Ch V.5] minimal Kodaira surfaces either have $b_2 = 4$ and $p_g = 1$ (primary Kodaira surfaces) or else $b_2 = 0, p_g = 0$ (secondary Kodaira surfaces). The former have signature $(2, 2)$ and since the form is even, it is isometric to $U \oplus U$. In particular, these need not be considered. Minimal secondary Kodaira surfaces have zero intersection form and so only non-minimal such surfaces have negative definite intersection form.
- Minimal non-Kähler elliptic surfaces. Since $c_1^2 = 0$ and $c_2 \geq 0$, the index theorem (1) shows that $\tau \leq 0$ and so only the negative definite case needs to be considered. Then $p_g = 0$, and thus $p_g - q + 1 = -q + 1 = \frac{1}{12}c_2 \geq 0$ implying $q = 1$, $b_1 = 1$, $c_2 = b_2 = 0$. Again only non-minimal such surfaces have negative definite diagonalizable intersection form.

Remark 3.2. As a consequence of this result, in the case of compact complex surfaces the intersection form is completely determinable from the Stiefel–Whitney class class $w_2 \equiv c_1 \bmod 2$ (this determines whether the form is odd or even), the signature of the surface, and the Euler number (or, equivalently, c_1^2 and c_2). So, the intersection form does not give supplementary topological information unlike for topological manifolds. It then follows from [Fre82] that the oriented homeomorphism type of a simply connected surface is uniquely determined by the invariants w_2, c_1^2 together with c_2. It is an open question whether this remains true for any compact complex surface by adding the fundamental group to the list of invariants. One can at least say that the latter determines whether the surface is Kähler or not so that the two classes (Kähler or not) can be dealt with separately.

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