New York Journal of Mathematics

New York J. Math. **26** (2020) 116–128.

# Indefinite Schwarz-Pick inequalities on the bidisk

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Abstract. Indefinite Schwarz-Pick inequalities for analytic self-maps of the bidisk are given as an application of the spectral theory on analytic Hilbert modules.

Dedicated to Professor Keiji Izuchi and Professor Takahiko Nakazi.

Contents



## 1. Introduction

The classical Schwarz-Pick inequality is fundamental in complex analysis and hyperbolic geometry, and also its functional analysis aspect has attracted a lot of interest. For example, Banach space theory related to the geometry derived from Schwarz-Pick inequality can be seen in Dineen [5]. In connection with operator theory, Schwarz-Pick type inequalities for analytic functions of one and several variables were discussed by Anderson-Rovnyak [3], Anderson-Dritschel-Rovnyak [2], Knese [12] and MacCluer-Stroethoff-Zhao [13, 14] in the context of Pick interpolation, realization formula, de Branges-Rovnyak space and composition operator. Now, the purpose of this paper is to give some variants of Schwarz lemma and Schwarz-Pick inequality for the bidisk. Here the author would like to emphasize the following three points:

(1) we deal with analytic self-maps of the bidisk,
(2) our inequalities are indefinite in a certain sense,
(3) our method is based on the theory of analytic Hilbert modules.

Received October 9, 2019.
2010 *Mathematics Subject Classification.* 46E22, 47B32.
*Key words and phrases.* Schwarz-Pick inequality, Hilbert module, Hardy space.

ISSN 1076-9803/2020

We shall introduce the language of the theory of Hilbert modules in the Hardy space over the bidisk. Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $H^2$ be the Hardy space over the bidisk $\mathbb{D}^2$, and $H^\infty$ be the Banach algebra consisting of all bounded analytic functions on $\mathbb{D}^2$. Then $H^2$ is a Hilbert module over $H^\infty$, that is, $H^2$ is a Hilbert space invariant under multiplication of functions in $H^\infty$. A closed subspace $\mathcal{M}$ of $H^2$ is called a submodule if $\mathcal{M}$ is invariant under the module action. Comparing with the theory of the Hardy space over the unit disk $\mathbb{D}$, structure of submodules in $H^2$ is very complicated. However, there are some well-behaved classes of submodules in $H^2$. One of those classes was introduced by Izuchi, Nakazi and the author in [9], and those members are called submodules of INS type. In this paper, as an application of spectral theory on submodules of INS type, the following Schwarz-Pick type inequalities will be given (Theorem 4.2 and Theorem 4.3): if $\psi = (\psi_1, \psi_2)$ is an analytic self-map on $\mathbb{D}^2$, then

$$0 \leq d(\psi(z), \psi(w)) \leq \sqrt{2}d(z, w) < \sqrt{2} \quad (z, w \in \mathbb{D}^2),$$

where we set

$$d(z, w) = \sqrt{\left|\frac{z_1 - w_1}{1 - \overline{w_1}z_1}\right|^2 + \left|\frac{z_2 - w_2}{1 - \overline{w_2}z_2}\right|^2 - \left|\frac{z_1 - w_1}{1 - \overline{w_1}z_1} \cdot \frac{z_2 - w_2}{1 - \overline{w_2}z_2}\right|^2}$$

for $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in $\mathbb{D}^2$. Further, if $\psi$ belongs to a certain class defined in Section 2, then

$$0 \leq d(\psi(z), \psi(w)) \leq d(z, w) < 1 \quad (z, w \in \mathbb{D}^2).$$

This paper contains four sections. Section 1 is this introduction. In Section 2, three classes of tuples of analytic functions on $\mathbb{D}^2$ are defined, and we show they are nontrivial. In Sections 3 and 4, as an application of the theory of analytic Hilbert modules, indefinite variants of Schwarz lemma and Schwarz-Pick inequality are given, respectively.

## 2. Schur-Drury-Agler class

Let $k_\lambda$ denote the reproducing kernel of $H^2$ at $\lambda$ in $\mathbb{D}^2$, that is,

$$k_\lambda(z) = \frac{1}{(1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)} \quad (z = (z_1, z_2), \lambda = (\lambda_1, \lambda_2) \in \mathbb{D}^2).$$

Then we set

$$\mathcal{D} = \left\{ \sum_\lambda c_\lambda k_\lambda \text{ (a finite sum)} : \lambda \in \mathbb{D}^2, c_\lambda \in \mathbb{C} \right\},$$

the linear space generated by all reproducing kernels of $H^2$. We shall consider unbounded Toeplitz operators with symbols in $H^2$. Let $f$ be a function in $H^2$. Then $T_f$ denotes the multiplication operator of $f$, where we fix $\mathcal{D}$ for the domain of $T_f$ . Then, since

$$\langle k_\lambda, T_f k_\mu \rangle = \langle \overline{f(\lambda)} k_\lambda, k_\mu \rangle \quad (\lambda, \mu \in \mathbb{D}^2),$$

$T_f^*$ is defined on $\mathcal{D}$ and we have

$$T_f^* k_\lambda = \overline{f(\lambda)} k_\lambda \quad (\lambda \in \mathbb{D}^2).$$

**Definition 2.1.** *Let $m$ and $n$ be non-negative integers. We consider a tuple*

$$\Phi_{m,n} = (\varphi_1, \dots, \varphi_m, \varphi_{m+1}, \dots, \varphi_{m+n})$$

*of $m+n$ analytic functions in $H^2$. Then $\mathcal{S}(\mathbb{D}; m, n)$ denotes the set of all $\Phi_{m,n}$ satisfying the following operator inequality on $\mathcal{D}$:*

$$0 \le \sum_{j=1}^{m} T_{\varphi_j} T_{\varphi_j}^* - \sum_{k=m+1}^{m+n} T_{\varphi_k} T_{\varphi_k}^* \le I.$$

*Equivalently, $\Phi_{m,n}$ belongs to $\mathcal{S}(\mathbb{D}; m, n)$ if and only if*

$$0 \le \frac{\sum_{j=1}^{m} \overline{\varphi_j(\lambda)} \varphi_j(z) - \sum_{k=m+1}^{m+n} \overline{\varphi_k(\lambda)} \varphi_k(z)}{(1 - \overline{\lambda_1} z_1)(1 - \overline{\lambda_2} z_2)} \le \frac{1}{(1 - \overline{\lambda_1} z_1)(1 - \overline{\lambda_2} z_2)}$$

*as kernel functions.*

Since the author has been influenced by Drury [6], in our paper, we would like to call $\mathcal{S}(\mathbb{D}^2; m, n)$ a Schur-Drury-Agler calss of $\mathbb{D}^2$. Here two remarks are given. First, unbounded functions are not excluded from $\mathcal{S}(\mathbb{D}^2; m, n)$ (cf. Definition 1 in Jury [11] for the Drury-Arveson space). Throughout this paper, a triplet $(\varphi_1, \varphi_2, \varphi_3)$ consisting of functions in $H^\infty$ will be said to be bounded. Second, $\mathcal{S}(\mathbb{D}^2; m, n)$ is more restricted than the class, which might be called a Schur-Agler class in some literatures, consisting of tuples of functions in $H^2$ satisfying the operator inequality

$$I - \sum_{j=1}^{m} T_{\varphi_j} T_{\varphi_j}^* + \sum_{k=m+1}^{m+n} T_{\varphi_k} T_{\varphi_k}^* \ge 0.$$

In this paper, we will focus on the case where $m = 2$ and $n = 1$, that is,

$$\mathcal{S}(\mathbb{D}^2; 2, 1) = \{(\varphi_1, \varphi_2, \varphi_3) \in (H^2)^3 : 0 \le T_{\varphi_1} T_{\varphi_1}^* + T_{\varphi_2} T_{\varphi_2}^* - T_{\varphi_3} T_{\varphi_3}^* \le I\}.$$

This class is closely related to submodules of rank 3 (see Wu-S-Yang [15] and Yang [16, 17]). Further, we define other two classes as follows:

$$\mathcal{P}(\mathbb{D}^2; 2, 1) = \{(\varphi_1, \varphi_2, \varphi_3) \in (H^2)^3 : T_{\varphi_1} T_{\varphi_1}^* + T_{\varphi_2} T_{\varphi_2}^* - T_{\varphi_3} T_{\varphi_3}^* \ge 0\},$$
$$\mathcal{Q}(\mathbb{D}^2; 2, 1) = \{(\varphi_1, \varphi_2, \varphi_3) \in (H^2)^3 : I - T_{\varphi_1} T_{\varphi_1}^* - T_{\varphi_2} T_{\varphi_2}^* + T_{\varphi_3} T_{\varphi_3}^* \ge 0\}.$$

Trivially, $\mathcal{P}(\mathbb{D}^2; 2, 1) \cap \mathcal{Q}(\mathbb{D}^2; 2, 1) = \mathcal{S}(\mathbb{D}^2; 2, 1)$. First, we shall give examples of elements of $\mathcal{S}(\mathbb{D}^2; 2, 1)$.

**Example 2.2.** Let $\varphi_1 = \varphi_1(z_1)$ and $\varphi_2 = \varphi_2(z_2)$ be analytic functions of single variable. If $\|\varphi_1\|_\infty \le 1$ and $\|\varphi_2\|_\infty \le 1$, then $(\varphi_1, \varphi_2, \varphi_1 \varphi_2)$ belongs to

$\mathcal{S}(\mathbb{D}^2;2,1)$. Indeed, since $T_{\varphi_1}$ and $T_{\varphi_2}$ are doubly commuting contractions,

$$\begin{aligned} & I - T_{\varphi_1}T_{\varphi_1}^* - T_{\varphi_2}T_{\varphi_2}^* + T_{\varphi_1\varphi_2}T_{\varphi_1\varphi_2}^* \\ &= (I - T_{\varphi_1}T_{\varphi_1}^*)(I - T_{\varphi_2}T_{\varphi_2}^*) \\ &= (I - T_{\varphi_1}T_{\varphi_1}^*)^{1/2}(I - T_{\varphi_2}T_{\varphi_2}^*)(I - T_{\varphi_1}T_{\varphi_1}^*)^{1/2} \\ &\geq 0, \end{aligned}$$

and

$$T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}T_{\varphi_2}^* - T_{\varphi_1\varphi_2}T_{\varphi_1\varphi_2}^* = T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}(I - T_{\varphi_1}T_{\varphi_1}^*)T_{\varphi_2}^* \geq 0.$$

In particular, $(z_1, z_2, z_1z_2)$ belongs to $\mathcal{S}(\mathbb{D}^2;2,1)$ and

$$T_{z_1}T_{z_1}^* + T_{z_2}T_{z_2}^* - T_{z_1z_2}T_{z_1z_2}^*$$

is the orthogonal projection of $H^2$ onto the submodule generated by $z_1$ and $z_2$.

**Example 2.3.** Let $\psi(z) = (\psi_1(z), \psi_2(z))$ be an analytic self-map of $\mathbb{D}^2$. Then, trivially, $\operatorname{ran} T_{\psi_1\psi_2/\sqrt{2}}$ is a subspace of $\operatorname{ran} T_{\psi_1}$. Hence, by the Douglas range inclusion theorem and $\|T_{\psi_j}\| \leq 1$, we have

$$0 \leq T_{\psi_1\psi_2/\sqrt{2}}T_{\psi_1\psi_2/\sqrt{2}}^* \leq \frac{1}{2}T_{\psi_1}T_{\psi_1}^* \leq T_{\psi_1}T_{\psi_1}^* + T_{\psi_2}T_{\psi_2}^* \leq 2I.$$

Therefore, we have

$$\begin{aligned} 0 &\leq \frac{1}{2}(T_{\psi_1}T_{\psi_1}^* + T_{\psi_2}T_{\psi_2}^* - T_{\psi_1\psi_2/\sqrt{2}}T_{\psi_1\psi_2/\sqrt{2}}^*) \\ &= T_{\psi_1/\sqrt{2}}T_{\psi_1/\sqrt{2}}^* + T_{\psi_2/\sqrt{2}}T_{\psi_2/\sqrt{2}}^* - T_{\psi_1\psi_2/2}T_{\psi_1\psi_2/2}^* \\ &\leq T_{\psi_1/\sqrt{2}}T_{\psi_1/\sqrt{2}}^* + T_{\psi_2/\sqrt{2}}T_{\psi_2/\sqrt{2}}^* \\ &\leq I. \end{aligned}$$

Thus $(\psi_1/\sqrt{2}, \psi_2/\sqrt{2}, \psi_1\psi_2/2)$ belongs to $\mathcal{S}(\mathbb{D}^2;2,1)$ for any analytic self-map $(\psi_1, \psi_2)$ of $\mathbb{D}^2$.

**Example 2.4.** Further non-trivial examples of elements in $\mathcal{S}(\mathbb{D}^2;2,1)$ related to the theory of Hilbert modules in $H^2$ can be obtained from Theorem 4.3 in Wu-S-Yang [15].

$\mathcal{P}(\mathbb{D}^2;2,1)$ and $\mathcal{Q}(\mathbb{D}^2;2,1)$ are closed under composition of elements in $\mathcal{Q}(\mathbb{D}^2;2,1)$ in the following sense (cf. Theorem 2 in Jury [11]).

**Theorem 2.5.** *Let $(\varphi_1, \varphi_2, \varphi_3)$ be a triplet in $\mathcal{P}(\mathbb{D}^2;2,1)$ (resp. $\mathcal{Q}(\mathbb{D}^2;2,1)$), and $\psi = (\psi_1, \psi_2)$ be an analytic self-map of $\mathbb{D}^2$. If $(\psi_1, \psi_2, \psi_1\psi_2)$ belongs to $\mathcal{Q}(\mathbb{D}^2;2,1)$, then $(\varphi_1 \circ \psi, \varphi_2 \circ \psi, \varphi_3 \circ \psi)$ belongs to $\mathcal{P}(\mathbb{D}^2;2,1)$ (resp. $\mathcal{Q}(\mathbb{D}^2;2,1)$).*

**Proof.** We set

$$\Phi(z,\lambda) = \overline{\varphi_1(\lambda)}\varphi_1(z) + \overline{\varphi_2(\lambda)}\varphi_2(z) - \overline{\varphi_3(\lambda)}\varphi_3(z).$$

If $(\varphi_1, \varphi_2, \varphi_3)$ belongs to $\mathcal{P}(\mathbb{D}^2; 2, 1)$, then, for any $\lambda_1, \ldots, \lambda_n$ in $\mathbb{D}^2$, we have

$$
\begin{aligned}
&\langle (T_{\varphi_1\circ\psi}T^*_{\varphi_1\circ\psi} + T_{\varphi_2\circ\psi}T^*_{\varphi_2\circ\psi} - T_{\varphi_3\circ\psi}T^*_{\varphi_3\circ\psi}) \sum_{i=1}^n c_i k_{\lambda_i}, \sum_{j=1}^n c_j k_{\lambda_j} \rangle \\
&= \sum_{i,j=1}^n c_i \overline{c_j} \Phi(\psi(\lambda_j), \psi(\lambda_i)) \langle k_{\lambda_i}, k_{\lambda_j} \rangle \\
&= \sum_{i,j=1}^n c_i \overline{c_j} \Phi(\psi(\lambda_j), \psi(\lambda_i)) \langle k_{\psi(\lambda_i)}, k_{\psi(\lambda_j)} \rangle \frac{\langle k_{\lambda_i}, k_{\lambda_j} \rangle}{\langle k_{\psi(\lambda_i)}, k_{\psi(\lambda_j)} \rangle} \\
&= \sum_{i,j=1}^n c_i \overline{c_j} \langle (T_{\varphi_1}T^*_{\varphi_1} + T_{\varphi_2}T^*_{\varphi_2} - T_{\varphi_3}T^*_{\varphi_3}) k_{\psi(\lambda_i)}, k_{\psi(\lambda_j)} \rangle \frac{\langle k_{\lambda_i}, k_{\lambda_j} \rangle}{\langle k_{\psi(\lambda_i)}, k_{\psi(\lambda_j)} \rangle}.
\end{aligned}
$$

Hence, by the definition of $\mathcal{Q}(\mathbb{D}^2; 2, 1)$ and Schur's theorem, we have

$$T_{\varphi_1\circ\psi}T^*_{\varphi_1\circ\psi} + T_{\varphi_2\circ\psi}T^*_{\varphi_2\circ\psi} - T_{\varphi_3\circ\psi}T^*_{\varphi_3\circ\psi} \geq 0.$$

Therefore, $(\varphi_1\circ\psi, \varphi_2\circ\psi, \varphi_3\circ\psi)$ belongs to $\mathcal{P}(\mathbb{D}^2; 2, 1)$. Similarly, considering $1 - \Phi$, we have the statement on $\mathcal{Q}(\mathbb{D}^2; 2, 1)$. □

**Corollary 2.6.** *Suppose that $\psi = (\psi_1, \psi_2)$ is an analytic self-map of $\mathbb{D}^2$ and $(\psi_1, \psi_2, \psi_1\psi_2)$ belongs to $\mathcal{S}(\mathbb{D}^2; 2, 1)$. Then $(\varphi_1 \circ \psi, \varphi_2 \circ \psi, \varphi_3 \circ \psi)$ belongs to $\mathcal{S}(\mathbb{D}^2; 2, 1)$ for any triplet $(\varphi_1, \varphi_2, \varphi_3)$ in $\mathcal{S}(\mathbb{D}^2; 2, 1)$.*

## 3. Indefinite Schwarz lemmas

In this section, we shall give inequalities which can be seen as variants of Schwarz lemma. We need several lemmas.

**Lemma 3.1.** *Let $T$ be a non-negative bounded linear operator, and $P$ be an orthogonal projection on a Hilbert space $\mathcal{H}$. If there exists some constant $c > 0$ such that $0 \leq T \leq cP$, then we may take $c = \|T\|$.*

**Proof.** By elementary theory of self-adjoint operators, we have the conclusion. □

**Lemma 3.2.** *Let $(\varphi_1, \varphi_2, \varphi_3)$ be a bounded triplet in $\mathcal{P}(\mathbb{D}^2; 2, 1)$. Then $\varphi_3$ belongs to $\varphi_1 H^2 + \varphi_2 H^2$.*

**Proof.** Applying the Douglas range inclusion theorem to the operator inequality

$$T_{\varphi_3}T^*_{\varphi_3} \leq T_{\varphi_1}T^*_{\varphi_1} + T_{\varphi_2}T^*_{\varphi_2},$$

we have

$$\operatorname{ran} T_{\varphi_3} \subseteq \operatorname{ran} \sqrt{T_{\varphi_1}T^*_{\varphi_1} + T_{\varphi_2}T^*_{\varphi_2}} = \operatorname{ran} T_{\varphi_1} + \operatorname{ran} T_{\varphi_2}$$

(see Theorem 2.2 attributed to Crimmins in Fillmore-Williams [7] or Theorem 3.6 in Ando [4]). This concludes the proof. □

**Lemma 3.3.** *Let* $(\varphi_1, \varphi_2, \varphi_3)$ *be a bounded triplet in* $\mathcal{P}(\mathbb{D}^2; 2, 1)$*. If*

$$\varphi_1(0,0) = \varphi_2(0,0) = 0,$$

*then*

$$0 \leq |\varphi_1(z)|^2 + |\varphi_2(z)|^2 - |\varphi_3(z)|^2 \leq \|T\|(|z_1|^2 + |z_2|^2 - |z_1 z_2|^2)$$

*for any* $z = (z_1, z_2)$ *in* $\mathbb{D}^2$*, where we set*

$$T = T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}T_{\varphi_2}^* - T_{\varphi_3}T_{\varphi_3}^*.$$

**Proof.** Suppose that $\varphi_1$, $\varphi_2$ and $\varphi_3$ are bounded and $\varphi_1(0,0) = \varphi_2(0,0) = 0$. Then, it follows from Lemma 3.2 that $\varphi_3(0,0) = 0$. Hence $\varphi_1, \varphi_2$ and $\varphi_3$ belong to the submodule $\mathcal{M}_0 = z_1 H^2 + z_2 H^2$. Then we have

$$\operatorname{ran}(T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}T_{\varphi_2}^* - T_{\varphi_3}T_{\varphi_3}^*) \subseteq \mathcal{M}_0.$$

Further, by elementary spectral theory, we have

$$\begin{aligned}\operatorname{ran}(T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}T_{\varphi_2}^* - T_{\varphi_3}T_{\varphi_3}^*)^{1/2} &\subseteq \overline{\operatorname{ran}}(T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}T_{\varphi_2}^* - T_{\varphi_3}T_{\varphi_3}^*) \\ &\subseteq \overline{\mathcal{M}_0} = \mathcal{M}_0.\end{aligned}$$

Hence, it follows from the Douglas range inclusion theorem that there exists a constant $c > 0$ such that

$$0 \leq T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}T_{\varphi_2}^* - T_{\varphi_3}T_{\varphi_3}^* \leq cP_{\mathcal{M}_0},$$

where $P_{\mathcal{M}_0}$ denotes the orthogonal projection of $H^2$ onto $\mathcal{M}_0$. By Lemma 3.1, we may take $c = \|T\|$. Hence we have

$$0 \leq T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}T_{\varphi_2}^* - T_{\varphi_3}T_{\varphi_3}^* \leq \|T\|P_{\mathcal{M}_0} = \|T\|(T_{z_1}T_{z_1}^* + T_{z_2}T_{z_2}^* - T_{z_1 z_2}T_{z_1 z_2}^*)$$

by Example 2.2. In particular,

$$\begin{aligned}&(|\varphi_1(\lambda)|^2 + |\varphi_2(\lambda)|^2 - |\varphi_3(\lambda)|^2)k_\lambda(\lambda) \\ &= \langle (T_{\varphi_1}T_{\varphi_1}^* + T_{\varphi_2}T_{\varphi_2}^* - T_{\varphi_3}T_{\varphi_3}^*)k_\lambda, k_\lambda\rangle \\ &\leq \langle \|T\|(T_{z_1}T_{z_1}^* + T_{z_2}T_{z_2}^* - T_{z_1 z_2}T_{z_1 z_2}^*)k_\lambda, k_\lambda\rangle \\ &= \|T\|(|\lambda_1|^2 + |\lambda_2|^2 - |\lambda_1\lambda_2|^2)k_\lambda(\lambda)\end{aligned}$$

for any $\lambda = (\lambda_1, \lambda_2)$ in $\mathbb{D}^2$. This concludes the proof. □

**Lemma 3.4.** *If* $(\psi_1, \psi_2)$ *is an analytic self-map on* $\mathbb{D}^2$*, then* $(\psi_1, \psi_2, \psi_1\psi_2)$ *belongs to* $\mathcal{P}(\mathbb{D}^2; 2, 1)$*.*

**Proof.** Since $\|\psi_j\|_\infty \leq 1$ for $j = 1, 2$, we have

$$T_{\psi_1}T_{\psi_1}^* + T_{\psi_2}T_{\psi_2}^* - T_{\psi_1\psi_2}T_{\psi_1\psi_2}^* = T_{\psi_1}T_{\psi_1}^* + T_{\psi_2}(I - T_{\psi_1}T_{\psi_1}^*)T_{\psi_2}^* \geq 0.$$

Hence $(\psi_1, \psi_2, \psi_1\psi_2)$ belongs to $\mathcal{P}(\mathbb{D}^2; 2, 1)$. □

The following theorem is a bidisk version of the Schwarz lemma.

**Theorem 3.5.** *If $\psi = (\psi_1, \psi_2)$ is an analytic self-map on $\mathbb{D}^2$ and $\psi(0,0) = (0,0)$, then*

$$0 \le |\psi_1(z)|^2 + |\psi_2(z)|^2 - |\psi_1(z)\psi_2(z)|^2 \le \|T\|(|z_1|^2 + |z_2|^2 - |z_1 z_2|^2)$$

*for any $z = (z_1, z_2)$ in $\mathbb{D}^2$, where we set*

$$T = T_{\psi_1}T_{\psi_1}^* + T_{\psi_2}T_{\psi_2}^* - T_{\psi_1\psi_2}T_{\psi_1\psi_2}^*.$$

**Proof.** By Lemma 3.3 and Lemma 3.4, we have the conclusion. □

**Proposition 3.6.** *Let $(\varphi_1, \varphi_2, \varphi_3)$ be a triplet in $\mathcal{S}(\mathbb{D}^2; 2, 1)$. If $\varphi_1(0,0) = \varphi_2(0,0) = 0$, then*

$$0 \le |\varphi_1(z)|^2 + |\varphi_2(z)|^2 - |\varphi_3(z)|^2 \le |z_1|^2 + |z_2|^2 - |z_1 z_2|^2$$

*for any $z = (z_1, z_2)$ in $\mathbb{D}^2$.*

**Proof.** If $(\varphi_1, \varphi_2, \varphi_3)$ is bounded, then we have the conclusion immediately by Lemma 3.3. Suppose that $(\varphi_1, \varphi_2, \varphi_3)$ is unbounded. Setting $\psi_r(z_1, z_2) = (rz_1, rz_2)$ for $0 < r < 1$, $(\varphi_1 \circ \psi_r, \varphi_2 \circ \psi_r, \varphi_3 \circ \psi_r)$ belongs to $\mathcal{S}(\mathbb{D}^2; 2, 1)$ by Corollary 2.6 and Example 2.2. Moreover, $\varphi_1 \circ \psi_r$, $\varphi_2 \circ \psi_r$ and $\varphi_3 \circ \psi_r$ are bounded on $\mathbb{D}^2$, and $\varphi_1 \circ \psi_r(0,0) = \varphi_2 \circ \psi_r(0,0) = 0$. Hence we have

$$\begin{aligned} 0 &\le |\varphi_1(rz)|^2 + |\varphi_2(rz)|^2 - |\varphi_3(rz)|^2 \\ &= |\varphi_1 \circ \psi_r(z)|^2 + |\varphi_2 \circ \psi_r(z)|^2 - |\varphi_3 \circ \psi_r(z)|^2 \\ &\le |z_1|^2 + |z_2|^2 - |z_1 z_2|^2 \end{aligned}$$

by Lemma 3.3. Letting $r$ tend to 1, we have the conclusion for unbounded triplets. □

**Theorem 3.7.** *Suppse that $\psi = (\psi_1, \psi_2)$ is an analytic self-map on $\mathbb{D}^2$ and $(\psi_1, \psi_2, \psi_1\psi_2)$ belongs to $\mathcal{Q}(\mathbb{D}^2; 2, 1)$. If $\psi(0,0) = (0,0)$, then $(\psi_1, \psi_2, \psi_1\psi_2)$ belongs to $\mathcal{S}(\mathbb{D}^2; 2, 1)$ and*

$$0 \le |\psi_1(z)|^2 + |\psi_2(z)|^2 - |\psi_1(z)\psi_2(z)|^2 \le |z_1|^2 + |z_2|^2 - |z_1 z_2|^2$$

*for any $z = (z_1, z_2)$ in $\mathbb{D}^2$. Moreover, equality*

$$|\psi_1(z)|^2 + |\psi_2(z)|^2 - |\psi_1(z)\psi_2(z)|^2 = |z_1|^2 + |z_2|^2 - |z_1 z_2|^2$$

*holds on some open set if and only if $\psi = (e^{i\theta_1}z_1, e^{i\theta_2}z_2)$ or $(e^{i\theta_2}z_2, e^{i\theta_1}z_1)$.*

**Proof.** First, by Lemma 3.4, $(\psi_1, \psi_2, \psi_1\psi_2)$ belongs to $\mathcal{S}(\mathbb{D}^2; 2, 1)$. Hence, we have the inequality by Theorem 3.5. Next, we suppose that

$$|\psi_1(z)|^2 + |\psi_2(z)|^2 - |\psi_1(z)\psi_2(z)|^2 = |z_1|^2 + |z_2|^2 - |z_1 z_2|^2$$

on an open set $V$. Then, by the polarization (see p. 28 in Agler-McCarthy [1] or p. 2762 in Knese [12]), we have

$$\overline{\psi_1(\lambda)}\psi_1(z) + \overline{\psi_2(\lambda)}\psi_2(z) - \overline{\psi_1(\lambda)\psi_2(\lambda)}\psi_1(z)\psi_2(z) = \overline{\lambda_1}z_1 + \overline{\lambda_2}z_2 - \overline{\lambda_1\lambda_2}z_1z_2$$

on $\overline{V} \times V$, and this identity can be extended to $\mathbb{D}^2 \times \mathbb{D}^2$. Then, for $j = 1, 2$, we have

$$\left|\frac{\partial \psi_1}{\partial z_j}\right|^2 + \left|\frac{\partial \psi_2}{\partial z_j}\right|^2 - \left|\frac{\partial \psi_1\psi_2}{\partial z_j}\right|^2 = \left|\frac{\partial z_1}{\partial z_j}\right|^2 + \left|\frac{\partial z_2}{\partial z_j}\right|^2 - \left|\frac{\partial z_1 z_2}{\partial z_j}\right|^2.$$

Hence we have

$$\left|\frac{\partial \psi_1}{\partial z_j}(0,0)\right|^2 + \left|\frac{\partial \psi_2}{\partial z_j}(0,0)\right|^2 = 1. \tag{3.1}$$

Similarly, we have

$$\left|\frac{\partial^2 \psi_1}{\partial z_j^2}(0,0)\right|^2 + \left|\frac{\partial^2 \psi_2}{\partial z_j^2}(0,0)\right|^2 - 4\left|\frac{\partial \psi_1}{\partial z_j}(0,0)\frac{\partial \psi_2}{\partial z_j}(0,0)\right|^2 = 0. \tag{3.2}$$

It follows from (3.1) that

$$\|\psi_1\|^2 + \|\psi_2\|^2 \geq \left|\frac{\partial \psi_1}{\partial z_1}(0,0)\right|^2 + \left|\frac{\partial \psi_1}{\partial z_2}(0,0)\right|^2 + \left|\frac{\partial \psi_2}{\partial z_1}(0,0)\right|^2 + \left|\frac{\partial \psi_2}{\partial z_2}(0,0)\right|^2 = 2.$$

Hence, $\|\psi_1\| = 1$ and $\|\psi_2\| = 1$ and

$$\psi_i = c_{i1}z_1 + c_{i2}z_2 \quad (|c_{i1}|^2 + |c_{i2}|^2 = 1).$$

Further, by (3.2), we have

$$\frac{\partial \psi_1}{\partial z_j}(0,0)\frac{\partial \psi_2}{\partial z_j}(0,0) = 0,$$

that is, $c_{1j}c_{2j} = 0$. This concludes the proof. □

**Corollary 3.8.** *Let $f$ be an analytic function on $\mathbb{D}^2$. If $\|f\|_\infty \leq 1$ and $f(0,0) = 0$, then*

$$0 \leq |f(z)|^2 \leq |z_1|^2 + |z_2|^2 - |z_1 z_2|^2$$

*for any $z = (z_1, z_2)$ in $\mathbb{D}^2$.*

**Proof.** Set $\psi = (\psi_1, \psi_2) = (f, 0)$. Then $\psi$ is an analytic self-map, $\psi(0,0) = (0,0)$ and $(\psi_1, \psi_2, \psi_1\psi_2) = (f, 0, 0)$ belongs to $\mathcal{Q}(\mathbb{D}^2; 2, 1)$. □

**Remark 3.9.** Suppose that $\psi = (\psi_1, \psi_2)$ is an analytic self-map on $\mathbb{D}^2$ and $(\psi_1, \psi_2, \psi_1\psi_2)$ belongs to $\mathcal{S}(\mathbb{D}^2; 2, 1)$. Then, the proof of Theorem 1 in Jury [10] can be applied and we have that the composition operator $C_\psi$ is contractive on $H^2$. As its corollary, the inequality in Theorem 3.7 is obtained.

**Remark 3.10** (Kreĭn space geometry and $\mathbb{D}^2$)**.** We introduce a Kreĭn space structure into $\mathbb{C}^3$ as follows:

$$\langle z, w\rangle_{\mathcal{K}} = z_1\overline{w_1} + z_2\overline{w_2} - z_3\overline{w_3} \quad (z = (z_1, z_2, z_3), w = (w_1, w_2, w_3) \in \mathbb{C}^3).$$

Let $\mathcal{K}$ denote the Kreĭn space $(\mathbb{C}^3, \langle\cdot,\cdot\rangle_{\mathcal{K}})$, and let $\Phi$ be the map defined as follows:

$$\Phi : \mathbb{D}^2 \to \mathcal{K}, \quad (z_1, z_2) \mapsto (z_1, z_2, z_1 z_2).$$

Moreover, we set

$$\begin{aligned}\Omega &= \{(z_1, z_2) \in \mathbb{C}^2 : 0 \le |z_1|^2 + |z_2|^2 - |z_1 z_2|^2 < 1\} \\ &= \{z \in \mathbb{C}^2 : 0 \le \langle \Phi(z), \Phi(z) \rangle_{\mathcal{K}} < 1\}.\end{aligned}$$

Then, since

$$|z_1|^2 + |z_2|^2 - |z_1 z_2|^2 = 1 - (1 - |z_1|^2)(1 - |z_2|^2),$$

$\mathbb{D}^2$ is the bounded connected component of $\Omega$, and $\partial\mathbb{D}^2$, the topological boundary of $\mathbb{D}^2$, is equal to the subset

$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 - |z_1 z_2|^2 = 1\} = \{z \in \mathbb{C}^2 : \langle \Phi(z), \Phi(z) \rangle_{\mathcal{K}} = 1\}.$$

## 4. Indefinite Schwarz-Pick inequalities

Let $q_1 = q_1(z_1)$ and $q_2 = q_2(z_2)$ be inner functions of single variable. Then

$$\mathcal{M} = q_1 H^2 + q_2 H^2$$

is a submodule of $H^2$. This submodule was introduced by Izuchi-Nakazi-S [9], and is called a submodule of INS-type. In this section, we shall give an application of spectral theory on submodules of INS type. In the general theory of Hilbert modules in $H^2$, the core (defect) operator of a submodule $\mathcal{M}$ in $H^2$ is defined as follows:

$$\Delta_{\mathcal{M}} = P_{\mathcal{M}} - T_{z_1} P_{\mathcal{M}} T_{z_1}^* - T_{z_2} P_{\mathcal{M}} T_{z_2}^* + T_{z_1 z_2} P_{\mathcal{M}} T_{z_1 z_2}^*,$$

where $P_{\mathcal{M}}$ denotes the orthogonal projection of $H^2$ onto $\mathcal{M}$. For a submodule of INS-type, it is known that

$$\Delta_{\mathcal{M}} = q_1 \otimes q_1 + q_2 \otimes q_2 - (q_1 q_2) \otimes (q_1 q_2),$$

where $\otimes$ denotes the Schatten form. Core operators were introduced and studied by Guo-Yang [8] and Yang [16] in detail, and which are devices connecting reproducing kernels and submodules. In particular, the following formula is useful:

$$k_\lambda(\Delta_{\mathcal{M}} k_\lambda) = P_{\mathcal{M}} k_\lambda. \tag{4.1}$$

Further, core operators of finite rank were discussed by Yang [17]. Let $\mathcal{M}$ be a submodule whose core operator $\Delta_{\mathcal{M}}$ is of finite rank. Then the rank of $\Delta_{\mathcal{M}}$ is odd. Moreover, if the rank of $\Delta_{\mathcal{M}}$ is $2n+1$, then the signature of $\Delta_{\mathcal{M}}$ is $(n+1, n)$. Hence $\Delta_{\mathcal{M}}$ has the following representation:

$$\Delta_{\mathcal{M}} = \sum_{j=1}^{n+1} \eta_j \otimes \eta_j - \sum_{j=n+2}^{2n+1} \eta_j \otimes \eta_j. \tag{4.2}$$

By application of those facts, Lemma 3.3 is generalized as follows.

**Lemma 4.1.** *Let $\mathcal{M}$ be a submodule of finite rank. We suppose that the core operator of $\mathcal{M}$ has the representation (4.2). If $(\varphi_1, \varphi_2, \varphi_3)$ is a bounded triplet in $\mathcal{P}(\mathbb{D}^2; 2, 1)$, and $\varphi_1$ and $\varphi_2$ belong to $\mathcal{M}$, then*

$$0 \le |\varphi_1(z)|^2 + |\varphi_2(z)|^2 - |\varphi_3(z)|^2 \le \|T\| \left( \sum_{j=1}^{n+1} |\eta_j(z)|^2 - \sum_{j=n+2}^{2n+1} |\eta_j(z)|^2 \right)$$

*for any $z$ in $\mathbb{D}^2$, where we set*

$$T = T_{\varphi_1} T_{\varphi_1}^* + T_{\varphi_2} T_{\varphi_2}^* - T_{\varphi_3} T_{\varphi_3}^*.$$

*In particular, if $\mathcal{M} = q_1 H^2 + q_2 H^2$ for inner functions $q_1 = q_1(z_1)$ and $q_2 = q_2(z_2)$ of single variable, then*

$$0 \le |\varphi_1(z)|^2 + |\varphi_2(z)|^2 - |\varphi_3(z)|^2 \le \|T\| (|q_1(z_1)|^2 + |q_2(z_2)|^2 - |q_1(z_1) q_2(z_2)|^2)$$

*for any $z = (z_1, z_2)$ in $\mathbb{D}^2$.*

**Proof.** By the same argument as the first half of the proof of Lemma 3.3, we have

$$0 \le T_{\varphi_1} T_{\varphi_1}^* + T_{\varphi_2} T_{\varphi_2}^* - T_{\varphi_3} T_{\varphi_3}^* \le \|T\| P_{\mathcal{M}}.$$

Then, for any $\lambda = (\lambda_1, \lambda_2)$ in $\mathbb{D}^2$, we have

$$\begin{aligned}
&(|\varphi_1(\lambda)|^2 + |\varphi_2(\lambda)|^2 - |\varphi_3(\lambda)|^2) k_\lambda(\lambda) \\
&= \langle (T_{\varphi_1} T_{\varphi_1}^* + T_{\varphi_2} T_{\varphi_2}^* - T_{\varphi_3} T_{\varphi_3}^*) k_\lambda, k_\lambda \rangle \\
&\le \langle \|T\| P_{\mathcal{M}} k_\lambda, k_\lambda \rangle \\
&= \|T\| \langle k_\lambda (\Delta_{\mathcal{M}} k_\lambda), k_\lambda \rangle \\
&= \|T\| \left\langle k_\lambda \left( \sum_{j=1}^{n+1} \eta_j \otimes \eta_j - \sum_{j=n+2}^{2n+1} \eta_j \otimes \eta_j \right) k_\lambda, k_\lambda \right\rangle \\
&= \|T\| \left( \sum_{j=1}^{n+1} |\eta_j(\lambda)|^2 - \sum_{j=n+2}^{2n+1} |\eta_j(\lambda)|^2 \right) k_\lambda(\lambda)
\end{aligned}$$

by (4.1). This concludes the proof. □

For $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in $\mathbb{D}^2$, we set

$$b_{w_j}(z_j) = \frac{z_j - w_j}{1 - \overline{w_j} z_j} \quad (j = 1, 2).$$

Then, we note that

$$\begin{aligned}
&|b_{w_1}(z_1)|^2 + |b_{w_2}(z_2)|^2 - |b_{w_1}(z_1) b_{w_2}(z_2)|^2 \\
&= 1 - (1 - |b_{w_1}(z_1)|^2)(1 - |b_{w_2}(z_2)|^2) > 0.
\end{aligned}$$

Hence

$$d(z, w) = \sqrt{|b_{w_1}(z_1)|^2 + |b_{w_2}(z_2)|^2 - |b_{w_1}(z_1) b_{w_2}(z_2)|^2}$$

is defined. It should be mentioned here that $d$ is a distance on $\mathbb{D}^2$ by Lemma 9.9 in Agler-McCarthy [1].

**Theorem 4.2.** *Let $\psi = (\psi_1, \psi_2)$ be an analytic self-map on $\mathbb{D}^2$. Then,*

$$0 \leq d(\psi(z), \psi(w)) \leq \sqrt{2} d(z, w) < \sqrt{2}$$

*for any $z$ and $w$ in $\mathbb{D}^2$.*

**Proof.** For $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in $\mathbb{D}^2$, we set

$$\varphi_j(z) = b_{\psi_j(w)} \circ \psi(z) = \frac{\psi_j(z) - \psi_j(w)}{1 - \overline{\psi_j(w)}\psi_j(z)}.$$

Then, $(\varphi_1, \varphi_2)$ is an analytic self-map on $\mathbb{D}^2$, and $(\varphi_1, \varphi_2, \varphi_1\varphi_2)$ belongs to $\mathcal{P}(\mathbb{D}^2; 2, 1)$ by Lemma 3.4. Further, since $\varphi_1(w) = \varphi_2(w) = 0$, $\varphi_1$ and $\varphi_2$ belong to the submodule $b_{w_1}(z_1)H^2 + b_{w_2}(z_2)H^2$. Hence, by Lemma 4.1, we have

$$\begin{aligned}
0 &\leq |\varphi_1(z)|^2 + |\varphi_2(z)|^2 - |\varphi_1(z)\varphi_2(z)|^2 \\
&\leq \|T\|(|b_{w_1}(z_1)|^2 + |b_{w_2}(z_2)|^2 - |b_{w_1}(z_1)b_{w_2}(z_2)|^2) \\
&\leq 2(|b_{w_1}(z_1)|^2 + |b_{w_2}(z_2)|^2 - |b_{w_1}(z_1)b_{w_2}(z_2)|^2) \\
&< 2.
\end{aligned}$$

This concludes the proof. □

**Theorem 4.3.** *Suppose that $\psi = (\psi_1, \psi_2)$ is an analytic self-map on $\mathbb{D}^2$ and $(\psi_1, \psi_2, \psi_1\psi_2)$ belongs to $\mathcal{Q}(\mathbb{D}^2; 2, 1)$. Then*

$$0 \leq d(\psi(z), \psi(w)) \leq d(z, w) < 1$$

*for any $z$ and $w$ in $\mathbb{D}^2$. Moreover, equality*

$$d(\psi(z), \psi(w)) = d(z, w) \quad (z, w \in V)$$

*holds on some open set $V$ if and only if $\psi$ belongs to $\mathrm{Aut}(\mathbb{D}^2)$.*

**Proof.** We shall use the same notations as those in the proof of Theorem 4.2. By the assumption and Corollary 2.6, $(\varphi_1, \varphi_2, \varphi_1\varphi_2)$ belongs to $\mathcal{S}(\mathbb{D}^2; 2, 1)$. Hence we have $\|T\| \leq 1$. Thus we have the first half. Next, suppose that

$$d(\psi(z), \psi(w)) = d(z, w) \quad (z, w \in V)$$

holds on some open set $V$. We fix a point $w$ in $V$. Then we have

$$|\varphi_1(z)|^2 + |\varphi_2(z)|^2 - |\varphi_1(z)\varphi_2(z)|^2 = |b_{w_1}(z_1)|^2 + |b_{w_2}(z_2)|^2 - |b_{w_1}(z_1)b_{w_2}(z_2)|^2$$

for any $z$ in $V$. Setting $\beta(z) = (b_{-w_1}(z_1), b_{-w_2}(z_2))$, $(\varphi_1 \circ \beta, \varphi_2 \circ \beta, (\varphi_1\varphi_2) \circ \beta)$ belongs to $\mathcal{S}(\mathbb{D}^2; 2, 1)$ by Corollary 2.6, $\varphi \circ \beta(0, 0) = (0, 0)$ by the definition of $\varphi$, and

$$|\varphi_1 \circ \beta(z)|^2 + |\varphi_2 \circ \beta(z)|^2 - |(\varphi_1\varphi_2) \circ \beta(z)|^2 = |z_1|^2 + |z_2|^2 - |z_1 z_2|^2$$

for any $z$ in $\beta^{-1}(V)$. Hence, by Theorem 3.7, we have

$$(\varphi_1 \circ \beta(z), \varphi_2 \circ \beta(z)) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2) \quad \text{or} \quad (e^{i\theta_2} z_2, e^{i\theta_1} z_1).$$

This concludes the second half. □

**Corollary 4.4.** *Let $f$ be an analytic function on $\mathbb{D}^2$. If $\|f\|_\infty \le 1$, then*

$$0 \le \left| \frac{f(z) - f(w)}{1 - \overline{f(w)} f(z)} \right| \le d(z,w)$$

*for any $z$ and $w$ in $\mathbb{D}^2$.*

**Proof.** In the proof of Corollary 3.8, we showed that $(f, 0, 0)$ belongs to $\mathcal{Q}(\mathbb{D}^2; 2, 1)$. □

## Acknowledgements

The author is grateful to the referee for valuable comments and suggestions which have contributed to the preparation of the paper.

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This paper is available via http://nyjm.albany.edu/j/2020/26-5.html.