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The v_1-periodic region in the cohomology of the $\mathbb{C}$-motivic Steenrod algebra

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Abstract. We establish a v_1-periodicity theorem in Ext over the $\mathbb{C}$-motivic Steenrod algebra. The element h_1 of Ext, which detects the homotopy class η in the motivic Adams spectral sequence, is non-nilpotent and therefore generates h_1-towers. Our result is that, apart from these h_1-towers, v_1-periodicity operators give isomorphisms in a range near the top of the Adams chart. This result generalizes well-known classical behavior.

Contents

1. Introduction

1.1. Background and Motivation. One of the primary tools for computing stable homotopy groups of spheres is the Adams spectral sequence. The E_2-page of the Adams spectral sequence is given by $\mathrm{Ext}^{*,*}_{\mathcal{A}^{cl}}(\mathbb{F}_2, \mathbb{F}_2) = H^{*,*}(\mathcal{A}^{cl})$, which we denote by Ext_{cl}, where $\mathcal{A}^{cl}$ is the classical Steenrod algebra. For Ext_{cl}, Adams [Ada1] showed that there is a vanishing line of slope $\frac{1}{2}$ and intercept $\frac{3}{2}$, and J. P. May showed there is a periodicity line of slope $\frac{1}{5}$ and intercept $\frac{12}{5}$, where the periodicity operation is defined by the Massey product $P_r(-) := \langle h_{r+1}, h_0^{2^r}, -\rangle$. This result has not been published by May, but can be found in the thesis of Krause:

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Theorem 1.1. [Kra, Theorem 5.14] *For $r \geq 2$, the Massey product operation $P_r(-) := \langle h_{r+1}, h_0^{2^r}, - \rangle$ is uniquely defined on $\mathrm{Ext}_{cl}^{s,f} = H^{s,f}(\mathcal{A}^{cl})$ when $s > 0$ and $f > \frac{1}{2}s + 3 - 2^r$, where s is the stem, and f is the Adams filtration.*

Furthermore, for $f > \frac{1}{5}s + \frac{12}{5}$, the operation

$$P_r \colon H^{s,f}(\mathcal{A}^{cl}) \xrightarrow{\cong} H^{s+2^{r+1}, f+2^r}(\mathcal{A}^{cl})$$

is an isomorphism.

The purpose of this article is to discuss an analog of the theorem above in the $\mathbb{C}$-motivic context. Motivic homotopy theory, also known as $\mathbb{A}^1$-homotopy theory, is a way to apply the techniques of algebraic topology, specifically homotopy, to algebraic varieties and, more generally, to schemes. The theory was formulated by Morel and Voevodsky [MV].

In this paper we analyze the case where the base field F is the complex numbers $\mathbb{C}$. Let $\mathbb{M}_2$ denote the bigraded motivic cohomology ring of Spec $\mathbb{C}$, with $\mathbb{F}_2 = \mathbb{Z}/2$-coefficients. Voevodsky [Voe] proved that $\mathbb{M}_2 \cong \mathbb{F}_2[\tau]$. Let $\mathcal{A}$ be the mod 2 motivic Steenrod algebra over $\mathbb{C}$. The motivic Adams spectral sequence is a trigraded spectral sequence with

$$E_2^{*,*,*} = \mathrm{Ext}_{\mathcal{A}}^{*,*,*}(\mathbb{M}_2, \mathbb{M}_2),$$

where the third grading is the motivic weight. (See Dugger and Isaksen [DI]). The $\mathbb{C}$-motivic E_2-page, which we denote by Ext, has a vanishing line computed by Guillou and Isaksen [GI1]. Quigley has a partial result that $\mathrm{Ext}^{s,f,w}$ has a periodicity line of slope $\frac{1}{3}$ under the condition $s \leq w$ in the case $r = 2$ [Qui, Corollary 5.4].

The multiplication by 2 map $S^{0,0} \xrightarrow{2} S^{0,0}$ is detected by h_0, and the Hopf map $S^{1,1} \xrightarrow{\eta} S^{0,0}$ is detected by h_1 in Ext. These elements have degrees $(0,1,0)$ and $(1,1,1)$ respectively. By an infinite h_1-tower we will mean a non-zero sequence of elements of the form $h_1^k x$ in Ext with $k \geq 0$, where x is not h_1 divisible. We will write h_1-towers for infinite h_1-towers, and refer to x as the base of the h_1-tower $h_1^k x$ $(k \geq 0)$. A short discussion on the h_1-towers can be found in subsection 1.2. Since all h_1-towers are τ-torsion, one might guess that the motivic Ext groups differ from the classical Ext^{cl} groups by only infinite h_1-towers. This is not true, but we may expect the h_1-torsion part of Ext to obtain a pattern similar to Ext^{cl}. Our result pertains solely to this h_1-torsion region.

Remark 1.2. Let $\mathcal{A}_*$ denote the dual Steenrod algebra. For Ext, we can work over $\mathcal{A}_*$ instead of $\mathcal{A}$. i.e.

$$E_2^{*,*,*} \cong \mathrm{Ext}_{\mathcal{A}_*}^{*,*,*}(\mathbb{M}_2, \mathbb{M}_2)_*.$$

Here we view $\mathbb{M}_2$ as the homology of the motivic sphere instead of the cohomology; this is an $\mathcal{A}_*$-comodule.

The goal of this paper is the following theorem:

Theorem 1.3. *For $r \geq 2$ the Massey product operation*

$$P_r(-) := \langle h_{r+1}, h_0^{2^r}, - \rangle$$

is uniquely defined on $\mathrm{Ext}^{s,f,w} = H^{s,f,w}(\mathcal{A})$ *when $s > 0$ and $f > \frac{1}{2}s + 3 - 2^r$. Furthermore, for $f > \frac{1}{5}s + \frac{12}{5}$, the restriction of P_r to the h_1-torsion*

$$P_r \colon [H^{s,f,w}(\mathcal{A})]_{h_1-torsion} \to [H^{s+2^{r+1},f+2^r,w+2^r}(\mathcal{A})]_{h_1-torsion}$$

is an isomorphism.

We first reduce the problem to establishing the vanishing region of certain Ext groups. Then we make an explicit computation for these Ext groups over the dual Steenrod subalgebra $\mathcal{A}(1)_*$ to get a starting vanishing region. We transport this vanishing region using the Cartan-Eilenberg spectral sequence along normal extensions of Hopf algebras and obtain the vanishing region of these groups over $\mathcal{A}(2)_*$, which is the same as the vanishing region of these Ext groups over $\mathcal{A}_*$.

There are close connections between the classical Adams spectral sequence and the motivic Adams spectral sequence. For instance, by inverting τ in Ext, we obtain Ext^{cl}. There are also abundant connections between the $\mathbb{C}$-motivic Ext groups, the $\mathbb{R}$-motivic Ext groups and the C_2-equivariant Ext groups. The ρ-Bockstein spectral sequence [Hil] takes the $\mathbb{C}$-motivic Ext groups as input and computes the $\mathbb{R}$-motivic Ext groups. The C_2-equivariant Ext groups can then be obtained [GHIR] by calculating $\mathbb{R}$-motivic Ext groups for a negative cone. Our periodicity results ought to be relevant for future computations in $\mathbb{R}$-motivic and C_2-equivariant homotopy theory.

1.2. Further Considerations. We study the h_1-torsion part of *Ext*; the h_1-periodic part has been entirely computed in [GI2].

Theorem 1.4. [GI2, Theorem 1.1] *The h_1-inverted algebra* $\mathrm{Ext}_{\mathcal{A}}[h_1^{-1}]$ *is a polynomial algebra over $\mathbb{F}_2[h_1^{\pm 1}]$ on generators v_1^4 and v_n for $n \geq 2$, where:*

(1) *v_1^4 is in the 8-stem and has Adams filtration 4 and weight 4.*
(2) *v_n is in the $(2^{n+1} - 2)$-stem and has Adams filtration 1 and weight $2^n - 1$.*

It is straightforward that P_r acts injectively on the h_1-inverted Ext; that is, P_r sends an h_1-tower $h_1^k x$ $(k \geq 0)$ to another h_1-tower $h_1^l y$ $(l \geq 0)$. But the base x might not be sent to the base y. As for surjectivity, there are h_1-towers not in the image of P_r, such as the h_1-tower on c_0; those are not multiples of v_1^4 in the h_1-inverted Ext. Partial results about the bases of those h_1-towers can be found in [Tha]. We expect that the determination of the bases of the h_1-towers will lead to a complete understanding of the region in which the v_1-periodicity operator acts as an isomorphism on *Ext*.

Analogously to the Massey product $P_2(-) := \langle h_3, h_0^4, - \rangle$, there is another Massey product $g(-) := \langle h_4, h_1^4, - \rangle$. This Massey product $g(-)$ is related to another periodicity operator called w_1 in motivic Ext, which does not exist classically. For many values of x, $P_2(x)$ is detected by Px, where $P = h_{20}^4$ has degree $(8, 4, 4)$ in the May spectral sequence. Similarly, for many values of x, $g(x)$ will be detected by $h_{21}^4 \cdot x$, where h_{21}^4 has degree $(20, 4, 12)$ in the May spectral sequence. The obstruction to studying w_1-periodicity is that g has a relatively low slope. Thus the method in this paper is not applicable. In addition, our method relies on a computation involving $\mathrm{Ext}_{\mathcal{A}(1)_*}$, but g restricts to zero in that group. Thus a strategy for studying g-periodicity would need to begin with $\mathrm{Ext}_{\mathcal{A}(2)_*}$, which is much more complicated [Isa].

1.3. Organization. We follow the approach of [Kra] primarily. In Section 2, we briefly introduce the stable (co)module category, in which we can consider the h_0 or h_1-torsion part of Ext by taking sequential colimits. In Section 3, we establish the existence of a homological self-map θ and use this to show that $P_r(-)$ is uniquely defined. In Section 4, we explicitly show where θ is an isomorphism over $\mathcal{A}(1)_*$, and obtain a region where it is an isomorphism over $\mathcal{A}_*$ by moving along the Cartan-Eilenberg spectral sequence. In Section 5, we combine the results of the previous two sections together to get the motivic periodicity theorem 1.3.

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2. Working environment: the stable (co)module category $\boldsymbol{Stab}(\Gamma)$

In order to restrict to working with only the h_1-torsion (also h_0-torsion) part, first we would like to choose a suitable working environment: a category with some nice properties that will serve our purposes. Usually Ext is defined in the derived category of $\mathcal{A}_*$-comodules, which we denote $D(\mathcal{A}_*)$. However, the coefficient ring $\mathbb{M}_2$ is not compact in $D(\mathcal{A}_*)$, which means that $\mathbb{M}_2$ does not interact well with colimits. The stable comodule category will better serve our purposes. That is a category $\mathscr{C}$ such that:

(1) If M is a $\mathcal{A}_*$-comodule that is free of finite rank over $\mathbb{M}_2$ and N is a $\mathcal{A}_*$-comodule, then $\mathrm{Hom}_{\mathscr{C}}(M, N) \cong \mathrm{Ext}_{\mathcal{A}_*}(M, N)$.

(2) If M is a $\mathcal{A}_*$-comodule that is free of finite rank over $\mathbb{M}_2$, then M is compact in $\mathscr{C}$. That is to say, for any sequential colimit in $\mathscr{C}$ of $\mathcal{A}_*$-comodules

$$\underset{i}{\mathrm{colim}} N_i := \mathrm{colim}(N_0 \xrightarrow{f_0} N_1 \to \cdots \to N_i \xrightarrow{f_i} \cdots),$$

we have $\underset{i}{\mathrm{colim}}\, \mathrm{Ext}_{\mathcal{A}_*}(M, N_i) \cong \mathrm{Hom}_{\mathscr{C}}(M, \underset{i}{\mathrm{colim}} N_i)$

The correct choice of $\mathscr{C}$ is called $Stab(\mathcal{A}_*)$. The category can be constructed in various ways (see [Bel, Sec. 2.1] for details), and has several useful properties for our case. The following proposition summarizes some of the discussion in [BHV, Sec. 4]:

Proposition 2.1. *The category $Stab(\mathcal{A}_*)$ satisfies conditions* (1) *and* (2) *above.*

Namely, for a Hopf algebra Γ and comodule M that is free of finite rank, we have a diagram

$$\begin{array}{ccccc} D(\Gamma) & \xleftarrow{\ i\ } & Comod_\Gamma & \overset{j}{\dashrightarrow} & Stab(\Gamma) \\ & \searrow^{\mathrm{Hom}_{D(\Gamma)}(iM,-)} & \downarrow{\scriptstyle \mathrm{Ext}_\Gamma(M,-)} & \swarrow_{\mathrm{Hom}_{Stab(\Gamma)}(jM,-)} & \\ & & \mathbf{grAb} & & \end{array}$$

where i is the canonical functor and j is well-defined only for comodules that are free of finite rank over $\mathbb{M}_2$. This diagram commutes. Because the stable comodule category cooperates nicely with taking colimits in the sense that the condition (2) holds, we can compute the colimit of a sequence of $\mathrm{Ext}_\Gamma(M, N)$.

Here we introduce notation that will be used in future sections.

Notation 2.2. For a motivic spectrum M such that $H_*(M)$ is free of finite rank over $\mathbb{M}_2$, let M also denote the embedded image of the homology of the spectrum M in the stable comodule category (i.e., $M = j(H_*(M))$). We use $[M,N]^{\Gamma}_{*,*,*}$ to denote $\mathrm{Hom}_{Stab(\Gamma)}(M,N)$, where M, $N \in Stab(\Gamma)$. For example, if $M = S$, then $H_*(S) = \mathbb{M}_2$, which we also denote by S. Thus $\mathrm{Ext}^{s,f,w}_{\mathcal{A}_*}(\mathbb{M}_2, \mathbb{M}_2) = [S,S]^{\mathcal{A}_*}_{s,f,w}$. When Γ is the motivic dual Steenrod algebra, we omit the superscript Γ. This notation is consistent with [Kra].

We use the grading (s, f, w), where s is the stem, f is the Adams filtration and w is the motivic weight. Notice that $t = s + f$ is the internal degree. Given a self-map θ: $\Sigma^{s_0,f_0,w_0}M \xrightarrow{\theta} M$ in $Stab(\mathcal{A}_*)$, we have a cofiber sequence $\Sigma^{s_0,f_0,w_0}M \xrightarrow{\theta} M \to M/\theta$ in $Stab(\mathcal{A}_*)$. The associated long exact sequence will be indexed as follows:

$$\cdots \to [M,N]_{s+s_0+1,f+f_0-1,w+w_0} \to [M/\theta,N]_{s,f,w} \to [M,N]_{s,f,w} \to [M,N]_{s+s_0,f+f_0,w+w_0} \to \cdots$$

Sometimes we omit indices when there is no risk of confusion.

3. Self-maps and Massey products

In this section, we show that the cofiber S/h_0^k admits a self-map and identify it with the Massey product in Theorem 1.3. Self-maps are maps of suspensions of an object to itself. For a dualizable object Y, self maps $\Sigma^n Y \to Y$ can also be described as elements of $\pi_*(Y \otimes DY)$, with DY the $\otimes$-dual of Y. In this paper we mainly deal with homological self-maps in $Stab(\mathcal{A}_*)$.

When considering the vanishing region and the periodicity region, we only work with the h_0-torsion part. (Of course, this is not much of a loss: as classically, the only h_0-local elements are in the 0-stem.) We next investigate the h_1-torsion part inside the h_0-torsion. For this purpose, we introduce the following notion.

Definition 3.1. Let F_0 be the fiber of $S \to S[h_0^{-1}]$, where $S[h_0^{-1}] := \operatorname{colim}(S \xrightarrow{h_0} S \xrightarrow{h_0} \cdots)$ in $Stab(\mathcal{A}_*)$. Similarly, let F_{01} be the fiber of $F_0 \to F_0[h_1^{-1}]$ with $F_0[h_1^{-1}]$ defined as an analogous colimit.

The group $[S, F_{01}]$ contains the subset of $[S, S]$ consisting of elements that are both h_0- and h_1-torsion, as well as the negative parts of those h_0 and h_1-towers in $F_0[h_1^{-1}]$. The regions we are considering are unaffected. We display the corresponding Ext groups in Figure 1 and 2.

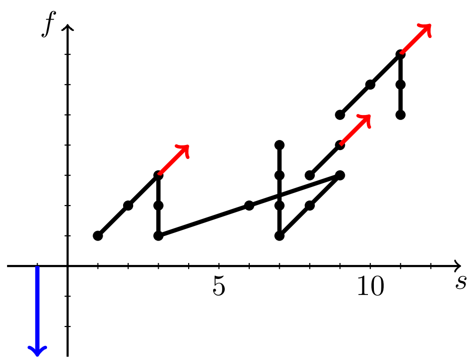

FIGURE 1. $[S, F_0]^{\mathcal{A}^\vee}_{*,*,*}$

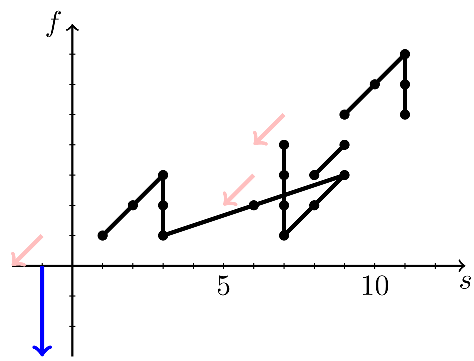

FIGURE 2. $[S, F_{01}]^{\mathcal{A}^\vee}_{*,*,*}$

The periodicity operator P corresponds to multiplying by the element h_{20}^4 of the May spectral sequence, meaning that for many values of x, $h_{20}^4 x \in \langle h_3, h_0^4, x\rangle$. However, h_{20}^4 does not survive to Ext. As a result, multiplying by P is not a map from $[S, S]$ to $[S, S]$. Luckily, [GI1, Figure 2] shows that P survives in $[S/h_0, S]$. Similarly, we have the following proposition:

Proposition 3.2 ([Ada2])**.** *The element $h_{20}^{2^r}$ survives the May spectral sequence to $[S/h_0^k, S]$ for $k \leq 2^r$, and thus gives a corresponding element $P^{2^{r-2}}$ in $[S/h_0^k, S/h_0^k]$, i.e. a self-map of S/h_0^k.*

If N is an $\mathcal{A}_*$-comodule in $Stab(\mathcal{A}_*)$, then $[S/h_0^k, S/h_0^k]$ acts on $[S/h_0^k, N]$. The corresponding element P (or some power of P) inside $[S/h_0^k, S/h_0^k]$ induces a map from $[S/h_0^k, N]$ to itself. We would like to show that for any $k \leq 2^r$ and $r \geq 2$, multiplying by $P^{2^{r-2}}$ on $[S/h_0^k, S]$ coincides with the Massey product $P_r(-) := \langle h_{r+1}, h_0^{2^r}, -\rangle$ in a certain region. In other words, we must show that there is zero indeterminacy.

The Massey product is defined on the kernel of $h_0^{2^r}$ on $[S, S]$, which we will denote $ker(h_0^{2^r})$. It lands in the cokernel of multiplication by h_{r+1}:

$$P_r(-) : ker(h_0^{2^r}) \to [S, S]/h_{r+1}.$$

Remark 3.3. Originally one would like to consider the following square and see that it commutes in a certain region

$$\begin{array}{ccc} [S/h_0^k, S] & \xrightarrow{-\cdot P^{2^{r-2}}} & [S/h_0^k, S] \\ \downarrow & & \downarrow \\ ker(h_0^{2^r}) & \xrightarrow[P_r(-)]{} & [S, S]/h_{r+1}. \end{array}$$

The vertical maps are induced by $S \to S/h_0^k$. However, since we lost the advantage of a vanishing region of $f > \frac{1}{2}s + \frac{3}{2}$ that we need in the classical setting, the region where the vertical maps are isomorphisms is not satisfactory. We solve this problem by restricting attention to the h_0 and h_1-torsion.

To better fit our purposes, consider the Massey product defined on $[S, F_{01}]$

$$P_r(-) : ker_{F_{01}}(h_0^{2^r}) \to [S, F_{01}]/h_{r+1}.$$

This gives the following squares, over which we have more control:

$$\begin{array}{ccc} [S/h_0^k, F_{01}] & \xrightarrow{-\cdot P^{2^{r-2}}} & [S/h_0^k, F_{01}] \\ \downarrow & & \downarrow \\ ker_{F_{01}}(h_0^{2^r}) & \xrightarrow[P_r(-)]{} & [S, F_{01}]/h_{r+1} \\ \downarrow & & \downarrow \\ ker_S(h_0^{2^r}) & \xrightarrow[P_r(-)]{} & [S, S]/h_{r+1} \end{array} \tag{1}$$

The canonical map $F_{01} \to S$ induces a map $[S, F_{01}] \to [S, S]$ given by inclusion on the h_0- and h_1-torsion elements and which sends negative towers to zero. The bottom square commutes for $s > 0$ and $f > 0$ modulo potential indeterminacy. We would like to show that the indeterminacy vanishes under some conditions.

Let $C(\eta)$ denote the cofiber of the first Hopf map

$$S^{1,1} \xrightarrow{\eta} S^{0,0}.$$

Writing C_η for the cohomology $H^{*,*}(C(\eta))$, we have the following result:

Theorem 3.4. [GI1, Theorem 1.1] *The group* $\mathrm{Ext}_{\mathcal{A}}^{s,f,w}(\mathbb{M}_2, C_\eta)$ *vanishes when* $s > 0$ *and* $f > \frac{1}{2}s + \frac{3}{2}$.

Theorem 3.4 gives us that $[S,C_\eta]_{s,f,w}$ vanishes when $s>0$ and $f>\frac{1}{2}s+\frac{3}{2}$. In other words, there are only h_1-towers when $s>0$ and $f>\frac{1}{2}s+\frac{3}{2}$ in $[S,S]_{s,f,w}$. Moreover, we have the following fact:

Proposition 3.5 (Corollary of [GI2, Theorem 1.1])**.** *For $r\geq 1$, h_{r+1} does not support an h_1-tower.*

Therefore the indeterminacy $(h_{r+1}[S,S])_{s,f,w}$ must vanish when $f>\frac{1}{2}s+3-2^r$, under the following two conditions: that h_{r+1} has $s=2^{r+1}-1$, and that there are only h_1-towers in $[S,S]_{s,f,w}$ when $s>0$ and $f>\frac{1}{2}s+\frac{3}{2}$, which are h_{r+1}-torsion groups.

Remark 3.6. It is easy to see that the indeterminacy $(h_{r+1}[S,F_{01}])_{s,f,w}$ also vanishes when $f>\frac{1}{2}s+3-2^r$.

The first row of the top square in (1) is multiplication by some power of the element P. We next determine when the vertical maps are isomorphisms.

Lemma 3.7 (Motivic version of [Kra, Lemma 5.2])**.** *Let $M,N\in Stab(\mathcal{A}_*)$. Assume that $[M,N]$ vanishes when $f>as+bw+c$ for some $a,b,c\in\mathbb{R}$, let $\theta:\Sigma^{s_0,f_0,w_0}M\to M$ be a map with $f_0>as_0+bw_0$, and let M/θ denote the cofiber of $\Sigma^{s_0,f_0,w_0}M\xrightarrow{\theta}M$. Then*

$$[M/\theta,N]\to[M,N]$$

is an isomorphism above a vanishing plane parallel with the one in $[M,N]$ but with f-intercept given by $c-(f_0-as_0-bw_0)$.

Proof. The result follows from the long exact sequence associated to the cofiber sequence $\Sigma^{s_0,f_0,w_0}M\xrightarrow{\theta}M\to M/\theta$:

$$\cdots\to[M,N]_{s+s_0+1,f+f_0-1,w+w_0}\to[M/\theta,N]_{s,f,w}\to[M,N]_{s,f,w}\to[M,N]_{s+s_0,f+f_0,w+w_0}\to\cdots$$

□

Remark 3.8. This approach could also apply to a vanishing region above several planes or even a surface. The vanishing condition of Lemma 3.7 could be rephrased as the following:

Assume that $[M,N]_{*,*,*}$ vanishes when $f>\varphi(s,w)$ where $\varphi:\mathbb{R}^2\to\mathbb{R}$ is a smooth function. Then the gradient $v(-,-)=(\frac{\partial\varphi}{\partial s}(-),\frac{\partial\varphi}{\partial w}(-))$ is a vector field. Let $d=\underset{(s_0,w_0)}{max}|v(s_0,w_0)|$, and assume both $\frac{f_0}{s_0}$ and $\frac{f_0}{w_0}$ are greater than d. The remaining proof would follow similarly, with the f-intercept given by $max\{c-(f_0-ds_0),c-(f_0-dw_0)\}$.

We have this as a corollary:

Corollary 3.9 (Motivic version of [Kra, Lemma 5.9])**.** *Let $k\geq 1$. For $f>\frac{1}{2}s+\frac{3}{2}-k$, the natural map $[S/h_0^k,F_{01}]_{s,f,w}\to[S,F_{01}]_{s,f,w}$ is an isomorphism.*

Proof. To determine this, we need to confirm that $[S, F_{01}]_{s,f,w}$ admits a vanishing region of $f > \frac{1}{2}s + \frac{3}{2}$. The fiber sequence $F_{01} \to F_0 \hookrightarrow F_0[h_1^{-1}]$ gives us an exact sequence:

$$\cdots \to [S, F_{01}]_{s,f,w} \to [S, F_0]_{s,f,w} \xrightarrow{h_1^{-1}} [S, F_0[h_1^{-1}]]_{s,f,w} \to [S, \Sigma^{1,-1,0}F_{01}]_{s,f,w} \to \cdots$$

Since $[S, F_0]$ differs from $[S, S]$ only in the 0-stem, there are only h_1-towers when $f > \frac{1}{2}s + \frac{3}{2}$. And by Theorem 3.4 again, $[S, C_\eta]_{s,f,w}$ vanishes when $s > 0$ and $f > \frac{1}{2}s + \frac{3}{2}$. In other words, above the plane $f = \frac{1}{2}s + \frac{3}{2}$, multiplying by h_1, which detects η, is an isomorphism from $[S, F_0]_{s,f,w}$ to $[S, F_0]_{s+1,f+1,w+1}$.

As a result, inverting h_1 would be an isomorphism from $[S, F_0]_{s,f,w}$ to $[S, F_0[h_1^{-1}]]_{s,f,w}$ when $f > \frac{1}{2}s + \frac{3}{2}$. Therefore, $[S, F_{01}]_{s,f,w}$ vanishes when $f > \frac{1}{2}s + \frac{3}{2}$. Applying Lemma 3.7 gives the corollary. □

The results in 3.2 and 3.6 locate the region where both squares commute, thus obtaining the first part of Theorem 1.3.

Theorem 3.10 (Motivic version of [Kra, Proposition 5.12])**.** *For $k \leq 2^r$ and $r \geq 2$, the cofiber S/h_0^k admits a self-map $P^{2^{r-2}}$ of degree $(2^{r+1}, 2^r, 2^r)$. Thus, for any $N \in Stab(\mathcal{A}_*)$, composition with $P^{2^{r-2}}$ defines a self-map on $[S/h_0^k, N]$.*

When $f > \frac{1}{2}s + 3 - k$, in the case $N = F_{01}$, the induced map coincides with the Massey product $P_r(-) := \langle h_{r+1}, h_0^{2^r}, - \rangle$ with zero indeterminacy.

4. Colimits and the Cartan-Eilenberg spectral sequence

We will obtain a vanishing region for $[S/(h_0, P), F_{01}]_{*,*,*}$ in this section. Consider the colimit

$$F_0/h_1^\infty := \operatorname*{colim}_i (\Sigma^{-1,-1,-1}F_0/h_1 \xrightarrow{h_1} \cdots \xrightarrow{h_1} \Sigma^{-i,-i,-i}F_0/h_1^i \xrightarrow{h_1} \cdots)$$

in $Stab(\mathcal{A}_*)$. As we show in the following result, it differs from F_{01} by a suspension in the region we are considering.

Proposition 4.1. *When $f > \frac{1}{2}s + \frac{3}{2}$,*

$$[S, \Sigma^{-1,1,0}F_0/h_1^\infty]_{s,f,w} \cong [S, F_{01}]_{s,f,w}$$

Proof. To see this, note that the colimit F_0/h_1^∞ is a union of all the h_1-torsion in F_0, while the fiber F_{01} detects the h_1-torsion together with those negative h_1-towers. □

Note that F_0 coincides with

$$\Sigma^{-1,1,0}S/h_0^\infty := \Sigma^{-1,1,0}\operatorname*{colim}_i (\Sigma^{0,1,0}S/h_0 \xrightarrow{h_0} \cdots \xrightarrow{h_0} \Sigma^{0,i,0}S/h_0^i \xrightarrow{h_0} \cdots),$$

if we ignore the negative h_0-tower. That is, we have $[S, \Sigma^{-1,1,0}S/h_0^\infty]_{s,f,w} \cong [S, F_0]_{s,f,w}$ when $f > 0$.

Remark 4.2. We have shown that the map

$$[S/h_0^k, F_0/h_1^\infty]_{s,f,w} \to [S, F_0/h_1^\infty]_{s,f,w}$$

is an isomorphism when $f > \frac{1}{2}s + 3 - k$. We consider this colimit because it is better for computational purposes (the fiber F_{01} is harder to deal with than the colimit F_0/h_1^∞).

Let θ be a self-map of S/h_0^k, and consider the cofiber sequence $S/h_0^k \xrightarrow{\theta} S/h_0^k \to S/(h_0^k, \theta)$. The vanishing region for $[S/(h_0^k, \theta), F_0/h_1^\infty]_{*,*,*}$ is the region where

$$[S/h_0^k, F_0/h_1^\infty]_{s,f,w} \xrightarrow{\theta} [S/h_0^k, F_0/h_1^\infty]_{s+s_0,f+f_0,w+w_0}$$

is an isomorphism. The goal of this section is to obtain a vanishing region for $[S/(h_0^k, \theta), F_0/h_1^\infty]_{*,*,*}$ in the case $k = 1$ and $\theta = P$.

The dual Steenrod algebra is too large to work with, so we would like to start with a smaller one, namely $\mathcal{A}(1)_* \cong \mathbb{M}_2[\tau_0, \tau_1, \xi_1]/(\tau_0^2 = \tau\xi_1, \tau_1^2, \xi_1^2)$. Then for $\mathcal{A}_*$-comodules M and N (thus also $\mathcal{A}(1)_*$-comodules), we can recover $[M, N]^{\mathcal{A}_*}$ from $[M, N]^{\mathcal{A}(1)_*}$ via infinitely many Cartan-Eilenberg spectral sequences along normal extensions of Hopf algebras, as we will explain later.

Let $N = F_0/h_1^\infty$. We will compute $[S/h_0, F_0/h_1^\infty]^{\mathcal{A}(1)_*}$ as an intermediate step before reaching our goal of $[S/(h_0, P), F_0/h_1^\infty]^{\mathcal{A}(1)_*}$. As a starting point, we can compute $[S/h_0, F_0]$ over $\mathcal{A}(1)_*$, via the cofiber sequence $S \xrightarrow{h_0} S \to S/h_0$.

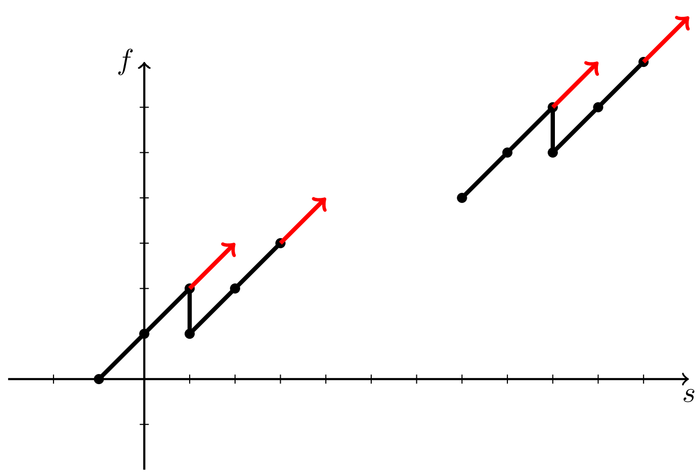

FIGURE 3. $[S/h_0, F_0]^{\mathcal{A}(1)_*}$

This is periodic,where the periodicity shifts degree by $(8, 4, 4)$. Since $[S/h_0, F_0/h_1^\infty]^{\mathcal{A}(1)_*}$ is a colimit, it is essential to know the maps over which we are taking the colimit. First let us take a look at the maps induced by multiplying by h_1 (we abbreviate $\Sigma^{-i,-i,-i}$ to Σ^{-i} in this diagram):

$$\begin{array}{ccccccc}
\xrightarrow{h_1} & [S/h_0, \Sigma^{-1}F_0] & \longrightarrow & [S/h_0, \Sigma^{-1}F_0/h_1] & \longrightarrow & \Sigma^{2,0,1}[S/h_0, \Sigma^{-1}F_0] & \xrightarrow{h_1} \\
 & \downarrow{\scriptstyle h_1\circ\Sigma^{-1}} & & \downarrow & & \downarrow{\scriptstyle id} & \\
\xrightarrow{h_1^2} & [S/h_0, \Sigma^{-2}F_0] & \longrightarrow & [S/h_0, \Sigma^{-2}F_0/h_1^2] & \longrightarrow & \Sigma^{3,1,2}[S/h_0, \Sigma^{-2}F_0] & \xrightarrow{h_1^2} \\
 & \downarrow{\scriptstyle h_1\circ\Sigma^{-1}} & & \downarrow & & \downarrow &
\end{array} \tag{2}$$

All rows are exact. From this we yield a more illuminating diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & coker(h_1) & \longrightarrow & [S/h_0, \Sigma^{-1}F_0/h_1] & \longrightarrow & ker(h_1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h_1\circ\Sigma^{-1}} & & \downarrow & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & coker(h_1^2) & \longrightarrow & [S/h_0, \Sigma^{-2}F_0/h_1^2] & \longrightarrow & ker(h_1^2) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h_1\circ\Sigma^{-1}} & & \downarrow & & \downarrow & &
\end{array}$$

The maps i on the right column are canonical inclusions, and passing to colimits gives

$$\operatorname*{colim}_k(coker(h_1^k)) \to [S/h_0, F_0/h_1^\infty] \to \operatorname*{colim}_k(ker(h_1^k)).$$

Working over the dual subalgebra $\mathcal{A}(1)_*$ we calculate $[S/h_0, \Sigma^{-1,1,0}F_0/h_1^\infty]^{\mathcal{A}(1)_*}_{*,*,*}$ directly. Furthermore we have:

Proposition 4.3. *For any $k \in \mathbb{Z}$, $k \geq 1$, the maps $[S/h_0, \Sigma^{-k}F_0/h_1^k]^{\mathcal{A}(1)_*} \to [S/h_0, \Sigma^{-k-1}F_0/h_1^{k+1}]^{\mathcal{A}(1)_*}$ are injective.*

The result of the calculation is shown in Figure 4. The shift in the figure appears as result of Proposition 4.1.

This is periodic, with a periodicity degree shift of $(8, 4, 4)$, just as with $[S/h_0, F_0]^{\mathcal{A}(1)_*}$. Note that $[S/h_0, \Sigma^{-1,1,0}F_0/h_1^\infty]^{\mathcal{A}(1)_*}_{*,*,*}$ differs from the classical $[S/h_0, S]^{\mathcal{A}_{cl}(1)_*}_{*,*}$ with two extra negative h_1-towers associated to each "lighting flash". The element in degree $(-1, 0, -1)$ in the first pattern is generated by τ with a shift.

Recall the self-map P on S/h_0 acts injectively as can be seen in Figure 4. Combining this with the long exact sequence:

$$\cdots \longrightarrow [S/(h_0,P), F_0/h_1^\infty]^{\mathcal{A}(1)_*}_{s,f,w} \longrightarrow [S/h_0, F_0/h_1^\infty]^{\mathcal{A}(1)_*}_{s,f,w} \xrightarrow{P}$$

$$\xrightarrow{P} [S/h_0, F_0/h_1^\infty]^{\mathcal{A}(1)_*}_{s+8,f+4,w+4} \longrightarrow [S/(h_0,P), F_0/h_1^\infty]^{\mathcal{A}(1)_*}_{s-1,f+1,w} \longrightarrow \cdots$$

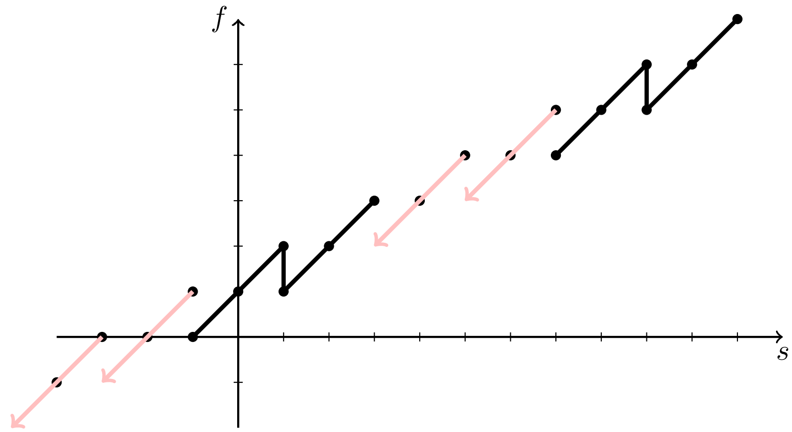

FIGURE 4. $[S/h_0, \Sigma^{-1,1,0}F_0/h_1^\infty]_{*,*,*}^{\mathcal{A}(1)_*}$

gives $[S/(h_0, P), \Sigma^{-1,1,0}F_0/h_1^\infty]_{*,*,*}^{\mathcal{A}(1)_*}$ as in Figure 5.

Remark 4.4. Analogously to Proposition 4.3, for any $k \in \mathbb{Z}$, $k \geq 1$, the following maps are also injective:

$$[S/(h_0, P), \Sigma^{-k}F_0/h_1^k]^{\mathcal{A}(1)_*} \to [S/(h_0, P), \Sigma^{-k-1}F_0/h_1^{k+1}]^{\mathcal{A}(1)_*}.$$

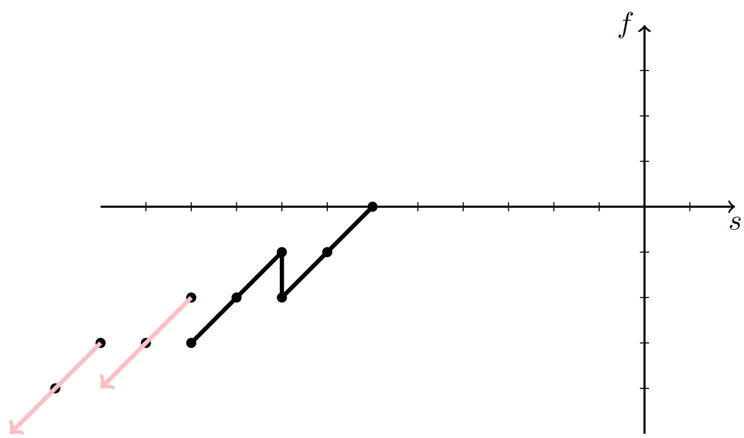

FIGURE 5. $[S/(h_0, P), \Sigma^{-1,1,0}F_0/h_1^\infty]_{*,*,*}^{\mathcal{A}(1)_*}$

Next we will use the Cartan-Eilenberg spectral sequence to bootstrap our result from $\mathcal{A}(1)_*$-homology to $\mathcal{A}_*$-homology. A brief introduction to the Cartan-Eilenberg spectral sequence (see [CE, Ch.XV] for details) is relevant at this point. Given an extension of Hopf algebras over $\mathbb{M}_2$

$$E \to \Gamma \to C$$

(so in particular $E \cong \Gamma \square_C \mathbb{M}_2$), the Cartan-Eilenberg spectral sequence computes $Cotor_\Gamma(M,N)$ for a Γ-comodule M and an E-comodule N. The spectral sequence arises from the double complex (Γ-resolution of M)$\square_\Gamma$(E-resolution of N), and we have $Cotor_\Gamma(M,N) \cong \mathrm{Ext}_\Gamma(M,N)$ when M and N are τ-free.

The Cartan-Eilenberg spectral sequence has the form

$$E_1^{s,t,*,*} = Cotor_C^{t,*}(M, \bar{E}^{\otimes s} \otimes N) \Rightarrow Cotor_\Gamma^{s+t,*}(M,N).$$

If E has trivial C-coaction, then we have $E_1^{s,t,*,*} \cong Cotor_C^{t,*}(M,N) \otimes \bar{E}^{\otimes s}$. Taking the cohomology we obtain the E_2-page:

$$E_2^{s,t,*,*} = Cotor_E^{s,*}(\mathbb{M}_2, Cotor_C^{t,*}(M,N)) \cong \mathrm{Ext}_E^{s,*}(\mathbb{M}_2,\mathbb{M}_2) \otimes \mathrm{Ext}_C^{t,*}(M,N).$$

The Cartan-Eilenberg spectral sequence converges when the input is a bounded-below $\mathcal{A}_*$-comodule. We will obtain a vanishing region for each finite stage $[S/(h_0,P), \Sigma^{-k}F_0/h_1^k]^{\mathcal{A}_*}$ and then deduce the vanishing region for $[S/(h_0,P), F_0/h_1^\infty]^{\mathcal{A}_*}$ by passing to the colimit.

$$\begin{array}{ccccc}
[S/(h_0,P),\Sigma^{-1}F_0/h_1]^{\mathcal{A}(1)_*} & \longrightarrow & [S/(h_0,P),\Sigma^{-2}F_0/h_1^2]^{\mathcal{A}(1)_*} & \longrightarrow \cdots \longrightarrow & [S/(h_0,P),F_0/h_1^\infty]^{\mathcal{A}(1)_*} \\
\rightsquigarrow & & CESS \rightsquigarrow & & \\
[S/(h_0,P),\Sigma^{-1}F_0/h_1]^{\mathcal{A}_*} & \longrightarrow & [S/(h_0,P),\Sigma^{-2}F_0/h_1^2]^{\mathcal{A}_*} & \longrightarrow \cdots \longrightarrow & [S/(h_0,P),F_0/h_1^\infty]^{\mathcal{A}_*}
\end{array}$$

We first calculate $[S/(h_0,P), F_0/h_1^\infty]^{\mathcal{A}(2)_*}$, where

$$\mathcal{A}(2)_* = \mathbb{M}_2[\tau_0,\tau_1,\tau_2,\xi_1,\xi_2]/(\tau_0^2 = \tau\xi_1, \tau_1^2 = \tau\xi_2, \tau_2^2, \xi_1^4, \xi_2^2).$$

To do this, we will use a sequence of normal maps of Hopf algebras:

$$\mathcal{A}(2)_* \to \mathcal{A}(2)_*/\xi_1^2 \to \mathcal{A}(2)_*/(\xi_1^2,\xi_2) \to \mathcal{A}(2)_*/(\xi_1^2,\xi_2,\tau_2) = \mathcal{A}(1)_*.$$

First we consider the Cartan-Eilenberg spectral sequence corresponding to the extension

$$E(\tau_2) \to \mathcal{A}(2)_*/(\xi_1^2,\xi_2) \to \mathcal{A}(1)_*.$$

The element τ_2, which has degree $(6,1,3)$, corresponds to h_{30} in the May spectral sequence. The $\mathcal{A}(1)_*$-coaction on $E(\tau_2)$ is trivial for degree reasons. So we start with the $E_1 = E_2$-page, and deduce a vanishing region on $[S/(h_0,P), F_0/h_1^\infty]^{\mathcal{A}(2)_*/(\xi_1^2,\xi_2)}$.

$$\begin{array}{ccc}
[S/(h_0,P),\Sigma^{-1}F_0/h_1]^{\mathcal{A}(1)_*} \otimes \mathbb{M}_2[h_{30}] & \longrightarrow \cdots \longrightarrow & [S/(h_0,P),F_0/h_1^\infty]^{\mathcal{A}(1)_*} \otimes \mathbb{M}_2[h_{30}] \\
\Downarrow & & \\
[S/(h_0,P),\Sigma^{-1}F_0/h_1]^{\mathcal{A}(2)_*/(\xi_1^2,\xi_2)} & \longrightarrow \cdots \longrightarrow & [S/(h_0,P),F_0/h_1^\infty]^{\mathcal{A}(2)_*/(\xi_1^2,\xi_2)}
\end{array}$$

For the normal extension $E(\beta) \to \Gamma \to C$ of Hopf algebras we state a motivic version of [Kra, Lemma 4.10], which gives a relationship between the vanishing region for $[M,N]^\Gamma$ and the vanishing condition of $[M,N]^C$ together with the two "slopes" associated to β. Note that if β has degree

(s_0, f_0, w_0), then $\frac{f_0}{s_0}$ and $\frac{f_0}{w_0}$ are the slopes of the projections of (s_0, f_0, w_0) onto the plane $w = 0$ and the plane $s = 0$.

Theorem 4.5. *Let $E(\alpha) \to \Gamma \xrightarrow{q} C$ be a normal extension of Hopf algebras and M,$N \in Stab(\Gamma)$. Suppose β is an element in $[S, S]^E$ of degree (s_0, f_0, w_0) with s_0, f_0, w_0 all positive. Its image in $[S, S]^\Gamma$ (which we also call β) acts on $[M, N]^\Gamma$. Suppose for some $a, b, c, m, c_0 \in \mathbb{R}$ with $a, b > 0$ and $m \geq \frac{f_0}{s_0} > 0$, the group $[q_*(M), q_*(N)]^C$ vanishes when $f > as + bw + c$ and also vanishes when $f > ms + c_0$. Then*

(1) *if $f_0 \leq as_0 + bw_0$, or β acts nilpotently on $[M, N]^\Gamma$, then $[M, N]^\Gamma$ has a parallel vanishing region. In other words, it vanishes when $f > as + bw + c'$ for some constant c' and also vanishes when $f > ms + c_0$.*
(2) *otherwise, $[M, N]^\Gamma$ vanishes when $f > \frac{mbw_0 - f_0(m-a)}{bw_0 - s_0(m-a)} s + \frac{bf_0 - mbs_0}{bw_0 - s_0(m-a)} w + c'$ and vanishes when $f > ms + c_0$.*

Remark 4.6. The additional vanishing plane $f > ms + c_0$ generalizes the bounded below condition. In the classical setting, we have that $[M, N]^\Gamma$ vanishes when $s < c_0$, but due to the negative h_1-towers we do not have a vertical vanishing plane. So we adjust the "∞-slope" plane to be $f = ms + c_0$ to fulfill our purpose. This bound does not affect the periodicity region we study here, so we omit it henceforth.

Proof of Theorem 4.5. If β has $f_0 \leq as_0 + bw_0$, then β multiples of classes in $[M, N]^C$ will lie under the existing vanishing planes.

If $f_0 > as_0 + bw_0$, then every infinite β tower will contain classes lying outside of the rigion $f > as + bw + c$. If β acts nilpotently, there exists an integer k such that $\beta^k x$ is zero for all $x \in [M, N]^\Gamma$. Then there is a maximum length for all β-towers, and so we can still get a parallel vanishing plane $f > as + bw + c'$ on $[M, N]^\Gamma$ by adjusting the f-intercept.

Now we turn to case (2). If $f_0 > as_0 + bw_0$ and β acts non-nilpotently, then there must exist an element $x \in [M, N]^\Gamma$ for which the classes $\beta^k x$ are not zero on the E_∞ page of the Cartan-Eilenberg spectral sequence for every k. Thus no matter how we move up the existing vanishing plane $f > as + bw + c$, some β multiples of x will lie above the plane. Instead, we will find a new vanishing plane $f > a's + b'w + c'$ for $a', b', c' \in \mathbb{R}$. The new vanishing region $f > a's + b'w + c'$ must satisfy the condition $f_0 \leq a's_0 + b'w_0 + c'$. This plane is spanned by the direction of β and the intersecting line of the two existing vanishing planes. Hence we can solve to obtain $a' = \frac{mbw_0 - f_0(m-a)}{bw_0 - s_0(m-a)}$ and $b' = \frac{bf_0 - mbs_0}{bw_0 - s_0(m-a)}$. □

Remark 4.7. In the relevant cases, the starting vanishing regions will have $b = 0$. In this case, the 3-dimensional conditions in Theorem 4.5 simplify to the following 2-dimensional conditions.

Suppose for some $a, c, m, c_0 \in \mathbb{R}$ with $a > 0$ and $m \geq \frac{f_0}{s_0} > 0$, the group $[q_*(M), q_*(N)]^C$ vanishes when $f > as + c$ and also vanishes when $f > ms + c_0$. Then:

(1) if $f_0 \leq as_0$, or β acts nilpotently on $[M, N]^\Gamma$, then $[M, N]^\Gamma$ has a parallel vanishing region. That is to say, it vanishes when $f > as + c'$ for some constant c', and also vanishes when $f > ms + c_0$,

(2) if otherwise, then $[M, N]^\Gamma$ vanishes when $f > \frac{f_0}{s_0} s + c'$ for some constant c', and vanishes when $f > ms + c_0$.

Remark 4.8. Similarly, we could generalize to the statement that the group $[q_*(M), q_*(N)]^C$ vanishes when $f > \varphi(s, w)$ where $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Then the gradient $v(-,-) = (\frac{\partial \varphi}{\partial s}(-), \frac{\partial \varphi}{\partial w}(-))$ is a vector field. Now we would like to consider $g = \underset{(s_0, w_0)}{Min} |v(s_0, w_0)|$ and compare g with $\frac{f_0}{s_0}$ and $\frac{f_0}{w_0}$. The conditions can be rewritten as follows:

(1) if $\frac{f_0}{s_0} \leq g$ or $\frac{f_0}{w_0} \leq g$, or β acts nilpotently, then $[M, N]^\Gamma$ has the same vanishing region translated vertically.

(2) if both $\frac{f_0}{s_0}$ and $\frac{f_0}{w_0} > g$, and β acts non-nilpotently, then we must modify the vanishing region of $[M, N]^\Gamma$. However, it takes some work to write down a precise modification, so we omit it here.

Remark 4.9. From the cofiber sequence $S \xrightarrow{h_0^k} S \to S/h_0^k$ we can take tensor duals to derive the fiber sequence $D(S/h_0^k) \to S \to S$. Since $D(S/h_0^k) \simeq \Sigma^{-1,1-k,0} S/h_0^k$, we have

$$[S/h_0^k, S]_{s,f,w} = [S, D(S/h_0^k)]_{s,f,w} = [S, S/h_0^k]_{s+1,f+k-1,w.}$$

Because S/h_0^k is compact in $Stab(\mathcal{A}_*)$, smashing with some $N \in Stab(\mathcal{A}_*)$, we get

$$[S/h_0^k, N]_{s,f,w} \cong [S, D(S/h_0^k) \wedge N]_{s,f,w} \cong [S, S/h_0^k \wedge N]_{s+1,f+k-1,w.}$$

As a result $\beta \in [S, S]^\Gamma$ acts on $[M, N]^\Gamma$ for compact $M \in Stab(\mathcal{A}_*)$, since β acts on $[S, DM \wedge N]^\Gamma$.

The group $[S/(h_0, P), \Sigma^{-1,1,0} F_0/h_1^\infty]^{\mathcal{A}(1)_*}_{*,*,*}$ has a single "lighting flash" pattern along with two negative h_1-towers (see Figure 5), so the vanishing region to start off with is $f > c$ (We obtain the same vanishing region of $[S/(h_0, P), \Sigma^{-1,1,0}(\Sigma^{-k} F_0/h_1^k)]^{\mathcal{A}(1)_*}_{*,*,*}$ for each k, since the maps we are taking colimit over are injections by Remark 4.4.) In our case, $[M, N]^\Gamma = [S/(h_0, P), \Sigma^{-1,1,0} F_0/h_1^\infty]^{\mathcal{A}(1)_*}_{*,*,*}$, and we will apply Theorem 4.5 in the following three cases: (i) β is τ_2 of degree $(6, 1, 3)$; (ii) β is ξ_2 of degree $(5, 1, 3)$; (iii) β is ξ_1^2 of degree $(3, 1, 2)$.

Recall that we are working with the Cartan-Eilenberg spectral sequence

$$[S/(h_0, P), \Sigma^{-1,1,0}(\Sigma^{-k} F_0/h_1^k)]^{\mathcal{A}(1)_*} \otimes \mathbb{M}_2[h_{30}] \Rightarrow [S/(h_0, P), \Sigma^{-1,1,0}(\Sigma^{-k} F_0/h_1^k)]^{\mathcal{A}(2)_*/(\xi_1^2, \xi_2)}$$

There cannot be any differentials for degree reasons. By Theorem 4.5 the element h_{30} will bring us a vanishing region $f > \frac{1}{6}s + c_1$ for each k, where c_1 is some constant (we obtain the same constant for all k). Passing to the colimit, we conclude that $[S/(h_0, P), \Sigma^{-1,1,0} F_0/h_1^\infty]^{\mathcal{A}(2)_*/(\xi_1^2,\xi_2)}$ shares the same vanishing region $f > \frac{1}{6}s + c_1$.

The second step is to consider the normal extension in which we add ξ_2, corresponding to the class h_{21}:

$$E(\xi_2) \to \mathcal{A}(2)_*/\xi_1^2 \to \mathcal{A}(2)_*/(\xi_1^2, \xi_2).$$

The $\mathcal{A}(2)_*/(\xi_1^2, \xi_2)$-coaction on $E(\xi_2)$ is trivial. We have E_2-pages as the first row:

$$\begin{array}{ccc} [S/(h_0,P),\Sigma^{-1}F_0/h_1]^{\mathcal{A}(2)_*/(\xi_1^2,\xi_2)} \otimes \mathbb{M}_2[h_{21}] & \longrightarrow \cdots \longrightarrow & [S/(h_0,P),F_0/h_1^\infty]^{\mathcal{A}(2)_*/(\xi_1^2,\xi_2)} \otimes \mathbb{M}_2[h_{21}] \\ \Downarrow & & \\ [S/(h_0,P),\Sigma^{-1}F_0/h_1]^{\mathcal{A}(2)_*/\xi_1^2} & \longrightarrow \cdots \longrightarrow & [S/(h_0,P),F_0/h_1^\infty]^{\mathcal{A}(2)_*/\xi_1^2} \end{array}$$

The spectral sequence collapses at the E_2-page. This is because in the May spectral sequence over $\mathcal{A}(2)$ or $\mathcal{A}$, there is a differential $d_1(h_{30}) = h_1 h_{21} + h_2 h_{20}$, but in the group $[S/(h_0, P), \Sigma^{-1,1,0}(\Sigma^{-k} F_0/h_1^k)]^{\mathcal{A}(2)_*/(\xi_1^2,\xi_2)}$, h_0 and h_2 are zero. As a result, h_{21} is also non-nilpotent. For some constant c_2, the vanishing region of $[S/(h_0, P), \Sigma^{-1,1,0}(\Sigma^{-k} F_0/h_1^k)]^{\mathcal{A}(2)_*/\xi_1^2}$ is $f > \frac{1}{5}s + c_2$ for each k according to Theorem 4.5, and the same is true for the colimit $[S/(h_0, P), \Sigma^{-1,1,0} F_0/h_1^\infty]^{\mathcal{A}(2)_*/\xi_1^2}$.

Next we consider the Cartan-Eilenberg spectral sequence corresponding to the extension:

$$E(\xi_1^2) \to \mathcal{A}(2)_* \to \mathcal{A}(2)_*/\xi_1^2.$$

Here the class ξ_1^2 corresponds to the class h_2 in the May spectral sequence. The $\mathcal{A}(2)_*/\xi_1^2$-coaction on $E(\xi_1^2)$ is trivial as well. We have E_2-pages as in the first row:

$$\begin{array}{ccc} [S/(h_0,P),\Sigma^{-1}F_0/h_1]^{\mathcal{A}(2)_*/\xi_1^2} \otimes \mathbb{M}_2[h_2] & \longrightarrow \cdots \longrightarrow & [S/(h_0,P),F_0/h_1^\infty]^{\mathcal{A}(2)_*/\xi_1^2} \otimes \mathbb{M}_2[h_2] \\ \Downarrow & & \\ [S/(h_0,P),\Sigma^{-1}F_0/h_1]^{\mathcal{A}(2)_*} & \longrightarrow \cdots \longrightarrow & [S/(h_0,P),F_0/h_1^\infty]^{\mathcal{A}(2)_*} \end{array}$$

We do have some non-zero differentials appear. In the previous steps, by introducing $[\tau_2] = (6, 1, 3)$ and $[\xi_2] = (5, 1, 3)$, which give rise to non-nilpotent elements in Ext, we arrived a vanishing region of $f > \frac{1}{5}s + c_3$, where c_3 is a constant. However $[\xi_1^2] = (3, 1, 2)$ is nilpotent since $h_2^4 = 0$ in $\mathrm{Ext}_{\mathcal{A}(2)_*}$ and Ext.

Moving from $\mathcal{A}(2)_*$ to $\mathcal{A}_*$, we have many more elements to introduce. However those elements won't satisfy $\frac{f}{s} > \frac{1}{5}$. By Theorem 4.5 (or Remark 4.7), for each k, $[S/(h_0, P), \Sigma^{-1,1,0}(\Sigma^{-k} F_0/h_1^k)]^{\mathcal{A}}$ vanishes if $f = \frac{1}{5}s + c_3$. Since the vanishing plane passes through the point $(-6, 0, -1) + 3 \cdot$

$(3,1,2) = (3,3,5)$, the constant c_3 is $\frac{12}{5}$ and the region $f > \frac{1}{5}s + \frac{12}{5}$ would be carried through to $\mathcal{A}_*$. We conclude that

Proposition 4.10. *The group $[S/(h_0, P), \Sigma^{-1,1,0}F_0/h_1^\infty]_{s,f,w}$ has a vanishing region of $f > \frac{1}{5}s + \frac{12}{5}$.*

Note that it is possible for many reasons that the vanishing region we have found is not optimal. First, we could consider the "slope" of the motivic weight side $\frac{f}{w}$ instead of $\frac{f}{s}$ under certain bounded below conditions. Second, if other elements were included, more differentials would occur, allowing for a larger vanishing region. More calculation is required to clarify these cases.

5. The motivic periodicity theorem

Let F_0 and F_{01} still be the same as in Definition 3.1, so that

$$[S, \Sigma^{-1,1,0}F_0/h_1^\infty]_{s,f,w} \cong [S, F_{01}]_{s,f,w}$$

when $f > \frac{1}{2}s + 3$. Given a self-map θ on S/h_0^k let us recall the diagram where the first row is exact:

$$\begin{array}{ccccccc}
[S/(h_0^k,\theta), \Sigma^{-1,1,0}F_0/h_1^\infty] & \longrightarrow & [S/h_0^k, \Sigma^{-1,1,0}F_0/h_1^\infty] & \xrightarrow{\theta} & [S/h_0^k, \Sigma^{-1,1,0}F_0/h_1^\infty] & \longrightarrow & \Sigma^{-1,1,0}[S/(h_0^k,\theta), \Sigma^{-1,1,0}F_0/h_1^\infty] \\
 & & \downarrow & & \downarrow & & \\
 & & [S, \Sigma^{-1,1,0}F_0/h_1^\infty] & \xrightarrow{P_r(-)} & [S, \Sigma^{-1,1,0}F_0/h_1^\infty] & &
\end{array}$$

The vertical maps are isomorphisms whenever $f > \frac{1}{2}s + \frac{3}{2} - k$ due to Corollary 3.9. We would like to further restrict the condition to $f > \frac{1}{2}s + 3 - k$ in order to eliminate the indeterminacy. The vanishing condition on $[S/(h_0^k, \theta), \Sigma^{-1,1,0}F_0/h_1^\infty]$, which is the same as the vanishing condition on $[S/(h_0^k, \theta), F_{01}]_{s,f,w}$, tells us whether θ is an isomorphism.

In the previous section, we established the case when $k = 1$, given in Proposition 4.10. We show in Figure 6 the $(2^{r+1}, 2^r, 2^r)$-periodic pattern for $[S/h_0^k, \Sigma^{-1,1,0}F_0/h_1^\infty]^{\mathcal{A}(1)_*}$, where $k \leq 2^r$. By an analogous computation, one can see that for a general positive integer $k \leq 2^r$, the groups $[S/(h_0^k, P^{2^{r-2}}), F_{01}]_{s,f,w}$ admit a parallel vanishing region as in the $k = 1$ case.

We have the following lemma for the f-intercept:

Lemma 5.1 (Corollary of [Kra, Lemma 5.4])**.** *Let $M, N \in Stable(\mathcal{A}_*)$ with M compact. Let $M_1 = M/\theta_1$ be the cofiber of the self-map $\Sigma^{s_1,f_1,w_1}M \xrightarrow{\theta_1} M$, and let $M_2 = M/(\theta_1, \theta_2)$ be the cofiber of the self-map $\Sigma^{s_2,f_2,w_2}M/\theta_1 \xrightarrow{\theta_2} M/\theta_1$. Define M_1' and M_2' with respect to the self-maps $\Sigma^{s_1',f_1',w_1'}M \xrightarrow{\theta_1'} M$ and $\Sigma^{s_2',f_2',w_2'}M/\theta_1' \xrightarrow{\theta_2'} M/\theta_1'$ in the same way. Suppose θ_i and θ_i' are parallel, i.e. $(s_i, f_i, w_i) = \lambda_i(s_i', f_i', w_i')$ where λ_i are non-zero real numbers and $i = 1, 2$.*

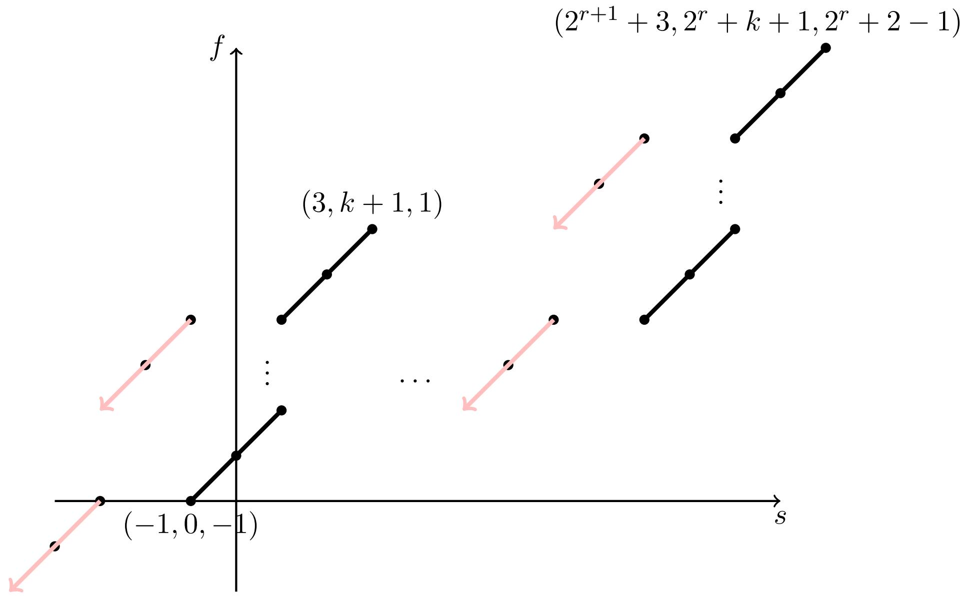

FIGURE 6. $[S/h_0^k, \Sigma^{-1,1,0}F_0/h_1^\infty]_{*,*,*}^{\mathcal{A}(1)_*}$

Further let $a, b \in \mathbb{R}$ and suppose $f_i > as_i + bw_i$ and $f'_i > as'_i + bw'_i$ for $i = 1, 2$. We make the convention that the f-intercept is ∞ if there is no such vanishing plane. Then the minimal f-intercepts of the vanishing planes parallel to $f = as + bw$ on $[M_2, N]$ and $[M'_2, N]$ agree.

Proof of Lemma 5.1. We construct the iterated cofiber $L_1 = M/(\theta_1, \theta'_1)$ and $L_2 = M/(\theta_1, \theta_2, \theta'_1, \theta'_2)$. Since $f_i > as_i + bw_i$ and $f'_i > as'_i + bw'_i$ for $i = 1, 2$, the minimal f-intercepts for the vanishing planes parellel to $f = as+bw$ agree on $[M_i, N]$, $[M'_i, N]$ and $[L_i, N]$ by inductively applying Lemma 3.7.

Note that the notation for L_1 and L_2 is ambiguous. The notation does not indicate that M/θ_1 should admit a θ'_1 self-map or vice versa. Because of the uniqueness of (homological) self-maps that Krause has shown in [Kra, Sec. 4], there is a self-map θ''_1 compatible with both θ_1 and θ'_1, which acts on M by a power of θ_1, and by a power of θ'_1. We will take L_1 to be the cofiber of the self-map θ''_1. Similarly, there exists a self-map θ''_2 on L_1 that acts on M_1 by a power of θ_2, and on M'_1 by a power of θ'_2. So we can set L_2 as the cofiber of the self-map θ''_2. □

Remark 5.2. Krause's proof of the uniqueness of self-maps is in the classical setting, yet for the $\mathbb{C}$-motivic case the proof is analogous.

Remark 5.3. The cofiber sequences arising from the Verdier's axiom and the 3×3 lemma offer an alternative way to view the vanishing condition of $[S/(h_0^k, P^{2^{r-2}}), F_{01}]_{s,f,w}$. Let $m, n, l, l' \in \mathbb{N}$ be positive with $m \leq 4l$ and

$m+n \leq 4(l+l')$. We have the following cofiber sequences:

$$S/h_0^m \to S/h_0^{m+n} \to S/h_0^n$$
$$S/(h_0^m, P^{l+l'}) \to S/(h_0^{m+n}, P^{l+l'}) \to S/(h_0^n, P^{l+l'})$$
$$S/(h_0^m, P^l) \to S/(h_0^m, P^{l+l'}) \to S/(h_0^m, P^{l'}).$$

Passing to the induced long exact sequences in homology, we conclude that for $k \leq 2^r$, the groups $[S/(h_0^k, P^{2^{r-2}}), F_{01}]_{s,f,w}$ admit the same vanishing condition as $[S/(h_0, P), F_{01}]_{s,f,w}$.

It follows that for any $k \leq 2^r$ and any self-map $\theta = P^{2^{r-2}}$ of S/h_0^k, the corresponding groups $[S/(h_0^k, \theta), F_{01}]$ have a vanishing region of $f > \frac{1}{5}s + \frac{12}{5}$. Combining with Theorem 3.10, we arrive at the motivic version of Theorem 1.1:

Theorem 5.4 (Another way of stating Theorem 1.3)**.** *For $r \geq 2$, the Massey product operation $P_r(-) := \langle h_{r+1}, h_0^{2^r}, - \rangle$ is uniquely defined on $\mathrm{Ext}^{s,f,w} = H^{s,f,w}(\mathcal{A})$ when $s > 0$ and $f > \frac{1}{2}s + 3 - 2^r$.*

Furthermore, for $f > \frac{1}{5}s + \frac{12}{5}$,

$$P_r \colon [S, F_{01}]_{s,f,w} \xrightarrow{P_r(-)} [S, F_{01}]_{s+2^{r+1}, f+2^r, w+2^r}.$$

is an isomorphism when restricted to the subgroup consisting of elements that are torsion with respect to both h_0 and h_1.

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