Computations of de Rham cohomology rings of classifying stacks at torsion primes

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Abstract. We compute the de Rham cohomology rings of $BG_2$ and $B\text{Spin}(n)$ for $7 \leq n \leq 11$ over base fields of characteristic 2.

Contents

Introduction 1002
Acknowledgments 1004
1. Preliminaries 1004
2. $G_2$ 1006
3. Spin groups 1010
References 1026

Introduction

Let $G$ be a smooth affine algebraic group over a commutative ring $R$. In [17], Totaro defines the Hodge cohomology group $H^i(BG,\Omega^j)$ for $i, j \geq 0$ to be the $i$th étale cohomology group of the sheaf of differential forms $\Omega^j$ over $R$ on the big étale site of the classifying stack $BG$. For $n \geq 0$, let $H^n_{\text{H}}(BG/R) := \oplus_j H^j(BG,\Omega^{n-j})$ denote the total Hodge cohomology group of degree $n$. De Rham cohomology groups $H^n_{\text{DR}}(BG/R)$ are defined to be the étale cohomology groups of the de Rham complex of $BG$. Let $\mathfrak{g}$ denote the Lie algebra associated to $G$ and let $O(\mathfrak{g}) = S(\mathfrak{g}^*)$ denote the ring of polynomial functions on $\mathfrak{g}$. In [17, Corollary 2.2], Totaro showed that the Hodge cohomology of $BG$ is related to the representation theory of $G$:

$$H^i(BG,\Omega^j) \cong H^{i-j}(G,S^j(\mathfrak{g}^*)) .$$

Let $G$ be a split reductive group defined over $\mathbb{Z}$. From the work of Bhatt-Morrow-Scholze in p-adic Hodge theory [1, Theorem 1.1], one might expect that
\[\text{dim}_{F_p} H^i_{\text{dR}}(BG_{F_p}/F_p) \geq \text{dim}_{F_p} H^i(BG_{\mathbb{C}}, F_p) \] for all primes \(p\) and \(i \geq 0\). The results from [1] do not immediately apply to \(BG\) since \(BG\) is not proper as a stack over \(\mathbb{Z}\). For \(p\) a non-torsion prime of a split reductive group \(G\) defined over \(\mathbb{Z}\), Totaro showed that

\[H^*_{\text{dR}}(BG_{F_p}/F_p) \cong H^*(BG_{\mathbb{C}}, F_p) \] \[\text{[17, Theorem 9.2]}\].

It remains to compare \(H^*_{\text{dR}}(BG_{F_p}/F_p)\) with \(H^*(BG_{\mathbb{C}}, F_p)\) for \(p\) a torsion prime of \(G\). For \(n \geq 3, 2\) is a torsion prime for the split group \(SO(n)\). Totaro showed that

\[H^*_{\text{dR}}(BSO(n)_{F_2}/F_2) \cong H^*(BSO(n)_{\mathbb{C}}, F_2) \cong \mathbb{F}_2[w_2, \ldots, w_n] \]
as graded rings where \(w_2, \ldots, w_n\) are the Stiefel-Whitney classes \[\text{[17, Theorem 11.1]}\]. In general, the rings \(H^*_{\text{dR}}(BG_{F_p}/F_p)\) and \(H^*(BG_{\mathbb{C}}, F_p)\) are different though. For example,

\[\text{dim}_{F_2} H^{32}_{\text{dR}}(BSpin(11)_{F_2}/F_2) > \text{dim}_{F_2} H^{32}(BSpin(11)_{\mathbb{C}}, F_2) \]

\[\text{[17, Theorem 12.1]}\].

In this paper, we verify inequality (1) for more examples. For the torsion prime \(2\) of the split reductive group \(G_2\) over \(\mathbb{Z}\), we show that

\[H^*_{\text{dR}}(B(G_2)_{F_2}/F_2) \cong H^*(B(G_2)_{\mathbb{C}}, F_2) \cong \mathbb{F}_2[y_4, y_6, y_7] \]
as graded rings where \(|y_i| = i\) for \(i = 4, 6, 7\). For the spin groups, we show that

\[H^*_{\text{dR}}(BSpin(n)_{F_2}/F_2) \cong H^*(BSpin(n)_{\mathbb{C}}, F_2) \]

\[\text{(3)}\]

for \(7 \leq n \leq 10\). Note that \(2\) is a torsion prime for \(Spin(n)\) for \(n \geq 7\). The isomorphism (3) holds for \(1 \leq n \leq 6\) by the “accidental” isomorphisms for spin groups along with (2).

For \(n = 11\), we make a full computation of the de Rham cohomology ring of \(BSpin(n)_{F_2}\):

\[H^*_{\text{dR}}(BSpin(11)_{F_2}/F_2) \cong \mathbb{F}_2[w_4, w_6, w_7, w_8, w_{10}, w_{11}, w_{64}]/(w_7w_{10} + w_6w_{11}, w_{11}^3 + w_1^2w_7w_4 + w_{11}w_8w_7^2)\]

where \(|w_i| = i\) for all \(i\). Equivalently,

\[H^*(BSpin(11)_{\mathbb{C}}, F_2) \cong H^*(BSO(11)_{\mathbb{C}}, F_2)/J \otimes \mathbb{F}_2[w_{64}]\]

where \(J\) is the ideal generated by the regular sequence

\[w_2, Sq^1(w_2), Sq^2 Sq^1(w_2), \ldots, Sq^{16} Sq^8 \cdots Sq^1 w_2.\]

Thus, the rings \(H^*_{\text{dR}}(BSpin(n)_{F_2}/F_2)\) and \(H^*(BSpin(n)_{\mathbb{C}}, F_2)\) are not isomorphic in general even though \(H^*_{\text{dR}}(BSO(n)_{F_2}/F_2) \cong H^*(BSO(n)_{\mathbb{C}}, F_2)\) for all \(n\). Steenrod squares on de Rham cohomology over a base field of
characteristic 2 have not yet been constructed. If they exist, our calculation suggests that their action on \( H^*_{dR}(BSO(n)_{F_2}/F_2) \cong H^*(BSO(n)_C,F_2) \) would have to be different from the action of the topological Steenrod operations.

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**1. Preliminaries**

In this section, we recall results from [17] that will be used in our computations. These results were also used by Totaro in [17, Theorem 11.1] to compute the de Rham cohomology of \( BSO(n)_k \) for \( k \) a field of characteristic 2.

The first result we mention [17, Proposition 9.3] is an analogue of the Leray-Serre spectral sequence from topology.

**Proposition 1.1.** Let \( G \) be a split reductive group defined over a field \( F \) and let \( P \) be a parabolic subgroup of \( G \) with Levi quotient \( L \) (this means that \( P \cong R_u(P) \times L \) where \( R_u(P) \) is the unipotent radical of \( P \) [2, 14.19]). There exists a spectral sequence of algebras

\[
E_2^{i,j} = H^i_H(BG/F) \otimes H^j_L((G/P)/F) \Rightarrow H^{i+j}_H(BL/F).
\]

Proposition 1.1 is the main tool that we will use to compute Hodge cohomology rings of classifying stacks. To apply Proposition 1.1, we will choose a parabolic subgroup \( P \) for which \( H^*_H(BL/F) \) is a polynomial ring.

To fill in the 0th column of the \( E_2 \) page in Proposition 1.1, we use a result of Srinivas [15].

**Proposition 1.2.** Let \( G \) be split reductive over a field \( F \) and let \( P \) be a parabolic subgroup of \( G \). The cycle class map

\[
CH^*(G/P) \otimes_{\mathbb{Z}} F \rightarrow H^*_H((G/P)/F)
\]

is an isomorphism.

Under the cycle class map, \( CH^i(G/P) \otimes_{\mathbb{Z}} F \) maps to \( H^i(G/P,\Omega^i) \). From the work of Chevalley [5] and Demazure [6], \( CH^*(G/P) \) is independent of the field \( F \) and is isomorphic to the singular cohomology ring \( H^*(G_C/P_C,\mathbb{Z}) \).

The last piece of information we will use to compute \( H^*_H(BG/F) \) is the ring of \( G \)-invariants \( O(\mathfrak{g})^G = \oplus_i H^i(BG,\Omega^i) \). Let \( T \) be a maximal torus in \( G \) with Lie algebra \( \mathfrak{t} \) and Weyl group \( W \). There is a restriction homomorphism

\[
O(\mathfrak{g})^G \rightarrow O(\mathfrak{t})^W.
\]
We will need the following theorem which is due to Chaput and Romagny [4, Theorem 1.1]. For the following theorem, a split algebraic group $G$ over a field $F$ is simple if every proper smooth normal connected subgroup of $G$ is trivial.

**Theorem 1.3.** Assume that $G$ is simple over a field $F$. Then the restriction homomorphism (4) is an isomorphism unless $\text{char}(F) = 2$ and $G_{\mathbb{F}}$ is a product of copies of $\text{Sp}(2n)$ for some $n \in \mathbb{N}$.

From the rings $O(g)^G, CH^*(G/P), H^*_H(BL/F)$, we will be able to determine the $E_\infty$ terms of the spectral sequence in Proposition 1.1. This will allow us to determine $H^*_H(BG/F)$ by using the following version of the Zeeman comparison theorem [12, Theorem VII.2.4].

**Theorem 1.4.** Fix a field $F$. Let $\{\bar{E}^{i,j}_r\}, \{E^{i,j}_r\}$ be first quadrant (cohomological) spectral sequences of $F$-vector spaces such that $\bar{E}^{i,j}_{2} = \bar{E}^{i,0}_2 \otimes_F E^{0,j}_2$ and $E^{i,j}_2 = E^{i,0}_2 \otimes_F E^{0,j}_2$ for all $i,j$. Let $\{f^{i,j}_r : \bar{E}^{i,j}_r \to E^{i,j}_r\}$ be a morphism of spectral sequences such that $f^{i,j}_{2} = f^{i,0}_{2} \otimes f^{0,j}_{2}$ for all $i,j$. Fix $N, Q \in \mathbb{N}$. Assume that $f^{i,j}_{2}$ is an isomorphism for all $i,j$ with $i+j < N$ and an injection for $i+j = N$. If $f^{0,i}_{2}$ is an isomorphism for all $i < Q$ and an injection for $i = Q$, then $f^{0,i}_{2}$ is an isomorphism for all $i < \min(N,Q+1)$ and an injection for $i = \min(N,Q+1)$.

We recall a result from [17, Section 11] on the degeneration of the Hodge spectral sequence for split reductive groups, under some assumptions. The result in [17, Section 11] was proved for the special orthogonal groups but the proof works more generally.

**Proposition 1.5.** Let $G$ be a split reductive group over a field $F$ and assume that the Hodge cohomology ring of $BG$ is generated as an $F$-algebra by classes in $\bigoplus_i H^{i+1}(BG, \Omega^i)$ and $\bigoplus_i H^i(BG, \Omega^i)$. Then the Hodge spectral sequence $E_1^{i,j} = H^j(BG, \Omega^i) \Rightarrow H^*_dR(BG/F)$ (5) for $BG$ degenerates at the $E_1$ page.

**Proof.** From [17, Lemma 8.2], there are natural maps

$$H^i(BG, \Omega^i) \to H^i_{dR}(BG/F)$$

and

$$H^{i+1}(BG, \Omega^i) \to H^{2i+1}_{dR}(BG/F)$$

for all $i \geq 0$. These maps are compatible with products. Let $T$ denote a maximal torus of $G$. From the group homomorphism $T \to G$, we have the commuting square
\[ \oplus_i H^i(BG, \Omega^i) \longrightarrow \oplus_i H^{2i}_{\text{dR}}(BG/F) \]
\[ \oplus_i H^i(BT, \Omega^i) \longrightarrow H^{2i}_{\text{dR}}(BT/F). \]

The restriction homomorphism (4) induces an injection
\[ \oplus_i H^i(BG, \Omega^i) \hookrightarrow \oplus_i H^i(BT, \Omega^i) \]
[17, Lemma 8.2]. Hence, from diagram (6), we get that the natural map
\[ \oplus_i H^i(BG, \Omega^i) \rightarrow \oplus_i H^{2i}_{\text{dR}}(BG/F) \]
is an injection. Hence, any differentials into the diagonal in the spectral sequence (5) must be 0. Then all classes in \( \oplus_i H^i(BG, \Omega^j) \) must be permanent cycles in the spectral sequence (5) since \( H^i(BG, \Omega^j) = 0 \) for \( i < j \) by [17, Corollary 2.2]. This proves that the Hodge spectral sequence for \( BG \) degenerates. \( \square \)

The following definition will be used later to describe the Hodge cohomology of flag varieties.

**Definition 1.6.** Let \( F \) be a field. For variables \( x_1, \ldots, x_n \) let \( \Delta(x_1, \ldots, x_n) \) denote the \( F \)-vector space with basis given by the products \( x_{i_1} \cdots x_{i_r} \) for \( 1 \leq i_1 < i_2 < \cdots < i_r \leq n \).

2. **\( G_2 \)**

Let \( k \) be a field of characteristic 2 and let \( G \) denote the split form of \( G_2 \) over \( k \).

**Theorem 2.1.** The Hodge cohomology ring of \( BG \) is freely generated as a commutative \( k \)-algebra by generators \( y_4 \in H^2(BG, \Omega^2) \), \( y_6 \in H^3(BG, \Omega^3) \), and \( y_7 \in H^4(BG, \Omega^3) \). The Hodge spectral sequence for \( BG \) degenerates at \( E_1 \) and we have
\[ H^*_{\text{dR}}(BG/k) \cong H^*_{\text{h}}(BG/k) = k[y_4, y_6, y_7]. \]

From the computation [12, Corollary VII.6.3] of the singular cohomology ring of \( B(G_2)_C \) with \( \mathbb{F}_2 \)-coefficients, we then have \( H^*(B(G_2)_C, k) \cong H^*_{\text{dR}}(BG/k) \).

**Proof.** We first choose a suitable parabolic subgroup of \( G \). Let \( P \) be the parabolic subgroup of \( G \) corresponding to inclusion of the long root.
From Proposition 1.2, $CH^*(G/P)$ is independent of the field $k$ and the characteristic of $k$. As discussed in [9, §23.3], if we consider $(G_2)_C$ over $\mathbb{C}$ along with the corresponding parabolic subgroup $P_\mathbb{C}$, $(G_2)_C/P_\mathbb{C}$ is isomorphic to a smooth quadric $Q_5$ in $\mathbb{P}^6$. Hence, by [8, Chapter XIII], $H^*_H((G/P)/k)$ is isomorphic to

$$CH^*(Q_5) \otimes \mathbb{Z} k \cong k[v, w]/(v^6, w^2, v^3 - 2w) = k[v, w]/(v^3, w^2)$$

where $|v| = 2$ and $|w| = 6$ in $H^*_H((G/P)/k)$.

We next show that the Levi quotient $L$ of $P$ is isomorphic to $GL(2)_k$. This can be seen by constructing an isomorphism from the root datum of $GL(2)_k$ to the root datum of the Levi quotient. Let $(X_1, R_1, X_1^\vee, R_1^\vee)$ be the usual root datum of $GL(2)_k$ where $X_1 = \mathbb{Z} \chi_1 + \mathbb{Z} \chi_2$, $R_1 = \mathbb{Z} (\chi_1 - \chi_2)$, and we take our torus to be the set of diagonal matrices in $GL(2)_k$. We take $(X_2, R_2, X_2^\vee, R_2^\vee)$ to be the root datum of $G$ as described in [3, Plate IX]. Here, $X_2 = \{(a, b, c) \in \mathbb{Z}^3 \mid a + b + c = 0\}$. The long root $\alpha$ for $G$ is then $(-2, 1, 1)$ and the root datum of $P/R_\mu(P)$ is $(X_2, \pm \alpha, X_2^\vee, \pm \frac{1}{2} \alpha)$. An isomorphism from the root datum of $GL(2)_k$ to the root datum of $G$ can then be obtained from the isomorphism

$$X_1 \to X_2$$

$$\chi_1 \mapsto (-1, 1, 0), \chi_2 \mapsto (1, 0, -1).$$

Thus, $L \cong GL(2)_k$.

We now analyze the spectral sequence

$$E^{i,j}_2 = H^i_H(BG/k) \otimes H^j_H((G/P)/k) \Rightarrow H^{i+j}_H(BL/k) \quad (7)$$

from Proposition 1.1. From [7, Proposition] and [10, II.4.22],

$$H^*_H(BL/k) = S^*(\mathfrak{g}_L)^{GL(2)_k} \cong S^*(t)^{S_2} = k[x_1, x_2]$$

where $x_1 \in H^1(BL, \Omega^1)$ and $x_2 \in H^2(BL, \Omega^2)$. Here, $t$ is the space of all diagonal matrices in $\mathfrak{g}_L$ and $S_2$ acts on $t$ by permuting the diagonal entries.

In order to compute $H^*_H(BG/k)$ from the spectral sequence above, we must first compute the ring of invariants of $S^*(\mathfrak{g}_2)^G$. From Theorem 1.3, $S^*(\mathfrak{g}_2)^G \cong S^*(t_0)^W$ where $t_0$ is the Lie algebra of a maximal torus $T$ in $G$ and $W$ is the corresponding Weyl group of $G$. By [17, Corollary 2.2],

$$H^i(BG, \Omega^1) \cong S^i(t_0)^W$$

for $i \geq 0$.

**Proposition 2.2.** The ring of invariants $S^*(t_0)^W$ is equal to $k[y_4, y_6]$ where $|y_4| = 2$ and $|y_6| = 3$ in $S^*(t_0)^W$.

**Proof.** Following the notation in [3, Plate IX], $W \cong \mathbb{Z}_2 \times S_3$ acts on the root lattice $X_2 = \{(a, b, c) \in \mathbb{Z}^3 \mid a + b + c = 0\}$ by multiplication by $-1$ and by permuting the coordinates. Hence, since we are working in characteristic $2$, $W$ acts on $S^*(t_0) = k[t_1, t_2, t_3]/(t_1 + t_2 + t_3)$ by permuting $t_1$, $t_2$, and $t_3$. We then have $S^*(t_0)^W = k[t_1 t_2 + t_1 t_3 + t_2 t_3, t_1 t_2 t_3] = k[y_4, y_6]$. \qed
We can now carry out the computation of $H^*_H(BG/k)$. First, we show that the class $v \in E^{2,0}_2$ is a permanent cycle. Consider the filtration on $H^*_H(BL/k) = k \cdot v$ given by (7): $H^*_H(BL/k) \hookrightarrow E^{2,0}_2$, where $H^*_H(BL/k)/E^{2,0}_\infty \cong E^{0,2}_\infty$. Here, $E^{1,1}_2 = 0$ and

$$E^{2,0}_\infty = E^{2,0}_2 = H^2_H(BG/k) = H^1(BG, \Omega^1)$$

(we have $H^2(BG, O) = 0$ since $H^2(BL, O) = 0$ and there are no differentials entering $E^{2,0}_2$) since $H^*_H((G/P)/k) = \oplus_i H^i(G/P, \Omega^i)$ is concentrated in even degrees. Hence,

$$E^{2,0}_\infty = H^3_H(BG/k) = H^1(BG, \Omega^1) = 0,$$

by Proposition 2.2. It follows that $E^{0,2}_\infty \cong E^{0,2}_2 = k \cdot v$ which implies that $d_3(v) = 0$. As (7) is a spectral sequence of algebras, it follows that $v$ and $v^2$ are permanent cycles. Using that $H^*_H(BL/k)$ is concentrated in even degrees, we then get that $H^3_H(BG/k) = E^{3,0}_2 = E^{3,0}_\infty = 0$ and $H^5_H(BG/k) = E^{5,0}_2 = E^{5,0}_\infty = 0$.

Next, we show that $w \in H^5_H((G/P)/k) = E^{0,6}_2$ is transgressive with $0 \not= d_7(w) \in E^{7,0}_7$. Note that $\dim_k H^6_H(BL/k) = 2$. As $v$ is a permanent cycle in $E_2$, we observe that $E^{4,2}_\infty \cong E^{4,2}_2 \cong k \cdot y_4 \otimes_k k \cdot v \cong k$ and $E^{6,0}_\infty \cong E^{6,0}_2 \cong k \cdot y_6 \cong k$. Hence, $\dim_k H^6_H(BL/k) = 2 = \dim_k E^{4,2}_\infty + \dim_k E^{6,0}_\infty$. From the filtration on $H^6_H(BL/k)$ given by the spectral sequence (7), it follows that $E^{0,6}_\infty = 0$. As $H^3_H(BG/k) = E^{3,0}_2 = E^{3,0}_\infty = 0$ and $H^5_H(BG/k) = E^{5,0}_2 = E^{5,0}_\infty = 0$, we then get that $0 \not= d_7(w) \in E^{7,0}_7$ and $d_7(w)$ lifts to a non-zero element $y_7 \in H^4(BG, \Omega^3) \subseteq H^7_H(BG/k)$.

```
|   k · w   | 0 | 0 | 0 | 0 | 0 | 0 |
|          |   |   |   |   |   |   |
| k · v^2  | 0 | 0 | 0 | 0 | 0 | 0 |
|          |   |   |   |   |   |   |
| k · v    | 0 | 0 | 0 | 0 | 0 | 0 |
|          |   |   |   |   |   |   |
| k        | 0 | 0 | 0 | 0 | 0 | 0 |
|          |   |   |   |   |   |   |
```
Now, we can determine the $E_\infty$ terms in (7). For $n$ odd, $E_\infty^{i,n-i} = 0$ since $H^n_H(BL/k)$ is concentrated in even degrees. Let $n \in \mathbb{N}$ be even. The $k$-dimension of $H^n_H(BL/k)$ is equal to the cardinality of the set

$$S_n = \{(a,b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : 2a + 4b = n\}.$$  

For $i = 0, 1, 2$, set $V_{i,n} := H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2})$. For $i = 0, 1, 2$, $\dim_k V_{i,n}$ is equal to the cardinality of the set $S_{i,n} = \{(a,b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : 4a + 6b = n - 2i\}$. As $v$ is a permanent cycle in (7), $E_2^{n-2i,2i} \cong E_7^{n-2i,2i}$ for $i = 0, 1, 2$. As $y_7 \in H^4(BG, \Omega^3)$ and $H^i(BG, \Omega^j) = 0$ for $i < j$,

$$y_7 : x \not\in \oplus_j H^j(BG, \Omega^j)$$

for all $x \in H^*_H(BG/k)$. Hence,

$$H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2}) \otimes_k k \cdot v^i \subseteq E_2^{n-2i,2i} \cong E_7^{n-2i,2i}$$

injects into $E_\infty^{n-2i,2i}$ for $i = 0, 1, 2$.

Define a bijection $f_n : S_n \to S_{0,n} \cup S_{1,n} \cup S_{2,n}$ by

$$f_n(a,b) = \begin{cases} 
(b,a/3) & \text{if } a \equiv 0 \mod 3, \\
(b,(a-1)/3) & \text{if } a \equiv 1 \mod 3, \\
(b,(a-2)/3) & \text{if } a \equiv 2 \mod 3. 
\end{cases}$$

Then

$$\dim_k H^n_H(BL/k) = |S_n| = |S_{0,n}| + |S_{1,n}| + |S_{2,n}|$$

$$\leq \dim_k E_\infty^{n,0} + \dim_k E_\infty^{n,-2,2} + \dim_k E_\infty^{n,-4,4}$$

where the inequality follows from the fact proved above that

$$H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2})$$

injects into $E_\infty^{n-2i,2i}$ for $i = 0, 1, 2$. From the filtration on $H^n_H(BL/k)$ defined by the spectral sequence (7), it follows that $H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2}) \cong E_\infty^{n-2i,2i}$ for $i = 0, 1, 2$ and $E_\infty^{n-2i,2i} = 0$ for $i \geq 3$.

We can now finish the computation of the Hodge cohomology of $BG$ by using Zeeman’s comparison theorem. Let $F_*$ denote the cohomological spectral sequence of $k$-vector spaces concentrated on the 0th column with $E_2$ page given by

$$F_2^{0,i} = \begin{cases} 
k & \text{if } i = 0, \\
k \cdot v & \text{if } i = 2, \\
k \cdot v^2 & \text{if } i = 4, \\
0 & \text{if } i \not\in 0, 2, 4. \end{cases}$$

As $v \in E_2^{0,2}$ in the spectral sequence (7) is transgressive with $d_r(v) = 0$ for all $r \geq 2$, there exists a map of of spectral sequences $F_* \to E_*$ that takes $v \in F_2^{0,2}$ to $v \in E_2^{0,2}$ and $v^2 \in E_2^{0,4}$ to $v^2 \in E_2^{0,4}$.

Fixing a variable $y$, let $H_*$ denote the cohomological spectral sequence with $E_2$ page given by $H_2 = \Delta(w) \otimes k[y]$ where $w$ is of bidegree $(0,6)$, $y$ is
of bidegree \((7,0)\), and \(w\) is transgressive with \(d_7(wy^i) = y^{i+1}\) for all \(i \geq 0\).

As \(w \in E_2^{0,6}\) is transgressive with \(d_7(w) = y_7 \in E_2^{7,0}\), there exists a map of spectral sequence \(H_* \to E_*\) such that \(w \in H_2^{0,6}\) maps to \(w \in E_2^{0,6}\) and \(y \in H_2^{7,0}\) maps to \(y_7 \in E_2^{7,0}\). Elements of the ring of \(G\)-invariants \(k[y_4, y_6]\) are permanent cycles in the spectral sequence (7) since they are concentrated on the 0th row. Thus, by tensoring the previous maps of spectral sequences, we get a map

\[
\alpha : I_* := F_* \otimes H_* \otimes k[y_4, y_6] \to E_*
\]

of spectral sequences.

As shown above, the map \(\alpha\) induces an isomorphism \(I_\infty \cong F_2 \otimes k[y_4, y_6] \to E_\infty\) on \(E_\infty\) pages. The 0th columns of the \(E_2\) pages of the spectral sequences \(I_*\) and \(E_*\) are both isomorphic to \(k[v, w]/(v^3, w^2)\) and \(\alpha\) induces an isomorphism on the 0th columns of the \(E_2\) pages. Thus, by Theorem 1.4, \(\alpha\) induces an isomorphism on the 0th rows of the \(E_2\) pages. Hence,

\[
H_*^*(BG/k) = k[y_4, y_6, y_7].
\]

From Proposition 1.5, the Hodge spectral sequence for \(BG\) degenerates.

\(\square\)

**Corollary 2.3.** Let \(G\) be a \(k\)-form of \(G_2\). Then

\[
H_*^*(BG/k) \cong k[x_4, x_6, x_7]
\]

where \(|x_i| = i\) for \(i = 4, 6, 7\).

**Proof.** Letting \(k_s\) denote the separable closure of \(k\), we have \(BG \times_k \text{Spec}(k_s) \cong B(G_2)_{k_s}\). From Theorem 2.1, \(H_*^*(B(G_2)_{k_s})/k_s) \cong k_s[x'_4, x'_6, x'_7]\) for some \(x'_4, x'_6, x'_7 \in H_7^*(B(G_2)_{k_s})/k_s)\) with \(|x'_i| = i\) for all \(i\). As Hodge cohomology commutes with extensions of the base field,

\[
H_*^*(BG \times_k \text{Spec}(k_s))/k_s) \cong H_*^*(BG/k) \otimes_k k_s.
\]

It follows that \(H_*^*(BG/k) \cong k[x_4, x_6, x_7]\) for some \(x_4, x_6, x_7 \in H_*^*(BG/k)\).

\(\square\)

3. **Spin groups**

Let \(k\) be a field of characteristic 2 and let \(G\) denote the split group \(\text{Spin}(n)_k\) over \(k\) for \(n \geq 7\).

Let \(P_0 \subset SO(n)_k\) denote a parabolic subgroup that stabilizes a maximal isotropic subspace. Let \(P \subset G\) denote the inverse image of \(P_0\) under the double cover map \(G \to SO(n)_k\). The Hodge cohomology of \(G/P\) is given by Proposition 1.2 and [12, Theorem III.6.11].

**Proposition 3.1.** There is an isomorphism

\[
H_*^*(G/P)/k) \cong k[e_1, \ldots, e_s]/(e_i^2 = e_{2i}),
\]

where \(s = [(n - 1)/2]\), \(e_m = 0\) for \(m > s\), and \(|e_i| = 2i\) for all \(i\).
The Levi quotient of $P_0$ is isomorphic to $GL(r)_k$ where $r = \lfloor n/2 \rfloor$. Hence, the Levi quotient $L$ of $P$ is a double cover of $GL(r)_k$.

**Proposition 3.2.** The torsion index of $L$ is equal to 1.

**Proof.** We show that the torsion index of the corresponding compact connected Lie group $M$ is equal to 1. As $M$ is a double cover of $U(r)$, $M$ is isomorphic to $(S^1 \times SU(r))/2\mathbb{Z}$ where $k \in \mathbb{Z}$ acts on $S^1 \times SU(r)$ by

$$ (z, A) \mapsto (ze^{2\pi ik/r}, e^{-2\pi ik/r} A). $$

Hence, the derived subgroup $[M, M]$ of $M$ is isomorphic to $SU(r)$. As $SU(r)$ has torsion index 1, $M$ has torsion index 1 by [16, Lemma 2.1]. Thus, $L$ has torsion index equal to 1. □

**Corollary 3.3.** We have

$$ H^{\text{H}}_*(BL/k) = O(l)^L = k[A, c_2, \ldots, c_r] $$

where $|c_i| = 2i$ in $H^*_H(BL/k)$ for all $i$ and $|A| = 2$.

**Proof.** From Proposition 3.2 and [17, Theorem 9.1],

$$ H^*_H(BL/k) = O(t)^L. $$

Let $T$ be a maximal torus in $L$ with Lie algebra $t$ and Weyl group $W$. From Theorem 1.3, $O(t)^L \cong O(t)^W$. To compute $O(t)^W$, we use that $L$ is a double cover of $GL(r)_k$. We have

$$ S(X^*(T) \otimes k) \cong \mathbb{Z}[x_1, \ldots, x_r, A]/(2A = x_1 + \cdots + x_r) \otimes k $$

$$ \cong k[x_1, \ldots, x_r, A]/(x_1 + \cdots + x_r). $$

The Weyl group $W$ of $L$ is isomorphic to the symmetric group $S_r$ and acts on $S(X^*(T) \otimes k)$ by permuting $x_1, \ldots, x_r$. From [13, Proposition 4.1],

$$ (k[x_1, \ldots, x_r, A]/(x_1 + \cdots + x_r))^{S_r} = k[A, c_2, \ldots, c_r] $$

where $c_1, \ldots, c_r$ are the elementary symmetric polynomials in the variables

$$ x_1, \ldots, x_r. $$

□

For our calculations, we will need to know the Hodge cohomology of $BSO(n)_k$ [17, Theorem 11.1].

**Theorem 3.4.** The Hodge spectral sequence for $BSO(n)_k$ degenerates and

$$ H^*_H(BSO(n)_k/k) = k[u_2, \ldots, u_n] $$

where $u_{2i} \in H^i(BSO(n)_k, \Omega^i)$ and $u_{2i+1} \in H^{i+1}(BSO(n)_k, \Omega^i)$ for all relevant $i$.

We'll also need to know the ring of invariants of $G = \text{Spin}(n)_k$ for all $n \geq 6$. This can be found in [17, Section 12].
Lemma 3.5. For $n \geq 6$,

$$O(g)^G = \begin{cases} k[c_2, \ldots, c_r, \eta_{r-1}] & \text{if } n = 2r + 1 \\ k[c_2, \ldots, c_r, \mu_{r-1}] & \text{if } n = 2r \text{ and } r \text{ is even} \\ k[c_2, \ldots, c_r, \mu_r] & \text{if } n = 2r \text{ and } r \text{ is odd} \end{cases}$$

where $|c_i| = i$, $|\eta_j| = 2^j$, and $|\mu_j| = 2^{j-1}$ in $O(g)^G$ for all $i$ and $j$.

Note that under the inclusion $O(g)^G \subset H^*_H(BG/k)$, the degree of an invariant function in $H^*_H(BG/k)$ is twice its degree in $O(g)^G$.

Theorem 3.6. Let $n = 7$. The Hodge spectral sequence for $BG$ degenerates and

$$H^i_{dR}(BG/k) \cong H^i_H(BG/k) = k[y_4, y_6, y_7, y_8]$$

where $|y_i| = i$ for $i = 4, 6, 7, 8$.

Proof. From Lemma 3.5,

$$O(g)^G = k[y_4, y_6, y_8]$$

where $|y_i| = i$ in $H^*_H(BG/k)$, viewing $O(g)^G$ as a subring of $H^*_H(BG/k)$. Consider the spectral sequence

$$E_2^{i,j} = H^i_H(BG/k) \otimes H^j_H(G/P)/k \Rightarrow H^{i+j}_H(BL/k)$$

(8)

from Proposition 1.1. From Proposition 3.1 and Corollary 3.3,

$$H^*_H((G/P)/k) \cong k[e_1, e_2, e_3]/(e_2^2 = e_2e_1) = k[e_1, e_3]/(e_1^3, e_3^2)$$

and

$$H^*_H(BL/k) \cong k[A, c_2, c_3].$$

First, we show that $e_1 \in E^{0,2}_2$ is a permanent cycle. From the filtration on $H^*_H(BL/k) = k \cdot A$ given by (8), we have

$$1 = \dim_k E^{0,2}_\infty + \dim_k E^{2,0}_\infty = \dim_k E^{0,2}_\infty + \dim_k E^{2,0}_2.$$ 

As $H^*_H(BL/k) = \oplus_i H^i(BL, \Omega^i)$, $E^{2,0}_2 = H^1(BG, \Omega^1) = 0$. Hence, $E^{0,2}_2 = E^0_{-2} = k \cdot e_1$ which implies that $e_1$ is a permanent cycle. As $e_2 = e_1^2$, it follows that $e_2$ is a permanent cycle. Hence, $E^{4,2}_\infty \cong E^{4,2}_2 \cong k \cdot (y_4 \otimes e_1)$ and $E^{6,0}_\infty \cong E^{6,0}_2 \cong k \cdot y_6$.

We next show that $e_3 \in E^{3,0}_2$ is transgressive with $d_7(e_3) \neq 0$. As $e_1$ is a permanent cycle and $H^*_H(BL/k) = 0$ for $i$ odd, the spectral sequence (8) implies that $E^{3,0}_2 = E^{5,0}_2 = 0$. Consider the filtration of (8) on $H^*_H(BL/k)$. We have

$$\dim_k H^6_H(BL/k) = 3 = \dim_k E^{6,0}_\infty + \dim_k E^{4,2}_\infty + \dim_k E^{0,6}_\infty = 2 + \dim_k E^{0,6}_2$$

which implies that $E^{0,6}_\infty \cong k \cdot e_1 e_2$. As $E^{3,0}_2 = E^{5,0}_2 = 0$, we must then have $e_3 \in E^{3,0}_7$ and $0 \neq d_7(e_3) \in E^{7,0}_2$. The class $d_7(e_3)$ lifts to a non-zero class $y_7 \in H^1(BG, \Omega^3) \subseteq E^{7,0}_2 = H^*_H(BG/k)$.
We can now determine the $E_\infty$ page of $(8)$. For $n$ odd, $E_2^{i,n-i} = 0$ since $H^*_H(BL/k)$ is concentrated in even degrees. Assume that $n \in \mathbb{N}$ is even. The $k$-dimension of $H^*_H(BL/k)$ is equal to the cardinality of the set

$$S_n = \{(a, b, c) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : 2a + 4b + 6c = n\}.$$ 

For $i = 0, 1, 2, 3$, set $V_{i,n} := H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2})$. For $i = 0, 1, 2, 3$, $\dim_k V_{i,n}$ is equal to the cardinality of the set $S_{i,n} = \{(a, b, c) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : 4a + 6b + 8c = n - 2i\}$. As $e_1$ is a permanent cycle in $(8)$,

$$V_{i,n} \cong V_{i,n} \otimes k \cdot e_1 \subseteq E_\infty^{n-2i,2i}$$

for $i = 0, 1, 2, 3$.

Define a bijection $f_n : S_n \to S_{0,n} \cup S_{1,n} \cup S_{2,n} \cup S_{3,n}$ by

$$f_n(a, b, c) = \begin{cases} 
(b, c, a/4) & \text{if } a \equiv 0 \mod 4, \\
(b, c, (a - 1)/4) & \text{if } a \equiv 1 \mod 4, \\
(b, c, (a - 2)/4) & \text{if } a \equiv 2 \mod 4, \\
(b, c, (a - 3)/4) & \text{if } a \equiv 3 \mod 4.
\end{cases}$$

Then

$$\dim_k H^*_H(BL/k) = |S_n| = |S_{0,n}| + |S_{1,n}| + |S_{2,n}| + |S_{3,n}|.$$

As

$$\dim_k H^*_H(BL/k) \geq E^{n,0}_\infty + E^{n-2,2}_\infty + E^{n-4,4}_\infty + E^{n-6,6}_\infty$$

and $V_{i,n} \subseteq E^{n-2i,2i}_\infty$ for $i = 0, 1, 2, 3$, it follows that $V_{i,n} \cong E^{n-2i,2i}_\infty$ for $i = 0, 1, 2, 3$ and $E^{n-2i,2i}_\infty = 0$ for $i \geq 4$.

We now use Theorem 1.4 to finish the computation of the Hodge cohomology of $BG$. Let $F_*$ denote the cohomological spectral sequence of $k$-vector spaces concentrated on the 0th column given by $F_2 = \Delta(e_1, e_2)$ where $e_i$ is of bidegree $(0, 2i)$ for $i = 1, 2$. As $e_1$ is a permanent cycle in $(8)$, there
is a map of spectral sequences $F_* \to E_*$ taking $e_i \in E^{0,2i}_2$ to $e_i \in E^{0,2i}_2$ for $i = 1, 2$. Fix a variable $y$. Let $H_\ast$ be the spectral sequence with $E_2$ page given by $H_2 = \Delta(e_3) \otimes k[y]$ where $e_3$ is of bidegree $(0, 6)$, $y$ is of bidegree $(7, 0)$, and $e_3$ is transgressive with $d_7(e_3^i) = y^{i+1}$ for all $i$. As $e_3 \in E^{0,6}_2$ is transgressive with $d_7(e_3) = y_7$, there exists a map of spectral sequences $H_* \to E_*$ taking $e_3 \in H^{0,6}_2$ to $e_3 \in E^{0,6}_2$ and $y \in H^{7,0}_2$ to $y_7 \in E^{7,0}_2$.

Elements in the ring of $G$-invariants $k[y_4, y_6, y_8]$ are permanent cycles in the spectral sequence (8). Tensoring maps of spectral sequences, we get a map

$$\alpha : I_* := F_* \otimes H_* \otimes k[y_4, y_6, y_8] \to E_*$$

of spectral sequences. As $I_\infty \cong F_2 \otimes k[y_4, y_6, y_8]$, $\alpha$ induces isomorphisms on $E_\infty$ terms and on the 0th columns of the $E_2$ pages. Hence, by Theorem 1.4, $\alpha$ induces an isomorphism on the 0th rows of the $E_2$ pages. Thus,

$$H^\ast_H(BG/k) = k[y_4, y_6, y_7, y_8].$$

The Hodge spectral sequence for $BG$ degenerates by Proposition 1.5.

As Hodge cohomology commutes with extensions of the base field, we have the following result.

**Corollary 3.7.** Let $k$ be a field of characteristic 2 and let $G$ be a $k$-form of $\text{Spin}(7)$. Then

$$H^\ast_H(BG/k) \cong k[x_4, x_6, x_7, x_8]$$

where $|x_i| = i$ for all $i$.

**Theorem 3.8.** Let $n = 8$. The Hodge spectral sequence for $BG$ degenerates and

$$H^n_H(BG/k) \cong H^n_H(BG/k) = k[y_4, y_6, y_7, y_8, y'_8]$$

where $|y_i| = i$ for $i = 4, 6, 7, 8$ and $|y'_8| = 8$.

**Proof.** From Lemma 3.5,

$$O(g)^G = k[y_4, y_6, y_8, y'_8]$$

where $|y_i| = i$ and $|y'_8| = 8$ in $H^\ast_H(BG/k)$, viewing $O(g)^G$ as a subring of $H^\ast_H(BG/k)$. Consider the spectral sequence

$$E^{i,j}_2 = H^i_H(BG/k) \otimes H^j_H((G/P)/k) \Rightarrow H^{i+j}_H(BL/k)$$

from Proposition 1.1. From Proposition 3.1 and Corollary 3.3,

$$H^i_H((G/P)/k) \cong k[e_1, e_2, e_3]/(e_1^2 = e_2) = k[e_1, e_3]/(e_1^2, e_3^2)$$

and

$$H^i_H(BL/k) \cong k[A, c_2, c_3, c_4].$$

Calculations similar to those performed in the proof of Proposition 3.6 show that $e_1$ is a permanent cycle in (9) and $e_3 \in E^{0,6}_2$ is transgressive with $0 \neq d_7(e_3) = y_7 \in H^4(BG, \Omega^3)$. We have $H^m_H(BG/k) \cong H^m_H(B\text{Spin}(7)/k/k)$ for $m < 8$ and $H^8_H(BG/k) = k \cdot y_8 \otimes k \cdot y'_8$. 


We can now determine the $E_\infty$ terms for (9). For $n$ odd, $E_i^{i,n-i} = 0$ since $H^n_{\Omega}(BL/k)$ is concentrated in even degrees. Assume that $n \in \mathbb{N}$ is even. The $k$-dimension of $H^n_{\Omega}(BL/k)$ is equal to the cardinality of the set

$$S_n = \{(a, b, c, d) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : 2a + 4b + 6c + 8d = n\}.$$  

For $i = 0, 1, 2, 3$, set $V_{i,n} := H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2})$. For $i = 0, 1, 2, 3$, $\dim_k V_{i,n}$ is equal to the cardinality of the set $S_{i,n} = \{(a, b, c, d) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : 4a + 6b + 8c + 8d = n - 2i\}$. As $e_1$ is a permanent cycle in (9),

$$V_{i,n} \cong V_{i,n} \otimes_k e_1^i \subseteq E_\infty^{n-2i,2i}$$

for $i = 0, 1, 2, 3$.

Define a bijection $f_n : S_n \to S_{0,n} \cup S_{1,n} \cup S_{2,n} \cup S_{3,n}$ by

$$f_n(a, b, c, d) = \begin{cases} (b, c, d, a/4) \in S_{0,n} & \text{if } a \equiv 0 \mod 4, \\ (b, c, d, (a - 1)/4) \in S_{1,n} & \text{if } a \equiv 1 \mod 4, \\ (b, c, d, (a - 2)/4) \in S_{2,n} & \text{if } a \equiv 2 \mod 4, \\ (b, c, d, (a - 3)/4) \in S_{3,n} & \text{if } a \equiv 3 \mod 4. \end{cases}$$

Then

$$\dim_k H^n_{\Omega}(BL/k) = |S_n| = |S_{0,n}| + |S_{1,n}| + |S_{2,n}| + |S_{3,n}|.$$  

As

$$\dim_k H^n_{\Omega}(BL/k) \geq E_\infty^{n,0} + E_\infty^{n-2,2} + E_\infty^{n-4,4} + E_\infty^{n-6,6}$$

and $V_{i,n} \subseteq E_\infty^{n-2i,2i}$ for $i = 0, 1, 2, 3$, it follows that $V_{i,n} \cong E_\infty^{n-2i,2i}$ for $i = 0, 1, 2, 3$ and $E_\infty^{n-2i,2i} = 0$ for $i \geq 4$.

Let $F_*$ denote the spectral sequence concentrated on the 0th column with $F_2 = \Delta(e_1, e_2, e_4)$ where $e_i$ is of bidegree $(0, 2i)$. There is a map of spectral sequences $F_* \to E_*$ taking $e_i$ to $e_i$ for $i = 1, 2, 4$. Fix a variable $y$. Let $H_*$ denote the spectral sequence with $E_2$ page $H_2 = \Delta(e_3) \otimes k[y]$ where $e_3$ is of bidegree $(0, 6)$, $y$ is of bidegree $(7, 0)$, and $e_3$ is transgressive with $d_7(e_3y^i) = y^{i+1}$ for all $i$. There is an obvious map of spectral sequences $H_* \to E_*$. Classes in the ring of $G$-invariants are permanent cycles in the spectral sequence (9). Tensoring these maps, we get a map of spectral sequences

$$\alpha : I_* := F_* \otimes H_* \otimes k[y_4, y_6, y_8, y_8'] \to E_*.$$  

The map $\alpha$ induces an isomorphism on $E_\infty$ terms and on the 0th columns of the $E_2$ pages. Theorem 1.4 then implies that $\alpha$ induces an isomorphism on the 0th rows of the $E_2$ pages. Thus,

$$H^n_{\Omega}(BG/k) = k[y_4, y_6, y_7, y_8, y_8'].$$

Proposition 1.5 implies that the Hodge spectral sequence for $BG$ degenerates.
Corollary 3.9. Let $k$ be a field of characteristic 2 and let $G$ be a $k$-form for $\text{Spin}(8)$. Then

$$H^*_\text{H}(BG/k) \cong k[y_4, y_6, y_7, y_8, y_8']$$

where $|y_i| = i$ for $i = 4, 6, 7, 8$ and $|y_8'| = 8$.

Theorem 3.10. Let $n = 9$. The Hodge spectral sequence for $BG$ degenerates and

$$H^*_\text{dR}(BG/k) \cong H^*_\text{H}(BG/k) = k[y_4, y_6, y_7, y_8, y_{16}]$$

where $|y_i| = i$ for $i = 4, 6, 7, 8, 16$.

Proof. From Lemma 3.5,

$$O(g)^G = k[y_4, y_6, y_8, y_{16}]$$

where $|y_i| = i$ in $H^*_\text{H}(BG/k)$, viewing $O(g)^G$ as a subring of $H^*_\text{H}(BG/k)$. Consider the spectral sequence

$$E_2^{i,j} = H^i_\text{H}(BG/k) \otimes H^j_\text{H}((G/P)/k) \Rightarrow H^{i+j}_\text{H}(BL/k)$$

from Proposition 1.1. From Proposition 3.1 and Corollary 3.3,

$$H^\ast_\text{H}((G/P)/k) \cong k[e_1, e_2, e_3, e_4]/(e_1^2 = e_2) = k[e_1, e_3]/(e_1^8, e_3^2)$$

and

$$H^*_\text{H}(BL/k) \cong k[A, c_2, c_3, c_4].$$

Calculations similar to those performed in the proof of Proposition 3.6 show that $e_1$ is a permanent cycle in (10) and $e_3 \in E_2^{b,6}$ is transgressive with $0 \neq d_7(e_3) = y_7 \in H^4(BG, \Omega^3)$. We have $H^n_\text{H}(BG/k) \cong H^n_\text{H}(B\text{Spin}(7)/k)$ for $m \leq 10$.

We now determine the $E_{\infty}$ terms for (10). For $n$ odd, $E_\infty^{n,n-i} = 0$ since $H^*_\text{H}(BL/k)$ is concentrated in even degrees. Assume that $n \in \mathbb{N}$ is even. The $k$-dimension of $H^*_\text{H}(BL/k)$ is equal to the cardinality of the set

$$S_n = \{(a, b, c, d) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : 2a + 4b + 6c + 8d = n\}.$$

For $0 \leq i \leq 7$, set $V_{i,n} := H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2})$. For $0 \leq i \leq 7$, $\dim_k V_{i,n}$ is equal to the cardinality of the set $S_{i,n} = \{(a, b, c, d) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : 4a + 6b + 8c + 16d = n - 2i\}$. As $e_1$ is a permanent cycle in (10),

$$V_{i,n} \cong V_{i,n} \otimes k \cdot e_1^i \subseteq E_{\infty}^{n-2i,2i}$$

for $0 \leq i \leq 7$.

Define a bijection $f_n : S_n \to \bigcup_{i=0}^{7} S_{i,n}$ by $f_n(a, b, c, d) = (b, c, d, (a - i)/8) \in S_{i,n}$ for $a \equiv i \mod (8)$. Then

$$\dim_k H^*_\text{H}(BL/k) = |S_n| = \sum_{i=0}^{7} |S_{i,n}|.$$
As
\[ \dim_k H^n_{\text{H}}(BL/k) \geq \sum_{i=0}^{7} E_{\infty}^{-2i,2i} \]
and \( V_{i,n} \subseteq E_{\infty}^{-2i,2i} \) for \( 0 \leq i \leq 7 \), it follows that \( V_{i,n} \cong E_{\infty}^{-2i,2i} \) for \( 0 \leq i \leq 7 \)
and \( E_{\infty}^{-2i,2i} = 0 \) for \( i \geq 8 \).

Let \( F_* \) denote the cohomological spectral sequence concentrated on the 0th column with \( E_2 \) page given by \( E_2 = \Delta(e_1, e_2, e_4) \) where \( e_1 \) has bidegree \((0,2i)\) for \( i = 1, 2, 4 \). As \( e_1 \) is a permanent cycle in the spectral sequence \( (10) \), there exists a map \( F_* \to E_* \) of spectral sequences taking \( e_i \) to \( e_i \) for \( i = 1, 2, 4 \). Let \( y \) be a variable and let \( H_* \) denote the spectral sequence with \( E_2 \) page \( H_2 = \Delta(e_3) \otimes k[y] \) where \( e_3 \) is of bidegree \((0,6)\), \( y \) is of bidegree \((7,0)\), and \( e_3 \) is transgressive with \( d_7(e_3 y^i) = y^{i+1} \) for all \( i \). As \( e_3 \) is transgressive in the spectral sequence \( (10) \) with \( d_7(e_3) = y_7 \), there exists a map of spectral sequences \( H_* \to E_* \) taking \( e_3 \) to \( e_3 \) and \( y \) to \( y_7 \).

Elements in the ring of \( G \)-invariants \( k[y_4, y_6, y_8, y_16] \) are permanent cycles in the spectral sequence \( (10) \). Tensoring maps of spectral sequences, we get a map
\[ \alpha : I_* := F_* \otimes H_* \otimes k[y_4, y_6, y_8, y_16] \to E_* . \]
The map \( \alpha \) induces an isomorphism on \( E_\infty \) terms and on the 0th columns of the \( E_2 \) pages. Hence, Theorem 1.4 implies that \( \alpha \) induces an isomorphism on the 0th rows of the \( E_2 \) pages. Thus,
\[ H^n_{\text{H}}(BG/k) = k[y_4, y_6, y_7, y_8, y_16] . \]
Proposition 5 implies that the Hodge spectral sequence for \( BG \) degenerates.

\[
\square
\]

**Corollary 3.11.** Let \( k \) be a field of characteristic 2 and let \( G \) be a \( k \)-form for \( \text{Spin}(9) \). Then
\[ H^n_{\text{H}}(BG/k) \cong k[y_4, y_6, y_7, y_8, y_16] \]
where \( |y_i| = i \) for \( i = 4, 6, 7, 8, 16 \).

**Remark 3.12.** Assume that \( k \) is perfect. Let \( \mu_2 \) denote the group scheme of the 2nd roots of unity over \( k \). For \( n \geq 10 \), the Hodge cohomology of \( BG \) is no longer a polynomial ring. To determine the relations that hold in \( H^n_{\text{H}}(BG/k) \), we will restrict cohomology classes to the classifying stack of a certain subgroup of \( G \) considered in [17, Section 12]. Let \( r = [n/2] \) and let \( T \cong G_r^n \) denote a split maximal torus of \( G \). Assume that \( n \not\equiv 2 \mod 4 \) so that the Weyl group \( W \) of \( G \) contains \(-1\), acting by inversion on \( T \). Then \(-1\) acts by the identity on \( T[2] \cong \mu_2^n \) (for \( n \in \mathbb{N}, T[n] \subset T \) is the kernel of the \( n \)th power map \( T \to T \)) and \( G \) contains a subgroup \( Q \cong \mu_2^n \times \mathbb{Z}/2 \). Under the double cover \( G \to SO(n)_k \), the image of \( Q \) is isomorphic to \( K \cong \mu_2^{n-1} \times \mathbb{Z}/2 \) and \( Q \to K \) is a split surjection. We will need to know the Hodge cohomology rings of the classifying stacks of these groups. For a
commutative ring $R$, we let $\text{rad} \subset R$ denote the ideal of nilpotent elements.

From [17, Proposition 10.1],

$$H^i_H(B\mu_2/k)/\text{rad} \cong k[t]$$

where $t \in H^1(B\mu_2, \Omega^1)$. From [17, Lemma 10.2],

$$H^i_*((B\mathbb{Z}/2)/k) = k[s]$$

where $s \in H^1(B\mathbb{Z}/2, \Omega^0)$. The Künneth formula [17, Proposition 5.1] then lets us calculate the Hodge cohomology ring of $B\mu_2^i \times B(\mathbb{Z}/2)^j$ for any $i, j \geq 0$. Fix $i, j > 0$. Then

$$H^i_*(B\mu_2^i \times B(\mathbb{Z}/2)^j)/\text{rad} \cong k[t_1, \ldots, t_i, s_1, \ldots, s_j]$$

where $t_i \in H^1(B\mu_2^i \times B(\mathbb{Z}/2)^j, \Omega^1)$ for all $l$ and $s_l \in H^1(B\mu_2^i \times B(\mathbb{Z}/2)^j, \Omega^0)$ for all $l$.

**Theorem 3.13.** Let $n = 10$. The Hodge spectral sequence for $BG$ degenerates and

$$H^*_K(BG/k) \cong H^*_H(BG/k) = k[y_4, y_6, y_7, y_8, y_{10}, y_{32}]/(y_7y_{10})$$

where $|y_i| = i$ for $i = 4, 6, 7, 8, 10, 32$.

**Proof.** We may assume that $k = \mathbb{F}_2$ so that Remark 3.12 applies. From Lemma 3.5,

$$O(g)^G = k[y_4, y_6, y_8, y_{10}, y_{32}]$$

Consider the spectral sequence

$$E_2^{i,j} = H^i_H(BG/k) \otimes H^j_H((G/P)/k) \Rightarrow H^{i+j}H(BL/k)$$

(11)

from Proposition 1.1. From Proposition 3.1 and Corollary 3.3,

$$H^i_H((G/P)/k) \cong k[e_1, e_2, e_3, e_4]/(e_1^2 = e_2) = k[e_1, e_3]/(e_1^8, e_3^2)$$

and

$$H^*_H(BL/k) \cong k[A, c_2, c_3, c_4, c_5].$$

Calculations similar to those performed in the proof of Proposition 3.6 show that $e_1$ is a permanent cycle in (11) and $e_3 \in E_2^{6,0}$ is transgressive with $0 \neq d_7(e_3) = y_7 \in H^4(BG, \Omega^3)$. We have $H^m_H(BG/k) \cong H^m_H(BSpin(9), k/k)$ for $m < 10$.

Let $F*$ be the spectral sequence concentrated on the 0th column with $E_2$ page given by $E_2 = \Delta(e_1, e_2, e_4)$ where $e_i$ has bidegree $(0, 2i)$ for all $i$. As $e_1$ is a permanent cycle in (11), there exists a map of spectral sequence $F_* \to E_*$ taking $e_i$ to $e_i$ for $i = 1, 2, 4$. Fix a variable $y$. Let $H_*$ denote the spectral sequence with $E_2$ page $H_2 = \Delta(e_3) \otimes k[y]$ where $e_3$ has bidegree $(0, 6)$, $y$ has bidegree $(7, 0)$, and $e_3$ is transgressive with $d_7(e_3y^i) = y^{i+1}$ for all $i$. As $e_3$ is transgressive in (11) with $d_7(e_3) = y_7$, there exists a map of spectral sequences $H_* \to E_*$ taking $e_3$ to $e_3$ and $y$ to $y_7$. Elements in the ring
of $G$-invariants $k[y_4, y_6, y_8, y_{10}, y_{32}]$ are permanent cycles in $(11)$. Tensoring maps of spectral sequences, we get a map

$$\alpha : I_* := F_* \otimes H_* \otimes k[y_4, y_6, y_8, y_{10}, y_{32}] \to E_*$$

which induces an isomorphism on the 0th columns of the $E_2$ pages.

Let $n$ be even. The $k$-dimension of $H^n_H(BL/k)$ is equal to the cardinality of the set

$$S_n = \{(a, b, c, d, e) \in \mathbb{Z}^5_{\geq 0} : 2a + 4b + 6c + 8d + 10e = n\}.$$

For $0 \leq i \leq 15$, set $V_{i,n} := H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2})$. For $0 \leq i \leq 15$, $\dim_k V_{i,n}$ is equal to the cardinality of the set $S_{i,n} = \{(a, b, c, d, e) \in \mathbb{Z}^5_{\geq 0} : 4a + 6b + 8c + 10d + 32e = n - 2i\}$. As $e_1 \in H^2_H((G/P)/k)$ is a permanent cycle in $(11)$,

$$V_{i,n} \cong V_{i,n} \otimes k \cdot e_1^i \subseteq E^{n-2i,2i}_\infty$$

for $0 \leq i \leq 7$. Hence, the map $\alpha$ in $(12)$ induces injections on all $E_\infty$ terms. For $n$ odd, $\alpha$ induces isomorphisms $0 = E_{i,n}^{n-i,i} \cong E_{i,n}^{n-i,i} = 0$ for all $i$ since $H^n_H(BL/k)$ is concentrated in even degrees.

Define a bijection $f_n : S_n \to \bigcup_{i=0}^{15} S_{i,n}$ by $f_n(a, b, c, d, e) = (b, c, d, e, (a - i)/16) \in S_{i,n}$ for $a \equiv i \mod (16)$. Then

$$\dim_k H^n_H(BL/k) = |S_n| = \sum_{i=0}^{15} |S_{i,n}| = \sum_{i=0}^{15} \dim_k V_{i,n}. \quad (13)$$

Now assume that $n \leq 14$. Then $f_n$ gives a bijection

$$S_n \to \bigcup_{i=0}^{7} S_{i,n}.$$

As

$$\dim_k H^n_H(BL/k) \geq \sum_{i=0}^{7} E_{i,n}^{n-2i,2i}$$

and $V_{i,n} \subseteq E^{n-2i,2i}_\infty$ for $0 \leq i \leq 7$, it follows that $V_{i,n} \cong E^{n-2i,2i}_\infty$ for $0 \leq i \leq 7$ and $E^{n-2i,2i}_\infty = 0$ for $i \geq 8$. As $\alpha$ induces injections on all $E_\infty$ terms, Theorem 1.4 implies that $\alpha$ in $(12)$ induces an isomorphism $I_{2}^{0,0} \to E_{2}^{0,0}$ for $n < 16$.

Now we consider the filtration on $H^H_16(BL/k)$ given by $(11)$. From the bijection $f_{16}$ defined in the previous paragraph, we have

$$\dim_k H^16_H(BL/k) = 1 + \sum_{i=0}^{7} |S_{i,n}| = 1 + \sum_{i=0}^{7} \dim_k V_{i,n} \otimes k \cdot e_1^i.$$

As $e_1$ is a permanent cycle and $\alpha$ induces isomorphisms on 0th row terms of the $E_2$ pages in degrees less than 16, we must then have

$$E^{10,0}_\infty \cong (H^10_H(BG/k) \otimes k \cdot e_3^3) \oplus (k \cdot z \otimes k \cdot e_3)$$
for some \(0 \neq z \in H^1(BG/k)\). Hence, \(y_7z = 0\) in \(H^*_H(BG/k)\). Write \(z = ay_4y_6 + by_{10}\) for some \(a, b \in k\).

We now show that \(a = 0\) by restricting \(y_7z = 0\) to the Hodge cohomology of the classifying stack of the subgroup \(\text{Spin}(8)_k\) of \(G\). Under the isomorphism

\[
H^*_H(B\text{Spin}(8)_k/k) \cong k[y_4, y_6, y_7, y_8, y_{10}]
\]

of Theorem 3.10, the pullback from \(H^*_H(BG/k)\) to \(H^*_H(B\text{Spin}(8)_k/k)\) maps \(y_4, y_6, y_{10} \in H^*_H(BG/k)\) to \(y_4, y_6, y_{10}\), and \(0\) respectively in \(H^*_H(B\text{Spin}(8)_k/k)\). Hence, to show that \(a = 0\), it suffices to show that \(y_7 \in H^*_H(BG/k)\) restricts to \(y_7 \in H^*_H(B\text{Spin}(8)_k/k)\). From the isomorphism

\[
H^*_H(B\text{SO}(m)_k/k) \cong k[u_2, \ldots, u_m]
\]

of Theorem 3.4 for \(m \geq 0\), the class \(u_7 \in H^*_H(B\text{SO}(10)_k/k)\) restricts to \(u_7 \in H^*_H(B\text{SO}(8)_k/k)\). Thus, we are reduced to showing that \(u_7 \in H^*_H(B\text{Spin}(8)_k/k)\) pulls back to a non-zero multiple of \(y_7 \in H^*_H(B\text{Spin}(8)_k/k)\).

Consider the subgroups \(\mu_2^4 \times \mathbb{Z}/2 \cong Q \subseteq \text{Spin}(8)_k\) and \(\mu_2^3 \times \mathbb{Z}/2 \cong K \subseteq \text{SO}(8)_k\) defined in Remark 3.12. As the morphism \(Q \to K\) is split surjective, if we can show that \(u_7\) restricts to a non-zero class in \(H^*_H(BK/k)\), then \(u_7\) would restrict to a non-zero class in \(H^*_H(B\text{Spin}(8)_k/k)\). From the inclusion \(O(2)_k \subset O(8)_k\), \(O(8)_k\) contains a subgroup of the form \(\mu_2^4 \times (\mathbb{Z}/2)^4\). As \(\text{SO}(8)_k\) is the kernel of the Dickson determinant (also called the Dickson invariant in some sources [11, §23]) \(O(8)_k \to \mathbb{Z}/2\), it follows that \(\text{SO}(8)_k\) contains a subgroup \(H \cong \mu_2^4 \times (\mathbb{Z}/2)^3\). Write

\[
H^*_H(BH/k)/\text{rad} \cong k[t_1, \ldots, t_4, s_1, \ldots, s_4]/(s_1 + s_2 + s_3 + s_4)
\]

using Remark 3.12. From the proof of [17, Lemma 11.4], the pullback of \(u_7\) to \(H^*_H(BK/k)/\text{rad}\) followed by pullback to

\[
H^*_H(BK/k)/\text{rad} \cong k[t_1, \ldots, t_4, s]((t_1 + \cdots + t_4)
\]

is given by

\[
\begin{align*}
u_7 \mapsto &\sum_{j=1}^{3} s_j (t_j + t_4) \sum_{1 \leq i_1 < i_2 \leq 3 \atop i_1, i_2 \neq j} t_{i_1} t_{i_2} \mapsto \sum_{j=1}^{3} s(t_j + t_4) \sum_{1 \leq i_1 < i_2 \leq 3 \atop i_1, i_2 \neq j} t_{i_1} t_{i_2} \\
&= s \sum_{1 \leq i_1 < i_2 \leq 3} (t_{i_1} + t_{i_2}) t_{i_1} t_{i_2} \neq 0.
\end{align*}
\]

Thus, \(u_7 \in H^*_H(B\text{SO}(8)_k/k)\) pulls back to a non-zero multiple of

\[
y_7 \in H^*_H(B\text{Spin}(8)_k/k)
\]
which implies that \( y_7y_{10} = 0 \) in \( H^*_H(BG/k) \).

\[
\begin{array}{c}
\sum_{j=1}^3 s(t_j + t_4) \sum_{1 \leq i_1 < i_2 \leq 3} t_{i_1} t_{i_2} \in H^7_H(BK/k) \rightarrow H^7_H(BQ/k)
\end{array}
\]

Using the relation \( y_7y_{10} = 0 \), we now modify the spectral sequence \( I_* \) defined above to define a new spectral sequence \( J_* \) that better approximates (and will actually be isomorphic to) the spectral sequence (11). Let

\[
(yy_{10}) := F_2 \otimes (\Delta(e_3) \otimes yk[y]) \otimes y_{10}k[y_4, y_6, y_8, y_{10}, y_{32}].
\]

Define the \( E_2 \) page of \( J_* \) by \( J_2 = I_2/(yy_{10}) \). Define the differentials \( d'_m \) of \( J_* \) so that \( I_2 \rightarrow J_2 \) induces a map \( I_* \rightarrow J_* \) of cohomological spectral sequences of \( k \)-vector spaces and \( d'_m = 0 \) for \( m > 7 \). This means that \( d'_7(f \otimes e_3 \otimes y_{10}g) = f \otimes y \otimes y_{10}g = 0 \) and \( d'_m(f \otimes e_3 \otimes y_{10}g) = 0 \) for \( m > 7 \), \( f \in F_2 \), and \( g \in k[y_4, y_6, y_8, y_{10}, y_{32}] \). The \( E_\infty \) page of \( J_* \) is given by

\[
J_\infty \cong (F_2 \otimes k[y_4, y_6, y_8, y_{10}, y_{32}]) \oplus (F_2 \otimes e_3 \otimes y_{10}k[y_4, y_6, y_8, y_{10}, y_{32}]).
\]

As \( y_7y_{10} = 0 \) in \( H^*_H(BG/k) \), \( \alpha \) induces a map \( \alpha' : J_* \rightarrow E_* \) of spectral sequences. To finish the calculation, we will show that \( \alpha' \) induces an isomorphism on \( E_\infty \) terms so that Theorem 1.4 will apply. For \( n \) odd, \( E^{n-1,i}_\infty = 0 \) for all \( i \) since \( H^*_H(BL/k) \) is concentrated in even degrees. Now assume that \( n \) is even. For \( 0 \leq i \leq 7 \),

\[
V_{i,n} \cong H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2}) \otimes e_1^i \subseteq E^{n-2i-2i}_\infty.
\]

For \( 8 \leq i \leq 15 \),

\[
V_{i,n} \cong y_{10}H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2}) \otimes e_1^{i-8} e_3 \subseteq E^{n-2i+10,2i-10}_\infty.
\]

Hence, from the description of the \( E_\infty \) terms of \( J_* \) given above, it follows that \( \alpha' \) induces an injection \( J^{n-2i,2i}_\infty \rightarrow E^{n-2i,2i}_\infty \) for all \( i \). Equation (13) then implies that \( J^{2i-2i,2i}_\infty \cong L^{2i-2i,2i}_\infty \) for all \( i \).

Thus, \( \alpha' \) induces an isomorphism on \( E_\infty \) pages and an isomorphism on the 0th columns of the \( E_2 \) pages of the 2 spectral sequences. Theorem 1.4 then implies that

\[
H^*_H(BG/k) \cong k[y_4, y_6, y_7, y_8, y_{10}, y_{32}] / (y_7y_{10}).
\]

From Proposition 5, the Hodge spectral sequence for \( BG \) degenerates.

\[ \square \]

**Corollary 3.14.** Let \( G \) be a \( k \)-form of \( \text{Spin}(10) \). Then

\[
H^*_H(BG/k) \cong k[y_4, y_6, y_7, y_8, y_{10}, y_{32}] / (y_7y_{10})
\]
where |y_i| = i for all i.

**Theorem 3.15.** Let n = 11. The Hodge spectral sequence for BG degenerates and

\[ H^*_d(BG/k) \cong H^*_H(BG/k) = k[y_4, y_6, y_7, y_8, y_{10}, y_{11}, y_{32}] / (y_7y_{10} + y_6y_{11}) \]

where |y_i| = i for i = 4, 6, 7, 8, 10, 11, 32.

**Proof.** We may assume that k = F_2 so that Remark 3.12 applies. From Lemma 3.5,

\[ O(g)^G \cong k[y_4, y_6, y_7, y_8, y_{32}] \]

where |y_i| = i in H^*_H(BG/k), viewing O(g)^G as a subring of H^*_H(BG/k).

Consider the spectral sequence

\[ E^{i,j}_2 = H^i_H(BG/k) \otimes H^j_H((G/P)/k) \Rightarrow H^{i+j}_H(BL/k) \] (14)

from Proposition 1.1. From Proposition 3.1 and Corollary 3.3,

\[ H^*_H((G/P)/k) \cong k[e_1, e_2, e_3, e_4, e_5] / (e_i^2 = e_{2i}) \]

and

\[ H^*_H(BL/k) \cong k[A, c_2, c_3, c_4, c_5] \]

Using Theorem 3.4, write H^*_H(BSO(11)_k/k) = k[u_2, \ldots, u_{11}]. From the inclusions O(2)^5 \subset O(10)_k \subset SO(11)_k, SO(11)_k contains a subgroup H \cong \mu_2^5 \times (\mathbb{Z}/2)^5. Write H^*_H(BH/k)/rad \cong k[t_1, \ldots, t_5, s_1, \ldots, s_5] as described in Remark 3.12. Under the pullback map H^*_H(BSO(11)_k/k) \to H^*_H(BH/k)/rad, u_{2m} pulls back to the mth elementary symmetric polynomial

\[ \sum_{1 \leq i_1 < \cdots < i_m \leq 5} t_{i_1} \cdots t_{i_m} \] (15)

and u_{2m+1} pulls back to

\[ \sum_{j=1}^{5} s_j \sum_{1 \leq i_1 < \cdots < i_m \leq 5 \text{ one equal to } j} t_{i_1} \cdots t_{i_m} \]

for 1 \leq m \leq 5 [17, Lemma 11.4]. To be concise, from now on we will write u_{2m} to denote the image of u_{2m} under pullback maps to H^*_H(BH/k)/rad or H^*_H(BK/k)/rad whenever we are dealing with these two rings.

Let Q \cong (\mu_2^5 \times \mathbb{Z}/2) \subset G and K \cong (\mu_2^5 \times \mathbb{Z}/2) \subset SO(11)_k be the subgroups described in Remark 3.12. Write H^*_H(BK/k)/rad \cong k[t_1, \ldots, t_5, s] / (t_1 + \cdots + t_5). Under the pullback map H^*_H(BSO(11)_k/k) \to H^*_H(BK/k)/rad, u_7 maps to s_{u_6} \neq 0 and u_{11} maps to s_{u_{10}} \neq 0. As Q \rightarrow K is split, it follows that u_7, u_{11} restrict to nonzero classes y_7 \in H^*_H(BG/k) and y_{11} \in H^*_H(BG/k).

Also, y_4y_7 and y_{11} are linearly independent in H^*_H(BG/k).

Returning to the spectral sequence (14), calculations similar to those performed in the proof of Proposition 3.6 show that e_1 is a permanent cycle in (14) and \( e_3 \in E^{0,6}_2 \) is transgressive with \( 0 \neq d_7(e_3) = y_7 \in H^4(BG, \Omega^3) \). We have \( H^{m}_H(BG/k) \cong H^{m}_H(Bspin(10)_k/k) \) for \( m \leq 10 \).
Let $F_*$ be the spectral sequence concentrated on the 0th column with $E_2$ page given by $\Delta(e_1, e_2, e_4)$ with $e_i$ of bidegree $(0, 2i)$ for $i = 1, 2, 4$. Fix a variable $y$ and let $H_*$ be the spectral sequence with $H_2 = \Delta(e_3) \otimes k[y]$ where $e_3$ is of bidegree $(0, 6)$, $y$ is of bidegree $(7, 0)$, and $e_3$ is transgressive with $d_7(e_3y^i) = y^{i+1}$ for all $i$. There exists a map of spectral sequence
\[
\alpha : I_* := F_* \otimes H_* \otimes k[y_4, y_6, y_8, y_{10}, y_{32}] \to E_*
\]
taking $e_i$ to $e_i$ for $i = 1, 2, 3, 4$ and taking $y$ to $y_7$. The $E_\infty$ page of $I_*$ is given by $I_\infty \cong F_2 \otimes k[y_4, y_6, y_8, y_{10}, y_{32}]$ and $\alpha$ induces an injection $I^{i,j}_\infty \to E^{i,j}_\infty$ for all $i,j$ with $i + j \leq 17$. For $n$ odd, $\alpha$ induces an isomorphism $0 = I^{n-i,i}_\infty \cong E^{n-i,i}_\infty = 0$ for all $i$ such that $H^n_\Omega(BL/k)$ is concentrated in even degrees.

Let $n$ be even. The $k$-dimension of $H^n_\Omega(BL/k)$ is equal to the cardinality of the set
\[
S_n = \{(a, b, c, d, e) \in \mathbb{Z}_{\geq 0}^5 : 2a + 4b + 6c + 8d + 10e = n\}.
\]
For $0 \leq i \leq 15$, set $V_{i,n} := H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2})$. For $0 \leq i \leq 15$, $\dim_k V_{i,n}$ is equal to the cardinality of the set $S_{i,n} = \{(a, b, c, d, e) \in \mathbb{Z}_{\geq 0}^5 : 4a + 6b + 8c + 10d + 32e = n - 2i\}$. As $e_1 \in H^1_\Omega((G/P)/k)$ is a permanent cycle in (14),
\[
V_{i,n} \cong V_{i,n} \otimes k \cdot e_1 \subseteq E^{n-2i,2i}_\infty
\]
for $0 \leq i \leq 7$ and $n \leq 16$.

Define a bijection $f_n : S_n \to \bigcup_{i=0}^{15} S_{i,n}$ by $f_n(a, b, c, d, e) = (b, c, d, e, (a - i)/16) \in S_{i,n}$ for $a \equiv i \mod (16)$. Then
\[
\dim_k H^n_\Omega(BL/k) = |S_n| = \sum_{i=0}^{15} |S_{i,n}| = \sum_{i=0}^{15} \dim_k V_{i,n}.
\]
Now assume that $n \leq 14$. Then $f_n$ gives a bijection
\[
S_n \to \bigcup_{i=0}^{7} S_{i,n}.
\]
As
\[
\dim_k H^n_\Omega(BL/k) \geq \sum_{i=0}^{7} E^{n-2i,2i}_\infty
\]
and $V_{i,n} \subseteq E^{n-2i,2i}_\infty$ for $0 \leq i \leq 7$, it follows that $V_{i,n} \cong E^{n-2i,2i}_\infty$ for $0 \leq i \leq 7$ and $E^{n-2i,2i}_\infty = 0$ for $i \geq 8$. In particular, $E^{0,10}_\infty \cong k \cdot e_1^5$. As mentioned above, we have $H^m_\Omega(BG/k) = 0$ for $m = 3, 5, 9$. After adding a $k$-multiple of $e_3e_1^2$ to $e_3$, we can assume that $d_7(e_3) = 0$. Then the isomorphism $E^{0,10}_\infty \cong k \cdot e_1^5$ implies that $d_{11}(e_5) \neq 0$. Hence, $e_5$ is transgressive in (14) and $y_{11} \in H^6(BG, \Omega^5)$ is a lifting of $d_{11}(e_5)$ to $E^{11,0}_2$. 


Fix a variable $x$. Let $J_*$ denote the spectral sequence with $E_2$ page $J_2 = \Delta(e_5) \otimes k[x]$ where $e_5$ has bidegree $(0, 10)$, $x$ has bidegree $(11, 0)$, and $e_5$ is transgressive with $d_{11}(e_5 x^i) = x^{i+1}$ for all $i$.

\[
\begin{array}{ccc}
  k \cdot e_5 & k \cdot e_5 x & \cdots \\
  \downarrow d_{11} & \downarrow d_{11} & \\
  0 & k \cdot x & k \cdot x^2 & \cdots 
\end{array}
\]

As $e_5$ is transgressive in (14), there exists a map of spectral sequences $J_* \to E_*$ taking $e_5$ to $e_5$ and $x$ to $y_{11}$. Tensoring with the map $\alpha$ defined above, we get a map

\[
\alpha' : K_* := I_* \otimes J_* \to E_*
\]

which induces an isomorphism on the 0th columns of the $E_2$ pages. The $E_\infty$ page of $K_*$ is given by

\[
K_\infty \cong I_\infty \cong F_2 \otimes k[y_4, y_6, y_8, y_{10}, y_{32}].
\]

As mentioned above, $\alpha$ and hence $\alpha'$ induce isomorphisms on $E_\infty^{i,j}$ terms for $n < 16$ and injections on all $E_\infty$ terms on or below the line $i + j = 17$. Theorem 1.4 implies that $\alpha'$ induces an isomorphism $K_2^{n,0} \to E_2^{n,0}$ for $n < 16$.

Next, we consider the filtration on $H_\ast^{16}(BL/k)$ given by (14). From (16),

\[
\dim_k H_\ast^{16}(BL/k) = 1 + \sum_{i=0}^{7} |S_{i,16}| = 1 + \sum_{i=0}^{7} \dim_k V_{i,16} \otimes k \cdot e_i^1.
\]

We must then have either $d_7(c_3 f) = y_7 f = 0 \in H_\ast^{17}(BG/k)$ for some $0 \neq f \in H_\ast^{16}(BG/k)$ or $d_{11}(e_5 g) = y_{11} g = d_7(e_5) h = y_7 h \in H_\ast^{17}(BG/k)$ for some $0 \neq g \in H_\ast^{6}(BG/k)$ and $h \in H_\ast^{10}(BG/k)$. Let $a, b, c, k, t, s, t_5$, not all zero, such that

\[
ay_{11}y_6 + by_7y_{10} + cy_7y_{10} = 0 \in H_\ast^{17}(BG/k).
\]

The class $au_{11}u_6 + bu_{10}u_6 + cu_{10}u_6 \in H_\ast^{17}(BSO(11)_k/k)$ pulls back to

\[
ay_{11}y_6 + by_7y_{10} + cy_7y_{10} = 0 \in H_\ast^{17}(BG/k).
\]

Under the pullback map \(H_\ast^{17}(BSO(11)_k/k) \to H_\ast^{17}(BK/k)/\rad \cong k[t_1, \ldots, t_5, s]/(t_1 + \cdots + t_5),\)

$au_{11}u_6 + bu_{10}u_6 + cu_{10}u_6$ maps to $asu_{10}u_6 + bsu_{6}u_{10} + csu_{6}u_{4}u_{6}$, which equals 0 since $Q \to K$ is split. Then $c = 0$ and $a = b$ since the elementary symmetric polynomials (15) in

\[
k[t_1, \ldots, t_5]/(t_1 + \cdots + t_5)
\]

generate a polynomial subring.

\[
\begin{array}{ccc}
  au_{11}u_6 + bu_{10}u_6 + cu_{10}u_6 \in H_\ast^{17}(BSO(11)_k/k) & \longrightarrow & 0 \in H_\ast^{17}(BG/k) \\
  \downarrow & & \downarrow \\
  asu_{10}u_6 + bsu_{6}u_{10} + csu_{6}u_{4}u_{6} \in H_\ast^{17}(BK/k)/\rad & \longrightarrow & 0 \in H_\ast^{17}(BQ/k)/\rad
\end{array}
\]
Thus, the relation $y_7y_{10} + y_6y_{11} = 0$ holds in $H^n_{\mathbb{H}}(BG/k)$ and $E^n_{\infty,10} \cong (k \cdot y_6 \otimes e_5) \oplus (k \cdot y_6 \otimes e_1^5)$. We now use the relation $y_7y_{10} + y_6y_{11} = 0$ to define a new spectral sequence $L_*$ from $K_*$. Let $(y_6x + y_7y_{10}) \subset K_2$ denote the ideal generated by $y_6x + y_7y_{10}$ and let $L_2 := K_2/(y_6x + y_7y_{10})$. Define the differentials $d'_m$ of $L_*$ so that $K_2 \to L_2$ induces a map of spectral sequences $K_* \to L_*$ and $d'_m = 0$ for $m > 11$. Then $\alpha' : K_* \to E_*$ induces a map of spectral sequences $\alpha'' : L_* \to E_*$. The $E_\infty$ page of $L_*$ is given by

$$L_\infty \cong (F_2 \otimes k[y_4, y_6, y_8, y_{10}, y_{32}]) \oplus (F_2 \otimes y_6k[y_4, y_6, y_8, y_{10}, y_{32}] \otimes e_5).$$

We now show by induction that $\alpha''$ induces an isomorphism $L_2^{n,0} \to E_2^{n,0}$ for all $n$. For $n < 16$, we have shown that $L_2^{n,0} \cong E_2^{n,0}$. Now let $n \geq 16$ and assume that $\alpha''$ induces an isomorphism $L_2^{m,0} \to E_2^{m,0}$ for all $m < n$. First, suppose that $n$ is even. As $L_2^{m,0} \cong E_2^{m,0}$ for $m < n$, $y_7g \neq 0 \in H^n_{\mathbb{H}}(BG/k)$ for all $0 \neq g \in H^n_{\mathbb{H}}(BG/k)$ with $|g| < n - 7$. Hence, for any $0 \neq g \in H^n_{\mathbb{H}}(BG/k)$ with $|g| = m < n - 7$, $g \otimes e_3e_5 \in E_2^{m,16} \cong E_{\infty}^{m,16}$ is not in the kernel of the differential $d_7 : E_7^{m,16} \to E_7^{m+7,10} \cong E_7^{m+7,10}$. As $y_{77} \in H^4(BG, \Omega^3)$ and $y_{11} \in H^6(BG, \Omega^5)$, $y_{77}z, y_{11}z \notin \oplus_i H^i(BG, \Omega^i)$ for all $z \in H^n_{\mathbb{H}}(BG/k)$. It follows that $\alpha''$ induces an injection $L_{i,j}^{n} \to E_{\infty}^{n,i,j}$ for all $i, j$ with $m = i + j \leq n$

$$L_{\infty}^{m-2i,2i} \cong V_{i,m} \otimes e_1^i \subseteq E_{\infty}^{m-2i,2i}$$

for $0 \leq i \leq 4$,

$$L_{\infty}^{m-2i,2i} \cong (V_{i,m} \otimes e_1^i) \oplus (y_6 V_{i+3,m} \otimes e_1^{i-5} e_5) \subseteq E_{\infty}^{m-2i,2i}$$

for $5 \leq i \leq 7$, and

$$L_{\infty}^{m-2i,2i} \cong y_6 V_{i+3,m} \otimes e_1^{i-5} e_5 \subseteq E_{\infty}^{m-2i,2i}$$

for $8 \leq i \leq 12$. The equality in (16) then implies that $\alpha''$ induces isomorphisms $L_{\infty}^{i,j} \to E_{\infty}^{i,j}$ for all $i, j$ with $i + j \leq n$. As mentioned above, $\alpha''$ induces isomorphisms $0 = L_{n+1-i,j}^{n+1-i,j} \to E_{\infty}^{n+1-i,j} = 0$ for all $i$ since $n + 1$ is odd. Theorem 1.4 then implies that $\alpha''$ induces an isomorphism $L_2^{n,0} \cong E_2^{n,0} = H^n_{\mathbb{H}}(BG/k)$.

Now assume that $n$ is odd. We have $0 = L_{\infty}^{i,j} \cong E_{\infty}^{i,j} = 0$ for all $i, j$ with $i + j = n$. An argument similar to the one used above for when $n$ is even shows that $\alpha''$ induces injections $L_{\infty}^{i,j} \to E_{\infty}^{i,j}$ for all $i, j$ with $i + j \leq n + 1$. Equation (16) then implies that $\alpha''$ induces isomorphisms $L_{\infty}^{i,j} \to E_{\infty}^{i,j}$ for all $i, j$ with $i + j \leq n + 1$. It follows that $\alpha''$ induces an isomorphism $L_2^{n,0} \cong E_2^{n,0} = H^n_{\mathbb{H}}(BG/k)$ by an application of Theorem 1.4. Thus, by induction, we have obtained that the 0th row of $L_2$ is isomorphic to the 0th row of $E_2$

$$H^n_{\mathbb{H}}(BG/k) = k[y_4, y_6, y_7, y_8, y_{10}, y_{11}, y_{32}] / (y_7 y_{10} + y_6 y_{11}).$$

The Hodge spectral sequence for $BG$ degenerates by Proposition 5.

□
Corollary 3.16. Let $G$ be a $k$-form of Spin(11). Then

$$H^*(BG/k) \cong k[y_4, y_6, y_7, y_8, y_{10}, y_{11}, y_{32}]/(y_7y_{10} + y_6y_{11})$$

where $|y_i| = i$ for $i = 4, 6, 7, 8, 10, 11, 32$.

References


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