Invariants, bitangents, and matrix representations of plane quartics with 3-cyclic automorphisms

Dun Liang

Abstract. In this work we compute the Dixmier invariants and bitangents of the plane quartics with 3, 6 or 9-cyclic automorphisms. We find that a quartic curve with 6-cyclic automorphism will have 3 horizontal bitangents which form an asyzygetic triple. We also discuss the linear matrix representation problem of such curves, and find a degree 6 equation of 1 variable which solves the symbolic solution of the linear matrix representation problem for the curve with 6-cyclic automorphism.

Contents

1. Introduction 636
2. Automorphisms of plane quartics 638
3. Dixmier invariants of $C_3$, $C_6$ and $C_9$ 639
4. The Bitangents of $C_3$, $C_6$ and $C_9$ 643
5. Discussion on the matrix representation problem 649
References 653

1. Introduction

The study of the geometry of plane quartics is one of the most beautiful achievements in classical algebraic geometry. Back to the late 19th and early 20th century, there were many studies on the existence and configurations of the 28 bitangents of a plane quartic such as [11], [8], [9], [23], and so on. For the invariants of plane quartics, Shioda computed the ring of invariants in [22]. However, the algebraic invariants of plane quartics were found much later by [1], [16] and [3]. In this work we compute the invariants and bitangents of plane quartics with 3-cyclic automorphism, and discuss the linear matrix representation problem (see [6],[24],[25]) of such curves. The...
classification of automorphism was given by [12] and [26]. There are many places one can see the full list, for example, in Section 6.5 of [2]. Explicitly, we consider the curves
\[ C_3 = C_3(r, s) : y^3 = x(x - 1)(x - r)(x - s) \text{ for } r, s \neq 0, 1 \text{ and } r \neq s \quad (1) \]
\[ C_6 = C_6(r) : y^3 = x(x - 1)(x - r)(x - 1 + r) \text{ for } r \neq 0, 1 \quad (2) \]
\[ C_9 : y^3 = x(x^3 - 1) \quad (3) \]
with automorphism group \( \mathbb{Z}/3, \mathbb{Z}/6 \) and \( \mathbb{Z}/9 \) respectively. The family \( C_3 \) is the famous Picard family of quartics (see [10], [17], [18]). Let \( R_3 = \text{End}(J_3) \) be the endomorphism ring of the Jacobian variety \( J_3 \) of \( C_3 \). Let \( \zeta_3 \) be the cubic root of the unity. The main property of the family \( C_3 \) is that \( R_3 \cong O_K \), the ring of integers of some number field \( K \) which contains \( \mathbb{Q}(\zeta_3) \).

We compute the invariants of these curves. The curves \( C_6 \) and \( C_9 \) are special cases of \( C_3 \). Thus we also compute the cutting equations of the invariants of \( C_6 \) and \( C_9 \) as special cases of \( C_3 \). In modern point of view, a smooth plane quartic is the canonical model of a smooth projective non-hyperelliptic curve of genus 3. Let \( M_3 \) be the moduli space of projective curves of genus 3, and let \( M_3^{\text{non}} \) be the non-hyperelliptic locus of \( M_3 \). Then the weight zero ratios of the Dixmier invariants \( I_3, I_6, I_9, I_{12}, I_{15}, I_{18}, I_{27} \), as functions of the coefficients of a given ternary quartic are like an analog of the \( j \)-invariant of a given cubic curve, and thus could be regarded as the coordinates of \( M_3^{\text{non}} \). Let \( G \) be a finite group. If we write \( X^G \) as the subvariety of \( M_3^{\text{non}} \) parametrizing curves with automorphism group containing \( X^G \), then we have \( X^{\mathbb{Z}/9} \subset X^{\mathbb{Z}/3} \subset M_3^{\text{non}} \) and \( X^{\mathbb{Z}/6} \subset X^{\mathbb{Z}/3} \subset M_3^{\text{non}} \). In this point of view, we are trying to find the “defining equation” of \( X^{\mathbb{Z}/3} \), \( X^{\mathbb{Z}/6} \) and \( X^{\mathbb{Z}/9} \) in \( M_3^{\text{non}} \).

The explicit formulae of the Dixmier invariants are listed in Section 3.1. We use Maxima to compute the Dixmier invariants. Summarizing Section 3.2, we have the following.

**Theorem 1.1.** The curve \( C_3 \) satisfies that
\[ I_3 = I_6 = I_{12} = I_{15} = 0, \]
and \( I_9, I_{18} \) are algebraically independent. For the curve \( C_6 \), the invariants \( I_9 \) and \( I_{18} \) satisfy a degree 8 affine equation. Furthermore, the curve \( C_9 \) is the curve on which all Dixmier invariants vanish.

The algebraic conditions between the invariants of \( C_6 \) are computed by Macaulay2 [5].

We use the idea in [19] to compute the bitangents of a plane quartic. This program is also realized by Macaulay2. We summarize Section 4.2 as the following theorem.

**Theorem 1.2.** The curve \( C_9 \) has all 28 explicit equations for the bitangents whose coefficients are radical expressions over \( \mathbb{Q} \). The curve \( C_6 \) has 3 horizontal explicit bitangents which form a triple of asyzygetic sets.
The definition of asyzygetic sets comes from the theory of theta characteristics. For details one can see [15]. Our definition in Section 4.1 is a geometric description as in [19].

For the linear matrix determinant representation problem of such curves, we use the idea in [20]. The problem asks whether the equation of a plane curve $C$ could be written of the form

$$\det(xA + yB + zC)$$

for some symmetric matrices $A, B, C$ of constants. Our result is Theorem 5.2 as follows:

**Theorem 1.3.** The matrix representation of $C_6$ could be explicitly written over an extension field of $K(r, s) = \mathbb{Q}(r, s)$ defined by a degree 6 polynomial $f(z) \in K(r, s)[z]$.

### 2. Automorphisms of plane quartics

We consider the algebraic varieties over the algebraic closure $K = \overline{\mathbb{Q}}$ of the rational field $\mathbb{Q}$ in the complex numbers $\mathbb{C}$ since we are interested in the geometric properties of such varieties. However, some of the algorithms we use later in this work will be realized over $\mathbb{Q}$ only. In this section, let $K = \overline{\mathbb{Q}}$.

Let $C$ be a smooth projective curve over $K$. If the genus $g(C)$ of $C$ is 3, and $C$ is non-hyperelliptic, then the canonical model of $C$ is a plane quartic and is isomorphic to $C$. Let $x, y, z$ be the coordinates of the projective plane $\mathbb{P}^2$. If we want to emphasize the coordinates, we also write $\mathbb{P}^2$ as $\mathbb{P}^2(x, y, z)$. Let $k[x, y, z]_d$ be the homogeneous degree $d$-part of the polynomial ring $K[x, y, z]$. Thus $k[x, y, z]_d \simeq \text{Sym}^d((K^\vee)^3)$, the 3rd symmetric product of $K^\vee = \text{Hom}_K(K, K) \simeq K$. We write $\mathbb{P}^2_n := \text{Sym}^d((K^\vee)^n)$. Thus, let $F_C = F_c(x, y, z)$ be the equation of $C$, we say both $F_C \in \mathbb{P}^2_3$ and $F_C \in K[x, y, z]_4$.

An element $F \in K[x, y, z]_4$ should be written as

$$F(x, y, z) = \sum_{i+j+k=4} a_{ijk} x^i y^j z^k.$$  

Let $C$, $D$ be two smooth non-hyperelliptic genus $g$ curves over $K$. The canonical models $\kappa_C$, $\kappa_D$ of $C$ and $D$ are closed subvarieties of degree $2g - 2$ in $\mathbb{P}^{g-1}$. Since $C$ and $D$ are non-hyperelliptic, we have $C \simeq \kappa_C$ and $D \simeq \kappa_D$. The theory of algebraic curves says that $C$ and $D$ are isomorphic as algebraic varieties if and only if $\kappa_C$ could be transformed to $\kappa_D$ by a non-degenerated projective linear transformation on the coordinates of $\mathbb{P}^{g-1}$. In particular, an automorphism of a non-hyperelliptic curve $C$ is a projective automorphism on the canonical model $\kappa_C$ of $C$.

In this work we consider non-hyperelliptic genus 3 curves with cyclic automorphism groups $\mathbb{Z}/3, \mathbb{Z}/6$ and $\mathbb{Z}/9$. 
The genus 3 non-hyperelliptic curves with \( \mathbb{Z}/3 \)-automorphisms form a 2-dimensional family

\[
C_3 = C_3(r, s) : \quad y^3 = x(x - 1)(x - r)(x - s).
\]

This is a family of smooth quartics written on the affine chart \( \{ z = 1 \} \) of the projective plane \( \mathbb{P}^2(x, y, z) \) with \( K \)-parameters \( r \) and \( s \).

Also we have the 1-dimensional family

\[
C_6 = C_6(r) : \quad y^3 = x(x - 1)(x - r)(x - 1 + r)
\]
of curves with automorphism group \( \mathbb{Z}/6 \) and the curve

\[
C_9 : \quad y^3 = x(x^3 - 1)
\]
whose automorphism group is \( \mathbb{Z}/9 \). Let \( \zeta_n \) be the \( n \)-th root of unity in \( \mathbb{C} \). According to [7] and [13], the action of \( \mathbb{Z}/3 \) on \( C_3 \) is given by the transformation \( y \mapsto \zeta_3 \cdot y \). For \( C_6 \), the \( \mathbb{Z}/6 \)-action is defined by \( x \mapsto x - r \) and \( y \mapsto \zeta_3 \cdot y \). For \( C_9 \), the \( \mathbb{Z}/9 \)-action is given by \( x \mapsto \zeta_3 \cdot x \) and \( y \mapsto \zeta_9 \cdot y \).

In the following sections we will compute the invariants and bitangents of \( C_3, C_6 \) and \( C_9 \).

3. Dixmier invariants of \( C_3, C_6 \) and \( C_9 \)

3.1. Dixmier invariants of plane quartics. First, we introduce some notation, following [4]. In general, let \( f \in K[x_1, \ldots, x_n] \) be a polynomial, we use \( D_f \) to denote the differential operator determined by \( f \). Explicitly, let

\[
f = f(x_1, \ldots, x_n) = \sum_{(i_1, \ldots, i_n) \in \mathbb{Z}_+^n} a_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n}, \tag{4}
\]

where \( a_{i_1, \ldots, i_n} \in K \) are coefficient of the monomial \( x_1^{i_1} \cdots x_n^{i_n} \) for \( (i_1, \ldots, i_n) \in \mathbb{Z}_+^n \) and (4) is a finite sum. For the rest of this paper, we will not emphasize that the powers \( i_1, \ldots, i_n \) are non-negative integers again.

The map \( D_f \) means

\[
D_f : \quad K[x_1, \ldots, x_n] \longrightarrow K[x_1, \ldots, x_n]
\]

\[
g(x_1, \ldots, x_n) \longmapsto \sum_{(i_1, \ldots, i_n) \in \mathbb{Z}_+^n} a_{i_1, \ldots, i_n} \frac{\partial^{i_1 + \cdots + i_n}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} g(x_1, \ldots, x_n).
\]

If we use \( D(f, g) \) to denote \( D_f(g) \) for all \( f, g \in K[x_1, \ldots, x_n] \), then the map

\[
D : \quad K[x_1, \ldots, x_n] \times K[x_1, \ldots, x_n] \longrightarrow K[x_1, \ldots, x_n]
\]

has some obvious properties as follows:

- \( D \) is bilinear.
- Let \( \deg(f) \) be the degree of \( f \) for all \( f \in K[x_1, \ldots, x_n] \). Let \( f, g \in K[x_1, \ldots, x_n] \). If \( \deg(f) > \deg(g) \), then \( D_f(g) = 0 \). If \( \deg(f) > \deg(g) \), then \( D_f(g) \leq \deg(g) - \deg(f) \). Let \( f = x_1^{i_1} \cdots x_n^{i_n} \) and
let $g = x_1^{j_1} \cdots x_n^{j_n}$ be two monomials such that $\deg(f) = \deg(g)$, then

$$D_f(g) = i_1! \cdots i_n! \delta_{fg}$$

where $\delta_{fg}$ is the Kronecker delta of $f$ and $g$.

For any $f \in K[x_1, \ldots, x_n]$, let $H(f)$ be the half Hessian matrix of $f$. For example, if $f \in K[x, y, z]$, then

$$(H(f))_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$\\n
Let $H^*(f)$ be the adjoint matrix of $H(f)$.

Another notation is the dot product of two matrices. Let $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n \times n}$ be two $n \times n$ matrices. Then the dot product “$\langle ., . \rangle$” is defined by

$$\langle A, B \rangle := \sum_{1 \leq i, j \leq n} a_{ij}b_{ji}.$$\\n
With these notations, we describe the Dixmier invariants of plane quartics.

Let $f, g \in K[x, y, z]$ be two quadratic homogeneous polynomials. Define

$$J_{1,1}(f, g) = \langle H(f), H(g) \rangle,$$

$$J_{2,2}(f, g) = \langle H^*(f), H^*(g) \rangle,$$

$$J_{3,0}(f, g) = J_{3,0}(f) = \det(H(f)),$$

$$J_{0,3}(f, g) = J_{0,3}(g) = \det(H(g)).$$

Let $F \in K[x, y]_r, G \in K[x, y]_s$ be two homogeneous polynomials of degree $r$ and $s$, respectively. For $k \leq \min\{r, s\}$, define $(F, G)^k$ as

$$\frac{(r-k)!(s-k)!}{r!s!} \left( \frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial y_1 \partial x_2} \right)^k F(x_1, y_1)G(x_2, y_2) \bigg|_{(x_1, y_1) = (x, y), i, 1, 2} \quad (5)$$

Let $P = P(x, y) \in K[x, y]_4$ be a quartic binary form. Let $Q = (P, P)^4$ defined as (5). Also we let

$$\Sigma(P) = \frac{1}{2}(P, P)^4, \quad \Psi(P) = \frac{1}{6}(P, Q)^4$$

$$\Delta(P) = \Sigma(P)^3 - 27\Psi(P)^2 \quad (6)$$

Then $\Delta(P)$ is the discriminant of $P$.

Let $u, v$ be two $K$-variables. For quartic $f \in K[x, y, z]_4$, let

$$g = g(x, y) = f(x, y, -ux - vy).$$

Then $g(x, y)$ is a homogeneous polynomial of degree 4 with respect to the variables $x$ and $y$, and the coefficients of $g$ are expressions of $u$ and $v$. Thus we can define $\Sigma(g)$ and $\Psi(g)$ as in (6). Since $\Sigma$ and $\Psi$ are expressions of the
coefficients, we have $\Sigma(g)$ and $\Psi(g)$ are expressions of $u$ and $v$. An explicit computation shows that $\Sigma(g)$ and $\Psi(g)$ are polynomials of degree 2 and 3 in the polynomial ring $K[u, v]$ respectively. Let $\sigma(u, v, w)$ and $\psi(u, v, w)$ be the homogenization of $\Sigma(g)$ for $w$, and $\psi(u, v, w)$ be the homogenization of $\Psi(g)$ for $w$. Then $\sigma(u, v, w) \in K[u, v, w]$ and $\psi(u, v, w) \in K[u, v, w]$. Finally, we substitute $u = x, v = y, w = z$ into $\sigma(u, v, w)$ and $\psi(u, v, w)$. For $f \in K[x, y, z]$, we define

$$
\sigma(f) = \sigma(x, y, z) \in K[x, y, z]
$$

(7)

$$
\psi(f) = \psi(x, y, z) \in K[x, y, z]
$$

Definition 3.1. Let $f \in K[x, y, z]$, let $\sigma, \psi$ defined as in (7). Let $\rho = Df(\psi)$ and $\tau = D\rho(f)$. The Dixmier invariants are defined as

$$
I_3 = D\sigma(f), \quad I_9 = J_{1,1}(\tau, \rho), \quad I_{15} = J_{3,0}(\tau),
$$

$$
I_6 = D\psi(H) - 8I_3^2, \quad I_{12} = J_{0,3}(\rho), \quad I_{18} = J_{2,2}(\tau, \rho)
$$

(8)

3.2. The Dixmier invariants of $C_3$, $C_6$ and $C_9$. We use Maxima to compute the Dixmier invariants of $C_3$, $C_6$ and $C_9$. And we use elimination in Macaulay2 to compute the conditions of the invariants with certain automorphisms.

Proposition 3.2. The Dixmier invariants of $C_3(r, s) : y^3 = x(x - 1)(x - r)(x - s)$ are

$$
I_3 = I_6 = I_{12} = I_{15} = 0
$$

$$
I_9 = \frac{s^3}{774} - \frac{s^2}{169} + \frac{s}{48922361856} - \frac{1}{97844723712} + \frac{1}{1207959552} + \frac{1}{5296}
$$

$$
I_6 = \frac{s^3}{774} - \frac{s^2}{169} + \frac{s}{48922361856} - \frac{1}{97844723712} + \frac{1}{1207959552} + \frac{1}{5296}
$$

$$
I_{12} = \frac{s^3}{774} - \frac{s^2}{169} + \frac{s}{48922361856} - \frac{1}{97844723712} + \frac{1}{1207959552} + \frac{1}{5296}
$$

$$
I_{15} = \frac{s^3}{774} - \frac{s^2}{169} + \frac{s}{48922361856} - \frac{1}{97844723712} + \frac{1}{1207959552} + \frac{1}{5296}
$$

$$
I_{18} = \frac{s^3}{774} - \frac{s^2}{169} + \frac{s}{48922361856} - \frac{1}{97844723712} + \frac{1}{1207959552} + \frac{1}{5296}
$$

$$
I_{27} = \Delta = \sigma^3 - 27\psi^2
$$
invariants of $C$ is the 0 ideal, which shows that $I$ are all zero.

The Dixmier invariants of $C$ are algebraically independent.

We can compute the invariants of $C_6$ by substitute $s = 1 - r$ into the invariants of $C_3$.

**Proposition 3.3.** The Dixmier invariants of

$$C_6(r): y^3 = x(x-1)(x-r)(x-1+r)$$

are

$$I_3 = I_6 = I_{12} = I_{15} = 0$$

$$I_9 = \frac{-65r^8-260r^7+1150r^6-2540r^5+3959r^4+2326r^3-712r+89}{331776}$$

$$I_{18} = \frac{25}{105737r^{11}} + \frac{25}{3057647616} + \frac{1325}{382205952} + \frac{1925}{3057647616} + \frac{79229}{1019215872} + \frac{12230590464}{817465r^4}$$

The elimination of the ideal generated by $I_9$ and $I_{18}$ with respect to $r$ is irreducible and generated by

$$40000000I_9^3 - 1998092052000I_9^7 - 676000000I_{18}^3 - 71500953768117831I_9^9 + 224328787434000I_{18}^3 - 3953613122539196237346I_9^9 + 8460248600243212740I_{18}^3 - 3873235651553250I_9^9 - 1206702500000I_{18}^3 + 36392104317997507611465I_9^9 + 31914880192757153442492I_{18}^3 - 33293697043611661050I_{18}^3 + 103850637726127500I_{18}^3 + 12745792515625I_{18}^4 - 826890695963630262273456I_9^9 - 98754399642472756663003440I_{18}^3 - 644187721569909674246640I_{18}^3 + 34362752394549791982000I_{18}^3 - 168880832609781468337056I_9^9 + 30826420907787244648372032I_{18}^3 + 474410438868202394564990304I_{18}^3 + 2545539129474384804480I_9 + 6939213188282316797541120I_{18}^3 + 9606056565900794374400.$$

For $C_9$, we have

**Proposition 3.4.** The Dixmier invariants of

$$C_9: y^3 = x(x^3 - 1)$$

are all zero.
4. The Bitangents of $C_3$, $C_6$ and $C_9$

4.1. The bitangents of plane quartics. The classical theory of plane quartics says that it has 28 bitangents. Recall that a line $L$ is a bitangent of a plane curve $C$ if it tangents $C$ at two points $p_1, p_2$ where $p_1$ and $p_2$ could coincide. Recall that a point is called an undulation point (see [21]) of a plane curve if a tangent line at that point meets the curve with multiplicity four or higher, this time the tangent line is called an undulation line of the curve. Thus, if $p_1$ and $p_2$ are coincide, then this point is an undulation point of $C$ and $L$ is an undulation line.

Explicitly, let $f = f(x, y, z) \in K[x, y, z]_4$ be the equation of a plane quartic $C$. Let $L : ax + by + cz = 0$, $a, b, c \in K$ be a line in $\mathbb{P}^2_{x,y,z}$. Thus the point $(a, b, c) \in \mathbb{P}^2_{x,y,z}$ determines the line $L$. Thus, in order to find all bitangents, we should consider all the affine charts $a \neq 0$, $b \neq 0$ and $c \neq 0$. For example, if we consider $c \neq 0$, and say $c = 1$. This time $L : ax + by + z = 0$ gives the condition $z = -ax - by$. Substitute this relation into $f(x, y, z)$ we have a quadratic form $f(x, y, -ax - by) \in R[x, y, z]_2$ where $R = K[a, b]$. If $L$ is a bitangent for some $a, b \in K$, then there exist $\lambda_0, \lambda_1, \lambda_2 \in K$ such that

$$f(x, y, -ax - by) = (\lambda_0 x^2 + \lambda_1 xy + \lambda_2 y^2)^2. \quad (9)$$

The other two affine charts $a \neq 0$, $b \neq 0$ should be considered in a similar way to find bitangents of the form $ax + by = 0$. From now on let us consider the equation (9).

**Definition 4.1.** For any quartic $f \in K[x, y, z]_4$, let $I(f)$ be the ideal of $K[a, b, \lambda_0, \lambda_1, \lambda_2]$ generated by comparing the coefficients of both sides of the monomials of $x, y$ in the expansion of (9). Let $J(f)$ be elimination ideal of $I$ with respect to $\lambda_0, \lambda_1, \lambda_2$ in $K[a, b]$.

The ideal $J(f)$ gives the conditions of $L$ being a bitangent of $C$. In general one cannot solve $a, b$ over $\mathbb{Q}$, and even there exists $L$ such that $a, b \in \mathbb{Q}$, the tangency points $p_1, p_2$ are not $\mathbb{Q}$-rational points of $C$.

There is a description of the relative positions of the bitangents of $C$. Let $L_1, \ldots, L_{28}$ be the bitangents of $C$, be careful that the number 28 counts the overlaps of the bitangents. Let $L_i, L_j, L_k$, where $i, j, k = 1, \ldots, 28$ are distinct, be a triple of bitangents. For each $L_i, \nu = 1, \ldots, 28$, let $p_{i\nu}, p_{j\nu}$ be the two tangency points of $L_i$ and $C$. Then $L_i, L_j, L_k$ determine 6 points on $C$. Generically a plane conic is determined by 5 points.

**Definition 4.2.** If the 6 points $p_{i1}, p_{i2}, p_{j1}, p_{j2}, p_{k1}, p_{k2}$ lie on a plane conic, then we say the triple $L_i, L_j, L_k$ are syzygetic, or else we say they are asyzygetic.

4.2. The Bitangents of $C_3$, $C_6$ and $C_9$. Before we use the computer to comply the algorithm above, let us observe an obvious bitangent of $C_3(r, s) : y^3 = x(x - 1)(x - r)(x - s)$. 
In the algorithm above, we considered the generic case on the affine chart \( z \neq 0 \). But if we expand \( C_3 \) and homogenize it with respect to \( z \), then we have

\[
F_3(r, s) : rs x z^3 - rs x^2 z^2 - s x^2 z^2 - r x^2 z^2 + y^3 z + s x^3 z + r x^3 z + x^3 z - x^4.
\]

Substitute \( z = 0 \) into (10) we get \( x^4 = (x^2)^2 \), which is a square. Thus \( z = 0 \) is a bitangent of \( C_3 \). To compute the tangent point, we observe that \( x^2 = 0 \) implies that \( x = 0 \). Substitute \( x = 0, z = 0 \) into (10) we get 0. This means that the intersection of \( C_3 \) and the line \( z = 0 \) is the point \((0, y, 0)\), or \((0, 1, 0) \in \mathbb{P}_2(x, y, z)\). This is the only undulation point of \( C_3 \).

In [21], the invariants of a generic plane quartic is constructed in order to determine if it has an undulation point. The expression is the determinant of a \( 21 \times 21 \) matrix. On the other hand, a quartic curve with homogeneous equation \( F(x, y, z) = 0 \) has an undulation point if and only if it could be written as the form

\[
F(x, y, z) = U_1(x, y, z)^4 + V_3(x, y, z)W_1(x, y, z)
\]

where \( U_1 \) and \( W_1 \) are linear forms and \( V_3 \) is a cubic form. But according to (10), let \( U_1 = x, W_1 = z, \) and \( V_3 = x(x - z)(x - rz)(x - sz) - x^4 - y^3 \), then

\[
F_3 = U_1^4 + V_3 W_1.
\]

So \( z = 0 \) is an undulation line of \( C_3 \).

Beyond this undulation line, there are another 27 bitangents of \( C_3 \). Let \( J(C_3) \) be the ideal defined as Definition 4.1. This time the coefficient list becomes \( \mathcal{K}[r, s] \), but we still can define \( J(C_3) \) by the same analogous. We can compute the primary decomposition of \( J(C_3) \) using Macaulay2. The inputs are as the following.

\[
R = \text{QQ}[r, a, b, k_0, k_1, k_2][x, y, z]
\]

\[
f = -r^2*x*z^3+r*x*z^3+r^2*x^2*z^2-r*x^2*z^2-x^2*z^2+y^3*z+2*x^3*z-x^4,
\]

\[
g = (k_0*x^2+k_1*x*y+k_2*y^2)^2
\]

\[
h = \text{substitute}(f, \{z => -a*x-b*y\})
\]

\[
H = h - g
\]

\[
\text{Coe} = \text{coefficients} \ H
\]

\[
L = \text{flatten} \ \text{entries} \ \text{Coe}#1
\]

\[
S = \text{QQ}[r,a,b,k_0,k_1,k_2]
\]

\[
I = \text{ideal} \ L
\]

\[
\psi = \text{map}(S, R)
\]

\[
\phi = \text{map}(R, S)
\]

\[
J = \psi I
\]

\[
E = \text{eliminate}(J, \{k_0, k_1, k_2\})
\]

\[
T = \text{QQ}[r, a, b]
\]

\[
\xi = \text{map}(T, S)
\]

\[
U = \xi E
\]
primaryDecomposition U

The primary decomposition of $J(C_3)$ has two components, one of them is the ideal $< a = 0, b = 0 >$, which gives the undulation line $z = 0$. Another component is irreducible in general. Let $J'$ be this component, and let $J'_a$ be the elimination of $J'$ with respect to $b$. Then one can see that $a$ satisfies the degree 9 equation

$$r^4 s^4 a^9 - 12 r^4 s^3 a^7 - 12 r^3 s^4 a^7 - 8 r^4 s^3 a^6 - 8 r^3 s^4 a^6 - 12 r^3 s^3 a^7 - 8 r^4 s^2 a^6 - 12 r^3 s^3 a^6 - 8 r^2 s^4 a^6 - 30 r^4 s^2 a^5 - 156 r^3 s^3 a^5 + 30 r^2 s^4 a^5 - 8 r^3 s^2 a^6 - 8 r^2 s^3 a^6 + 48 r^4 s^2 a^4 - 96 r^3 s^3 a^4 + 48 r^2 s^4 a^4 - 156 r^3 s^2 a^5 - 156 r^2 s^3 a^5 + 16 r^4 s^2 a^3 - 32 r^3 s^3 a^3 + 16 r^2 s^4 a^3 + 48 r^4 s^2 a^4 - 168 r^3 s^2 a^4 - 168 r^2 s^3 a^4 + 48 r^4 s^3 a^5 + 30 r^2 s^4 a^5 + 68 r^4 s^2 a^5 - 68 r^3 s^2 a^5 - 68 r^2 s^3 a^5 + 68 r^4 s^4 a^5 - 168 r^3 s^2 a^4 - 96 r^3 s^3 a^4 + 24 r^4 s^2 a^5 - 24 r^3 s^2 a^5 - 24 r^2 s^3 a^5 + 24 r^4 s^4 a^5 - 24 r^3 s^3 a^5 - 24 r^2 s^4 a^5 = 0.$$  

which is able to be output by Macaulay2. This equation is irreducible over $\mathbb{Q}$. In the following cases, we try to find explicit bitangents for special cases of $C_3(r, s)$.

**Theorem 4.3.** The curve

$$C_9 : y^3 = x(x^3 - 1) \quad (11)$$

has all 28 explicit equations for the bitangents whose coefficients are radical expressions over $\mathbb{Q}$, the group $\mathbb{Z}/9$ acts on the configuration of the bitangents.

**Proof.** Let $J(C_9)$ be the ideal of $K[a, b]$ defined as Definition 4.1. Let $J'$ be the component of $J(C_9)$ beyond $< a = 0, b = 0 >$. Let $J'_a$ be the elimination of $J'$ with respect to $b$. Then $a$ satisfies the following equation.

$$a^9 - 96 a^6 + 48 a^3 + 64. \quad (12)$$

Let $u = a^3$, then $u$ satisfies the cubic equation

$$u^3 - 96 u^2 + 48 u + 64. \quad (13)$$

This equation is solvable. For example, using Maxima, we have
\[
\begin{align*}
    u_1 &= -\frac{(\sqrt{3}i + 1) \left(32 \cdot 3^2 i + 31968\right)^{\frac{2}{3}} - 64 \left(32 \cdot 3^2 i + 31968\right)^{\frac{1}{3}} - 112 \cdot 3^2 i + 1008}{2 \left(32 \cdot 3^2 i + 31968\right)^{\frac{1}{3}}}, \\
    u_2 &= \frac{(\sqrt{3}i - 1) \left(32 \cdot 3^2 i + 31968\right)^{\frac{2}{3}} + 64 \left(32 \cdot 3^2 i + 31968\right)^{\frac{1}{3}} - 112 \cdot 3^2 i - 1008}{2 \left(32 \cdot 3^2 i + 31968\right)^{\frac{1}{3}}}, \\
    u_3 &= \frac{\left(32 \cdot 3^2 i + 31968\right)^{\frac{2}{3}} + 32 \cdot \left(32 \cdot 3^2 i + 31968\right)^{\frac{1}{3}} + 1008}{\left(32 \cdot 3^2 i + 31968\right)^{\frac{1}{3}}}.
\end{align*}
\]

Taking the cube root of each \(u_i\) we can get all 9 solutions of \(a\).

Similarly we have an equation
\[
    b^{27} - 29496b^{18} + 401808b^9 - 64 = 0 \quad (14)
\]
and let \(v = b^9\) we have a cubic equation
\[
    v^3 - 29496v^2 + 401808v - 64 = 0.
\]
This time one has to take the ninth root of all the three solutions \(v_i\)’s ,
\(i = 1, 2, 3\) of this equation. At the end, one has to judge which pairs \((a, b)\)
among the solutions give a bitangent \(ax + by + z = 0\) of the original curve.

We list the Macaulay2 input as the following.

```plaintext
R = QQ[r,s,b,k_0,k_1,k_2][x,y,z]
f = r*s*x*z^3 - r*s*x^2*z^2 - s*x^2*z^2 - r*x^2*z^2 + y^3*z + s*x^3*z + r*x^3*z + x^3*z - x^4
g = (k_0*x^2 + k_1*x*y + k_2*y^2)^2
h = substitute(f,{z => -b*y})
H= h-g
Coe = coefficients H
L = flatten entries Coe#1
S = QQ[r,s,b,k_0,k_1,k_2]
I = ideal L
psi=map(S,R)
phi=map(R,S)
J = psi I
E=eliminate(J,{k_0,k_1,k_2})
T = QQ[r,s,b]
xi=map(T,S)
U = xi E
primaryDecomposition U
```

The equations (12) and (14) contain terms of degree 3\(n\) and 9\(n\) for \(a\) and \(b\), respectively. Thus if \(ax + by + z = 0\) is a bitangent, so is \(\zeta_{3n} \cdot ax + \zeta_{9n} \cdot by + z = \frac{1}{n} i\).
0. But this means \(a(\zeta_3 \cdot x) + b(\zeta_9 \cdot y) + z = 0\), which means this bitangent is in the orbit of the \(\mathbb{Z}/9\)-action. This means the group \(\mathbb{Z}/9\) acts on the configuration of the bitangents. \(\square\)

There is no canonical method to find explicit bitangents for special cases. Our observation is that we can try to find \(r,s \in K\) such that the bitangent is “horizontal”, that is, for those bitangents such that \(a = 0\). The equation of the bitangent becomes \(bx + z = 0\). Repeat the same idea in Section 4.1, we get the following result.

**Theorem 4.4.** The family \(C_3\) has a horizontal bitangent when \(r-s = \pm 1\) or \(r+s = 1\). In each of these cases, the slope \(b\) satisfies a cubic equation whose coefficients are polynomials of \(s\), thus there are 3 horizontal bitangents.

**Proof.** Let \(F_3\) be the polynomial defined in (10). Generically, \(a = 0\) is not a solution to the degree 9 equation of \(a\). However, when \(a = 0\), we have \(L : by + z = 0\). Then \(z = -by\). Using the same idea as in Section 4.1, we have the equation

\[
F_3(x, y, -by) = (\lambda_0 x^2 + \lambda_1 xy + \lambda_2 y^2)^2. \tag{15}
\]

Let \(I(F_3)\) be the ideal of \(R[b, \lambda_0, \lambda_1, \lambda_2]\) generated by comparing the coefficients of both sides of the monomials of \(x, y\) in the expansion of (15). Let \(J(F_3)\) be elimination ideal of \(I(F_3)\) with respect to \(\lambda_0, \lambda_1, \lambda_2\) in \(R[b]\). Then the primary decomposition of \(J(F_3)\) as an ideal in \(K[r, s, b]\) is

\[
\langle b \rangle, \quad \langle r - s - 1, s^2 b^3 - 4 \rangle
\]

\[
\langle r + s - 1, s^4 b^3 - 2s^3 b^3 + s^2 b^3 - 4 \rangle
\]

\[
\langle r - s + 1, s^2 b^3 - 2s b^3 + b^3 - 4 \rangle. \tag{16}
\]

The first ideal of (16) corresponds to the bitangent \(z = 0\). The third ideal of (16) gives \(r + s - 1 = 0\), which implies \(s = r - 1\), this is the family \(C_6\). Furthermore, we have a result on the positions of the horizontal bitangents of \(C_6\).

**Theorem 4.5.** The three horizontal bitangents of \(C_6\) form an asyzygetic triple. Furthermore, the automorphism group \(\mathbb{Z}/6\) acts on this asyzygetic triple.

**Proof.** Let \(F_6 : -r^2 x z^3 + r x z^3 + r^2 x^2 z^2 - r x^2 z^2 - x^2 z^2 + y^3 z + 2x^3 z - x^4 \in R[x, y, z]\) be the homogenization of \(C_6\) with respect to \(z\) where \(R = K[r]\). As before, we have the equation

\[
F_6(x, y, -by) = (\lambda_0 x^2 + \lambda_1 xy + \lambda_2 y^2)^2 \tag{17}
\]

Let \(I(F_6)\) be the ideal of \(R[b, \lambda_0, \lambda_1, \lambda_2]\) generated by comparing the coefficients of both sides of the monomials of \(x, y\) in the expansion of (17). In
Theorem 4.4 we have proved that for $C_6$ the condition of being a horizontal bitangent for the line $bx + z = 0$ is given by the ideal 

$$\langle r + s - 1, s^4b^3 - 2s^3b^3 + s^2b^3 - 4 \rangle.$$

Substitute $s = 1 - r$ into the second generator of this ideal, we have a relation 

$$p(r, b) = b^3 r^4 - 2b^3 r^3 + b^3 r^2 - 4$$

This time, let $\mathcal{J}(F_6)$ be intersection of the elimination ideal of $I(F_6)$ with respect to $r, b$ in $K[\lambda_0, \lambda_1, \lambda_2]$ and the ideal $(p(r, b))$. Macaulay2 outputs 

$$\mathcal{J}(F_6) = \langle \rangle,$$

which means that generically there is no conic $\lambda_0x^2 + \lambda_1xy + \lambda_2y^2$ satisfies the conditions of passing through the 6 tangent points at the same time.

Consider the action $x \mapsto -x - r$ and $y \mapsto \zeta_3 \cdot y$ on $C_6$. The equation $p(r, b)$ only contains degree $3n$ terms, so as we have seen, the transformation $y \mapsto \zeta_3 \cdot y$ will transform a bitangent to another. On the other hand, since $a = 0$, a transformation $x \mapsto -x - r$ will fix a horizontal bitangent $z = -by$. Thus $\mathbb{Z}/6$ acts on the configuration of this asyzygetic triple. \hfill \Box

**Remark 4.6.** In general, there is another way to check whether 6 points lie on a common conic in $\mathbb{P}^2$. Let $p_i = (x_i, y_i, z_i) \in \mathbb{F}_3(x, y, z)$, $i = 1, \ldots, 6$ be 6 points in the projective plane. Let $V$ be the Veronese map 

$$V : \mathbb{P}^2(x, y, z) \longrightarrow \mathbb{P}^5$$

$$(x, y, z) \longmapsto (x^2, y^2, z^2, xy, yz, zx).$$

If we regard $V(p)$ as a row matrix for any $p = (x, y, z) \in \mathbb{P}^2$, then for the given 6 points $p_1, \ldots, p_6$, we have a $6 \times 6$ matrix 

$$V := \begin{pmatrix} V(p_1) \\ V(p_2) \\ V(p_3) \\ V(p_4) \\ V(p_5) \\ V(p_6) \end{pmatrix} = \begin{pmatrix} x_1^2 & y_1^2 & z_1^2 & x_1y_1 & y_1z_1 & z_1x_1 \\ x_2^2 & y_2^2 & z_2^2 & x_2y_2 & y_2z_2 & z_2x_2 \\ x_3^2 & y_3^2 & z_3^2 & x_3y_3 & y_3z_3 & z_3x_3 \\ x_4^2 & y_4^2 & z_4^2 & x_4y_4 & y_4z_4 & z_4x_4 \\ x_5^2 & y_5^2 & z_5^2 & x_5y_5 & y_5z_5 & z_5x_5 \\ x_6^2 & y_6^2 & z_6^2 & x_6y_6 & y_6z_6 & z_6x_6 \end{pmatrix}.$$

For our problem, let $p_1, \ldots, p_6$ be the 6 points of tangency of the three horizontal bitangents in Theorem 4.5. From the proof of Theorem 4.5 we see that there is a symbolic solution of these three bitangents, and since the algorithm of finding the points of tangency is essentially solving a quadratic equation, we can find the symbolic solutions of the points of tangency. But this algorithm costs too much for a popular processor. We can compute it in special values. For example, let $r = \frac{1}{8}$, we can compute the determinant using Maxima, the result is 

$$V = -\frac{\sqrt{25\sqrt{3}i - 25 \sqrt{25\sqrt{3}i - 25}}}{2^{\frac{37}{2}}\sqrt{\frac{3}{1} + 2^{\frac{37}{2}}}} \left(120052^{\frac{10}{7}} 3^{\frac{3}{2}} 4^{\frac{2}{3}} i + 3241352^{\frac{10}{7}} 4^{\frac{2}{3}} \right)$$

\footnote{This value could be simplified, we put the original result from Maxima.}
which is not zero.

5. Discussion on the matrix representation problem

We discuss the matrix representation problem of the curves \( C_3 \) and \( C_6 \) using the idea in [20]. In order to coincide the notations with respect to [20], we exchange \( y \) and \( z \), and write \( C_3 \) as

\[
C_3 : \quad z^3 = x(x-1)(x-r)(x-s).
\]

Homogenize \( C_3 \) with respect to \( y \) we have

\[
C_3 : \quad f(x,y,z) := F_3(r,s) = x(x-y)(x-ry)(x-sy) - yz^3 = 0. \tag{18}
\]

This time we have

\[
f(x,0,0) = x^4 \quad \text{and} \quad f(x,y,0) = \prod_{i=1}^{4} (x + \beta_i y) \tag{19}
\]

where \( \beta_1 = 0, \beta_2 = -1, \beta_3 = -r, \beta_4 = -s \). The matrix representation problem for \( C_3 \) asks whether the polynomial \( f(x,y,z) \) in (18) could be written of the form

\[
f(x,y,z) = \det(xA + yB + zC)
\]

where \( A, B, C \) are symmetric matrices. Here the entries of the matrices \( A, B \) and \( C \) belong to the algebraic closure of the rational function field \( K(r,s) \). According to Section 2 in [20], if (19) holds, then one can assume that

\[
A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 & -r & -s \\ -1 & -1 & -1 & -1 \\ -r & -r & -r & -r \\ -s & -s & -s & -s \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{12} & c_{22} & c_{23} & c_{24} \\ c_{13} & c_{23} & c_{33} & c_{34} \\ c_{14} & c_{24} & c_{34} & c_{44} \end{pmatrix}.
\]

and we also have that

\[
c_{ii} = \beta_i \cdot \frac{\partial f}{\partial y}(-\beta_i,1,0) = \frac{\partial f}{\partial y}(-\beta_i,1,0), \quad i = 1, 2, 3, 4. \tag{20}
\]

But for (18) we have \( \frac{\partial f}{\partial z} = -3yz^2 \), which implies if \( z = 0 \), then \( c_{ii} = 0 \) for \( i = 1, 2, 3, 4 \) by (20).

For convinience we denote

\[
D = \begin{pmatrix} c_{12} & c_{13} & c_{14} \\ c_{23} & c_{24} & c_{34} \\ c_{34} & c_{34} & c_{34} \end{pmatrix} = \begin{pmatrix} a & b & d \\ c & e & f \end{pmatrix},
\]

then \( C = D + \text{^t}D \) where \( \text{^t}D \) is the matrix transpose of \( D \) since \( c_{ii} = 0 \) for \( i = 1, 2, 3, 4. \)
Using Maxima, we directly compute the coefficients of

\[
\det(xA + yB + zC) = \det \begin{pmatrix}
x & az & bz & dz \\
az & x - y & cz & ez \\
bz & cz & x - ry & fz \\
dz & ez & fz & x - sy
\end{pmatrix}
\]

and compare the coefficients with \( f(x, y, z) \) in (18), the output is a system of equations

\begin{align*}
-c^2s - b^2s - a^2s - e^2r - d^2r - a^2r - f^2 - d^2 - b^2 &= 0, \\
a^2rs + b^2s + d^2r &= 0, \\
2abcs + 2ader + 2bdf - 1 &= 0, \\
f^2 + e^2 + d^2 + c^2 + b^2 + a^2 &= 0, \\
-2cef - 2bdf - 2ade - 2abc &= 0, \\
-a^2f^2 + 2abf + 2acdf - b^2e^2 + 2bcde - c^2d^2 &= 0.
\end{align*}

We add the first equation with the fourth one, and rewrite the system of as 6 equations

\begin{align*}
a^2rs + b^2s + d^2r &= 0, \\
2abcs + 2ader + 2bdf - 1 &= 0, \\
f^2 + e^2 + d^2 + c^2 + b^2 + a^2 &= 0, \\
-2cef - 2bdf - 2ade - 2abc &= 0, \\
-a^2f^2 + 2abf + 2acdf - b^2e^2 + 2bcde - c^2d^2 &= 0.
\end{align*}

of the 6 variables \( a, b, c, d, e, f \).

It is too complicated to solve this entire system. Our computation are proceeded under the following principle:

- We only seek for one solution to the equation system (27)-(32), thus if there is an “either-or” argument in any step, we can choose one of them as our solution.

We eliminate \( a, f \), and get a system of 4 equations with respect to the 4 variables \( b, c, d, e \).
Proposition 5.1. The equation system
\[
\frac{b^2}{r} + \frac{c^2}{r-1} + \frac{d^2}{s} + \frac{e^2}{s-1} = 0, \tag{33}
\]
\[
\frac{(b-e)^4}{be} = \frac{(e+d)^4}{cd}, \tag{34}
\]
\[
(bc+de)(bd+ce) = \left(2(\sqrt{bd} + \sqrt{cd}) \cdot \left| \begin{array}{cc}
bc + de & bd \\
bc(1-s) + de(1-r) & ce
\end{array} \right| \right)^2, \tag{35}
\]
\[
\frac{(bd+ce)^2 + (bc+de)^2}{bc(1-s) + de(1-r)} \cdot \frac{bc}{ce} + (b^2 + c^2 + d^2 + e^2) = 0 \tag{36}
\]
with respect to the variables \(b, c, d, e\) give solutions to the equation system (27)-(32) where
\[
a = \begin{vmatrix}
bd + ce \\
bc + de
\end{vmatrix}, \quad f = -\begin{vmatrix}
bd + ce \\
bc + de
\end{vmatrix}. \tag{37}
\]

Proof. First, the equation (33) is simply from \(\frac{1}{rs}(27) - \frac{1}{(1-r)(1-s)}(28)\).

Next we regard \(a, f\) as unknowns and \(b, c, d, e, r, s\) as constants. The solution (37) is the solution to the linear system (29) and (31). From (31) we also have
\[
\frac{a}{f} = \frac{bd + ce}{bc + de}, \quad \frac{f}{a} = -\frac{bc + de}{bd + ce}. \tag{38}
\]
Substitute (38) into (30) we have (36).

Let \(g = af\), then (32) becomes a quadratic equation
\[
g^2 - 2(bc + cd)g + (bc - cd)^2 = 0
\]
of \(g\) whose solution is
\[
a = \frac{(\sqrt{bd} \pm \sqrt{cd})^2}{\sqrt{be} + \sqrt{cd}}
\]
As before, for “±” we choose +, which is
\[
a = \frac{(\sqrt{bd} \pm \sqrt{cd})^2}{\sqrt{be} + \sqrt{cd}} \tag{39}
\]
Substitute (37) into (39) we get (35).

Last, let us prove (34). The quadratic equation (30) and the linear equation (31) have an solution
\[
a = -\frac{\sqrt{-1}(ce + bd) \sqrt{e^2 + d^2 + c^2 + b^2} \sqrt{(d^2 + c^2)} c^2 + 4bcde + b^2 d^2 + b^2 c^2}{(d^2 + c^2)} e^2 + 4bcde + b^2 d^2 + b^2 c^2, \tag{40}
\]
\[
f = \frac{\sqrt{-1}(bc + de) \sqrt{e^2 + d^2 + c^2 + b^2} \sqrt{(d^2 + c^2)} e^2 + 4bcde + b^2 d^2 + b^2 c^2}{(d^2 + c^2)} e^2 + 4bcde + b^2 d^2 + b^2 c^2.
From (40) we have

\[ af = \frac{P}{(d^2 + c^2) e^2 + 4bcde + b^2 d^2 + b^2 c^2} \]

where the numerator \( P \) equals to minus the product of

\[ cde^4 - 4bcde^3 + 6b^2 cde^2 - 6bd^2 e - 6bc^3 d e - 4bc^3 de - 4b^3 cde - bc^4 e + b^3 cd \] (41)

and

\[ cde^4 + 4bcde^3 + 6b^2 cde^2 - 6bd^2 e + 4bc^3 d e - 4bc^3 de + 4b^3 cde - bc^4 e + b^3 cd. \]

Dividing (41) by \( bcde \) and regrouping the terms, we prove (34). \qed

As we reminded, it is hard to continue solving this equation system. Our observation is that for (34), we have an obvious solution

\[ e = b, \quad d = -c. \] (42)

From (24) we have \( b^2 + d^2 + f^2 = -a^2 - c^2 - e^2 \), thus we can rewrite (21) as

\[ (a^2 + b^2 + d^2)r + (a^2 + b^2 + c^2)s = a^2 + c^2 + e^2. \]

Substitute (42) into this equation we have

\[ r + s = 1 \]

which means the curve \( C_3 \) becomes \( C_6 \) in this situation.

Next, we substitute (42) into the equation system (27)-(32), then (31) is trivial, and (27) is the same as (28). We have a system of 4 equations

\[ a^2 rs + b^2 s + c^2 r = 0 \] (43)
\[ 2abc(s - r) - 2bcf - 1 = 0 \] (44)
\[ a^2 + f^2 + 2(b^2 + c^2) = 0 \] (45)
\[ a^2 f^2 - 2af(b^2 - c^2) + (b^2 + c^2)^2 = 0 \] (46)

of the 4 variables \( a, b, c, f \).

**Theorem 5.2.** The matrix representation of \( C_6 \) could be explicitly written over an extension field of \( K(r,s) = \overline{\mathbb{Q}}(r,s) \) defined by a degree 6 polynomial \( f(z) \in K(r,s)[z] \).

**Proof.** From (44) we have

\[ (a(s - r) - f) = \frac{1}{2bc}, \]

thus we have

\[ a^2(s - r)^2 - 2af(s - r) + f^2 = \frac{1}{4b^2 e^2}. \] (47)

From (45) we have \( f^2 = -2(b^2 + c^2) - a^2 \), substitute it into (47) we have

\[ a^2[(s - r)^2 - 1] - 2af(s - r) - 2(b^2 + c^2) = \frac{1}{4b^2 e^2}. \] (48)
From (43) we have
\[ a^2 = -\frac{b^2}{r} - \frac{c^2}{s} \]  
(49)
and from (46) we have
\[ af = (b + \sqrt{-1}c)^2 \]  
(50)
if we take one of the solutions of the quadratic equation with respect to \( af \).
Substitute them into (48), we have
\[ 4(b^2s + c^2r) - 2(s-r)(b + \sqrt{-1}c)^2 - 2(b^2 + c^2) = \frac{1}{4b^2c^2}. \]  
(51)
This is a degree 6 equation with respect to \( b \) and \( c \). Thus, if we know \( q = b/c \),
then the theorem is proved. From (45) and (49) we can solve
\[ f^2 = -2(b^2 + c^2) + \frac{b^2}{r} + \frac{c^2}{s}. \]  
(52)
The trivial equation
\[ (af)^2 = a^2 \cdot f^2 \]
implies that \((50)^2 = (49) \cdot (52)\), which is
\[ (b + \sqrt{-1}c)^4 = \left( -\frac{b^2}{r} - \frac{c^2}{s} \right) \cdot \left( -2(b^2 + c^2) + \frac{b^2}{r} + \frac{c^2}{s} \right) \]  
(53)
This equation is homogeneous of degree 4 with respect to \( b \) and \( c \), thus if we set \( q = b/c \), it will become a degree 4 equation of \( q \), which is solvable. \( \square \)

References


(Dun Liang) School of Mathematics and Statistics, Hengyang Normal University, Hengyang, Hunan 421002, China

liangdun@hynu.edu.cn

This paper is available via http://nyjm.albany.edu/j/2020/26-29.html.