A construction of pseudo-Anosov braids with small normalized entropies

Susumu Hirose and Eiko Kin

Abstract. Let $b$ be a pseudo-Anosov braid whose permutation has a fixed point and let $M_b$ be the mapping torus by the pseudo-Anosov homeomorphism defined on the genus 0 fiber $F_b$ associated with $b$. We prove that there is a 2-dimensional subcone $C_0$ contained in the fibered cone $C$ of $F_b$ such that the fiber $F_\alpha$ for each primitive integral class $\alpha \in C_0$ has genus 0. We also give a constructive description of the monodromy $\phi_\alpha : F_\alpha \rightarrow F_\alpha$ of the fibration on $M_b$ over the circle, and consequently provide a construction of many sequences of pseudo-Anosov braids with small normalized entropies. As an application we prove that the smallest entropy among skew-palindromic braids with $n$ strands is comparable to $1/n$, and the smallest entropy among elements of the odd/even spin mapping class groups of genus $g$ is comparable to $1/g$.

Contents

1. Introduction 563
2. Preliminaries 568
3. $i$-increasing braids and Theorem 3.2 572
4. Proof of Theorem 3.2 575
5. Sequences of pseudo-Anosov braids with small normalized entropies 583
6. Stable foliation for the monodromy 584
7. Properties of $F$-surfaces and $E$-surfaces 586
8. Application 589
References 595

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1. Introduction

Let $\Sigma = \Sigma_{g,n}$ be an orientable surface of genus $g$ with $n$ punctures for $n \geq 0$. We set $\Sigma_g = \Sigma_{g,0}$. By mapping class group $\text{Mod}(\Sigma_g)$, we mean the group of isotopy classes of orientation preserving self-homeomorphisms on $\Sigma_{g,n}$ preserving punctures setwise. By Nielsen-Thurston classification, elements in $\text{Mod}(\Sigma)$ are classified into three types: periodic, reducible, pseudo-Anosov [30, 9]. For $\phi \in \text{Mod}(\Sigma)$ we choose a representative $\Phi \in \phi$ and consider the mapping torus $M_\phi = \Sigma \times \mathbb{R} / \sim$, where $\sim$ identifies $(x, t + 1)$ with $(\Phi(x), t)$ for $x \in \Sigma$ and $t \in \mathbb{R}$. Then $\Sigma$ is a fiber of a fibration on $M_\phi$ over the circle $S^1$ and $\phi$ is called the monodromy. A theorem by Thurston [31] asserts that $M_\phi$ admits a hyperbolic structure of finite volume if and only if $\phi$ is pseudo-Anosov.

For a pseudo-Anosov element $\phi \in \text{Mod}(\Sigma)$ there is a representative $\Phi : \Sigma \to \Sigma$ of $\phi$ called a pseudo-Anosov homeomorphism with the following property: $\Phi$ admits a pair of transverse measured foliations $(F^u, \mu^u)$ and $(F^s, \mu^s)$ and a constant $\lambda = \lambda(\phi) > 1$ depending on $\phi$ such that $F^u$ and $F^s$ are invariant under $\Phi$, and $\mu^u$ and $\mu^s$ are uniformly multiplied by $\lambda$ and $\lambda^{-1}$ under $\Phi$. The constant $\lambda(\phi)$ is called the dilatation and $F^u$ and $F^s$ are called the unstable and stable foliation. We call the logarithm $\log(\lambda(\phi))$ the entropy, and call

$$\text{Ent}(\phi) = |\chi(\Sigma)| \log(\lambda(\phi))$$

the normalized entropy of $\phi$, where $\chi(\Sigma)$ is the Euler characteristic of $\Sigma$. Such normalization of the entropy is suited for the context of 3-manifolds [8, 22].

Penner [27] proved that if $\phi \in \text{Mod}(\Sigma_{g,n})$ is pseudo-Anosov, then

$$\frac{\log 2}{12g - 12 + 4n} \leq \log(\lambda(\phi)).$$

(1.1)

See also [22, Corollary 2]. For a fixed surface $\Sigma$, the set

$$\{ \log \lambda(\phi) \mid \phi \in \text{Mod}(\Sigma) \text{ is pseudo-Anosov} \}$$

is a closed, discrete subset of $\mathbb{R}$ ([1]). For any subgroup or subset $G \subset \text{Mod}(\Sigma)$ let $\delta(G)$ denote the minimum of $\lambda(\phi)$ over all pseudo-Anosov elements $\phi \in G$. Then $\delta(G) \geq \delta(\text{Mod}(\Sigma))$. We write $f \asymp h$ if there is a universal constant $P > 0$ such that $1/P \leq f/h \leq P$. It is proved by Penner [27] that the minimal entropy among pseudo-Anosov elements in $\text{Mod}(\Sigma_g)$ on the closed surface of genus $g$ satisfies

$$\log \delta(\text{Mod}(\Sigma_g)) \asymp \frac{1}{g}.$$ 

See also [16, 32, 33] for other sequences of mapping class groups.

For any $P > 0$, consider the set $\Psi_P$ consisting of all pseudo-Anosov homeomorphisms $\Phi : \Sigma \to \Sigma$ defined on any surface $\Sigma$ with the normalized entropy $|\chi(\Sigma)| \log \lambda(\Phi) \leq P$. This is an infinite set in general (take $P > 2 \log(2 + \sqrt{3})$ for example) and is well-understood in the context of
hyperbolic fibered 3-manifolds. The universal finiteness theorem by Farb-Leininger-Margalit [8] states that the set of homeomorphism classes of mapping tori of pseudo-Anosov homeomorphisms $\Phi^\circ : \Sigma^0 \to \Sigma^0$ is finite, where $\Phi^\circ : \Sigma^0 \to \Sigma^0$ is the fully punctured pseudo-Anosov homeomorphism obtained from $\Phi \in \Psi_P$. (Clearly $\lambda(\Phi^\circ) = \lambda(\Phi)$.) In other words such $\Phi^\circ : \Sigma^0 \to \Sigma^0$ is a monodromy of a fiber in some fibered cone for a hyperbolic fibered 3-manifold in the finite list determined by $P$. Thus 3-manifolds in the finite list govern all pseudo-Anosov elements in $\Psi_P$. It is natural to ask the dynamics and a constructive description of elements in $\Psi_P$. There are some results about this question by several authors [4, 15, 20, 21, 33], but it is not completely understood. In this paper we restrict our attention to the pseudo-Anosov elements in $\Psi_P$ defined on the genus 0 surfaces, and provide an approach for a concrete description of those elements.

Let $B_n$ be the braid group with $n$ strands. The group $B_n$ is generated by the braids $\sigma_1, \ldots, \sigma_{n-1}$ as in Figure 1. Let $S_n$ be the symmetric group, the group of bijections of $\{1, \ldots, n\}$ to itself. A permutation $\mathcal{P} \in S_n$ has a fixed point if $\mathcal{P}(i) = i$ for some $i$. We have a surjective homomorphism $\pi : B_n \to S_n$ which sends each $\sigma_j$ to the transposition $(j, j + 1)$.
The closure $\text{cl}(b)$ of a braid $b \in B_n$ is a knot or link in the 3-sphere $S^3$. The braided link
$$\text{br}(b) = \text{cl}(b) \cup A$$
is a link in $S^3$ obtained from $\text{cl}(b)$ with its braid axis $A$ (Figure 2). Let $M_b$ denote the exterior of $\text{br}(b)$ which is a 3-manifold with boundary. It is easy to find an $(n+1)$-holed sphere $F_b$ in $M_b$ (Figure 2(3)). Clearly $F_b$ is a fiber of a fibration on $M_b \to S^1$ and its monodromy $\phi_b : F_b \to F_b$ is determined by $b$. We call $F_b$ the $F$-surface for $b$.

A braid $b \in B_n$ is periodic (resp. reducible, pseudo-Anosov) if the associated mapping class $f_b \in \text{Mod}(\Sigma_{0,n+1})$ is of the corresponding type (Section 2.3). If $b$ is pseudo-Anosov, then the dilatation $\lambda(b)$ is defined by $\lambda(f_b)$ and the normalized entropy $\text{Ent}(b)$ is defined by $\text{Ent}(f_b)$. The following theorem is due to Hironaka-Kin [16, Proposition 3.36] together with the observation by Kin-Takasawa [21, Section 4.1].

**Theorem 1.1.** There is a sequence of pseudo-Anosov braids $z_n \in B_n$ such that $\text{Ent}(z_n) \neq 2 \log(2 + \sqrt{3})$, $M_{z_n} \simeq M_{\sigma_1^2 \sigma_2^{-1}}$ for each $n \geq 3$ and $\text{Ent}(z_n) \to 2 \log(2 + \sqrt{3})$ as $n \to \infty$.

Here $\simeq$ means they are homeomorphic to each other. The limit point $2 \log(2 + \sqrt{3})$ is equal to $\text{Ent}(\sigma_1^2 \sigma_2^{-1})$. By the lower bound (1.1), Theorem 1.1 implies that
$$\log \delta(\text{Mod}(\Sigma_{0,n})) \asymp \frac{1}{n}.$$In particular, the hyperbolic fibered 3-manifold $M_{\sigma_1^2 \sigma_2^{-1}}$ admits an infinitely family of genus 0 fibers of fibrations over $S^1$.

Let $z_n$ be a pseudo-Anosov braid with $d_n$ strands. We say that a sequence $\{z_n\}$ has a small normalized entropy if $d_n \asymp n$ and there is a constant $P > 0$ which does not depend on $n$ such that $\text{Ent}(z_n) \leq P$. By (1.1) a sequence $\{z_n\}$ having a small normalized entropy means $\log(\lambda(z_n)) \asymp 1/n$. One of the aims in this paper is to give a construction of many sequences of pseudo-Anosov braids with small normalized entropies. The following result generalizes Theorem 1.1.

**Theorem A.** Suppose that $b$ is a pseudo-Anosov braid whose permutation has a fixed point. There is a sequence of pseudo-Anosov braids $\{z_n\}$ with small normalized entropy such that $\text{Ent}(z_n) \to \text{Ent}(b)$ as $n \to \infty$ and $M_{z_n} \simeq M_b$ for $n \geq 1$.

The proof of Theorem A is constructive. In fact one can describe braids $z_n$ explicitly. For a more general result see Theorems 5.1, 5.2. Let $C \subset H_2(M_b, \partial M_b)$ be the fibered cone containing $[F_b]$. A theorem by Thurston [29] states that for each primitive integral class $a \in C$ there is a connected fiber $F_a$ with the pseudo-Anosov monodromy $\phi_a : F_a \to F_a$ of a fibration on the hyperbolic 3-manifold $M_b$ over $S^1$. The following theorem states a structure of $C$. 
Theorem B. Suppose that \( b \) is a pseudo-Anosov braid whose permutation has a fixed point. Then there are a 2-dimensional subcone \( C_0 \subset C \) and an integer \( u \geq 1 \) with the following properties.

1. The fiber \( F_a \) for each primitive integral class \( a \in C_0 \) has genus 0.
2. The monodromy \( \phi_a : F_a \to F_a \) for each primitive integral class \( a \in C_0 \) is conjugate to
   \[
   (\omega_1 \psi) \cdots (\omega_{u-1} \psi)(\omega_u \psi)\psi^{m-1} : F_a \to F_a,
   \]
   where \( m \geq 1 \) depends on the class \( a \), \( \psi \) is periodic and each \( \omega_j \) is reducible. Moreover there are homeomorphisms \( \hat{\omega}_j : S_0 \to S_0 \) on a surface \( S_0 \) for \( j = 1, \ldots, u \) determined by \( b \) and an embedding \( h : S_0 \hookrightarrow F_a \) such that \( h(S_0) \) is the support of each \( w_j \) and
   \[
   w_j|_{h(S_0)} = h \circ \hat{\omega}_j \circ h^{-1}.
   \]

Theorem B gives a constructive description of \( \phi_a \). Also it states that each \( w_j : F_a \to F_a \) is reducible supported on a uniformly bounded subsurface \( h(S_0) \subset F_a \). It turns out from the proof that the type of the periodic homeomorphism \( \psi : F_a \to F_a \) does not depend on \( a \in C_0 \) (Remark 3.3), see Figure 3(1). Theorem B reminds us of the symmetry conjecture in [23] by Farb-Leininger-Margalit.

Clearly the permutation of each pure braid has a fixed point. For any pseudo-Anosov braid \( b \), a suitable power \( b^k \) becomes a pure braid and one can apply Theorems A, B for \( b^k \).

We have a remark about Theorem A. While the existence of a sequence \((F_n, \phi_n)\) of fibers and monodromies in \( C \) for which \( \text{Ent}(\phi_n) \to \text{Ent}(b) \) is guaranteed by McMullen [25, Theorem 10.2], it does not say anything about the genera of fibers \( F_n \). Theorem B has the extra (constructive) information that each fiber \( F_n \) along \( C_0 \) is genus 0.
A CONSTRUCTION OF PSEUDO-ANOSOV BRAIDS

Figure 4. Illustration of braids (1) $b$, (2) $\text{rev}(b)$, (3) $\text{skew}(b)$.

Figure 5. (1) $\mathcal{I} : \Sigma_g \to \Sigma_g$. (2) A basis $\{x_1, y_1, \ldots, x_g, y_g\}$ of $H_1(\Sigma_g; \mathbb{Z}_2)$.

As an application we will determine asymptotic behaviors of the minimal dilatations of a subset of $B_n$ consisting of braids with a symmetry. A braid $b \in B_n$ is palindromic if $\text{rev}(b) = b$, where $\text{rev} : B_n \to B_n$ is a map such that if $w$ is a word of letters $\sigma_j^{\pm 1}$ representing $b$, then $\text{rev}(b)$ is the braid obtained from $b$ reversing the order of letters in $w$. A braid $b \in B_n$ is skew-palindromic if $\text{skew}(b) = b$, where $\text{skew}(b) = \Delta \text{rev}(b) \Delta^{-1}$ and $\Delta$ is a half twist (Section 2.2). See Figure 4. We will prove that dilatations of palindromic braids have the following lower bound.

**Theorem C.** If $b \in B_n$ is palindromic and pseudo-Anosov for $n \geq 3$, then

$$\lambda(b) \geq \sqrt{2 + \sqrt{5}}.$$

In contrast with palindromic braids we have the following result.

**Theorem D.** Let $PA_n$ be the set of skew-palindromic elements in $B_n$. We have

$$\log \delta(PA_n) \propto \frac{1}{n}.$$

The hyperelliptic mapping class group $\mathcal{H}(\Sigma_g)$ is the subgroup of $\text{Mod}(\Sigma_g)$ consisting of elements with representative homeomorphisms that commute with some fixed hyperelliptic involution $\mathcal{I} : \Sigma_g \to \Sigma_g$ as in Figure 5(1). It is shown in [16] that $\log \delta(\mathcal{H}(\Sigma_g)) \propto 1/g$. See also [7, 15, 19] for other subgroups of $\text{Mod}(\Sigma_g)$. As an application we will determine the asymptotic behavior of the minimal dilatations of the odd/even spin mapping class groups of genus $g$. To define these subgroups let $(\cdot, \cdot)_2$ be the mod-2 intersection form on $H_1(\Sigma_g; \mathbb{Z}_2)$. A map $q : H_1(\Sigma_g; \mathbb{Z}_2) \to \mathbb{Z}_2$ is a quadratic form if $q(v + w) = q(v) + q(w) + (v, w)_2$ for $v, w \in H_1(\Sigma_g; \mathbb{Z}_2)$. For a quadratic
form $q$, the *spin mapping class group* $\Mod_g[ q ]$ is the subgroup of $\Mod( \Sigma_g )$ consisting of elements $\phi$ such that $q \circ \phi_* = q$. To define the two quadratic forms $q_0$ and $q_1$ we choose a basis \{ $x_1, y_1, \ldots, x_g, y_g$ \} of $H_1(\Sigma_g; \mathbb{Z}_2)$ as in Figure 5(2). Let $q_0$ be the quadratic form such that $q_0(x_i) = q_0(y_i) = 0$ for $1 \leq i \leq g$. Let $q_1$ be the quadratic form such that $q_1(x_i) = q_1(y_i) = 1$ and $q_1(x_i) = q_1(y_i) = 0$ for $2 \leq i \leq g$. A result of Dye [5] tells us that $\Mod_g[ q ]$ for any $q$ is conjugate to either $\Mod_g[ q_0 ]$ or $\Mod_g[ q_1 ]$ in $\Mod( \Sigma_g )$. We call $\Mod_g[ q_0 ]$ and $\Mod_g[ q_1 ]$ the *even spin* and *odd spin mapping class group* respectively. It is known that $\Mod_g[ q_1 ]$ attains the minimum index for a proper subgroup of $\Mod( \Sigma_g )$ and $\Mod_g[ q_0 ]$ attains the secondary minimum, see Berrick-Gebhardt-Paris [2].

**Theorem E.** We have

1. $\log \delta( \Mod_g[ q_1 ] \cap \mathcal{H}( \Sigma_g ) ) \geq \frac{1}{g}$ and
2. $\log \delta( \Mod_g[ q_0 ] \cap \mathcal{H}( \Sigma_g ) ) \geq \frac{1}{g}$.

In particular $\log \delta( \Mod_g[ q ] ) \approx 1/g$ for each quadratic form $q$.

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2. Preliminaries

2.1. Links. Let $L$ be a link in the 3-sphere $S^3$. Let $\mathcal{N}(L)$ denote a tubular neighborhood of $L$ and let $\mathcal{E}(L)$ denote the exterior of $L$, i.e. $\mathcal{E}(L) = S^3 \setminus \text{int}(\mathcal{N}(L))$.

Oriented links $L$ and $L'$ in $S^3$ are *equivalent*, denoted by $L \sim L'$ if there is an orientation preserving homeomorphism $f : S^3 \to S^3$ such that $f(L) = L'$ with respect to the orientations of the links. Furthermore for components $K_i$ of $L$ and $K'_i$ of $L'$ with $i = 1, \ldots, m$ if $f$ satisfies $f(K_i) = K'_i$ for each $i$, then $(L, K_1, \ldots, K_m)$ and $(L', K'_1, \ldots, K'_m)$ are *equivalent* and we write $(L, K_1, \ldots, K_m) \sim (L', K'_1, \ldots, K'_m)$.

2.2. Braid groups $B_n$ and spherical braid groups $SB_n$. Let us set

\[
\delta_j = \sigma_1 \sigma_2 \cdots \sigma_{j-1} \quad \text{and} \quad \rho_j = \sigma_1 \sigma_2 \cdots \sigma_{j-2} \sigma_{j-1}^2.
\]

The half twist $\Delta_j$ is given by

\[
\Delta_j = \delta_j \delta_{j-1} \cdots \delta_2.
\]

We often omit the subscript $n$ in $\Delta_n$, $\delta_n$ and $\rho_n$ when they are precisely $n$-braids.
We put indices $1, 2, \ldots, n$ from left to right on the bottoms of strands, and give an orientation of strands from the bottom to the top (Figure 1). The closure $\text{cl}(b)$ is oriented by the strands. We think of $\text{br}(b) = \text{cl}(b) \cup A$ as an oriented link in $S^3$ choosing an orientation of $A = A_b$ arbitrarily. (In Section 3 we assign an orientation of the braid axis for $i$-monotonic braids).

If two braids are conjugate to each other, then their braided links are equivalent. Morton proved that the converse holds if their axes are preserved.

**Theorem 2.1** (Morton [26]). If $(\text{br}(b), A_b)$ is equivalent to $(\text{br}(c), A_c)$ for braids $b, c \in B_n$, then $b$ and $c$ are conjugate in $B_n$.

Let us turn to the spherical braid group $SB_n$ with $n$ strands. We also denote by $\sigma_i$, the element of $SB_n$ as shown in Figure 1(1). The group $SB_n$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$. For a braid $b \in B_n$ represented by a word of letters $\sigma_i^{\pm 1}$, let $S(b)$ denote the element in $SB_n$ represented by the same word as $b$.

For a braid $b$ in $B_n$ or $SB_n$ the degree of $b$ means the number $n$ of the strands, denoted by $d(b)$.

### 2.3. Mapping classes and mapping tori from braids.

Let $D_n$ be the $n$-punctured disk. Consider the mapping class group $\text{Mod}(D_n)$, the group of isotopy classes of orientation preserving self-homeomorphisms on $D_n$ preserving the boundary $\partial D$ of the disk setwise. We have a surjective homomorphism

$$\Gamma : B_n \rightarrow \text{Mod}(D_n)$$

which sends each generator $\sigma_i$ to the right-handed half twist $t_i$ between the $i$th and $(i + 1)$st punctures. The kernel of $\Gamma$ is an infinite cyclic group generated by the full twist $\Delta^2$.

Collapsing $\partial D$ to a puncture in the sphere we have a homomorphism

$$c : \text{Mod}(D_n) \rightarrow \text{Mod}(\Sigma_{0,n+1})$$

We say that $b \in B_n$ is *periodic* (resp. *reducible*, *pseudo-Anosov*) if $f_b := c(\Gamma(b))$ is of the corresponding Nielsen-Thurston type. The braids $\delta, \rho \in B_n$ are periodic since some power of each braid is the full twist: $\Delta^2 = \delta^0 = \rho^{n-1} \in B_n$.

We also have a surjective homomorphism

$$\widehat{\Gamma} : SB_n \rightarrow \text{Mod}(\Sigma_{0,n})$$

sending each generator $\sigma_i$ to the right-handed half twist $t_i$. We say that $\eta \in SB_n$ is *pseudo-Anosov* if $\widehat{\Gamma}(\eta) \in \text{Mod}(\Sigma_{0,n})$ is pseudo-Anosov. In this case $\lambda(\eta)$ is defined by the dilatation of $\widehat{\Gamma}(\eta)$. 


2.4. Stable foliations $\mathcal{F}_b$ for pseudo-Anosov braids $b$. Recall the surjective homomorphism $\pi: B_n \to S_n$. We write $\pi_b = \pi(b)$ for $b \in B_n$. Consider a pseudo-Anosov braid $b \in B_n$ with $\pi_b(i) = i$. Removing the $i$th strand $b(i)$ from $b$, we get a braid $b - b(i) \in B_{n-1}$. Taking its spherical element, we have $S(b - b(i)) \in SB_{n-1}$. Note that $b - b(i)$ and $S(b - b(i))$ are not necessarily pseudo-Anosov. A well-known criterion uses the stable foliation $\mathcal{F}_b$ for the monodromy $\phi_b: \mathcal{F}_b \to \mathcal{F}_b$ of a fibration on $M_b \to S^1$ as we recall now. Such a fibration on $M_b$ extends naturally to a fibration on the manifold obtained from $M_b$ by Dehn filling a cusp along the boundary slope of the fiber $F_b$ which lies on the torus $\partial \mathcal{N}(\text{cl}(b(i)))$. Also $\phi_b$ extends to the monodromy defined on $\mathcal{F}_b^\bullet$ of the extended fibration, where $\mathcal{F}_b^\bullet$ is obtained from $\mathcal{F}_b$ by filling in the boundary component of $F_b$ which lies on $\partial \mathcal{N}(\text{cl}(b(i)))$ with a disk. Then $b - b(i)$ is the corresponding braid for the extended monodromy defined on $\mathcal{F}_b^\bullet$. Suppose that $\mathcal{F}_b$ is not 1-pronged at the boundary component in question. (See Figure 6 in the case where $F_b$ is 1-pronged at a boundary component.) Then $\mathcal{F}_b$ extends to the stable foliation for $b - b(i)$, and hence $b - b(i)$ is pseudo-Anosov with the same dilatation as $b$. Furthermore if $\mathcal{F}_b$ is not 1-pronged at the boundary component of $F_b$ which lies on $\partial \mathcal{N}(A)$, then $S(b - b(i))$ is still pseudo-Anosov with the same dilatation as $b$.

2.5. Thurston norm. Let $M$ be a 3-manifold with boundary (possibly $\partial M = \emptyset$). If $M$ is hyperbolic, i.e. the interior of $M$ possess a complete hyperbolic structure of finite volume, then there is a norm $\| \cdot \|$ on $H_2(M, \partial M; \mathbb{R})$, now called the Thurston norm [29]. The norm $\| \cdot \|$ has the property such that for any integral class $a \in H_2(M, \partial M; \mathbb{R})$, $\|a\| = \min_S \{-\chi(S)\}$, where the minimum is taken over all oriented surface $S$ embedded in $M$ with $a = [S]$ and with no components of non-negative Euler characteristic. The surface $S$ realizing this minimum is called a norm-minimizing surface of $a$.

**Theorem 2.2** (Thurston [29]). *The norm $\| \cdot \|$ on $H_2(M, \partial M; \mathbb{R})$ has the following properties.*

1. There are a set of maximal open cones $\mathcal{C}_1, \cdots, \mathcal{C}_k$ in $H_2(M, \partial M; \mathbb{R})$ and a bijection between the set of isotopy classes of connected fibers of fibrations $M \to S^1$ and the set of primitive integral classes in the union $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k$. 
The restriction of $\| \cdot \|$ to $C_j$ is linear for each $j$.

If we let $F_a$ be a fiber of a fibration $M \to S^1$ associated with a primitive integral class $a$ in each $C_j$, then $\|a\| = -\chi(F_a)$.

We call the open cones $C_j$ fibered cones and call integral classes in $C_j$ fibered classes.

**Theorem 2.3** (Fried [11]). For a fibered cone $C$ of a hyperbolic 3-manifold $M$, there is a continuous function $\text{ent}: C \to \mathbb{R}$ with the following properties.

1. For the monodromy $\phi_a: F_a \to F_a$ of a fibration $M \to S^1$ associated with a primitive integral class $a \in C$, we have $\text{ent}(a) = \log(\lambda(\phi_a))$.
2. $\text{Ent} = \| \cdot \| \text{ent}: C \to \mathbb{R}$ is a continuous function which becomes constant on each ray through the origin.
3. If a sequence $\{a_n\} \subset C$ tends to a point $\neq 0$ in the boundary $\partial C$ as $n$ tends to $\infty$, then $\text{ent}(a_n) \to \infty$. In particular $\text{Ent}(a_n) = \|a_n\| \text{ent}(a_n) \to \infty$.

We call $\text{ent}(a)$ and $\text{Ent}(a)$ the entropy and normalized entropy of the class $a \in C$.

For a pseudo-Anosov element $\phi \in \text{Mod}(\Sigma)$ we consider the mapping torus $M_\phi$. The vector field $\frac{\partial}{\partial t}$ on $\Sigma \times \mathbb{R}$ induces a flow $\phi^t$ on $M_\phi$ called the suspension flow.

**Theorem 2.4** (Fried [10]). Let $\phi$ be a pseudo-Anosov mapping class defined on $\Sigma$ with stable and unstable foliations $\mathcal{F}^s$ and $\mathcal{F}^u$. Let $\widehat{\mathcal{F}}^s$ and $\widehat{\mathcal{F}}^u$ denote the suspensions of $\mathcal{F}^s$ and $\mathcal{F}^u$ by $\phi$. If $C$ is a fibered cone containing the fibered class $[\Sigma]$, then we can modify a norm-minimizing surface $F_a$ associated with each primitive integral class $a \in C$ by an isotopy on $M_\phi$ with the following properties.

1. $F_a$ is transverse to the suspension flow $\phi^t$, and the first return map $\phi_a: F_a \to F_a$ is precisely the pseudo-Anosov monodromy of the fibration on $M_\phi \to S^1$ associated with $a$. Moreover $F_a$ is unique up to isotopy along flow lines.
2. The stable and unstable foliations for $\phi_a$ are given by $\widehat{\mathcal{F}}^s \cap F_a$ and $\widehat{\mathcal{F}}^u \cap F_a$.

**2.6. Disk twist.** Let $L$ be a link in $S^3$. Suppose an unknot $K$ is a component of $L$. Then the exterior $\mathcal{E}(K)$ (resp. $\partial \mathcal{E}(K)$) is a solid torus (resp. torus). We take a disk $D$ bounded by the longitude of a tubular neighborhood $N(K)$ of $K$. We define a mapping class $T_D$ defined on $\mathcal{E}(K)$ as follows. We cut $\mathcal{E}(K)$ along $D$. We have resulting two sides obtained from $D$, and reglue two sides by twisting either of the sides 360 degrees so that the mapping class defined on $\partial \mathcal{E}(K)$ is the right-handed Dehn twist about $\partial D$. Such a mapping class on $\mathcal{E}(K)$ is called the disk twist about $D$. For simplicity we also call a self-homeomorphism representing the mapping class $T_D$ the disk twist about $D$, and denote it by the same notation $T_D: \mathcal{E}(K) \to \mathcal{E}(K)$.
Figure 7. Disk twist $T_D$.

Clearly $T_D$ equals the identity map outside a neighborhood of $D$ in $E(K)$. We observe that if $u + 1$ segments of $L - K$ pass through $D$ for $u \geq 1$, then $T_D(L - K)$ is obtained from $L - K$ by adding the full twist near $D$. In the case $u = 1$, see Figure 7. We may assume that $T_D$ fixes one of these segments, since any point in $D$ becomes the center of the twisting about $D$.

For any integer $\ell$, consider a homeomorphism

$$T_D^\ell : E(K) \to E(K).$$

Observe that $T_D^\ell$ converts $L$ into a link $K \cup T_D^\ell(L - K)$ such that $S^3 \setminus L$ is homeomorphic to $S^3 \setminus (K \cup T_D^\ell(L - K))$. Then $T_D^\ell$ induces a homeomorphism between the exteriors of links

$$h_{D,\ell} : E(L) \to E(K \cup T_D^\ell(L - K)).$$

(2.1)

We use the homeomorphism in (2.1) in later section.

3. $i$-increasing braids and Theorem 3.2

Definitions of $i$-increasing braids, signs and intersection numbers.

Let $L$ be an oriented link in $S^3$ with a trivial component $K$. We take an oriented disk $D$ bounded by the longitude of $N(K)$ so that the orientation of $D$ agrees with the orientation of $K$. For each component $K'$ of $L - K$ such that $D$ and $K'$ intersect transversally with $D \cap K' \neq \emptyset$, we assign each point of intersection $+1$ or $-1$ as shown in Figure 8.
Let $b$ be a braid with $\pi_b(i) = i$. We consider an oriented disk $D = D_{(b,i)}$ bounded by the longitude $\ell_i$ of $N(\text{cl}(b(i)))$. Such a disk $D$ is unique up to isotopy on $\mathcal{E}(\text{cl}(b(i)))$. We say that a braid $b \in B_n$ with $\pi_b(i) = i$ is $i$-increasing (resp. $i$-decreasing) if there is a disk $D = D_{(b,i)}$ as above with the following conditions.

(D1) There is at least one component $K'$ of $\text{cl}(b - b(i))$ such that $D \cap K' \neq \emptyset$.

(D2) Each component of $\text{cl}(b - b(i))$ and $D$ intersect with each other transversally, and every point of intersection has the sign $+1$ (resp. $-1$).

We set $\epsilon(b,i) = 1$ (resp. $\epsilon(b,i) = -1$), and call it the sign of the pair $(b,i)$. We also call $D$ the associated disk of the pair $(b,i)$. We say that $b$ is $i$-monotonic if $b$ is $i$-increasing or $i$-decreasing. Then we set

$$I(b,i) = D \cap \text{cl}(b - b(i))$$

and let $u(b,i) \geq 1$ be the cardinality of $I(b,i)$. We call $u(b,i)$ the intersection number of the pair $(b,i)$. If the pair $(b,i)$ is specified, then we simply denote $\epsilon(b,i)$ and $u(b,i)$ by $\epsilon$ and $u$ respectively. For example $\sigma_1^2\sigma_2^{-1}$ is 1-increasing with $u(\sigma_1^2\sigma_2^{-1},1) = 1$.

A braid $b$ is positive if $b$ is represented by a word in letters $\sigma_j$, but not $\sigma_j^{-1}$. A braid $b$ is irreducible if the Nielsen-Thurston type of $b$ is not reducible.

**Lemma 3.1.** Let $b$ be a positive braid with $\pi_b(i) = i$. Then $b$ is $i$-increasing if $b$ is irreducible.

**Proof.** Suppose that a positive braid $b$ with $\pi_b(i) = i$ is irreducible. Since $b$ is positive, there is a disk $D = D_{(b,i)}$ with the condition (D2). Assume that $D$ fails in (D1). Let $\partial D_n$ be the boundary of the disk $D_n$ containing $n$ punctures. Consider a neighborhood of $\partial D_n \cup (D_n \cap D)$ in $D_n$ which is an annulus. One of the boundary components of this annulus is an essential
simple closed curve in $D_n$ preserved by $\Gamma(b) \in \text{Mod}(D_n)$. This means that $b$ is reducible, a contradiction. Thus $D$ satisfies (D1), and $b$ is $i$-increasing. \hfill \Box

**Orientation of the axis $A$.** Let $b$ be $i$-monotonic with $\epsilon(b,i) = \epsilon$ and $u(b,i) = u$. Consider the braided link $\text{br}(b) = \text{cl}(b) \cup A$. The associated disk $D$ has a unique point of intersection with $A$, and the cardinality of $I(b,i) \cup (D \cap A)$ is $u(b,i) + 1$. To deal with $\text{br}(b) = \text{cl}(b) \cup A$ as an oriented link, we consider an orientation of $\text{cl}(b)$ as we described before, and assign an orientation of $A$ so that the sign of the intersection between $D$ and $A$ coincides with $\epsilon(b,i)$. See Figure 2(2).

Recall that $M_b = \mathcal{E}(\text{br}(b))$ is the exterior of $\text{br}(b)$ which is a surface bundle over $S^1$. We consider an orientation of the $F$-surface $F_b$ which agrees with the orientation of $A$.

**E-surface.** We now define an oriented surface $E_{(b,i)}$ of genus 0 embedded in $M_b$. Consider small $u(b,i) + 1$ disks in the oriented disk $D = D_{(b,i)}$ whose centers are points of $I(b,i) \cup (D \cap A)$. Then $E_{(b,i)}$ is a sphere with $u(b,i) + 2$ boundary components obtained from $D$ by removing the interiors of those small disks. We choose the orientation of $E_{(b,i)}$ so that it agrees with the orientation of $D$. We call $E_{(b,i)}$ the $E$-surface for $b$. For example, the 1-increasing braid $\sigma_1^2 \sigma_2^{-1}$ has the $E$-surface $E_{(\sigma_1^2 \sigma_2^{-1},1)}$ homeomorphic to a 3-holed sphere.

**Subcone $C_{(b,i)}$.** Consider the 2-dimensional subcone of $H_2(M_b, \partial M_b; \mathbb{R})$ spanned by $[F_b]$ and $[E_{(b,i)}]$ (Figure 9):

$$C_{(b,i)} = \{x[F_b] + y[E_{(b,i)}] \mid x > 0, \ y > 0\}.$$  

Let $\overline{C_{(b,i)}}$ denote the closure of $C_{(b,i)}$. We write $(x,y) = x[F_b] + y[E_{(b,i)}]$. We prove the following theorem in Section 4.

**Theorem 3.2.** For a pseudo-Anosov, $i$-increasing braid $b$ with $u(b,i) = u$, let $C$ be the fibered cone containing $[F_b]$. We have the following.

1. $C_{(b,i)} \subset C$.
2. The fiber $F(x,y)$ for each primitive integral class $(x,y) \in C_{(b,i)}$ has genus 0.
3. The monodromy $\phi(x,y) : F(x,y) \to F(x,y)$ for each primitive integral class $(x,y) \in C_{(b,i)}$ is conjugate to

$$(\omega_1 \psi) \cdots (\omega_{u-1} \psi)(\omega_u \psi)^{m-1} : F(x,y) \to F(x,y),$$

where $m \geq 1$ depends on $(x,y)$, $\psi$ is periodic and each $\omega_j$ is reducible. Moreover there are homeomorphisms $\tilde{\omega}_j : S_0 \to S_0$ for $j = 1, \ldots, u$ on a surface $S_0$ determined by $b$ and an embedding $h : S_0 \hookrightarrow F(x,y)$ such that the subsurface $h(S_0)$ of $F(x,y)$ is the support of each $w_j$ and

$$w_j|_{h(S_0)} = h \circ \tilde{\omega}_j \circ h^{-1}.$$
The conclusion of Theorem 3.2 holds for \(i\)-decreasing braids as well. We now claim that Theorem 3.2 implies Theorem B.

**Proof of Theorem B.** Suppose that Theorem 3.2 holds. Let \(b \in B_n\) be a pseudo-Anosov braid such that \(\pi_b(i) = i\). We consider the braid \(b\Delta^{2k} \in B_n\) for \(k \geq 1\). The full twist \(\Delta^2\) is an element in the center \(Z(B_n)\) and \(\Delta^2 = \sigma_j P_j\) holds for each \(1 \leq j \leq n - 1\), where \(P_j\) is positive. Such properties imply that \(b\Delta^{2k}\) is positive for \(k\) large. We fix such large \(k\). Since \(\Gamma(b) = \Gamma(b\Delta^{2k})\) in \(\text{Mod}(D_n)\), the braid \(b\Delta^{2k}\) is certainly pseudo-Anosov. Hence it is \(i\)-increasing by Lemma 3.1. One can apply Theorem 3.2 for this braid, and obtains the subcone \(C(b\Delta^{2k},i)\). Consider the \(k\)th power of the disk twist about the disk \(D_A\) bounded by the longitude of \(N(A)\):

\[
T^k_{D_A} : \mathcal{E}(A) \to \mathcal{E}(A).
\]

Since \(A \cup T^k_{D_A}(\text{cl}(b)) = A \cup \text{cl}(b\Delta^{2k}) = \text{br}(b\Delta^{2k})\), we have \(S^3 \setminus \text{br}(b) \simeq S^3 \setminus \text{br}(b\Delta^{2k})\). Let us set

\[
f_k := h_{D_A,k} : M_b \to M_{b\Delta^{2k}},
\]

where \(h_{D_A,k}\) is the homeomorphism in (2.1). The isomorphism

\[
f_{k*} : H_2(M_b, \partial M_b) \to H_2(M_{b\Delta^{2k}}, \partial M_{b\Delta^{2k}})
\]

sends \([F_b]\) to \([F_{b\Delta^{2k}}]\). (Here we note that the above \(k\) is suppose to be large, but the homeomorphism \(f_k\) makes sense for all integer \(k\).) The pullback of the subcone \(C(b\Delta^{2k},i)\) into \(H_2(M_b, \partial M_b)\) is a desired subcone contained in \(C\). \(\square\)

**Remark 3.3.** If \(F_{(x,y)}\) is a \((d+1)\)-holed sphere, then the periodic homeomorphism \(\psi : F_{(x,y)} \to F_{(x,y)}\) in Theorem 3.2 is determined by the periodic braid \(\rho = \sigma_1 \sigma_2 \ldots \sigma_d \Delta^{-2} \Delta_{d-1}^2 \in B_d\). See the proof of Theorem 3.2(3) in Section 4.3.

### 4. Proof of Theorem 3.2

We fix integers \(n \geq 3\) and \(1 \leq i \leq n\). Throughout Section 4, we assume that \(b \in B_n\) is pseudo-Anosov and \(i\)-increasing with \(u(b,i) = u\). We now choose an associated disk about the pair \((b,i)\) suitably. Let \(D\) denote the unit disk with the center \((0,0)\) in the plane \(\mathbb{R}^2\). Let \(J = (-1,1) \times \{0\} \subset D\) be the interval and let \(A_0 = (-2,0)\) be a point in \(\mathbb{R}^2\). We denote by \(D_n\), the disk \(D\) with equally spaced \(n\) points in \(J\). Let us denote these \(n\) points by \(A_1, \ldots, A_n\) from left to right. We take a point \(Q_i \neq A_i \in J\) between \(A_{i-1}\) and \(A_i\) so that the Euclidean distance \(d(Q_i, A_i)\) is sufficiently small (e.g. \(d(Q_i, A_i) < \frac{1}{n+1}\)). Let \(r_i\) denote the closed interval in \([-2,1] \times \{0\}\) with endpoints \(A_0\) and \(Q_i\). (Figure 10(1).) We regard \(b\) as a braid contained in the cylinder \(D \times [0,1] \subset \mathbb{R}^3\) and \(b\) is based at \(n\) points \(A_1 \times \{0\}, \ldots, A_n \times \{0\}\). Since \(\pi_b(i) = i\), one can take a representative of \(b\) such that \(b(i)\) is an interval in the cylinder:
\[ \partial D = \ell_i \text{ is a union of four segments. } U_i \text{ is an annulus in the figure.} \]

\[ \diamond 1. \ b(i) = \bigcup_{0 \leq t \leq 1} A_i \times \{t\}. \]

Furthermore we may assume that \( \partial D(= \ell_i) \) of an associated disk \( D \) of \( (b, i) \) is a union of the following four segments as a set (Figure 10):

\[ \diamond 2. \ ( \bigcup_{-1 \leq t \leq 2} A_0 \times \{t\} ) \cup ( r_i \times \{-1\} ) \cup ( \bigcup_{-1 \leq t \leq 2} Q_i \times \{t\} ) \cup ( r_i \times \{2\} ). \]

Preserving \( \diamond 1, 2 \) we may further assume the following (Figures 10(2), 11(1)):

\[ \diamond 3. \text{ For a regular neighborhood } U_i \text{ of } \ell_i \text{ in } D, \text{ we have } I(b,i) \subset U_i. \]

This is because every point \( x \in D \cap K' \), where \( K' \) is a component of \( \text{cl}(b - b(i)) \), one can slide \( x \) along \( K' \) so that the resulting point on \( K' \) is in \( U_i \). Said differently, preserving \( \partial D \) pointwise, we can modify a small neighborhood of \( D \) near \( K' \) so that the resulting associated disk satisfies \( \diamond 3 \).

Under the conditions \( \diamond 1, 2, 3 \) we have the following. For each \( x \in D \cap K' \subset U_i \), there is a segment \( h' \subset K' \) through \( x \) such that \( h' \) passes over \( b(i) \) since \( b \) is \( i \)-increasing. See Figure 11(1). Such a local picture of \( \text{cl}(b) \) is used in the the next section. Hereafter we assume that associated disks possess conditions \( \diamond 1, 2, 3 \).

### 4.1. Proof of Theorem 3.2(1)

Let \( s \) be the open segment (1-dimensional simplex) in \( H_2(M_b, \partial M_b; \mathbb{R}) \) with the endpoints \( \frac{n-1}{u}[E(b,i)] = (0, \frac{n-1}{u}) \) and \( |F_b| = (1,0) \):

\[ s = \{(x,y) \in C_{(b,i)} \mid y = -\frac{n-1}{u}x + \frac{n-1}{u}, \ 0 < x < 1\}. \tag{4.1} \]

The ray of each point in \( C_{(b,i)} \) through the origin intersects with \( s \). Thus for the proof of (1), it suffices to prove that \( s \subset C \).

We now introduce a sequence of braided links \( \{\text{br}(b_p)\} \) from an \( i \)-increasing braid \( b \in B_n \) such that \( M_{b_p} \simeq M_b \) for each \( p \geq 1 \). (We use the
Figure 11. Case: $b$ is $i$-increasing. (1) Associated disk $D$ with conditions ♦1,2,3. (2) $\text{br}(b_1)$. Circles $\circ$ indicate points of intersection between $D$ and components of $\text{br}(b - b(i))$. See also Figure 12.

Figure 12. Braided links for (1) 1-increasing $\sigma_1^2 \sigma_2^{-1}$, (2) 2-increasing ($\sigma_1^2 \sigma_2^{-1}$)$_1$ and (3) 3-increasing ($\sigma_1^2 \sigma_2^{-1}$)$_2$.

1-increasing braid $\sigma_1^2 \sigma_2^{-1} \in B_3$ to illustrate the idea.) Let $D$ be an associated disk of the pair $(b, i)$. We take a disk twist

$$T_D : \mathcal{E}(\text{cl}(b(i))) \to \mathcal{E}(\text{cl}(b(i)))$$

so that the point of intersection $D \cap A$ becomes the center of the twisting about $D$, i.e. $T_D(D \cap A) = D \cap A$. We may assume that $T_D(A) = A$ as a set. Figure 11 illustrates the image of the segment $h'$ under $T_D$. The condition ♦3 ensures that $T_D$ equals the identity map outside a neighborhood of $U_i$ in $\mathcal{E}(\text{cl}(b(i)))$. Then by ♦1,2, it follows that

$$T_D(\text{br}(b - b(i)) \cup \text{cl}(b(i)))$$

is a braided link of some $(i + u)$-increasing braid with $(n + u)$ strands. We define $b_1 \in B_{n+u}$ to be such a braid. The trivial knot $T_D(A)(= A)$ becomes
a braid axis of $b_1$. By definition of the disk twist, we have $M_{b_1} \simeq M_b$. See Figure 12 for $\text{br}((\sigma_1^2 \sigma_2^{-1})_1)$.

As discussed below, there is some ambiguity in defining $b_1$. As we will see, the ambiguity is irrelevant for the study of pseudo-Anosov monodromies defined on fibers of fibrations on the mapping torus. Suppose that both $D$ and $D'$ are the associated disks of the pair $(b, i)$ with conditions $\Diamond 1, 2, 3$. We consider the disk twists $T_D$ and $T_{D'}$ with the above condition, i.e. both $D \cap A$ and $D' \cap A$ become the center of the twisting about $D$ and $D'$ respectively. Observe that the resulting two links obtained from $D$ and $D'$ are equivalent:

$$T_D(\text{br}(b - b(i))) \cup \text{cl}(b(i)) \sim T_{D'}(\text{br}(b - b(i))) \cup \text{cl}(b(i)).$$

They are braided links, say $\text{br}(b_1)$ and $\text{br}(b'_1)$ of some braids $b_1, b'_1 \in B_{n+u}$ respectively with the same axis $T_D(A) = A = T_{D'}(A)$. This means that a more stronger claim holds:

$$(\text{br}(b_1), A) \sim (\text{br}(b'_1), A).$$

Thus $b_1$ and $b'_1$ are conjugate in $B_{n+u}$ by Theorem 2.1. In particular both $b_1$ and $b'_1$ are pseudo-Anosov (since the initial braid $b$ is pseudo-Anosov and $M_b$ is hyperbolic) and they have the same dilatation.

To define $b_p$ for $p \geq 1$, we consider the $p$th power

$$T_D^p : \mathcal{E}(\text{cl}(b(i))) \to \mathcal{E}(\text{cl}(b(i)))$$

using the above $T_D$. As in the case of $p = 1$,

$$T_D^p(\text{br}(b - b(i))) \cup \text{cl}(b(i))$$

is a braided link of some $i + pu$-increasing braid with $(n + pu)$ strands. We define $b_p \in B_{n + pu}$ to be such a braid. Then $M_{b_p} \simeq M_b$. As in the case of $p = 1$, such a braid $b_p$ is well-defined up to conjugate. We say that $b_p$ is obtained from $b$ by the disk twist. Clearly $u(b_p, i + pu) = u(b, i)$ for $p \geq 1$. See Figure 12.

Let us set

$$g_p := h_{D,p} : M_b \to M_{b_p},$$

where $h_{D,p}$ is the homeomorphism in (2.1). We consider the isomorphism

$$g_{p*} : H_2(M_b, \partial M_b) \to H_2(M_{b_p}, \partial M_{b_p}).$$

Lemma 4.1. For each integer $p \geq 1$, $g_{p*}$ sends $(0, 1) \in C_1(b,i)$ to $(0, 1) \in C_1(b_{p,i+pu})$, and sends $(1, p) \in C_1(b,i)$ to $(1, 0) \in C_1(b_{p,i+pu})$. In particular for integers $x, y \geq 1$ with $y = xp + r$ for $0 \leq r < p$, $g_{p*}$ sends $(x, y) \in C_1(b,i)$ to $(x, r) \in C_1(b_{p,i+pu})$.

Proof. We consider the oriented sum $F_{(x,y)} := xF_b + yE_{(b,i)}$. This is an oriented surface embedded in $M_b$, and is obtained from the cut and past construction of parallel $x$ copies of $F_b$ and parallel $y$ copies of $E_{(b,i)}$. The orientation of $F_{(x,y)}$ agrees with those of $F_b$ and $E_{(b,i)}$. We have $[F_{(x,y)}] = (x, y) \in C_1(b,i)$. Then $g_p$ sends $E_{(b,i)}$ to $E_{(b_{p,i+pu})}$, and sends $F_{(1,p)}$ to $F_{b_p}$. 


Thus $g_p$ sends $(0,1)$ to $(0,1)$, and sends $(1,p)$ to $(1,0)$. This completes the proof. □

By the proof of Lemma 4.1, $g_1$ sends $F_{(1,1)} = F_b + E_{(b,i)}$ to the fiber $F_{b_1}$ of a fibration on $M_b$ associated with $(1,1) \in C_{(b,i)}$. Since the fibers $F_{(1,1)\ast}$ and $F_b$ are norm-minimizing, $E_{(b,i)}$ is also norm-minimizing.

**Proof of Theorem 3.2(1).** We have $\|F_b\| = n-1$ and $\|[F_{b_1}]\| = n+pu-1$ since $F_b$ and $F_{b_1}$ are fibers, and $\|[E_{(b,i)}]\| = u$ since $E_{(b,i)}$ is norm-minimizing. By Lemma 4.1, $[F_{b_1}] = (1,p) \in C_{(b,i)}$. Consider the rational class

$$c_p := \frac{n-1}{n+pu-1}[F_{bp}] = \left( \frac{n-1}{n+pu-1}, \frac{p(n-1)}{n+pu-1} \right).$$

The classes $c_p$ that are all projectively fibered, and they lie on the 1-dimensional linear simplex $s$ given by (4.1). Note that the closure of $s$ contains $[F_b]$. Moreover, the Thurston norm of all $c_p$ equals that of $[F_b]$ (and it is $n-1$). This is only possible if the simplex $s$ is projectively contained in a single fibered face. The corresponding fibered cone has to contain $[F_b]$ from the above discussion, and hence it is $\mathcal{C}$. Thus $s \subset \mathcal{C}$. This completes the proof. □

**Remark 4.2.** From the proof of Theorem 3.2(1), one sees the following: If $[E_{(b,i)}] \notin \mathcal{C}_{(b,i)}$ is a fibered class, then $[E_{(b,i)}] \notin \mathcal{C}$. Otherwise $[E_{(b,i)}] \notin \partial \mathcal{C}$. See Figure 9(2)/(3).

### 4.2 Proof of Theorem 3.2(2).

We start with a simple observation: $\Delta^2 \in B_n$ is $j$-increasing for each $1 \leq j \leq n$, and $u(\Delta^2,j) = n-1$ holds. The following lemma is immediate.

**Lemma 4.3.** If $b \in B_n$ is $i$-increasing, then $b\Delta^2 \in B_n$ is $i$-increasing with $u(b\Delta^2,i) = u(b,i) + n - 1$.

We explain the idea of Theorem 3.2(2). Let $D$ be the associated disk of the pair $(b,i)$. We have two types of the disk twist. One is $T_{\Delta^2}^D : \mathcal{E}(A) \to \mathcal{E}(A)$ which appears in the proof of Theorem B in Section 3 and the other is $T_D^{\partial A} : \mathcal{E}(\text{cl}(b(i))) \to \mathcal{E}(\text{cl}(b(i)))$. If $k$ and $p$ are positive, then we obtain the $i$-increasing $b\Delta^{2k}$ from the former type $T_{\Delta^2}^D$, and another increasing braid $b_p$ from the latter type $T_D^{\partial A}$. Since both resulting braids are increasing, we can further apply two types of the disk twist for the resulting braid. This is a key of the proof. Choosing two types of the disk twist alternatively, we get a sequence of increasing and pseudo-Anosov braids (since the initial braid $b$ is pseudo-Anosov). We shall see that the desired monodromies associated with primitive classes in $C_{(b,i)}$ are given by these braids.

Let $p_1, \ldots, p_j$ be integers such that $p_1 \geq 0$ and $p_2, \ldots, p_j \geq 1$. Given an $i$-increasing braid $b \in B_n$ with $u(b,i) = u$, we define an integer $i[p_1, \ldots, p_j] \geq 1$ and an $i[p_1, \ldots, p_j]$-increasing braid $b[p_1, \ldots, p_j]$ inductively as follows.
If \( j = 1 \) and \( p_1 = 0 \), then \( i[0] = i \) and \( b[0] = b \). If \( j = 1 \) and \( p_1 = p \geq 1 \), then \( i[p] = i + pu \) and \( b[p] = b_p \).

If \( j > 1 \) is even, then
\[
\begin{align*}
i[p_1, \ldots, p_{j-1}, p_j] & = i[p_1, \ldots, p_{j-1}], \\
b[p_1, \ldots, p_{j-1}, p_j] & = (b[p_1, \ldots, p_{j-1}]) \Delta^{2p}.
\end{align*}
\]

The right-hand side is \( i[p_1, \ldots, p_{j-1}] \)-increasing by Lemma 4.3.

If \( j > 1 \) is odd, then
\[
\begin{align*}
i[p_1, \ldots, p_{j-1}, p_j] & = i[p_1, \ldots, p_{j-1}] + p_j u(b[p_1, \ldots, p_{j-1}], i[p_1, \ldots, p_{j-1}]), \\
b[p_1, \ldots, p_{j-1}, p_j] & = (b[p_1, \ldots, p_{j-1}]) p_j.
\end{align*}
\]

We say that \( b[p_1, \ldots, p_j] \) has length \( j \).

**Example 4.4.**

1. \( b[p] = b_p \) by definition.
2. Let \( \beta = b^2 \Delta \). Then \( b[0, 1] = \beta \) and \( b[0, 1, p] = \beta_p \).
3. We have \( b[0, p] = b^2 \Delta p \) and \( b[0, p, 1] = (b^2 \Delta p)_1 \), where \( (b^2 \Delta p)_1 \) is obtained from \( i \)-increasing \( b^2 \Delta p \) by the disk twist.

For each \( k \geq 1 \), let \( f_k : M_b \to M_{b^2 \Delta 2^k} \) be the homeomorphism which in the proof of Theorem B. Consider the isomorphism \( f_{k*} : H_2(M_b, \partial M_b) \to H_2(M_{b^2 \Delta 2^k}, \partial M_{b^2 \Delta 2^k}) \). We have the following property.

**Lemma 4.5.** For each integer \( k \geq 1 \), \( f_{k*} \) sends \( (1, 0) \in C_{(b,i)} \) to \( (1, 0) \in C_{(b^2 \Delta 2^k,i)} \), and sends \( (k, 1) \in C_{(b,i)} \) to \( (0, 1) \in C_{(b^2 \Delta 2^k,i)} \). In particular for integers \( x, y \geq 1 \) with \( x = yk + r \) for \( 0 \leq r < k \), then \( f_{k*} \) sends \( (x, y) \in C_{(b,i)} \) to \( (r, y) \in C_{(b^2 \Delta 2^k,i)} \).

**Proof.** The homeomorphism \( f_k \) sends \( F_b \) to \( F_{b^2 \Delta 2^k} \), and sends \( F_{(k,1)} = kF_b + E_{(b,i)} \) to \( E_{(b^2 \Delta 2^k,i)} \). This implies that the claim holds.

**Proof of Theorem 3.2(2).** Let \( (x, y) \in C_{(b,i)} \) be a primitive integral class. (Hence \( x, y \) are positive integers with \( \gcd(x, y) = 1 \).) We consider the continued fraction of \( y/x \) by the Euclidean algorithm
\[
\frac{y}{x} = p_1 + \frac{1}{p_2 + \frac{1}{p_3 + \cdots + \frac{1}{p_{j-1} + \frac{1}{p_j}}}} := p_1 + \frac{1}{p_2 + \frac{1}{p_3 + \cdots + \frac{1}{p_{j-1} + \frac{1}{p_j}}}}
\]

with length \( j \) and \( p_j \geq 2 \) and \( p_1 = 0 \) if \( 0 < y < x \). There is another expression
\[
\frac{y}{x} = p_1 + \frac{1}{p_2 + \frac{1}{p_3 + \cdots + \frac{1}{p_{j-1} + \frac{1}{p_j}}}} = \frac{1}{(p_j - 1) + \frac{1}{\cdots + \frac{1}{p_j}}}
\]
with length $j + 1$. We choose one of the two expressions with odd length $\ell$:

$$\frac{y}{x} = \frac{1}{p_1} + \frac{1}{p_2 + p_3 + \cdots + p_{\ell-1} + p_\ell}.$$ 

This encodes the fiber $F(x,y)$ and its monodromy $\phi_{(x,y)}$. In fact Lemmas 4.1, 4.5 ensure that

$$(g_{p_1} f_{p_{\ell-1}} g_{p_{\ell-2}} \cdots f_{p_2} g_{p_1}) : H_2(M_b, \partial M_b) \to H_2(M_{b[p_1,\ldots,p_\ell]}, \partial M_{b[p_1,\ldots,p_\ell]})$$

sends $(x,y) = [xF_b + yE_{b(i)}]$ to $(1,0)$ which is the integral class of the $F$-surface of $b[p_1,\ldots,p_\ell]$. $(g_{p_1} = id : M_b \to M_b$ if $p_1 = 0.)$ Thus $F(x,y)$ has genus 0. Moreover this means that one can take $F_{b[p_1,\ldots,p_\ell]}$ as a representative of $(x,y) \in C_{b(i)}$ and the monodromy $\phi(x,y) : F_{(x,y)} \to F_{(x,y)}$ is determined by $b[p_1,\ldots,p_\ell]$. This completes the proof. $\square$

We denote by $b_{(x,y)}$ the braid $b[p_1,\ldots,p_\ell]$ which determines $\phi_{(x,y)}$. Here is an example: If $(x,y) = (5,14)$, then $\frac{14}{5} = 2 + \frac{1}{1 + \frac{1}{5}}$ and $\phi_{(5,14)}$ is determined by $b_{(5,14)} = b[2,1,4]$. If $(x,y) = (14,5)$, then $\frac{5}{14} = 0 + \frac{1}{2 + \frac{1}{3 + \frac{1}{5}}}$ and $\phi_{(14,5)}$ is determined by $b_{(14,5)} = b[0,2,1,3,1]$.

4.3. Proof of Theorem 3.2(3). We begin with the following lemma.

Lemma 4.6 (Standard form). If $b \in B_n$ is $i$-increasing with $u(b,i) = u$, then $b$ is conjugate to an $n$-increasing braid $b'$ of the form

$$b' = (w_1\sigma_{n-1}^2) \cdots (w_u\sigma_{n-1}^2),$$

where each $w_k$ is a word of $\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}$, but not $\sigma_{n-1}^{\pm 1}$, possibly $w_k = \emptyset$ for some $k$.

Figure 13(1) shows the form of $b'$ in Lemma 4.6 in case $u = 2$.

Proof. We regard $b$ as a braid in $\mathbb{D} \times [0,1]$. By $\diamondsuit 1$, $b(i)$ is an interval in $\mathbb{D} \times [0,1]$. If $i = n$, then $b$ is $n$-increasing and it is not hard to see that a representative of $b$ is of the desired form in Lemma 4.6. Suppose that $b$ is $i$-increasing for $1 \leq i < n$. We set $\sigma = \sigma_{n-1}\sigma_{n-2} \cdots \sigma_{i}$ if $1 \leq i < n - 1$ and $\sigma = \sigma_{n-1}$ if $i = n - 1$. We consider the $n$-braid $b' = \sigma b \sigma^{-1}$ which is $n$-increasing with $u(b',n) = u$. We pull $b'(n)$ tight in $\mathbb{D} \times [0,1]$ and make it straight. Then a representative of $b'$ is of the desired form. $\square$

Proof of Theorem 3.2(3). Since each $i$-increasing braid is conjugate to an $n$-increasing braid of a standard form in Lemma 4.6, we may assume that $b \in B_n$ is an $n$-increasing braid of the form $b = (w_1\sigma_{n-1}^2) \cdots (w_u\sigma_{n-1}^2)$. Since $b \in B_n$ is the periodic braid such that $\rho = \sigma_1\sigma_2 \cdots \sigma_{n-2}\sigma_{n-1}^2$ we have $\sigma_{n-1}^2 = (\sigma_1 \cdots \sigma_{n-2})^{-1}\rho$. Then $b$ is expressed as follows.

$$b = (\nu_1 \rho) \cdots (\nu_u \rho),$$

where $\nu_i = w_i(\sigma_1 \cdots \sigma_{n-2})^{-1}$ is written by a word of $\sigma_1^{\pm 1}, \cdots, \sigma_{n-2}^{\pm 1}$, but not $\sigma_{n-1}^{\pm 1}$. Each $\nu_j$ in $b$ is a reducible braid and $\rho$ in $b$ is the periodic braid.
Let $\omega_j : F_b \to F_b$ denote a reducible representative whose mapping class is determined by $\nu_j$, and let $\psi : F_b \to F_b$ denote a periodic representative whose mapping class determined by $\rho$. The monodromy $\phi_b$ defined on $F_b$ is written by $\phi_b = (\omega_1 \psi) \cdots (\omega_d \psi)$.

Recall that $\mathbb{D}_{n-1}$ is the disk $\mathbb{D}$ with marked points $A_1, \cdots, A_{n-1}$. Let $S_0$ be an $n$-holed sphere obtained from $\mathbb{D}_{n-1}$ by removing the interiors of small $(n-1)$ disks with centers $A_1, \cdots, A_{n-1}$. Each $\nu_j$ as an $(n-1)$-braid determines a homeomorphism $\tilde{\omega}_j : S_0 \to S_0$. We may assume that $\tilde{\omega}_j$ fixes one of the boundary components corresponding to $\partial \mathbb{D}$ pointwise. It is clear that we have an embedding $h : S_0 \hookrightarrow F_b$ such that each $\omega_j$ in $\phi_b$ is reducible supported on the subsurface $h(S_0)$ and the restriction of $\omega_j$ to $h(S_0)$ is given by $h \circ \tilde{\omega}_j \circ h^{-1}$.

By the proof of Theorem 3.2(2), $\phi_{(x,y)} : F_{(x,y)} \to F_{(x,y)}$ associated with each primitive class $(x,y) \in C_{(b,i)}$ is determined by the braid of the form $b[p_1, \ldots, p_{\ell}]$. We now prove by the induction on length $\ell$ that

$$b[p_1, \ldots, p_{\ell}] = (\nu_1 \rho) \cdots (\nu_{\ell-1} \rho)(\nu_\ell \rho)^{m-1} = (\nu_1 \rho) \cdots (\nu_{\ell-1} \rho)(\nu_\ell \rho)^m$$

for some $m \geq 1$ depending on $(x,y)$. Here each $\nu_j$ in $b[p_1, \ldots, p_{\ell}]$ is a reducible braid which is an extension of $\nu_j$ in $b$ and $\rho$ is the periodic braid with the degree of $b[p_1, \ldots, p_{\ell}]$. If this holds, then $\phi_{(x,y)}$ has a desired property as in Theorem 3.2(3). Suppose that $\ell = 1$. If $p_1 = 0$, then $b[0] = b$ and we are done. If $p_1 \geq 1$, then $b[p_1] = b[p_1]$. Using the above expression of $b$ we observe that $b[p_1]$ is written by

$$b[p_1] = (\nu_1 \rho) \cdots (\nu_p \rho) \in B_{n+p_1 u}$$

(see Figure 13). We are done.

For $\ell \geq 2$, suppose that $b[p_1, \ldots, p_{\ell-1}] = (\nu_1 \rho_d) \cdots (\nu_{\ell-1} \rho_d)(\nu_\ell \rho_d^m)$ for some $m$, where $d$ is the degree of $b[p_1, \ldots, p_{\ell-1}]$. Consider $b[p_1, \ldots, p_{\ell}]$ with

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure13.png}
\caption{The figure illustrates how an initial braid $b$ generates $\{b_p\}$. (1) $b = w_1 \sigma_2^2 w_2 \sigma_3^2 = (\nu_1 \rho)(\nu_2 \rho) \in B_4$, where $\nu_j = w_j(\sigma_1 \sigma_2)^{-1}$. (2) $b_1 = (\nu_1 \rho)(\nu_2 \rho) \in B_6$. (3) $b_2 = (\nu_1 \rho)(\nu_2 \rho) \in B_8$.}
\end{figure}
length $\ell$. If $\ell$ is even, then by induction hypothesis
\[ b[p_1, \ldots, p_\ell] = (b[p_1, \ldots, p_{\ell-1}])\Delta_d^{2\ell} = (\nu_1\rho_d) \cdots (\nu_u-1\rho_d)(\nu_u\rho_d^m)\Delta_d^{2\ell}. \]
Since $\Delta_d^2 = \rho_d^{d-1}$ we have $(\nu_u\rho_d^m)\Delta_d^{\ell} = \nu_u\rho_d^{m+p(\ell-1)}$. Thus $b[p_1, \ldots, p_\ell]$ has a desired expression and we are done. If $\ell$ is odd, then by induction hypothesis again
\[ b[p_1, \ldots, p_\ell] = (b[p_1, \ldots, p_{\ell-1}])_{p_i} = ((\nu_1\rho_d) \cdots (\nu_u-1\rho_d)(\nu_u\rho_d^m))_{p_i}. \]
As in the case of $\ell = 1$, the braid in the right-hand side is expressed as
\[ (\nu_1\rho_d) \cdots (\nu_u-1\rho_d)(\nu_u\rho_d^m)_{p_i} = (\nu_1\rho_t) \cdots (\nu_u-1\rho_t)(\nu_u\rho_t^m), \]
where $\dagger$ is the degree of $b[p_1, \ldots, p_\ell]$. This completes the proof.\hfill\Box

5. Sequences of pseudo-Anosov braids with small normalized entropies

In this section we prove Theorem A. We begin with an observation. Let
\[ \Omega \subset \{ a \in \mathcal{C} \mid \|a\| = 1 \} \]
be a compact set in $H_2(M_b, \partial M_b; \mathbb{R})$ and let $\mathcal{C}_\Omega \subset \mathcal{C}$ denote the cone over $\Omega$ through the origin. By Theorem 2.3(2) there is a constant $P = P(\Omega) > 0$ depending on $\Omega$ such that $\text{Ent}(a) < P$ for any $a \in \mathcal{C}_\Omega$. This observation provides us many sequences of pseudo-Anosov braids with small normalized entropies from a single pseudo-Anosov braid $b$.

**Theorem 5.1.** Suppose that $b$ is a pseudo-Anosov braid whose permutation has a fixed point. We fix any $0 < \ell < \infty$. Let $\{(x_p, y_p)\}$ be a sequence of primitive integral classes in $C_{(b, i)}$ such that $y_p/x_p < \ell$ and $\|(x_p, y_p)\| \asymp p$. Then the sequence of pseudo-Anosov braids $\{b_{(x_p, y_p)}\}$ has a small normalized entropy.

**Proof.** If $\{(x_p, y_p)\}$ is the sequence under the assumption, then we have $d(b_{(x_p, y_p)}) \asymp \|(x_p, y_p)\| \asymp p$. (Recall that $d(\cdot)$ denotes the degree of the braid, i.e., the number of the strands.) Since $(1, 0) \in C_{(b, i)} \subset \mathcal{C}$ and the slope of $y_p/x_p$ is bounded by $\ell$ from above, the set of projective classes $(x_p, y_p)$ is contained in some compact set in $\{ a \in \mathcal{C} \mid \|a\| = 1 \}$ (Figure 9). Thus there is a constant $P = P(\ell) > 1$ such that $\text{Ent}(b_{(x_p, y_p)}) < P$ for any $p$. This completes the proof.\hfill\Box

Let us discuss three sequences coming from Example 4.4. They are $\{b_p\}$, $\{\beta_p\}$ and $\{(b\Delta^{2p})_1\}$ varying $p$. It is not hard to see that $d(b_p)$, $d(\beta_p)$, $d((b\Delta^{2p})_1) \asymp p$.

**Theorem 5.2.** For an $i$-increasing and pseudo-Anosov $b \in B_n$, we have the following on the sequences of pseudo-Anosov braids.

1. $\{b_p\}$ has a small normalized entropy if and only if $[E_{(b, i)}]$ is a fibered class.
(2) For $\beta = b\Delta^2 \in B_n$, $\{\beta_p\}$ has a small normalized entropy and $\text{Ent}(\beta_p) \to \text{Ent}((1,1))$ as $p \to \infty$.

(3) $\{(b\Delta^{2p})_1\}$ has a small normalized entropy and $\text{Ent}((b\Delta^{2p})_1) \to \text{Ent}(b)$ as $p \to \infty$.

**Proof of Theorem 5.2.** For $a = (x, y) \in C_{(b,i)}$, let $a = (x, y)$ denote its projective class. We have $[F_{\beta_p}] = (1, p) \to [E_{(b,i)}] = (0,1)$ as $p \to \infty$. If $[E_{(b,i)}] \in \mathcal{C}$ by Remark 4.2 and $\text{Ent}(b) \to \text{Ent}([E_{(b,i)}])$ as $p \to \infty$ by Theorem 2.3(2). If $[E_{(b,i)}]$ is a non-projective class, then $[E_{(b,i)}] \in \partial \mathcal{C}$ by Remark 4.2 and $\text{Ent}(b) \to \infty$ as $p \to \infty$ by Theorem 2.3(3). We finish the proof of (1). We turn to (2). Since $[F_{\beta_p}] = (p+1, p) \in C_{(b,i)}$, its projective class goes to $(1,1)$ as $p \to \infty$. Since $(1,1) \in C_{(b,i)} \subset \mathcal{C}$ by Theorem 3.2(1), $\text{Ent}(\beta_p) \to \text{Ent}((1,1))$ as $p \to \infty$ by Theorem 2.3(2). This completes the proof of (2). Finally we prove (3). The fibered class of $F$-surface of $(b\Delta^{2p})_1$ is given by $(p+1, 1) \in C_{(b,i)}$. Its projective class goes to $[F_b] = (1,0)$ as $p \to \infty$. Thus $\text{Ent}((b\Delta^{2p})_1) \to \text{Ent}(b)$ as $p \to \infty$. This completes the proof. \[\square\]

We use Theorem 5.2(1)(2) in Section 8. For an application using (3), see [19].

**Proof of Theorem A.** Suppose that $b \in B_n$ is pseudo-Anosov with $\pi_b(i) = i$. Let $\beta(k)$ denote $b\Delta^{2k} \in B_n$ for $k \geq 1$. Clearly $\beta(k)$ is pseudo-Anosov with the same dilatation as $b$ (for any $k$) and $\beta(k)$ is positive for $k$ large. We fix such large $k$. By Lemma 3.1 $\beta(k)$ is $i$-increasing. If we let $z_p = (\beta(k)\Delta^{2p})_1$, then $M_{z_p} \simeq M_{\beta(k)} \simeq M_b$ holds for $p \geq 1$. By Theorem 5.2(3), $\{z_p\}$ has a small normalized entropy and $\text{Ent}(z_p) \to \text{Ent}(\beta(k)) = \text{Ent}(b)$ as $p \to \infty$. \[\square\]

Let $b^*_p$ denote the braid obtained from $(i + pu)$-increasing $b_p$ by removing the strand of the index $i + pu$. Taking its spherical element we have $S(b^*_p)$. A mild generalization of the sequence $\{b_p\}$ is the ones $\{b^*_p\}$ and $\{S(b^*_p)\}$ varying $p$. Although $b^*_p$, $S(b^*_p)$ may not be pseudo-Anosov, they are frequently pseudo-Anosov. To be more precise, we need to consider the number of prongs of singularities in the stable foliation $F_{b_p}$ for $b_p$ as we explained in Section 2.3. This is the motivation of the study in Section 6.

6. Stable foliation for the monodromy

Let $b$ be pseudo-Anosov and $i$-monotonic with the sign $\epsilon(b,i) = \epsilon$. For any primitive integral class $(x, y) \in C_{(b,i)}$, the oriented sum $F_{(x,y)} = xF_b + yE_{(b,i)}$ is connected. Let $T_{(b,A)}$ and $T_{(b,i)}$ denote the tori $\partial N(A)$ and $\partial N(\text{cl}(b(i)))$ respectively. Let us set

$$\partial_{(b,A)}F_{(x,y)} = \partial F_{(x,y)} \cap T_{(b,A)} \quad \text{and} \quad \partial_{(b,x)}F_{(x,y)} = \partial F_{(x,y)} \cap T_{(b,i)},$$

each of which is a single simple closed curve on the torus (since $\text{gcd}(x, y) = 1$). Recall that we chose the orientation of the axis for the $i$-monotonic $b$
in Section 3. We use the meridian and longitude basis \{m_A, \ell_A\} for \(T_{(b,A)}\) to represent a homology class of a disjoint union of simple closed curves on \(T_{(b,A)}\). We also use the meridian and the longitude basis \{m_i, \ell_i\} for \(T_{(b,i)}\).

Observe that the homology classes \([\partial_{(b,A)}F_{(x,y)}]\) and \([\partial_{(b,i)}F_{(x,y)}]\) are given by the pairs of integers

\[\begin{align*}
[\partial_{(b,A)}F_{(x,y)}] &= (\epsilon y, x) \\
[\partial_{(b,i)}F_{(x,y)}] &= (\epsilon x, y).
\end{align*}\]  

They are called boundary slopes of \(F_{(x,y)}\). See Figure 14.

Figure 14. Case: \(b\) is \(i\)-increasing. (1) Meridian and longitude basis. (2) Two boundary slopes \(\partial_{(b,A)}F_{(1,1)}\) (in green) on \(T_{(b,A)}\) and \(\partial_{(b,i)}F_{(1,1)}\) (in red) on \(T_{(b,i)}\) when \((x,y) = (1,1)\).

Let \(\phi_b : F_b \to F_b\) be the pseudo-Anosov monodromy of a fiber \(F_b\) of the fibration on \(M_b \to S^1\). The stable foliation \(F_b\) of \(\phi_b\) has singularities on each boundary component of \(F_b\). Now we consider the suspension flow \(\phi_{b,t}\) on the mapping torus \(M_b\). We obtain a disjoint union of simple closed curves \(c_A = c_{(b,A)}\) on \(T_{(b,A)}\) (possibly a single simple closed curve) which is a union of closed orbits for singularities in \(\partial_{(b,A)}F_b\) under the flow. Similarly we have a disjoint union of simple closed curves \(c_i = c_{(b,i)}\) on \(T_{(b,i)}\) (possibly a single simple closed curve again) which is a union of closed orbits for singularities in \(\partial_{(b,i)}F_b\). (Figure 17 depicts these closed curves for some pseudo-Anosov 3-braid.) A useful tool is train track maps which encode those data \(\phi_b, F_b\). They also enable us to compute homology classes \([c_A]\) and \([c_i]\).

The following lemma is a consequence of Theorem 2.4(2) by Fried.

**Lemma 6.1.** Let \(\phi_{(x,y)} : F_{(x,y)} \to F_{(x,y)}\) be the monodromy of a fibration on \(M_b \to S^1\) associated with a primitive integral class \((x,y) \in C_{(b,i)}\). Then the stable foliation \(F_{(x,y)}\) for \(\phi_{(x,y)}\) is \(i([c_A], [\partial_{(b,A)}F_{(x,y)}])\)-pronged at \(\partial_{(b,A)}F_{(x,y)}\), and is \(i([c_i], [\partial_{(b,i)}F_{(x,y)}])\)-pronged at \(\partial_{(b,i)}F_{(x,y)}\), where \(i(\cdot, \cdot)\) means the geometric intersection number between homology classes of closed curves.

**Remark 6.2.** Every closed orbit of the suspension flow \(\phi_{b,t}\) on the mapping torus \(M_b\) travels around \(S^1\) direction at least once. This implies that
[c_A] has a non-zero first coordinate of the meridian and longitude basis for \( T(b,A) \), i.e., we have \([c_A] = (k, \ell) \in \mathbb{Z}^2\) with \( k \neq 0\), since the meridian for \( T(b,A) \) corresponds to the flow direction. Similarly, \([c_i]\) has a non-zero second coordinate of the meridian and longitude basis for \( T(b,i) \), that is we have \([c_i] = (k', \ell') \in \mathbb{Z}^2\) with \( \ell' \neq 0\), since the longitude for \( T(b,i) \) corresponds to the flow direction in this case.

Recall that given a braid \( b \in B_n \), we denote by \( S(b) \in SB_n \), the spherical \( n\)-braid with the same word as \( b \). For an \( i\)-increasing braid \( b \) of pseudo-Anosov type, consider the braid \( (b\Delta^2)^1 = b[0,p,1] \) in Example 4.4(3). This is an \( i[0,p,1]\)-increasing braid. Then we have its spherical braid \( S((b\Delta^2)^1) \).

We now define other braids obtained from \( (b\Delta^2)^1 \). Let \((b\Delta^2)^1\) denote the braid obtained from \((b\Delta^2)^1\) by removing the strand of the index \( i[0,p,1]\). Let \( S((b\Delta^2)^1) \) and \( S((b\Delta^2)^1) \) be the spherical braids corresponding to \((b\Delta^2)^1\) and \((b\Delta^2)^1\) respectively. Then we have the following result.

**Lemma 6.3.** Suppose that \( b \) is an \( i\)-increasing braid of pseudo-Anosov type. For \( p \) large, the braid \((b\Delta^2)^1\) and the spherical braids \( S((b\Delta^2)^1) \), \( S((b\Delta^2)^1) \) are all pseudo-Anosov with the same dilatation as \((b\Delta^2)^1\).

Before proving Lemma 6.3, we recall a formula of the geometric intersection number \( i([c], [c']) \) between two homology classes of simple closed curves \( c, c' \) on a torus. Let \((p, q)\) and \((p', q')\) be primitive elements of \( \mathbb{Z}^2 \) which represent \([c]\) and \([c']\) respectively. Then

\[
i([c], [c']) = |pq' - p'q|.
\]

**Proof of Lemma 6.3.** The fibered class of \( F\)-surface of \((b\Delta^2)^1\) is \((p + 1, 1) \in C(b,i)\). We have \([\partial(b,A)F_{(p+1,1)}] = (-1, p + 1)\) and \([\partial(b,i)F_{(p+1,1)}] = (-p+1, 1)\), see (6.1). By Remark 6.2, one can write \([c_A] = (k, \ell)\) with \( k \neq 0\) and \([c_i] = (k', \ell')\) with \( \ell' \neq 0\). Then \( i([c_A], [\partial(b,A)F_{(p+1,1)}]) = |k(p + 1) + \ell|\) and \( i([c_A], [\partial(b,i)F_{(p+1,1)}]) = |k' + \ell'(p + 1)|\). Since \( k \neq 0\) and \( \ell' \neq 0\), these intersection numbers are increasing with respect to \( p \) and they are clearly greater than 1 when \( p \) is large. Then Lemma 6.1 says that when \( p \) is large, the stable foliation \( F_{(p+1,1)} \) for the monodromy \( \phi_{(p+1,1)} \) is not 1-pronged at each component of \( \partial(b,A)F_{(p+1,1)} \cup \partial(b,i)F_{(p+1,1)} \). By the discussion in Section 2.4, we are done. \( \Box \)

7. Properties of \( F\)-surfaces and \( E\)-surfaces

The aim of this section is to study properties of \( E\), \( F\)-surfaces and to present the technique used in the last section.

**Lemma 7.1.** For an \( i\)-increasing braid \( b \in B_n \) with \( u(b,i) = u \), we set \( \beta = b\Delta^2 \in B_n \). Then there is an \( n\)-increasing braid \( \gamma \in B_{n+u} \) such that

\[
(br(\beta), cl(\beta(i)), A_\beta) \sim (br(\gamma), A_\gamma, cl(\gamma(n))).
\]

In particular \( M_\beta \simeq M_\gamma \simeq M_\gamma \) and \( E(\beta,i) = F_\gamma \), \( F_\beta = E(\gamma,\gamma) \) up to isotopy in \( M_\beta \). Moreover if \( b \) is pseudo-Anosov, then \( \gamma \) is also pseudo-Anosov.
A similar claim holds for \( i \)-decreasing braids.

**Proof.** By Lemma 4.6 we may assume that \( b \in B_n \) is an \( n \)-increasing braid of a standard form \( b = (w_1 \sigma_{n-1}^2 \cdots w_u \sigma_{n-1}^2) \) containing \( u \) subwords \( \sigma_{n-1}^2 \).

Using the identity
\[
\Delta^2 = \Delta_{n-1}^2 \sigma_{n-1} \cdots \sigma_2 \sigma_1 \sigma_1 \sigma_2 \cdots \sigma_{n-1} \in B_n,
\]
This expression says that $M_{\gamma}$ is representative of $\beta$ for the deformation as in (5)(6) of Figure 15. We can take the following $F$-surface for $\beta$ and its braided axis, namely a braided link, see Figure 15(3)(4)(5). As a result, 

$$\text{br}(\beta), \text{cl}(\gamma(n)), A_{\beta}) \sim \text{br}(\gamma), A_\gamma, \text{cl}(\gamma(n)),$$

This expression says that $M_{\beta} \simeq M_{\gamma}$ and the $E_4$ $F$-surfaces for $\beta$ are equal to the $F_4$ $E$-surfaces for $\gamma$. Since $M_b \simeq M_{\beta}$ we are done. \hfill \Box

Here we introduce a simple representative of $\gamma \in B_{n+u}$ in Lemma 7.1. By the deformation as in (5)(6) of Figure 15, we can take the following representative of $\gamma$.

$$\gamma = \kappa_0 \kappa_1 \cdots \kappa_{u+1} \Delta^2_{n-1}, \text{ where}$$

$$\kappa_0 = \sigma_n^{-1} \sigma_{n-1} \cdots \sigma_{n-u+1},$$

$$\kappa_j = w_j \sigma_{n-u-j}^{-1} \sigma_{n+u-j-1} \sigma_{n+u-j-2} \cdots \sigma_{n-1}^{-1} \text{ if } 1 \leq j \leq u-1,$$

$$\kappa_u = w_u \sigma_{n-1},$$

$$\kappa_{u+1} = \sigma_n^{-1} \text{ if } u = 1,$$

$$\kappa_{u+1} = \sigma_{n-u+1}^{-1} \sigma_{n-u+2} \cdots \sigma_n^{-1} \text{ if } u \geq 2.$$  

For example if $(n, u) = (3, 2)$, then 

$$\gamma = \kappa_0 \kappa_1 \kappa_2 \kappa_3 \Delta^2_2 = \sigma_2 \sigma_1^2 \sigma_2 \sigma_3 \sigma_4 w_1 \sigma_2 \sigma_3 \sigma_2^{-1} w_2 \sigma_2 \sigma_4^{-1} \sigma_3^{-1} \sigma_1^2.$$  

(7.1)

If $(n, u) = (3, 3)$, then $\gamma = \kappa_0 \kappa_1 \kappa_2 \kappa_3 \kappa_4 \Delta^2_2$, that is 

$$\gamma = \sigma_2 \sigma_1^2 \sigma_2 \sigma_3 \sigma_4 \sigma_5 w_1 \sigma_2 \sigma_3 \sigma_2^{-1} \sigma_4^{-1} \sigma_3^{-1} \sigma_2^{-1} \sigma_4 \sigma_5^{-1}.$$  

(7.2)

Lemma 7.1 is used in the following situation. Suppose that $\alpha \in B_{n+u}$ is a $j$-increasing braid and our task is to prove that $\alpha$ is pseudo-Anosov and its $E$-surface $E_{(\alpha, j)}$ is a fiber of a fibration on $M_\alpha \to S^1$. (The conditions are needed to apply Theorem 5.2(1) for $\alpha$.) To do this, we need to find an $i$-increasing and pseudo-Anosov braid $b \in B_n$ with $u = u(b, i)$ and need to check the resulting $n$-increasing braid $\gamma \in B_{n+u}$ in Lemma 7.1 satisfies the property

$$(\text{br}(\gamma), A_{\gamma}, \text{cl}(\gamma(n))) \sim (\text{br}(\alpha), A_\alpha, \text{cl}(\alpha(j))).$$

i.e. $\gamma$ is conjugate to $\alpha$ preserving the corresponding strand. If this equivalence holds, then by Lemma 7.1 together with the above equivalence $\sim$, our task is done. As a result $\{\alpha_p\}$ has a small normalized entropy by Theorem 5.2(1).
8. Application

In the last section we prove Theorems C, D and E. We first recall a study of pseudo-Anosov 3-braids [14, 24]. Let $w$ be a word in $\sigma_1^{-1}$ and $\sigma_2$. If both $\sigma_1^{-1}$ and $\sigma_2$ occur at least once in $w$, then we say that $w$ is a pA word. It is known that the 3-braid represented by a pA word is pseudo-Anosov. Conversely a 3-braid $b$ is pseudo-Anosov, then there is a pA word $w$ such that the braid represented by $w$ is conjugate to $b$ up to a power of the full twist.

The stable foliation $\mathcal{F}_b$ is 1-pronged at each boundary component of $F_b$ for each pseudo-Anosov 3-braid $b$. Figure 17(3) exhibits a train track automaton. A train track map for the 3-braid represented by a pA word $w$ is obtained from the closed loop corresponding to $w$ in the automaton. For more details, see Ham-Song [13].

8.1. Palindromic/Skew-palindromic braids. We define a map

$$rev : B_n \rightarrow B_n$$

$$\sigma_{i_1}^{\mu_1} \sigma_{i_2}^{\mu_2} \cdots \sigma_{i_k}^{\mu_k} \mapsto \sigma_{i_k}^{\mu_k} \cdots \sigma_{i_2}^{\mu_2} \sigma_{i_1}^{\mu_1}, \quad \mu_j = \pm 1,$$

which is an anti-homomorphism. A braid $b \in B_n$ is palindromic if $rev(b) = b$. Clearly $b \cdot rev(b)$ is palindromic for any $b \in B_n$. Let us consider another anti-homomorphism

$$skew : B_n \rightarrow B_n$$

$$\sigma_{i_1}^{\mu_1} \sigma_{i_2}^{\mu_2} \cdots \sigma_{i_k}^{\mu_k} \mapsto \sigma_{n-i_k}^{\mu_k} \cdots \sigma_{n-i_2}^{\mu_2} \sigma_{n-i_1}^{\mu_1}, \quad \mu_j = \pm 1.$$

A braid $b \in B_n$ is skew-palindromic if $skew(b) = b$. Clearly $b \cdot skew(b)$ is skew-palindromic for any $b \in B_n$.

We now prove Theorems C and D which indicate the asymptotic behaviors of minimal entropies among these subsets are quite distinct.

**Proof of Theorem C.** For the surjective homomorphism $\pi : B_n \rightarrow S_n$ we write $\pi_j = \pi(\sigma_j)$. Suppose that an $n$-braid $b = \sigma_{i_1}^{\mu_1} \sigma_{i_2}^{\mu_2} \cdots \sigma_{i_k}^{\mu_k}$ is palindromic. Since $rev(b) = b$ we have

$$(\pi_{rev(b)} = \pi_{i_k} \cdots \pi_{i_2} \pi_{i_1} = \pi_{i_1} \pi_{i_2} \cdots \pi_{i_k} (= \pi_b).$$

Multiply the both side by $\pi_{i_1} \pi_{i_2} \cdots \pi_{i_k}$ from the left:

$$(\pi_{i_1} \pi_{i_2} \cdots \pi_{i_k}) \cdot (\pi_{i_k} \cdots \pi_{i_2} \pi_{i_1}) = (\pi_{i_1} \pi_{i_2} \cdots \pi_{i_k}) \cdot (\pi_{i_1} \pi_{i_2} \cdots \pi_{i_k}) = \pi_{i_k}^2.$$

Since $\pi_{i_k}^2 = id$ the left-hand side equals $id$. Hence $id = \pi_{i_k}^2$ which means that the square $b^2$ is pure. A theorem by Song [28] states that for a pseudo-Anosov pure element $b' \in B_n$, its dilatation has a uniform lower bound $2 + \sqrt{5} \leq \lambda(b')$. In particular if $b' = b^2$, then $2 + \sqrt{5} \leq \lambda(b^2) = (\lambda(b))^2$. This completes the proof.
Proof of Theorem D. We separate the proof into two cases, depending on the parity of the braid degree. We first prove \( \log \delta(PA_{2n}) \approx 1/n \). Let us take \( \xi = \sigma_1^2 \sigma_2 \sigma_3 \sigma_4 \in B_5 \) (Figure 16). The braid \( \xi \) is 3-increasing with \( u(\xi, 3) = 2 \). We consider the disk twist about \( D(\xi, 3) \). We obtain the braid \( \xi_p \) which is \( (3 + 2p) \)-increasing for each \( p \geq 1 \). Observe that \( \xi_p \) is a skew-palindromic braid with even degree for each \( p \geq 1 \):

\[
\xi_p = (\sigma_1 \cdots \sigma_{1+2p})(\sigma_3 \cdots \sigma_{3+2p}) \in B_{4+2p}.
\]

(For the definition of \( \xi_p \), see Section 5.) By the lower bound of dilatations by Penner, it is enough to prove that the sequence \( \{\xi_p\} \) has a small normalized entropy. We prove this in the following two steps. In Step 1 we prove that \( \{\xi_p\} \) has a small normalized entropy. In Step 2 we prove that the stable foliation \( F_{\xi_p} \) is not 1-pronged at \( \partial(\xi_p, 3+2p)F_{\xi_p} \) for \( p \geq 1 \). This tells us that \( \xi_p \)

Figure 16. (1) \( \text{br}(\xi) \). (2) Skew-palindromic \( \xi^* \in B_{4+2p} \).
is pseudo-Anosov with the same dilatation as \( \xi_p \). By Step 1 it follows that \( \{\xi^*_p\} \) has a small normalized entropy.

**Step 1.** The sequence \( \{\xi_p\} \) has a small normalized entropy.

By Theorem 5.2(1) it suffices to prove that \( \xi \) is pseudo-Anosov and \( [E_{(\xi,3)}] \) is a fibered class. Consider a pseudo-Anosov braid \( b = \sigma_1^{-1}\sigma_2^{3}\sigma_1^{-1}\sigma_2^{3} \in B_3 \). It is 3-increasing with \( u(b,3) = 2 \). For \( \beta = b\Delta^2 \) we have \( M_b \simeq M_{\beta} \). By Lemma 7.1 \( (\text{br}(\beta), \text{cl}(\beta(3)), \xi, A) ) \sim (\text{br}(\gamma), \xi, \gamma(3)) \), where \( \gamma \in B_3 \) is the braid in (7.1) substituting \( \sigma_1^{-1} \) for \( w_1 \) and \( \sigma_1^{-1} \) for \( w_2 \). It is not hard to check that \( 1, \gamma \) is conjugate to \( \xi \) in \( B_3 \) and their permutations have a common fixed point 3. Hence

\[
(\text{br}(\beta), \text{cl}(\beta(3)), A) \sim (\text{br}(\xi), A, \text{cl}(\xi(3))).
\]

In particular \( E_{(\xi,3)} = F_\beta \) which means that \( E_{(\xi,3)} \) is a fiber of a fibration on the hyperbolic mapping torus \( M_\beta \simeq M_{\xi} \) over \( S^1 \). Thus \( \xi \) is pseudo-Anosov.

**Step 2.** \( F_{\xi_p} \) is \( (p + 1) \)-pronged at \( \partial_{(\xi_p,3+2p)}F_{\xi_p} \) for \( p \geq 1 \).

We read the singularity data of \( F_{\xi_p} \) from the monodromy \( \phi_{\beta} : F_{\beta} \rightarrow F_{\beta} \) of the fibration on \( M_{\beta} \rightarrow S^1 \). First consider the suspension flow \( \phi_{\beta}^1 \) on the mapping torus \( M_\beta \). Since \( F_{\beta} \) is 1-pronged at each component of \( F_{\beta} \), we have simple closed curves \( c_A \subset T_{(b,a)} \) and \( c_3 \subset T_{(b,3)} \) such that \( \{c_A\} = (1,0), \{c_3\} = (2,1) \in Z^2 \) (Figure 17(1)(2)).

Next we turn to \( \beta = b\Delta^2 \in B_3 \) and the suspension flow \( \phi_{\beta}^1 \) on \( M_{\beta} \simeq M_{\xi} \). We have simple closed curves \( c_{(\beta,3)} \subset T_{(\beta,3)} \) and \( c_{(3,3)} \subset T_{(3,3)} \). Since \( \beta \) is the product of \( b \) and \( \Delta^2 \), we get \( \{c_{(\beta,3)}\} = (1,0) + (0,1) = (1,1) \). The first term \( (1,0) \) comes from \( \{c_A\} \) and the second one \( (0,1) \) comes from \( \Delta^2 \). Similarly we have \( \{c_{(3,3)}\} = (2,1) + (1,0) = (3,1) \). By (8.1) we have \( F_{\beta} = E_{(\xi,3)} \) and \( E_{(3,3)} = F_{\xi} \). We also have \( T_{(\beta,3)} = T_{(\xi,3)} \) and \( T_{(3,3)} = T_{(\xi,A)} \).

Since

\[
p[F_{\beta}] + [E_{(3,3)}] = [F_{\xi}] + p[E_{(\xi,3)}] = [F_{\xi} + pE_{(\xi,3)}] = (1, p) \in C_{(\xi,3)},
\]

the stable foliation \( F_{(1,p)} \) associated with an integral class \( (1,p) \in C_{(\xi,3)} \) is the stable foliation associated with \( (p,1) \in C_{(\beta,3)} \). By (6.1) for \( (x,y) = (p,1) \)

\[
[\partial_{(\beta,A)}(F_{\xi} + pE_{(\xi,3)})] = (-1,p),
\]

From \( i[\{c_{(\beta,A)}\}, \{\partial_{(3,3)}(F_{\xi} + pE_{(\xi,3)})\}] = p + 1 \) together with Lemma 6.1, one sees that \( F_{(1,p)} \) associated with \( (1,p) \in C_{(\xi,3)} \) is \( (p+1) \)-pronged at \( \partial_{(\beta,A)}F_{(1,p)}(= \partial_{(\xi,3)}F_{(1,p)}) \), and is \( (p+3) \)-pronged at \( \partial_{(3,3)}F_{(1,p)}(= \partial_{(\xi,A)}F_{(1,p)}) \).

\[1\] There is a solution for the conjugacy problem on \( B_n [6] \). The software *Braiding [12]* can be used to determine whether two braids are conjugate.
Since \( g_p : M_{\xi} \to M_{\xi_p} \) sends \( F_{(1,p)} \) to \( F_{\xi_p} \), the stable foliation \( F_{(1,p)} \) associated with \((1,p) \in C_{(\xi,3)} \) is identified with \( F_{\xi_p} \) via \( g_p \). The boundary components \( \partial_{\xi(A)} F_{(1,p)} \) and \( \partial_{\xi(A)} F_{\xi_p} \) correspond to \( \partial_{\xi(1,p)} F_{\xi_p} \) and \( \partial_{\xi_p,3+2p} F_{\xi_p} \) respectively via \( g_p \). Thus \( F_{\xi_p} \) is \((p+1)\)-pronged at \( \partial_{\xi_p,3+2p} F_{\xi_p} \). This completes the proof of Step 2.

Next we prove \( \log \delta(PA_{2n+1}) \asymp 1/n \) following the above arguments in Steps 1,2. Take an initial braid

\[
\eta = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_3 \sigma_4 \sigma_1 \sigma_4 \sigma_5 \sigma_6 \sigma_7 \in B_8.
\]

It is 4-increasing with \( u(\eta,4) = 2 \). Consider \( \eta_p \in B_{8+2p} \) obtained from \( \eta \) by the disk twist. Then \( \eta_p^* \) is a skew-palindromic braid with odd degree for each \( p \geq 1 \):

\[
\eta_p^* = (\sigma_1 \sigma_2 \cdots \sigma_4 \cdot \sigma_6 + 2)(\sigma_3 \sigma_4 \cdots \sigma_6 + 2p) \in B_{7+2p}.
\]

For our purpose it suffices to prove that \( \{\eta_p^*\} \) has a small normalized entropy. Following Step 1 we first prove that \( \eta \) is pseudo-Anosov and \([E_{(\eta,4)}]\) is a fibered class. Consider a pseudo-Anosov braid \( b = \sigma^{-1} \sigma_2^6 \Delta^2 \in B_3 \) which is 3-increasing with \( u(b,3) = 5 \). For \( \beta = b \Delta^2 \) Lemma 7.1 tells us that \((\text{br}(\beta), \text{cl}(\beta(3)), A_{\beta}) \sim (\text{br}(\gamma), A_{\gamma}, \text{cl}(\gamma(3)))\), where \( \gamma = \kappa_0 \kappa_1 \cdots \kappa_6 \Delta^2 \in B_8 \).

One sees that \( \gamma \) is conjugate to \( \eta \) in \( B_8 \). Since the permutation \( \pi_\eta \) has a unique fixed point it follows that \((\text{br}(\beta), \text{cl}(\beta(3)), A_{\beta}) \sim (\text{br}(\eta), A_{\eta}, \text{cl}(\eta(4)))\).

This expression says that \( E_{(\eta,4)} = F_\beta \) is a fiber of a fibration on the hyperbolic \( M_b \simeq M_\eta \) over \( S^1 \). Hence \( \eta \) is pseudo-Anosov. We conclude that \( \{\eta_p\} \) has a small normalized entropy.

Following Step 2 one sees that \( F_{\eta_p} \) is \((p+2)\)-pronged at \( \partial_{(\eta_p,4+2p)} F_{\eta_p} \) for \( p \geq 1 \). Thus \( \eta_p^* \) is pseudo-Anosov with the same dilatation as \( \eta_p \). This completes the proof.

8.2. Spin mapping class groups. In this section we prove Theorem E. We first recall a connection between \( \mathcal{H}(\Sigma_g) \) and \( \text{Mod}(\Sigma_{0,2g+2}) \). Let \( t_j \in \text{Mod}(\Sigma_g) \) for \( 1 \leq j \leq 2g + 1 \) be the right-handed Dehn twist about the simple closed curve \( C_j \) as in Figure 18. Birman-Hilden [3] proved that \( \mathcal{H}(\Sigma_g) \) is generated by \( t_1, t_2, \ldots, t_{2g+1} \). In fact they prove that

\[
\begin{align*}
Q : \mathcal{H}(\Sigma_g) & \to \text{Mod}(\Sigma_{0,2g+2}) \\
 t_j & \mapsto t_j
\end{align*}
\]

Figure 18. Simple closed curve \( C_j \) on \( \Sigma_g \).
Proof. We prove the lemma by the induction on \( p \). When \( p = 1 \)

\[ t_2 t_3 (t_4 t_5 \cdots t_{5+2p})^2 t_{5+2p} \in \text{Mod}_g[q_1] \text{ for any } g \geq p + 2. \]
Assume that \( t_2t_3(t_4t_5 \cdots t_{5+2(p-1)})^2 t_{5+2(p-1)} \in \text{Mod}_g[q_1] \) for \( g \geq p - 1 + 2 \). By the braid relations, \( t_2t_3(t_4t_5 \cdots t_{4+2(p-1)}t_{5+2(p-1)}t_{4+2p}^2) t_{5+2p} \) is equal to
\[
t_2t_3(t_4t_5 \cdots t_{5+2(p-1)})^2 t_{5+2(p-1)} \cdot t_{5+2(p-1)}^{-2} \cdot t_{5+2(p-1)}^{-2} \cdot t_{4+2p} \cdot t_{5+2p}^{-2} \cdot t_{5+2p}^{-2}.
\]

Note that \( t_j t_{j+1} t_{j-1} t_j = (t_j t_{j+1}^{-1})(t_j t_{j-1}^{-1}) t_j^2 \). Then the assumption together with Lemma 8.1(1) implies that \( t_2t_3(t_4t_5 \cdots t_{5+2p}) t_{5+2p} \in \text{Mod}_g[q_1] \) for \( g \geq p + 2 \).

Let us turn to (2). When \( p = 1 \)
\[
(t_2t_3t_4t_5t_6t_7)^2 = t_2t_3t_4^{-1} \cdot t_4^2 \cdot t_4t_5t_4^{-1} \cdot t_4^2 \cdot t_6t_7t_6^{-1} \cdot t_6^2 \cdot t_7^2 \cdot t_7^2,
\]
which is an element of \( \text{Mod}_g[q_0] \) for \( g \geq 3 \).

Assume that \( (t_2t_3 \cdots t_{5+2(p-1)})^2 t_{5+2(p-1)} \in \text{Mod}_g[q_0] \) for any \( g \geq p - 1 + 2 \). By the braid relations again, we have
\[
(t_2t_3 \cdots t_{4+2(p-1)}t_{5+2(p-1)}t_{4+2p}^{-1})^2 t_{5+2p}^{-2} = (t_2t_3 \cdots t_{5+2(p-1)})^2 t_{5+2(p-1)}^{-4} \cdot t_{4+2p} \cdot t_{5+2p}^{-2} \cdot t_{5+2p}^{-2}.
\]

The assumption together with Lemma 8.1(2) says that \( (t_2t_3 \cdots t_{5+2})^2 t_{5+2p} \in \text{Mod}_g[q_0] \) for \( g \geq p + 2 \). This completes the proof. \( \square \)

The shift map \( sh : B_n \to B_{n+1} \) is an injective homomorphism sending \( \sigma_j \) to \( \sigma_{j+1} \) for \( 1 \leq j \leq n - 1 \). Suppose that \( b \in B_n \) is pseudo-Anosov. Then \( S(sh(b)) \in SB_{n+1} \) is pseudo-Anosov with the same dilatation as \( b \) since \( \tilde{\Gamma}(S(sh(b))) \) is conjugate to \( f_b = c(\Gamma(b)) \) in \( \text{Mod}(\Sigma_{o,n+1}) \). (See Section 2.3 for definitions \( \Gamma, \tilde{\Gamma} \).) We finally prove Theorem E.

**Proof of Theorem E(1).** Consider \( o = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_3^2 \sigma_4 \sigma_5 \sigma_3 \sigma_5 \in B_6 \). It is a 4-increasing braid with \( u(o,4) = 2 \) (Figure 19). The braid \( o_p \) is obtained from \( o \) by disk twist for each \( p \geq 1 \). Then
\[
S(sh(o_p^*)) = \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_3^2 \sigma_4 \sigma_5 \sigma_3 \sigma_5 \in B_6.
\]

By Lemma 8.2(1) \( t_2t_3(t_4t_5 \cdots t_{5+2})^2 t_{5+2} \in \text{Mod}_{p+2}[q_1] \) for \( p \geq 1 \), and it is pseudo-Anosov if \( S(sh(o_p^*)) \) is pseudo-Anosov. In this case they have the same dilatation. Thus by the relation between \( o_p^* \) and \( S(sh(o_p^*)) \) it is enough to prove that \( \{ o_p^* \} \) has a small normalized entropy. We first claim that \( \{ o_p \} \) has a small normalized entropy. By Theorem 5.2(1) it suffices to prove that \( o \) is a pseudo-Anosov and \( [E(o,4)] \) is a fibered class. Consider a 3-braid \( b = \sigma_1^2 \sigma_2^2 \cdot \sigma_3^2 \cdot \sigma_4^2 \) which is 3-increasing with \( u(b,3) = 3 \). Let \( \beta \) denote \( b\Delta^2 \).

By Lemma 7.1 \( (br(\beta), cl(\beta(3)), A_\beta) \sim (br(\gamma), A_\gamma, cl(\gamma(3))) \), where \( \gamma \in B_6 \) is the braid in (7.2) substituting \( \sigma_1^2, \emptyset, \emptyset \) for \( w_1, w_2, w_3 \) respectively. In this case \( \gamma \) is conjugate to \( o \) in \( B_6 \). Since the permutation \( \pi_o \) has a unique fixed point 4, it follows that \( (br(\beta), cl(\beta(3)), A_\beta) \sim (br(o), A_o, cl(o(4))) \). This tells us that \( M_\beta \simeq M_o \) and \( [E(o,4)] = [F_\beta] \) is a fibered class. On the other hand \( \beta \)
is conjugate to $\sigma_1^4 \sigma_2^{-2} \Delta^4$ in $B_3$ which means that $\beta$ is pseudo-Anosov. Thus $M_\beta \simeq M_o$ is hyperbolic and $o$ is pseudo-Anosov.

Next we prove that $o_p^*$ is pseudo-Anosov with the same dilatation as $o_p$ for $p \geq 1$. By the same argument as in the proof of Theorem D one sees that $F_{o_p}$ is $(p+2)$-pronged at $\partial(o_p,4+2p)F_{o_p}$. Thus $o_p^*$ has the desired property for $p \geq 1$. We finish the proof of (1).

We turn to (2). Let us consider $v = (\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5)^2 \sigma_1 \sigma_2 \sigma_3^3 \in B_6$ which is $3$-increasing with $u(v,3) = 2$. Let $v_p \in B_{6+2p}$ be the braid obtained from $v$ by the disk twist. Then $v_p$ is $(3+2p)$-increasing and

$$v_p^* = (\sigma_1 \sigma_2 \cdots \sigma_{4+2p})^2 \sigma_{4+2p}^3 \in B_{5+2p},$$

$$S(sh(v_p^*)) = (\sigma_2 \sigma_3 \cdots \sigma_{5+2p})^2 \sigma_{5+2p}^3 \in SB_{6+2p}.$$  

By Lemma 8.2(2) it is enough to prove that $\{v_p^*\}$ has a small normalized entropy. To do this we first prove that $\{v_p\}$ has a small normalized entropy. Consider a pseudo-Anosov 3-braid

$$b = \sigma_1^2 \sigma_2^{-2} \Delta^4 = \sigma_1^3 \sigma_2^3 \sigma_1 \Delta^2 = \sigma_1^3 \sigma_2^2 \cdot \sigma_1 \sigma_2^2$$

which is 3-increasing with $u(b,3) = 3$. Lemma 7.1 tells us that for $\beta = b \Delta^2$ we have $(br(\beta), cl(\beta(3)), A_\beta) \sim (br(\gamma), A_\gamma, cl(\gamma(3)))$, where $\gamma \in B_6$ is the braid in (7.2) substituting $\sigma_1^2$ for $w_1$, $\sigma_1^2$ for $w_2$ and $\sigma_1$ for $w_3$. One sees that $\gamma$ is conjugate to $v$ in $B_6$. Thus $(br(\beta), cl(\beta(3)), A_\beta) \sim (br(v), A_v, cl(v(3)))$. This implies that $[E_{(v,3)}] = [F_{\beta}]$ is a fibered class of the hyperbolic $M_\beta \simeq M_v$, and hence $v$ is pseudo-Anosov. By Theorem 5.2(1), $\{v_p\}$ has a small normalized entropy.

One sees that $F_{v_p}$ is $(p+3)$-pronged at $\partial(v_p,3+2p)F_{v_p}$. Thus $v_p^*$ is pseudo-Anosov with the same dilatation as $v_p$ for $p \geq 1$. This completes the proof. 

□

References


A CONSTRUCTION OF PSEUDO-ANOSOV BRAIDS


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