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Analyticity and kernel stabilization of unbounded derivations on C^*-algebras

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Abstract. We first show that a derivation studied recently by E. Christensen has a set of analytic elements which is strong operator topology-dense in the algebra of bounded operators on a Hilbert space, which strengthens a result of Christensen. Our second main result shows that this derivation has kernel stabilization, that is, no elements have derivative eventually equal to 0 unless their first derivative is 0. As applications, we (1) show that a family of derivations on C*-algebras studied by Bratteli and Robinson has kernel stabilization, and (2) we provide sufficient conditions for when two operators which satisfy the Heisenberg Commutation Relation must both be unbounded.

Contents

1. Introduction

Given an algebra $\mathcal{A}$ with involution and a fixed element $a \in \mathcal{A}$ such that $a = a^*$, the map $\delta_a : \mathcal{A} \to \mathcal{A}$ by $\delta_a(b) := [ia, b]$ (where $[x, y] = xy - yx$) is a $*$-derivation, that is, $\delta_a(b^*) = \delta_a(b)^*$ for all $b \in \mathcal{A}$. Conversely, for an arbitrary $*$-derivation $\delta : \mathcal{A} \to \mathcal{A}$, certain conditions on the algebra can imply $\delta = \delta_a$ for some $a \in \mathcal{A}$. The correspondence between derivations on algebras and their representation as commutators has a rich history and is deeply connected to the mathematical formulation of quantum mechanics.

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To illustrate, a quantum system can be modeled by a Hilbert space H and the associated Hamiltonian of that quantum system is given by a self-adjoint operator D whose domain is a dense subspace of H. Despite the potential for D to be unbounded, we wish to consider commutators of D with elements of $B(H)$. As not every $x \in B(H)$ will result in the commutator $[D, x]$ being defined and bounded on a dense subspace of H, the definition of the derivation "δ_D" is ambiguous. A plethora of literature is dedicated to exploring the various definitions of δ_D and their corresponding domains, and in each situation, if D is unbounded then the domain of δ_D is a proper subspace of $B(H)$. In turn, further research has been dedicated to the study of unbounded derivations on an abstract C^*-algebra. The unboundedness of such a derivation creates complexities that are not found with derivations defined on the entire C^*-algebra. In [6], Kadison summarizes three of the many significant results pertaining to bounded derivations:

(1) Every such derivation on a commutative C^*-algebra is 0. (This follows from the Singer-Wermer Theorem from 1955 in [12].)
(2) Sakai (1959) showed in [10] that every derivation on a C^*-algebra is automatically bounded, thus affirmatively settling a 1953 conjecture of Kaplansky.
(3) In [7], Kaplansky showed every bounded derivation δ of a type I von Neumann algebra M is *inner*, i.e., there exists $a \in M$ such that $\delta = \delta_a$.

We turn our attention to densely-defined derivations on C^*-algebras. Our primary setting of interest is a $*$-derivation δ_w^D on $B(H)$ defined by commutation with a fixed (possibly unbounded) self-adjoint operator D. In Section 2 we give a formal definition of δ_w^D, its domain, domains of its higher powers, and state its desirable properties. All of these can be found in [3]. In particular, Christensen shows that the domain of δ_w^D is strong operator topology (SOT)-dense in $B(H)$. We strengthen this property in Theorem 3.15, stated as the following theorem.

Theorem. *The set of analytic elements for δ_w^D is SOT-dense in $B(H)$.*

Our second main result, Theorem 4.6, shows δ_w^D has a property called *kernel stabilization.*

Theorem. *If H is a Hilbert space and D is a (possibly unbounded) self-adjoint operator on H, then $\ker(\delta_w^D)^n = \ker \delta_w^D$ for all $n \in \mathbb{N}$.*

The proof requires use of Christensen's work in [4] and [3]. Let D be an unbounded self-adjoint operator on H. Seeking to formalize the connection between commutators and unbounded derivations on $B(H)$ of the form δ_D, Christensen showed in [3] that $x \in B(H)$ makes $[D, x]$ defined and bounded on a core for D if and only if for every $h, k \in H$, the map $t \mapsto \langle e^{itD} x e^{-itD} h, k \rangle$ is continuously differentiable. If x satisfies this, we say x is *weakly D-differentiable*, denoted $x \in \text{dom } \delta_w^D$. Define $\delta_w^D(x)$ to be the bounded extension of $[iD, x]$ to all of H. Christensen defines higher weak D-differentiability in [4] and extends the aforementioned equivalence.

In Section 4, we prove Theorem 4.6, and in Section 5, we give two applications. The first extends the property of kernel stabilization to a class of unbounded $*$-derivations on C^*-algebras described in the following theorem.

Theorem 1.1 (Bratteli-Robinson, Theorem 4 [1])**.** *Let δ be a derivation of a C^*-algebra $\mathcal{A}$, and assume there exists a state ω on $\mathcal{A}$ which generates a faithful cyclic representation (π, H, f) satisfying*

$$\omega(\delta(a)) = 0, \quad \forall a \in dom\, \delta.$$

Then δ is closable and there exists a symmetric operator S on H such that

$$dom\, S = \{h \in H : h = \pi(a)f \quad for\ some\ a \in \mathcal{A}\}$$

and $\pi(\delta(a))h = [S, \pi(a)]h$, for all $a \in dom\, \delta$ and all $h \in dom\, S$. Moreover, if the set $\mathsf{A}(\delta)$ of analytic elements for δ is dense in $\mathcal{A}$, then S is essentially self-adjoint on dom S. For $x \in B(H)$ and $t \in \mathbb{R}$, define

$$\alpha_t(x) := e^{i\overline{S}t} x e^{-i\overline{S}t}$$

where $\overline{S}$ denotes the self-adjoint closure of S. It follows that $\alpha_t(\pi(\mathcal{A})) = \pi(\mathcal{A})$ for all $t \in \mathbb{R}$, and $\{\alpha_t\}_{t\in\mathbb{R}}$ is a strongly continuous group of automorphisms with closed infinitesimal generator $\widetilde{\delta}$ equaling the closure of $\pi \circ \delta|_{\mathsf{A}(\delta)}$.

Physically, we interpret ω as an *invariant state* of the quantum system whose observables lie in $\mathcal{A}$. Also, we interpret the condition $\omega(\delta(x)) = 0$ for all $x \in \text{dom}\ \delta$ as saying ω is an *equilibrium state* for the system. For more details, see the introduction of [2]. We state our application formally below.

Application 1. *Let $\mathcal{A}$ be a C^*-algebra, δ a derivation on $\mathcal{A}$, and ω a state on $\mathcal{A}$ which satisfy the hypotheses of Theorem 1.1. For every $n \in \mathbb{N}$,* $\ker \delta^n = \ker \delta$.

As a second application of Theorem 4.6, we provide sufficient conditions for when two operators satisfying the Heisenberg Commutation Relation must both be unbounded.

Definition 1.2. Let A and B be two (possibly unbounded) self-adjoint operators on a Hilbert space H, with domains dom A and dom B, respectively. We say A and B *satisfy the Heisenberg Commutation Relation* if there is a dense subspace K of H satisfying

$$K \subseteq \text{dom}\,[A, B] := \{h \in \text{dom}\, A \cap \text{dom}\, B : Ah \in \text{dom}\, B, Bh \in \text{dom}\, A\}$$

and $[A, B]k = ik$ for all $k \in K$.

The classical example of such a pair is the *Schrödinger pair*, which we define in Example 5.8. Note both operators in this pair are unbounded. A large body of research has been committed to finding sufficient conditions for when two operators satisying the Heisenberg Commutation Relation must be unitarily equivalent to a direct sum of copies of the Schrödinger pair, thus implying that the two operators are unbounded. We provide a sufficient condition for when two operators satisfying the HCR must be unbounded without proving they are unitarily equivalent to a direct sum of copies of the Schrödinger pair.

Application 2. *Let A and B be self-adjoint operators on a Hilbert space H which satisfy the Heisenberg Commutation Relation on a dense subspace $K \subseteq H$. If K is a core for both A and B, then A and B must be unbounded.*

As an outline of the rest of the paper, Section 2 is devoted to providing background and summarizing some of Christensen's results from [4] and [3]. In Section 3, we prove SOT-density of the analytic elements in $B(H)$ for δ_w^D, in Section 4 we prove kernel stabilization of δ_w^D, and in Section 5 we provide applications of kernel stabilization.

2. Definition and properties of weak D-differentiability

Let D be a self-adjoint operator with domain dom $D \subseteq H$. For any $t \in \mathbb{R}$, the operator e^{itD} is unitary, and the one-parameter family $\{e^{itD}\}_{t\in\mathbb{R}}$ is strongly continuous. For $x \in B(H)$ and $t \in \mathbb{R}$, define $\alpha_t(x) := e^{itD}xe^{-itD}$. Then $\{\alpha_t\}_{t\in\mathbb{R}}$ defines a flow on $B(H)$, and more specifically, is a one-parameter automorphism group on $B(H)$. While the *infinitesimal generator* of this automorphism group in the norm topology of $B(H)$ is a natural derivation to consider, we focus instead on a related derivation with a larger domain.

Definition 2.1. An operator $x \in B(H)$ is *weakly D-differentiable* if there exists $y \in B(H)$ such that for every $h, k \in H$,

$$\lim_{t\to 0}\left|\left\langle\left(\frac{\alpha_t(x)-x}{t}-y\right)h,k\right\rangle\right| = 0.$$

Equivalently, for every $h, k \in H$ the function $t \mapsto \langle \alpha_t(x)h, k\rangle$ is continuously differentiable.

Theorem 2.2 (Christensen, 3.8 [3])**.** *Let x be a bounded operator on H. The following properties are equivalent:*

(i) x is weakly D-differentiable.

(ii) There exists $y \in B(H)$ such that for every $h \in H$,

$$\lim_{t\to 0}\left\|\left(\frac{\alpha_t(x)-x}{t}-y\right)h\right\| = 0.$$

(iii) There exists $c > 0$ such that for all $t \in \mathbb{R}$,

$$\|\alpha_t(x) - x\| \leq c\,|t|\,.$$

(iv) The commutator $[iD, x]$ is defined and bounded on the domain of D.

(v) The commutator $[iD, x]$ is defined and bounded on a core for D.

(vi) *The sesquilinear form on dom* $D \times$ *dom* D *given by*

$$(h,k) \mapsto i\langle xh, Dk\rangle - i\langle xDh, k\rangle$$

is bounded.

(vii) *The matrix* $m([iD,x])_{rc} = i(DP_r x P_c - P_r x P_c D)$ *defines a bounded operator on* H*, where* $(P_n)_{n\in\mathbb{Z}}$ *are the spectral projections of the intervals* $(n-1,n]$.

If any of the above conditions hold, then

$$x(\text{dom } D) \subseteq \text{dom } D, \qquad \delta_w^D(x)|_{\text{dom } D} = i[D,x].$$

We write $x \in$ *dom* δ_w^D *and the* y *in item (ii) satisfies* $y = \delta_w^D(x)$*. Moreover, for any* $h,k \in H$, $\frac{d}{dt}\langle \alpha_t(x)h,k\rangle = \langle \alpha_t(\delta_w^D(x))h,k\rangle$.

Theorem 2.3 (Christensen, 3.9 [3])**.** *The domain of definition dom* δ_w^D *is a strongly dense* $*$*-subalgebra of* $B(H)$ *and* δ_w^D *is a* $*$*-derivation into* $B(H)$*. The graph of* δ_w^D *is weak operator topology closed.*

In Theorem 3.15 we strengthen the first statement of Theorem 2.3 by proving that the analytic elements for δ_w^D are SOT-dense in $B(H)$

Definition 2.4. An operator $x \in B(H)$ is *n-times weakly D-differentiable* if for every $k = 0,...,n-1$, $(\delta_w^D)^k(x) \in \text{dom } \delta_w^D$. We denote this by $x \in \text{dom } (\delta_w^D)^n$.

Proposition 2.5 (Christensen, 2.6 [4])**.** *A bounded operator* x *on* H *is* n*-times weakly* D*-differentiable if and only if for any pair* $h,k \in H$ *the function* $t \mapsto \langle \alpha_t(x)h,k\rangle$ *is* n*-times continuously differentiable. If* x *is* n*-times weakly* D*-differentiable, then*

$$\frac{d^n}{dt^n}\langle \alpha_t(x)h,k\rangle = \langle \alpha_t((\delta_w^D)^n(x))h,k\rangle.$$

Analogous to Theorem 2.2, Christensen shows in [4] that higher order weak D-differentiability is directly tied to iterated commutators $[iD,...,[iD,x]]$.

Proposition 2.6 (Christensen, 3.3 [4])**.** *Let* $x \in$ *dom* $(\delta_w^D)^n$*. Then for* $k = 1,...,n$,

(i) $(\delta_w^D)^{k-1}(x)(\text{dom } D) \subseteq \text{dom } D$

(ii) $x(\text{dom } D^k) \subseteq \text{dom } D^k$

(iii) $\text{dom } \underbrace{[iD,...,[iD,x]]}_{k \text{ times}} = \text{dom } D^k$

(iv) $(\delta_w^D)^k(x)|_{\text{dom } D^k} = \underbrace{[iD,...,[iD,x]]}_{k \text{ times}}$

(v) $(\delta_w^D)^k(x)$ *is the bounded extension of* $\underbrace{[iD,...,[iD,x]]}_{k \text{ times}}$ *from dom* D^k *to all of* H.

Theorem 2.7 (Christensen, 4.1 [4])**.** *Let* $x \in B(H)$ *and* n *be a natural number. The following are equivalent:*

(i) $x \in dom\,(\delta_w^D)^n$.
(ii) x *is* n *times weakly* D*-differentiable.*
(iii) *For all* $k = 1, ..., n$, $x(dom\,D^k) \subseteq dom\,D^k$ *and* $\underbrace{[iD, ..., [iD, x]]}_{k\ times}$ *is defined and bounded on dom* D^k *with closure* $(\delta_w^D)^k(x)$.
(iv) *There exists a core* $\mathcal{X}$ *for* D *such that for any* $k = 1, ..., n$, *the operator* $\underbrace{[iD, ..., [iD, x]]}_{k\ times}$ *is defined and bounded on* $\mathcal{X}$.

Notation 2.8. *For notational convenience, we define*

$$d^k(x) := \underbrace{[iD, ..., [iD, x]]}_{k\ times}$$

for each $k \in \mathbb{N}$.

3. Density of the analytic elements for δ_w^D

Definition 3.1. Let S be an operator on a Banach space X. An element $x \in X$ is an *analytic element for* S if

(1) $x \in \text{dom}\, S^n$ for all $n \in \mathbb{N}$ and
(2) there exists $t_x > 0$ such that for all $0 \leq t < t_x$, the following series converges:
$$\sum_{n=0}^{\infty} \frac{\|S^n x\|}{n!} t^n.$$

Notation 3.2. *Let* $\mathsf{A}(S)$ *denote the set of analytic elements for* S*.*

By Nelson's Analytic Vector Theorem in [8], a symmetric operator S on a Hilbert space H is essentially self-adjoint if and only if $\mathsf{A}(S)$ is dense in H. In particular, if D is a self-adjoint operator, then the set $\mathsf{A}(D)$ is dense in H. An analogous statement for δ_w^D spurs our investigation. To relate the analytic elements for D and δ_w^D, we exploit an equivalent notion of analyticity for the one-parameter families for which D and δ_w^D are infinitesimal generators: $\{e^{itD}\}_{t\in\mathbb{R}}$ and $\{\alpha_t\}_{t\in\mathbb{R}}$, respectively. We first introduce the notion of analytic elements for a general one-parameter family on a Banach space, and then we specialize to our setting.

Definition 3.3. Let X be a Banach space and let Y be a closed subspace of X^*. A one-parameter family $\{\tau_t\}_{t\in\mathbb{R}}$ of bounded linear maps of X into itself is called a $\sigma(X,Y)$*-continuous group of isometries of* X if

(1) $\tau_0 = I$,
(2) $\tau_{s+t} = \tau_s\tau_t$ for all $s, t \in \mathbb{R}$,
(3) $\|\tau_t x\| = \|x\|$ for all $t \in \mathbb{R}$, $x \in X$,
(4) $t \mapsto \tau_t(x)$ is $\sigma(X,Y)$-continuous for all $x \in X$, i.e.,
$$t \mapsto \psi(\tau_t(x))$$
is continuous for all $x \in X$ and $\psi \in Y$, and

(5) $x \mapsto \tau_t(x)$ is $\sigma(X,Y)$-$\sigma(X,Y)$ continuous for all $t \in \mathbb{R}$.

Definition 3.4. Given a $\sigma(X,Y)$-continuous group of isometries $\{\tau_t\}_{t\in\mathbb{R}}$, an element $x \in X$ is *analytic for* $\{\tau_t\}_{t\in\mathbb{R}}$ if there exist $\lambda > 0$, a strip $I_\lambda := \{z \in \mathbb{C} : |\text{Im } z| < \lambda\}$, and a function $\varphi : I_\lambda \to X$ such that

(1) $\varphi(t) = \tau_t(x)$ for all $t \in \mathbb{R}$ and
(2) $z \mapsto \psi(\varphi(z))$ is analytic on I_λ for all $\psi \in Y$.

Proposition 3.6 states that Definition 3.1 and Definition 3.4 are equivalent when S is the *infinitesimal generator* of the family $\{\tau_t\}_{t\in\mathbb{R}}$.

Definition 3.5. Given a $\sigma(X,Y)$-continuous group of isometries $\{\tau_t\}_{t\in\mathbb{R}}$, the *infinitesimal generator* S for $\{\tau_t\}_{t\in\mathbb{R}}$ is the operator whose domain consists of all elements $x \in X$ such that there exists $x' \in X$ which satisfies

$$\lim_{t\to 0} \psi\left(\frac{\tau_t(x) - x}{t} - x'\right) = 0 \text{ for all } \psi \in Y.$$

If $x \in \text{dom } S$ with corresponding difference quotient limit x', set $Sx := x'$.

Proposition 3.6 (Bratteli-Robinson, [2]). *If $\{\tau_t\}_{t\in\mathbb{R}}$ is a $\sigma(X,Y)$-continuous group of isometries with infinitesimal generator S, then x is analytic for $\{\tau_t\}_{t\in\mathbb{R}}$ if and only if $x \in \mathsf{A}(S)$.*

Consider $X = B(H)$, the one-parameter group of $*$-automorphisms $\{\alpha_t\}_{t\in\mathbb{R}}$, and the closed subspace of $B(H)^*$ defined by

$$Y := \{\psi_{f,g} : f, g \in H,\ \psi_{f,g}(x) = \langle xf, g\rangle\}.$$

Note that $\sigma(X,Y)$ is precisely the weak operator topology (WOT) on $B(H)$.

Proposition 3.7. *The family $\{\alpha_t\}_{t\in\mathbb{R}}$ is a WOT-continuous group of automorphisms with infinitesimal generator δ_w^D.*

The WOT-continuity of $\{\alpha_t\}_{t\in\mathbb{R}}$ is a simple computation and showing δ_w^D is the corresponding infinitesimal generator is immediate by the definition of weak D-differentiability. As a corollary of Propositions 3.6 and 3.7, we have the following:

Corollary 3.8. *An operator $x \in B(H)$ is analytic for $\{\alpha_t\}_{t\in\mathbb{R}}$ if and only if $x \in \mathsf{A}(\delta_w^D)$.*

Notation 3.9. Given $h, k \in H$, define the rank-one operator $h \otimes k^* \in B(H)$ by

$$(h \otimes k^*)(f) := \langle f, k\rangle\, h \text{ for all } f \in H.$$

Notation 3.10. Given subsets $S_1, S_2 \subseteq H$, let

$$\mathsf{F}(S_1, S_2) := \text{span}\{h \otimes k^* : h \in S_1, k \in S_2\}.$$

We simply denote $\mathsf{F}(S_1, S_1)$ by $\mathsf{F}(S_1)$.

Lemma 3.11. *If $S_1, S_2 \subseteq H$ are dense subspaces, then $\mathsf{F}(S_1, S_2)$ is norm-dense in $K(H)$.*

The proof of Lemma 3.11 is just an "$\frac{\varepsilon}{3}$"-argument using norm-density of $\mathsf{F}(H)$ in $K(H)$. Our initial method for proving SOT-density of the set of analytic elements for δ_w^D in $B(H)$ was to show that any rank-one operator belonging to the set $\mathsf{F}(\mathsf{A}(D))$ is analytic for δ_w^D. We were successful in proving the inclusion

$$\mathsf{F}(\text{dom } D^n) \subseteq \text{dom } (\delta_w^D)^n \text{ for every } n \in \mathbb{N}$$

but extending this argument to show $\mathsf{F}(\mathsf{A}(D)) \subseteq \mathsf{A}(\delta_w^D)$ fails. To remediate this argument, we chose to consider the set of finite-rank operators $\mathsf{F}\left(\mathsf{A}(D), R^{-1}[\mathsf{A}(D^{\#})]\right)$, where $D^{\#}$ is conjugate to D via the antiunitary Riesz map, $R : H \to H^*$ given by

$$\text{for each } h \in H, \ \ [Rh](f) := \langle f, h \rangle \ \text{ for all } f \in H.$$

Lemma 3.12. *The map $D^{\#} := RDR^{-1}$ is self-adjoint.*

Proof. To show $D^{\#} = (D^{\#})^*$, we must show dom $(D^{\#})^* =$ dom $D^{\#}$ and $D^{\#}\xi = (D^{\#})^*\xi$ for all $\xi \in$ dom $D^{\#}$. We first show $D^{\#}$ is a linear symmetric operator and then relate its adjoint's domain to the domain of D. By definition, dom $D^{\#} = R($dom $D)$. Thus, given $h \in$ dom D and $\lambda \in \mathbb{C}$, observe

$$\begin{aligned} D^{\#}(\lambda Rh) &= [RDR^{-1}](\lambda Rh) = [RD](\overline{\lambda}h) = R\left(\overline{\lambda}Dh\right) \\ &= \lambda[RDR^{-1}]Rh = \lambda D^{\#}(Rh). \end{aligned}$$

As $h \in$ dom D was arbitrary and dom $D^{\#} = R($dom $D)$, we have $D^{\#}(\lambda\xi) = \lambda D^{\#}\xi$ for all $\xi \in$ dom $D^{\#}$ and $\lambda \in \mathbb{C}$. It's easy to check additivity of $D^{\#}$, so $D^{\#}$ is linear. For $f, h \in$ dom D,

$$\begin{aligned} \left\langle D^{\#}Rh, Rf \right\rangle &= \langle RDR^{-1}Rh, Rf \rangle \\ &= \langle RDh, Rf \rangle \\ &= \langle f, Dh \rangle \\ &= \langle Df, h \rangle \\ &= \langle Rh, RDf \rangle \\ &= \left\langle Rh, D^{\#}Rf \right\rangle. \end{aligned}$$

As $f, h \in$ dom D were arbitrary and dom $D^{\#} = R($dom $D)$, we have

$$\left\langle D^{\#}\xi, \eta \right\rangle = \left\langle \xi, D^{\#}\eta \right\rangle \text{ for all } \xi, \eta \in \text{dom } D^{\#}.$$

Hence, $D^{\#}$ is symmetric. By symmetry of $D^{\#}$, we have

$$\text{dom } D^{\#} \subseteq \text{dom } (D^{\#})^* \text{ and } D^{\#}\xi = (D^{\#})^*\xi \text{ for all } \xi \in \text{dom } D^{\#}.$$

Thus, it suffices to prove dom $(D^{\#})^* \subseteq$ dom $D^{\#}$. The domain of the adjoint of $D^{\#}$ is the set

$$\begin{aligned}
&\text{dom } (D^{\#})^* \\
&= \{\eta \in H^* : \text{the map dom } D^{\#} \to \mathbb{C};\ \xi \mapsto \left\langle D^{\#}\xi, \eta \right\rangle \text{ is bounded}\} \\
&= \{\eta \in H^* : \text{the map } R(\text{dom } D) \to \mathbb{C};\ \ Rh \mapsto \left\langle D^{\#}(Rh), \eta \right\rangle \text{ is bounded}\}. \\
&= \{\eta \in H^* : \text{the map } R(\text{dom } D) \to \mathbb{C}; Rh \mapsto \left\langle R^{-1}\eta, R^{-1}D^{\#}(Rh) \right\rangle \text{ is bounded}\}. \\
&= \{\eta \in H^* : \text{the map } R(\text{dom } D) \to \mathbb{C};\ \ Rh \mapsto \left\langle R^{-1}\eta, Dh \right\rangle \text{ is bounded}\}.
\end{aligned}$$

Hence, given $\eta \in$ dom $(D^{\#})^*$, the map $R(\text{dom } D) \to \mathbb{C}$ defined by

$$Rh \mapsto \left\langle R^{-1}\eta, Dh \right\rangle \text{ for all } h \in \text{dom } D$$

is a bounded linear functional. Then, as R is isometric, the composition

$$\text{dom } D \to R(\text{dom } D) \to \mathbb{C} \quad \text{given by} \quad h \mapsto Rh \mapsto \left\langle R^{-1}\eta, Dh \right\rangle$$

defines a bounded linear functional on the domain of D. By the definition of the domain of D^*, this implies $R^{-1}\eta$ belongs to dom D^*. Further, self-adjointness of D implies $R^{-1}\eta \in$ dom D. Since R is bijective, we conclude $\eta \in R(\text{dom } D) =$ dom $D^{\#}$. Therefore, $D^{\#}$ is self-adjoint. □

Another application of Nelson's Analytic Vector Theorem in [8] implies that the set of analytic elements for $D^{\#}$, denoted $\mathsf{A}(D^{\#})$, are dense in H^*. As $R^{-1} : H^* \to H$ is antiunitary, it follows that $R^{-1}[\mathsf{A}(D^{\#})]$ is dense in H. By Lemma 3.11, we obtain norm-density of $\mathsf{F}\left(\mathsf{A}(D), R^{-1}[\mathsf{A}(D^{\#})]\right)$ in the compact operators.

Proposition 3.13. *If $h \in \mathsf{A}(D)$ and $k \in R^{-1}[\mathsf{A}(D^{\#})]$, then $h \otimes k^*$ is analytic for $\{\alpha_t\}_{t\in\mathbb{R}}$.*

Proof. Let $h \in \mathsf{A}(D)$ and $k \in R^{-1}[\mathsf{A}(D^{\#})]$. To prove $h \otimes k^*$ is analytic for $\{\alpha_t\}_{t\in\mathbb{R}}$ in the WOT, we must find $\lambda > 0$ and a function $\varphi : I_\lambda \to B(H)$ such that

(1) $\varphi(t) = \alpha_t(h \otimes k^*)$ for all $t \in \mathbb{R}$ and
(2) $z \mapsto \langle \varphi(z)f, g\rangle$ is analytic on I_λ for all $f, g \in H$.

We shall construct φ using functions obtained from analytic properties of h and k. As $h \in \mathsf{A}(D)$, Proposition 3.6 implies h is analytic for $\{e^{itD}\}_{t\in\mathbb{R}}$. Thus, there exist $\lambda_h > 0$ and a function $\varphi_h : I_{\lambda_h} \to H$ such that

(1) $\varphi_h(t) = e^{itD}h$ for all $t \in \mathbb{R}$ and
(2) $z \mapsto \langle \varphi_h(z), g\rangle$ is analytic on I_{λ_h} for all $g \in H$.

As $k \in R^{-1}[\mathsf{A}(D^{\#})]$, there exists a unique $\eta \in \mathsf{A}(D^{\#})$ such that $k = R^{-1}\eta$. Since η is analytic for $D^{\#}$, it is analytic for $\{e^{itD^{\#}}\}_{t\in\mathbb{R}}$ by Proposition 3.6. Thus, there exist $\lambda_\eta > 0$ and a function $\varphi_\eta : I_{\lambda_\eta} \to H^*$ such that

(1) $\varphi_\eta(t) = e^{itD^{\#}}\eta$ for all $t \in \mathbb{R}$ and

(2) $z \mapsto \langle \varphi_\eta(z), Rf \rangle$ is analytic on I_{λ_η} for all $f \in H$.

Note that in (2) for η, we simply identified H^* with $R(H)$.

Set $\lambda := \min\{\lambda_h, \lambda_\eta\}$, and fix $z \in I_\lambda$. Define a map

$$[\cdot,\cdot] : H \times H \to \mathbb{C} \text{ by } [f,g] := \langle \varphi_h(z), g\rangle \langle \varphi_\eta(z), Rf\rangle \text{ for all } f,g \in H.$$

Sesquilinearity of the inner products on H and H^* and antilinearity of R establishes that $[\cdot,\cdot]$ is a sesquilinear form. Moreover, for any $f,g \in H$,

$$|[f,g]| = |\langle \varphi_h(z), g\rangle| \, |\langle \varphi_\eta(z), Rf\rangle| \le \|\varphi_h(z)\| \, \|g\| \, \|\varphi_\eta(z)\| \, \|f\| \, .$$

As h, η, and z are all fixed, $[\cdot,\cdot]$ defines a bounded sesquilinear form on H. Hence, for each $z \in I_\lambda$, the Riesz Representation Theorem provides an operator $\varphi(z) \in B(H)$ such that

$$\langle \varphi(z)f, g\rangle = [f,g] = \langle \varphi_h(z), g\rangle \langle \varphi_\eta(z), Rf\rangle \text{ for all } f,g \in H.$$

As the two maps $z \mapsto \langle \varphi_h(z), g\rangle$ and $z \mapsto \langle \varphi_\eta(z), Rf\rangle$ are analytic on I_λ for all $f,g \in H$, their product $z \mapsto \langle \varphi(z)f, g\rangle$ is analytic on I_λ for all $f,g \in H$. Furthermore, for each $t \in \mathbb{R}$,

$$\begin{aligned} \langle \varphi(t)f, g\rangle &= \left\langle e^{itD}h, g\right\rangle \left\langle e^{itD^\#}\eta, Rf\right\rangle = \left\langle e^{itD}h, g\right\rangle \left\langle f, e^{itD}k\right\rangle \\ &= \langle \alpha_t(h \otimes k^*)f, g\rangle \, . \end{aligned}$$

As $f,g \in H$ were arbitrary, we have $\varphi(t) = \alpha_t(h \otimes k^*)$ for all $t \in \mathbb{R}$. Therefore, $h \otimes k^*$ is analytic for $\{\alpha_t\}_{t\in\mathbb{R}}$ in the WOT. □

Lemma 3.14. *If S is a subspace of $B(H)$ such that $S \cap \mathsf{F}(H)$ is norm-dense in $K(H)$, then S is SOT-dense in $B(H)$.*

Theorem 3.15. *The set of analytic elements for δ_w^D are SOT-dense in $B(H)$.*

Proof. By Proposition 3.6, the set of analytic elements for δ_w^D is precisely the set of analytic elements for $\{\alpha_t\}_{t\in\mathbb{R}}$. Since the set of analytic elements for $\{\alpha_t\}_{t\in\mathbb{R}}$ is a linear space, Proposition 3.13 implies $\mathsf{F}(\mathsf{A}(D), R^{-1}[\mathsf{A}(D^\#)])$ is contained in $\mathsf{A}(\delta_w^D)$. In particular,

$$\mathsf{F}(\mathsf{A}(D), R^{-1}[\mathsf{A}(D^\#)]) \subseteq \mathsf{A}(\delta_w^D) \cap \mathsf{F}(H).$$

By Lemma 3.11 and Nelson's Analytic Vector Theorem, we know that $\mathsf{F}(\mathsf{A}(D), R^{-1}[\mathsf{A}(D^\#)])$ is norm-dense in $K(H)$. Thus, by the above inclusion, we then have that $\mathsf{A}(\delta_w^D) \cap \mathsf{F}(H)$ is norm-dense in $K(H)$. From Lemma 3.14 we obtain SOT-density of $\mathsf{A}(\delta_w^D)$ in $B(H)$. □

4. Kernel stabilization of δ_w^D

In this section, we show for any self-adjoint operator D on a Hilbert space, $\ker(\delta_w^D)^n = \ker \delta_w^D$ for all $n \in \mathbb{N}$. We call this property *kernel stabilization.*

We now present the motivating example for Theorem 4.6. Given a σ-finite measure space (X, μ), define

$$\operatorname{diag} : L^\infty(X,\mu) \to B(L^2(X,\mu))$$

$$\operatorname{diag}(f) := M_f,$$

where $M_f g = fg$ for each $g \in L^2(X, \mu)$. Throughout, we denote the standard orthonormal basis for $\ell^2(\mathbb{Z})$ by $\{\epsilon_j : j \in \mathbb{Z}\}$, and we denote the matrix representation of an operator $x \in B(\ell^2(\mathbb{Z}))$ with respect to the standard orthonormal basis by $[x_{rc}]$ where

$$x_{rc} := \langle x\epsilon_c, \epsilon_r \rangle .$$

Example 4.1. Define $(Df)(j) := jf(j)$ for $f \in \operatorname{dom} D$, where

$$\operatorname{dom} D := \{f \in \ell^2(\mathbb{Z}) : \sum_{j \in \mathbb{Z}} j^2 |f(j)|^2 < \infty\}.$$

Then,

(a) the operator D is self-adjoint.

(b) an operator $x \in B(\ell^2(\mathbb{Z}))$ is n-times weakly D-differentiable if and only if for every $k \leq n$, $x(\operatorname{dom} D^k) \subseteq \operatorname{dom} D^k$ and the matrix $[i^k(r-c)^k x_{rc}]$ with dense domain $\operatorname{dom} D^k$ extends to a bounded operator on $\ell^2(\mathbb{Z})$. When either condition is satisfied,

$$[(\delta_w^D)^n(x)_{rc}]|_{\operatorname{dom} D^n} = [i^n(r-c)^n x_{rc}].$$

(c) for any $g \in \ell^\infty(\mathbb{Z})$, $\delta_w^D(M_g) = 0$.

(d) for all $n \in \mathbb{N}$, $\ker(\delta_w^D)^n = \operatorname{diag}(\ell^\infty(\mathbb{Z}))$.

Proof. (a) See Example 7.1.5 of [11].

(b) Matrix multiplication shows for any $r, c \in \mathbb{Z}$,

$$d^k(x)_{rc} = i^k(r-c)^k x_{rc}.$$

Given $x \in B(\ell^2(\mathbb{Z}))$ such that $x(\operatorname{dom} D^k) \subseteq \operatorname{dom} D^k$ for each $k \leq n$, the domain of $d^k(x)$ is $\operatorname{dom} D^k$. Theorem 2.7 states x is n-times weakly D-differentiable if and only if for every $k \leq n$, $x(\operatorname{dom} D^k) \subseteq \operatorname{dom} D^k$ and $d^k(x)$ is bounded on $\operatorname{dom} D^k$. It follows that x is n-times weakly D-differentiable if and only if $x(\operatorname{dom} D^k) \subseteq \operatorname{dom} D^k$ and $[d^k(x)_{rc}] = [i^k(r-c)^k x_{rc}]$ is bounded on $\operatorname{dom} D^k$. As D is self-adjoint, $\operatorname{dom} D^k$ is dense in $\ell^2(\mathbb{Z})$ for all $k \in \mathbb{N}$. Therefore, $[d^k(x)_{rc}]$ extends to a bounded matrix on all of $\ell^2(\mathbb{Z})$. By Theorem 2.7, the closure $(\delta_w^D)^n(x)$ is the extension of $[i^n(r-c)^n x_{rc}]$ to all of $\ell^2(\mathbb{Z})$.

(c) Fix $g \in \ell^\infty(\mathbb{Z})$, and let $f \in \operatorname{dom} D$. We show $M_g f \in \operatorname{dom} D$. Observe

$$\sum_{j \in \mathbb{Z}} |j(M_g f)(j)|^2 = \sum_{j \in \mathbb{Z}} |jg(j)f(j)|^2 \leq \|g\|_\infty^2 \left(\sum_{j \in \mathbb{Z}} |jf(j)|^2 \right) < \infty.$$

As $f \in \operatorname{dom} D$ was arbitrary, $M_g(\operatorname{dom} D) \subseteq \operatorname{dom} D$, and hence, the commutator $[iD, M_g]$ is a well-defined linear operator on $\operatorname{dom} D$. Furthermore, iD and M_g are diagonal matrices with complex entries (which commute), so the commutator $[iD, M_g]$ is simply a restriction of the 0 operator to $\operatorname{dom} D$. Theorem 2.2 implies $M_g \in \operatorname{dom} \delta_w^D$ and $\delta_w^D(M_g)$ is the extension of $[iD, M_g]$ to all of H. In particular, $\delta_w^D(M_g) = 0$. Hence, $M_g \in \ker \delta_w^D$, and since $g \in \ell^\infty(\mathbb{Z})$ was arbitrary, $\operatorname{diag}(\ell^\infty(\mathbb{Z})) \subseteq \ker \delta_w^D$.

(d) Part (c) quickly implies $\operatorname{diag}(\ell^\infty(\mathbb{Z})) \subseteq \ker(\delta_w^D)^n$ for all $n \in \mathbb{N}$. We now show if $(\delta_w^D)^n(x) = 0$, then $x \in \operatorname{diag}(\ell^\infty(\mathbb{Z}))$. If $x \in \operatorname{dom} (\delta_w^D)^n$ and $(\delta_w^D)^n(x) = 0$, then $x \in B(\ell^2(\mathbb{Z}))$ and $(\delta_w^D)^n(x)_{rc} = 0$ for every $r, c \in \mathbb{Z}$. By part (b),
$$[(\delta_w^D)^n(x)_{rc}]|_{\operatorname{dom} D^n} = [i^n(r-c)^n x_{rc}],$$
thus, $i^n(r-c)^n x_{rc} = 0$ for every $r, c \in \mathbb{Z}$. If $r \neq c$, it must be that $x_{rc} = 0$, i.e., x must be zero off the diagonal. As $x \in B(\ell^2(\mathbb{Z}))$, we conclude $x \in \operatorname{diag}(\ell^\infty(\mathbb{Z}))$. Therefore, $\ker(\delta_w^D)^n = \operatorname{diag}(\ell^\infty(\mathbb{Z}))$ for all $n \in \mathbb{N}$.

□

This kernel stabilization phenomenon initially appears unique to the setting of Example 4.1; the self-adjoint operator is multiplicity-free (the von Neumann algebra generated by its spectral projections is a maximal abelian self-adjoint subalgebra of $B(\ell^2(\mathbb{Z}))$) and its eigenvectors constitute our choice of orthonormal basis. Below, we show our example is not unique; kernel stabilization holds for every self-adjoint operator on any Hilbert space.

Proposition 4.2. *Let H be a Hilbert space and D a self-adjoint operator. Then $\ker \delta_w^D$ is a von Neumann algebra.*

Proof. The identity I of $B(H)$ is easily shown to be in $\ker \delta_w^D$. Let $x \in \ker \delta_w^D$. As $\operatorname{dom} \delta_w^D$ is a $*$-algebra by Theorem 2.3, $x^* \in \operatorname{dom} \delta_w^D$. Since δ_w^D is a $*$-derivation, $\delta_w^D(x^*) = \delta_w^D(x)^* = 0$. Therefore, $x^* \in \ker \delta_w^D$. Finally, if $x, y \in \ker \delta_w^D$, then $xy \in \operatorname{dom} \delta_w^D$ and $\delta_w^D(xy) = \delta_w^D(x)y + x\delta_w^D(y) = 0$, so $xy \in \ker \delta_w^D$.

Let $(x_\lambda) \subset \ker \delta_w^D$ be a net converging in the weak operator topology to some $x \in B(H)$. We show $x \in \operatorname{dom} \delta_w^D$ and $\delta_w^D(x) = 0$. Because $\delta_w^D(x_\lambda) = 0$ for all λ, we trivially have $\delta_w^D(x_\lambda) \overset{\text{WOT}}{\to} 0$. By Theorem 2.3, the graph of δ_w^D is weak operator topology closed. Therefore, $x \in \operatorname{dom} \delta_w^D$ and $\delta_w^D(x) = 0$. We conclude $\ker \delta_w^D$ is a von Neumann algebra. □

Notation 4.3. Let $\mathcal{P}_D$ denote the collection of all spectral projections for D obtained through the spectral theorem for unbounded self-adjoint operators. Also, let
$$\mathcal{M}_D := \mathcal{P}_D''.$$

We give further description of the structure $\ker \delta_w^D$ in terms of of $\mathcal{M}_D$ in the following lemma and proposition.

Lemma 4.4. *Suppose $x \in B(H)$ satisfies $x(\text{dom } D) \subseteq \text{dom } D$. If $P \in \mathcal{P}_D$, then*
$$[P, [D, x]]h = [D, [P, x]]h$$
for all $h \in \text{dom } D$.

Proof. Let $B(\mathbb{R})$ denote the bounded Borel functions on $\mathbb{R}$, and for each $R \in \mathbb{R}$, define $\mathrm{id}_R : \mathbb{R} \to \mathbb{R}$ by $\mathrm{id}_R(t) = t$ whenever $-R \leq t \leq R$ and $\mathrm{id}_R(t) = 0$ otherwise. The spectral theorem, stated as in Theorem 7.2.8 [11], provides a bounded Borel functional calculus for D, that is, a $*$-homomorphism $\Phi_D : B(\mathbb{R}) \to B(H)$ satisfying $\Phi_D(1) = I$,

$$\mathrm{dom}\ D = \{h \in H : \lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R)h\| < \infty\},$$

and

$$Dh = \lim_{R\to\infty} \Phi_D(\mathrm{id}_R)h$$

for all $h \in \mathrm{dom}\ D$. We claim for each $P \in \mathcal{P}_D$, $P(\mathrm{dom}\ D) \subseteq \mathrm{dom}\ D$ and $PDh = DPh$ for all $h \in \mathrm{dom}\ D$. Given $P \in \mathcal{P}_D$, $P = \Phi_D(\chi_E)$ for some Borel set $E \subseteq \mathbb{R}$. Note that $(\mathrm{id}_R \cdot \chi_E)(t) = 0$ if $t \notin E \cap [-R, R]$, and otherwise $(\mathrm{id}_R \cdot \chi_E)(t) = t$. Thus, for any $h \in \mathrm{dom}\ D$,

$$\lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R)Ph\| = \lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R \cdot \chi_E)h\| \leq \lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R)h\| < \infty.$$

Therefore, $Ph \in \mathrm{dom}\ D$, and as $h \in \mathrm{dom}\ D$ was arbitrary, $P(\mathrm{dom}\ D) \subseteq \mathrm{dom}\ D$. Furthermore,

$$\begin{aligned}
\|DPh - PDh\| &= \lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R)\Phi_D(\chi_E)h - \Phi_D(\chi_E)\Phi_D(\mathrm{id}_R)h\| \\
&= \lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R \cdot \chi_E)h - \Phi_D(\chi_E \cdot \mathrm{id}_R)h\| \\
&= \lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R \cdot \chi_E)h - \Phi_D(\mathrm{id}_R \cdot \chi_E)h\| \\
&= 0.
\end{aligned}$$

Let $x \in B(H)$ and suppose $x(\mathrm{dom}\ D) \subseteq \mathrm{dom}\ D$. For $h \in \mathrm{dom}\ D$, observe

$$\begin{aligned}
[P, [D, x]]h &= P(Dx - xD)h - (Dx - xD)Ph \\
&= PDxh - PxDh - DxPh + xDPh \\
&= DPxh - PxDh - DxPh + xPDh \\
&= DPxh - DxPh + xPDh - PxDh \\
&= D(Px - xP)h + (xP - Px)Dh \\
&= D(Px - xP)h - (Px - xP)Dh \\
&= [D, [P, x]]h
\end{aligned}$$

Hence, $[P, [D, x]]h = [D, [P, x]]h$ for all $h \in \mathrm{dom}\ D$, and as $P \in \mathcal{P}_D$ was arbitrary, this equality holds for any spectral projection of D. □

Proposition 4.5. $\mathcal{M}_D \subseteq \ker \delta_w^D = \mathcal{M}_D'$.

Proof. Let $P \in \mathcal{P}_D$. By the previous lemma, $[D, P] = 0$ on $\mathrm{dom}\ D$, so $P \in \mathrm{dom}\ \delta_w^D$ by Theorem 2.2. Moreover, $\delta_w^D(P)$ is the bounded extension of $i(DP - PD)$ to all of H, which is 0. Therefore, $P \in \ker \delta_w^D$. Proposition 4.2 implies $\mathcal{M}_D \subseteq \ker \delta_w^D$.

Let $x \in \ker \delta_w^D$. By Theorem 2.7, $x(\mathrm{dom}\ D) \subseteq \mathrm{dom}\ D$ and $\delta_w^D(x)|_{\mathrm{dom}\ D} = [D, x]|_{\mathrm{dom}\ D} = 0$. Then, by Theorem X.4.11 [5], $xf(D) \subseteq f(D)x$ for any

$f \in B(\mathbb{R})$. In particular, when $f = \chi_E$ for some Borel subset $E \subseteq \mathbb{R}$ and P denotes the corresponding spectral projection for D, $xP = Px$. Hence, x commutes with all projections in $\mathcal{P}_D$, and as $\mathcal{M}_D$ is generated as a von Neumann algebra by these projections, it follows that $x \in \mathcal{M}_D'$.

Let $x \in \mathcal{M}_D'$. For each $t \in \mathbb{R}$, $e^{itD} \in \mathcal{M}_D$. Thus, $\alpha_t(x) = e^{itD}xe^{-itD} = x$ for all $t \in \mathbb{R}$. In particular, for any $h, k \in H$, the function $t \mapsto \langle \alpha_t(x)h, k\rangle = \langle xh, k\rangle$ is constant, and thus is continuously differentiable with derivative 0. Therefore, $x \in \ker \delta_w^D$. □

We now present our kernel stabilization result.

Theorem 4.6. *If D is any self-adjoint operator on a Hilbert space H, then for every $n \in \mathbb{N}$,*

$$\ker(\delta_w^D)^n = \ker \delta_w^D.$$

Proof. We first show $\ker(\delta_w^D)^2 = \ker \delta_w^D$. The inclusion $\ker \delta_w^D \subseteq \ker(\delta_w^D)^2$ is clear. Let $x \in \ker(\delta_w^D)^2$. Proposition 4.5 states $\ker \delta_w^D = \mathcal{M}_D'$. Thus, it suffices to prove $x \in \mathcal{M}_D'$, which holds if and only if $[P, x] = 0$ for every $P \in \mathcal{P}_D$. By Proposition 2.6, if $x \in \text{dom } (\delta_w^D)^2$, then $x(\text{dom } D) \subseteq \text{dom } D$, $\delta_w^D(x)(\text{dom } D) \subseteq \text{dom } D$, and $(\delta_w^D)^2(x)|_{\text{dom } D} = [iD, \delta_w^D(x)]$. Since $(\delta_w^D)^2(x) = 0$, it must be that $[iD, \delta_w^D(x)] = 0$. Thus, Theorem X.4.11 of [5] implies $\delta_w^D(x)$ commutes with the bounded Borel functional calculus for D, so, in particular, $[P, \delta_w^D(x)] = 0$ for every $P \in \mathcal{P}_D$. Because $\delta_w^D(x)$ and P both preserve the domain of D, so does the commutator $[P, \delta_w^D(x)]$. Thus, Lemma 4.4 implies

$$0 = [P, \delta_w^D(x)]|_{\text{dom } D} = [P, [iD, x]]|_{\text{dom } D} = [iD, [P, x]]|_{\text{dom } D}.$$

As $[P, x] \in B(H)$, $[P, x](\text{dom } D) \subseteq \text{dom } D$, and $[iD, [P, x]]$ is bounded on the domain of D, Theorem 2.7 implies $[P, x] \in \ker \delta_w^D$. Hence, by Proposition 4.5, $[P, x] \in \mathcal{M}_D'$. Therefore,

$$\begin{aligned}
[P, x] &= (P + P^\perp)[P, x](P + P^\perp) \\
&= P[P, x]P + P[P, x]P^\perp + P^\perp[P, x]P + P^\perp[P, x]P^\perp \\
&= P[P, x]P + PP^\perp[P, x] + P^\perp P[P, x] + P^\perp[P, x]P^\perp \\
&= P(Px - xP)P + 0 + 0 + P^\perp(Px - xP)P^\perp \\
&= PxP - PxP + 0 + 0 + 0 \\
&= 0.
\end{aligned}$$

As $P \in \mathcal{P}_D$ was arbitrary, $x \in \mathcal{M}_D'$. By Proposition 4.5, $x \in \ker \delta_w^D$.

We proceed by induction on n. The case when $n = 1$ is vacuous. Suppose $\ker(\delta_w^D)^k = \ker \delta_w^D$ for some $k \in \mathbb{N}$. Let $x \in \ker(\delta_w^D)^{k+1}$. Then $\delta_w^D(x) \in \ker(\delta_w^D)^k$, which equals $\ker \delta_w^D$ by the inductive hypothesis. Hence, $x \in \ker(\delta_w^D)^2$. Since we have already shown $\ker(\delta_w^D)^2 = \ker \delta_w^D$, we have $x \in \ker \delta_w^D$. Therefore, $\ker(\delta_w^D)^n = \ker \delta_w^D$ for all $n \in \mathbb{N}$. □

Remark 4.7. Let $n \in \mathbb{N}$ be arbitrary. Kernel stabilization of δ_w^D is equivalent to the following statement: Suppose $x \in B(H)$, the domains of the iterated commutators $d^k(x)$ for $k = 1, ..., n$ contain a common core $\mathcal{X}$ for D, and $d^k(x)$ is bounded on $\mathcal{X}$ for all $k = 1, ..., n$. If the continuous bounded extension of $d^n(x)$ to all of H belongs to M_D', then $[iD, x]|_{\mathcal{X}} = 0$. Less formally, if $\underbrace{[iD, ..., [iD, x]]}_{n \text{ times}}$ and all lower commutators are well-defined and bounded on a core for D, then

$$\underbrace{[iD, ..., [iD, x]]}_{n \text{ times}} = 0 \text{ implies } [iD, x] = 0.$$

We are grateful to the referee's hunch that this rephrasing of Theorem 4.6 in the case when $n = 2$ is similar to a theorem of C.R. Putnam's in [9]. Upon investigation, we found that when $n = 2$, this statement is in fact equivalent to Theorem 1.6.3 of [9] in the self-adjoint setting. Putnam's proof relies on techniques in the proof of Fuglede's Theorem, whereas our proof is direct. Establishing the equivalence of these statements requires use of Christensen's work in [4].

5. Applications of kernel stabilization (Theorem 4.6)

The first application is in the context of Theorem 1.1, which we copy below for convenience.

Theorem 5.1 (Bratteli-Robinson, Theorem 4 [1])**.** *Let δ be a derivation of a C^*-algebra $\mathcal{A}$, and assume there exists a state ω on $\mathcal{A}$ which generates a faithful cyclic representation (π, H, f) satisfying*

$$\omega(\delta(a)) = 0, \quad \forall a \in dom\, \delta.$$

Then δ is closable and there exists a symmetric operator S on H such that

$$dom\, S = \{h \in H : h = \pi(a)f \quad for\ some\ a \in \mathcal{A}\}$$

and $\pi(\delta(a))h = [S, \pi(a)]h$, for all $a \in dom\, \delta$ and all $h \in dom\, S$. Moreover, if the set $\mathsf{A}(\delta)$ of analytic elements for δ is dense in $\mathcal{A}$, then S is essentially self-adjoint on $dom\, S$. For $x \in B(H)$ and $t \in \mathbb{R}$, define

$$\alpha_t(x) := e^{i\overline{S}t} x e^{-i\overline{S}t}$$

where $\overline{S}$ denotes the self-adjoint closure of S. It follows that $\alpha_t(\pi(\mathcal{A})) = \pi(\mathcal{A})$ for all $t \in \mathbb{R}$, and $\{\alpha_t\}_{t\in\mathbb{R}}$ is a strongly continuous group of automorphisms with closed infinitesimal generator $\widetilde{\delta}$ equaling the closure of $\pi \circ \delta|_{\mathsf{A}(\delta)}$.

We relate the infinitesimal generator $\widetilde{\delta}$ to a derivation δ_u^D studied by Christensen. Since the one-parameter automorphism group in Bratteli and Robinson's Theorem given by $\alpha_t(x) := e^{itD} x e^{-itD}$ for each $t \in \mathbb{R}$ is strongly continuous, $\widetilde{\delta}$ and δ_u^D are precisely the same derivations.

Definition 5.2. An operator $x \in B(H)$ is *uniformly D-differentiable* if there exists $y \in B(H)$ such that

$$\lim_{t\to 0}\left\|\frac{\alpha_t(x)-x}{t}-y\right\|=0.$$

We denote this by $x \in \operatorname{dom}\,\delta_u^D$ and $\delta_u^D(x)=y$.

Proposition 5.3. $\ker\delta_u^D=\ker\delta_w^D$.

Proof. Theorem 4.1 [3] states $x \in \operatorname{dom}\,\delta_u^D$ if and only if $x \in \operatorname{dom}\,\delta_w^D$ and $t \mapsto \alpha_t(\delta_w^D(x))$ is norm continuous. Moreover, δ_w^D extends δ_u^D. Thus, $\ker\delta_u^D \subseteq \ker\delta_w^D$.

Let $x \in \ker\delta_w^D$. Then $t \mapsto \alpha_t(\delta_w^D(x))=0$ is norm continuous, and hence, $x \in \operatorname{dom}\,\delta_u^D$. Moreover, $\delta_u^D(x)=\delta_w^D|_{\operatorname{dom}\,\delta_u^D}(x)=0$. Therefore, $x \in \ker\delta_u^D$. □

Corollary 5.4. *For all $n \in \mathbb{N}$, $\ker(\delta_u^D)^n=\ker\delta_u^D$.*

Proof. Fix $n>1$ and let $x \in \ker(\delta_u^D)^n$. Then $(\delta_u^D)^{n-1}(x) \in \operatorname{dom}\,\delta_u^D$. Hence, $(\delta_u^D)^{n-1}(x) \in \operatorname{dom}\,\delta_w^D$. Further, as $x \in \operatorname{dom}\,\delta_u^D$, we have $x \in \operatorname{dom}\,\delta_w^D$ and $\delta_w^D(x)=\delta_u^D(x)$. Hence, $x \in \operatorname{dom}\,(\delta_w^D)^n$ and $(\delta_w^D)^n(x)=(\delta_u^D)^n(x)=0$. By Theorem 4.6, $x \in \ker\delta_w^D$. By Proposition 5.3, $x \in \ker\delta_u^D$. □

Given a self-adjoint operator D, our proof of kernel stabilization of δ_w^D relied on the relationship between δ_w^D and commutation with D. Intuitively, then, kernel stabilization is likely to occur for a derivation δ on an abstract C^*-algebra that can be implemented, under an appropriate representation, as commutation with a self-adjoint operator. Bratteli and Robinson provide sufficient conditions for when a derivation on a C^*-algebra has such a representation.

Under this representation π, Bratteli and Robinson construct an essentially self-adjoint operator S which implements the derivation's action as commutation with S. Once this essentially self-adjoint operator is in play, we use its self-adjoint closure $D=\overline{S}$ to generate the weak-D derivation δ_w^D. We show δ_w^D extends $\delta\circ\pi$ and apply Theorem 4.6 (kernel stabilization of δ_w^D) to obtain kernel stabilization of δ.

Definition 5.5. Given a one-parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ of maps on $B(H)$, let $\operatorname{dom}\,\widetilde{\delta}$ be the set of all $x \in B(H)$ so that there exists $y \in B(H)$ satisfying

$$\lim_{t\to 0}\left\|\frac{\alpha_t(x)-x}{t}-y\right\|=0.$$

For $x \in \operatorname{dom}\,\widetilde{\delta}$, let $\widetilde{\delta}(x)=y$ where y is the uniform limit described above. We call $\widetilde{\delta}$ the *infinitesimal generator* for $\{\alpha_t\}_{t\in\mathbb{R}}$.

Remark. When $\alpha_t(x):=e^{itD}xe^{-itD}$ for some self-adjoint operator D, Definition 5.5 is identical to the derivation δ_u^D in Definition 5.2.

Lemma 5.6. *If δ, $\mathcal{A}$, π, and $\widetilde{\delta}$ are as in Theorem 1.1, then*

$$\ker \widetilde{\delta}^n \cap \pi(\mathsf{A}(\delta)) = \pi(\ker \delta^n)$$

for all $n \in \mathbb{N}$.

Proof. Recall if $a \in \mathsf{A}(\delta)$, then Theorem 1.1 provides $\widetilde{\delta}(\pi(a)) = \pi(\delta(a))$. It follows by analyticity of a that $\widetilde{\delta}^n(\pi(a)) = \pi(\delta^n(a))$ for every $n \in \mathbb{N}$. Suppose $\widetilde{\delta}^n(\pi(a)) = 0$. Then $\pi(\delta^n(a)) = \widetilde{\delta}^n(\pi(a)) = 0$, and since π is faithful, $\delta^n(a) = 0$. Therefore, $\pi(a) \in \pi(\ker \delta^n)$.

Conversely, suppose $a \in \ker \delta^n$. Then $a \in \mathsf{A}(\delta)$ because $\delta^j(a) = 0$ for all $j \geq n$ and $\sum_{k=0}^{\infty} \frac{t^k}{k!} \left\|\delta^k(a)\right\| = \sum_{k=0}^{n-1} \frac{t^k}{k!} \left\|\delta^k(a)\right\| < \infty$ for any choice of $t > 0$. Similar to above, $\widetilde{\delta}^n(\pi(a)) = \pi(\delta^n(a)) = \pi(0) = 0$. Therefore, $\pi(a) \in \ker \widetilde{\delta}^n \cap \pi(\mathsf{A}(\delta))$. The desired equality holds for all $n \in \mathbb{N}$. □

Theorem 5.7. *If δ, $\mathcal{A}$, π, $\widetilde{\delta}$, and S are as in Theorem 1.1, then $\ker \delta^n = \ker \delta$.*

Proof. Fix $n \in \mathbb{N}$, and let $a \in \ker \delta^n$. Then, $a \in \mathsf{A}(\delta)$ and $\pi(a) \in \ker \widetilde{\delta}^n$ by Lemma 5.6. Note $\widetilde{\delta} = \delta_u^D$ where $D = \overline{S}$, so Proposition 5.4 implies $\ker \widetilde{\delta}^n = \ker \widetilde{\delta}$ for all $n \in \mathbb{N}$. Hence, $\pi(a) \in \ker \widetilde{\delta} \cap \pi(\mathsf{A}(\delta))$. By another application of Lemma 5.6, we get $a \in \ker \delta$. Therefore, $\ker \delta^n = \ker \delta$ for all $n \in \mathbb{N}$. □

The second application of Theorem 4.6 is related to the Heisenberg Commutation Relation, defined in Definition 1.2.

Example 5.8. The classical example of a pair satisfying the Heisenberg Commutation Relation is the *Schrödinger pair*, the quantum mechanical position operator Q and momentum operator P on $L^2(\mathbb{R})$. Let

$$\text{dom } Q = \{f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |xf(x)|^2 \, dx < \infty\}$$

and, for $g \in \text{dom } Q$, define $(Qg)(x) = xg(x)$ for a.e. $x \in \mathbb{R}$. It is shown in Example 7.1.5 of [11] that Q defines a self-adjoint operator. If a function f is absolutely continuous, denote its almost-everywhere defined derivative by f'. Now, let

$$\text{dom } P = \{f \in L^2(\mathbb{R}) : f \text{ is absolutely continuous and } f' \in L^2(\mathbb{R})\},$$

and for $h \in \text{dom } P$, define $Ph := ih'$. It is shown in Theorem 6.30 of [13] that P defines a self-adjoint operator. Let $S(\mathbb{R})$ denote the Schwartz space on $\mathbb{R}$, that is,

$$S(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \forall m, n \in \mathbb{N}, \ \|Q^m P^n f\|_\infty < \infty\}.$$

Proposition X.6.5 of [5] shows $S(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, and it is clear from its definition that $S(\mathbb{R})$ is contained in $\text{dom } Q \cap \text{dom } P$ and is invariant under both Q and P. Hence, $S(\mathbb{R}) \subseteq \text{dom } [P, Q]$. Furthermore, $[P, Q]g = ig$

for all $g \in S(\mathbb{R})$. Therefore, P and Q satisfy the Heisenberg Commutation Relation.

If two operators are unitarily equivalent to a direct sum of copies of the Schrödinger pair, then they are certainly both unbounded. There are, however, examples of operators satisfying the Heisenberg Commutation Relation where one operator is bounded.

Example 5.9. For $f \in L^2[0,1]$, define $(Bf)(x) = xf(x)$ for a.e. $x \in [0,1]$. In contrast to its unbounded analogue Q, the operator B is contractive. Let $AC[0,1]$ denote the set of functions which are absolutely continuous on $[0,1]$, and let

$$\text{dom } A = \{f \in AC[0,1] : f' \in L^2[0,1],\ f(0) = f(1)\}.$$

For $g \in \text{dom } A$, define $Ag = ig'$. Example X.1.12 of [5] shows the operator A with this particular domain is self-adjoint. Due to boundedness of B,

$$\text{dom } [A,B] = \{f \in \text{dom } A : Bf \in \text{dom } A\}.$$

Choose

$$K := \{f \in AC[0,1] : f' \in L^2[0,1],\ f(0) = f(1) = 0\}.$$

Example X.1.11 of [5] shows K is dense in $L^2[0,1]$ as it contains all polynomials p on $[0,1]$ satisfying $p(0) = p(1) = 0$. Furthermore, we claim K is invariant for B. Indeed, products of absolutely continuous functions are again absolutely continuous, so $(Bg)(x) = xg(x)$ for a.e. $x \in [0,1]$ defines an absolutely continuous function. The a.e.-defined derivative of Bg is equivalent to $Bg' + g$ by the product rule. Moreover, $Bg' + g$ belongs to $L^2(\mathbb{R})$ as $g' \in L^2(\mathbb{R})$ and $B \in B(L^2[0,1])$. Lastly,

$$(Bg)(0) = 0 \cdot g(0) = 0 = 1 \cdot 0 = 1 \cdot g(1) = (Bg)(1).$$

Thus, $BK \subseteq K$. As a result, $K \subseteq \text{dom } [A,B]$. For $k \in K$, observe

$$[A,B]k = i\left(\frac{d}{dx}(Bk) - B(k')\right) = i(Bk' + k - Bk') = ik.$$

Therefore, A and B satisfy the Heisenberg Commutation Relation.

We claim the boundedness of the operators in Examples 5.8 and 5.9 differs due the relative size of dom $[P,Q]$ in $L^2(\mathbb{R})$ versus dom $[A,B]$ in $L^2[0,1]$. In particular, dom $[A,B]$ does not contain a core for A or B, while dom $[P,Q]$ contains $S(\mathbb{R})$, which is a core for both P and Q.

Theorem 5.10. *Let $A : dom\ A \to H$ and $B : dom\ B \to H$ be self-adjoint operators which satisfy the Heisenberg Commutation Relation on a dense subspace $K \subseteq H$. If K is a core for both A and B, then A and B are both unbounded.*

Proof. Suppose that K is a core for both A and B. It is well-known that A and B cannot both be bounded and satisfy the Heisenberg Relation. Thus, without loss of generality, the only possibilities are that A is bounded and

B is unbounded, or both A and B are unbounded. Suppose that $A \in B(H)$. By the Heisenberg Commutation Relation, $[A, B]k = ik$ for all $k \in K$, or, equivalently, $[iB, A]k = k$ for all $k \in K$.

As K is a core for B and $\|[iB, A]|_K\| = 1$, we have that $A \in \text{dom } \delta_w^B$. Furthermore, $\delta_w^B(A)$ is the continuous extension of the bounded and densely-defined operator $[iB, A]|_K$ to all of H, and thus, $\delta_w^B(A) = I$. Trivially, $I \in \text{dom } \delta_w^B$ and $\delta_w^B(I) = 0$, so $A \in \text{dom } (\delta_w^B)^2$ and $(\delta_w^B)^2(A) = 0$. By Theorem 4.6, $A \in \ker(\delta_w^B)^2 = \ker \delta_w^B$. But then

$$0 = \delta_w^B(A)|_K = [iB, A]|_K = I|_K,$$

which is absurd. Therefore, A cannot be bounded. We conclude that if A and B satisfy the Heisenberg Commutation Relation on a common core for A and B, then A and B must both be unbounded. □

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