New York Journal of Mathematics

New York J. Math. **25** (2019) 723–744.

On configuration spaces and simplicial complexes

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Abstract. The n-point configuration space of a space M is a well-known object in topology, geometry, and combinatorics. We introduce a generalization, the simplicial configuration space M_S, which takes as its data a simplicial complex S on n vertices, and explore the properties of its homology, considered as an invariant of S.

As in Eastwood-Huggett's geometric categorification of the chromatic polynomial, our construction gives rise to a polynomial invariant of the simplicial complex S, which generalizes and shares several formal properties with the chromatic polynomial.

Contents

1. Introduction

The configuration space of n points in the space M,

$$\mathrm{Conf}_n(M) = M^n \setminus \bigcup_{i\neq j} \{x^i = x^j\}$$

was introduced by Fadell and Neuwirth [14]. Its topology has since been the subject of extensive study (e.g. [16, 18, 23]), with particular focus on

Received August 15, 2018.

2010 *Mathematics Subject Classification.* 55U10, 05C15, 05C31.

Key words and phrases. simplicial complex, homology, chromatic polynomial, categorification.

ISSN 1076-9803/2019

how to relate the homology and cohomology of this (noncompact) space to the homology or cohomology of the space M, which may be taken to be a closed manifold or algebraic variety.

Bendersky-Gitler [5] approached the problem of computing the cohomology of $\mathrm{Conf}_n(M)$ via a spectral sequence starting from a certain *simplicial space*, that is, a space with stratification data encoded by a simplicial complex.

There are several interesting ways to generalize the notion of configuration space. In particular, Eastwood-Huggett [12] generalized the definition to remove only certain diagonals, as determined by the edges of a graph. Baranovsky-Sazdanović [4] observed that this *graph configuration space* M_G admits a Bendersky-Gitler-type spectral sequence which allows for the computation of its cohomology directly from the combinatorics of the graph [15].

Eastwood-Huggett consider the graph configuration spaces $M_{G/e}$ and M_{G-e} corresponding to the graphs G/e and $G - e$ obtained from G by contracting and deleting an edge, respectively. These spaces satisfy a long exact sequence in homology, which descends, by taking Euler characteristics, to the familiar deletion-contraction identity satisfied by the the chromatic polynomial P_G [6]. Thus the spaces M_G and their homology groups can be regarded as a categorification of (evaluations of) the chromatic polynomial of G.

The present paper extends Eastwood-Huggett's construction by taking a simplicial complex S as data instead of a graph G. The (co)homology of this *simplicial configuration space* is a homology-theory invariant of the simplicial complex. The Euler characteristic of this homology theory, in turn, yields a novel polynomial invariant χ_c of the simplicial complex which satisfies a more general *addition-contraction* identity and generalizes the chromatic polynomial of graphs in the sense that $\chi_c(I(G), t) = P_G(t)$, where $I(G)$ is the independence complex of G.

We provide generalizations of several of the results of Fadell-Neuwirth appropriate to our setting, as well as giving a notion of *addition-contraction*, and explore some of the basic properties of the homology theory as an invariant of the simplicial complex. The main results and outline of the paper are:

- **§2:** The construction of the configuration spaces M_S and the notions of *deletion* and *contraction* appropriate to it.
- **§3:** The homology and cohomology of M_S satisfy *addition-contraction long exact sequences.*
- **§4:** Simplicial operations on S induce topological operations on M_S. There is an *orphan point spectral sequence* relating $H^*_c(M_{\{pt\}\sqcup S})$, $H^*_c((M \setminus \{pt\})_S)$, and $H^*_c(M)$.

- **§5:** Some sample computations of $H_c^*(\mathbb{CP}^1_S)$ and $H_c^*(\mathbb{CP}^2_S)$, demonstrating the computational theorems of §4 and showing that the choice of M matters.
- **§6:** From $H_c^*(M_S)$ we obtain a polynomial invariant, the *simplicial chromatic polynomial* $\chi_c(S)$.
- **§7:** $H_c^*(M_S)$ and $\chi_c(S)$ detect different information from other Tutte-type invariants.

2. The simplicial configuration space

Eastwood-Huggett [12] introduced the graph configuration space of a graph G on n vertices,

$$M_G = M^n \setminus \bigcup_{e\in E(G)} D_e.$$

where for any $e = [ij] \in E(G)$, $D_e = \left\{(x^1,\dots,x^n)\middle| x^i = x^j\right\}$. The homology of these spaces, $EH_*(G,M) := H_*(M_G)$, categorifies the evaluations of the chromatic polynomial of G:

Theorem 2 of [12]**.** *For any graph G and any closed, orientable manifold M,*

$$\chi(EH_*(G,M)) = P_G(\chi(M))$$

where P_G is the chromatic polynomial of G and $\chi(M)$ is the Euler characteristic of M.

We generalize this construction to simplicial complexes.

2.1. The construction.

Definition 2.1. Let S be a simplicial complex whose 0-skeleton is given by a vertex set $V = V(S) = \{v_1,\dots,v_n\}$. Let M be a topological space. For each simplex $\sigma = [v_{i_1}\cdots v_{i_k}]$, define the diagonal corresponding to σ to be

$$D_\sigma = \left\{(x^1,\dots,x^n)\middle| x^{i_1} = \cdots = x^{i_k}\right\} \subseteq M^n$$

We define the simplicial configuration space to be

$$M_S = M^n \setminus \bigcup_{\sigma\in\Delta^V\setminus S} D_\sigma$$

where Δ^V is the simplex spanned by the vertex set $V = V(S)$.

Remark 1. We interpret $\sigma \in \Delta^V \setminus S$ in the combinatorial rather than the geometric sense: if σ is a simplex on a subset of the vertices $v_1,\dots,v_n$, and σ does not appear among the set of simplices that occur in S, then $\sigma \in \Delta^V \setminus S$.

Remark 2. Our notation is similar to that of Eastwood-Huggett. This has the inconvenient side-effect that for a simple graph G, the symbol M_G has two possible interpretations: it could be the Eastwood-Huggett space, or the simplicial configuration space obtained by considering G as a one-dimensional simplicial complex. Generally the proper interpretation will be clear from context, but we will specify where there is the possibility of confusion.

Lemma 2.1. *If τ is a face of σ, then $D_\sigma \subseteq D_\tau$.*

Given a graph G, its *independence complex* $I(G)$ is the simplicial complex whose faces are the independent sets of vertices of G. That is, $\sigma \in I(G)$ if and only if no pair of vertices in σ are adjacent in G. We have the following relationship between our construction and graph configuration spaces

Proposition 2.2. *For any graph G, the simplicial configuration space with data $I(G)$ is the graph configuration space with data G; that is*

$$M_{I(G)} = M_G$$

where the left-hand side denotes a simplicial configuration space and the right-hand side denotes a graph configuration space.

Proof. Suppose e is an edge in G. Then $e \notin I(G)$. So $D_e \subseteq \bigcup_{\sigma \in \Delta^V \setminus I(G)} D_\sigma$. On the other hand, suppose $\sigma \in \Delta^V \setminus I(G)$. Some edge e of σ is in $E(G)$. So

$$D_\sigma \subseteq D_e \subseteq \bigcup_{e \in G} D_e$$

Thus we have

$$\bigcup_{e \in G} D_e = \bigcup_{\sigma \in \Delta^V \setminus I(G)} D_\sigma$$

and the result follows. □

Proposition 2.3. *The configuration space for a 0-dimensional simplicial complex V with n vertices is the ordinary configuration space of n ordered points in M, i.e. $M_V = \mathrm{Conf}_n(M)$.*

Proposition 2.4. *Let Δ^{n-1} be a simplex on n vertices. Then $M_{\Delta^{n-1}} = M^n$.*

Therefore given a graph G, we can think of M_S as giving an interpolation between graph configuration spaces and ordinary configuration space, as S ranges from the 0-dimensional simplicial complex $V(G)$ to the independence complex $I(G)$. The same can be said of Eastwood-Huggett's construction, which interpolates between $\mathrm{Conf}_n(M) = M_{K_n}$ and M^n as G ranges from the complete graph K_n to the graph on n vertices with no edges; the present construction offers a finer interpolation than Eastwood-Huggett's.

2.2. Deletion, addition, and contraction. The motivation behind Eastwood-Huggett's construction is to geometrically categorify the *deletion-contraction formula* for the chromatic polynomial, which says that, for a simple graph G any edge $e \in E(G)$,

$$P_G(\lambda) = P_{G-e}(\lambda) - P_{G/e}(\lambda) \tag{1}$$

where $G-e$ is the graph with $V(G-e) = V(G)$ and $E(G-e) = E(G) \setminus \{e\}$, and G/e is the graph obtained by identifying the endpoints of e and removing e (see Chapter V of [7]). The deletion-contraction formula can be rephrased as an *addition-contraction formula*: for any edge $e \notin E(G)$,

$$P_G(\lambda) = P_{G+e}(\lambda) + P_{G/e}(\lambda) \tag{2}$$

where G is the graph with $V(G+e) = V(G)$ and $E(G+e) = E(G) \cup \{e\}$.

We refer to the operations $G \mapsto G-e$, $G \mapsto G+e$, and $G \mapsto G/e$ as *deletion*, *addition*, and *contraction*, respectively. Formula (1) is called the *deletion-contraction formula* and formula (2) is called the *addition-contraction formula*.

We introduce corresponding definitions for simplicial complexes. The primary difficulty is ensuring that the space which results from contraction is a simplicial complex. Wang [25] and Bajo-Burdick-Chmutov [3] generalized the deletion-contraction relation for the Tutte polynomial to the Bott and Tutte-Renardy-Krushkal polynomials, respectively, of cell complexes; the contraction operation is used for subcomplexes which are *bridges*, *loops*, or *boundary-regular*. Contraction in the Wang-Bajo-Burdick-Chmutov sense does not alter the top homology group. As another example, Ehrenborg-Hetyei [13] consider a class of simplicial complexes they call *constrictive*, which are simple-homotopic to either a single vertex or the boundary complex of a simplex, and the operation of *simplicial collapse*. Our notion of tidied contraction differs from these in that it applies to any face of any simplicial complex to yield a simplicial complex.

Definition 2.2. Given a face $\sigma \in S$, define the *deletion of* σ to be the complex $S \setminus \sigma = S \setminus \mathrm{St}(\sigma)$ to be the complement of the open star of σ; that is, the complex obtained by removing σ and every simplex of S which contains σ.

Given a simplicial complex S on n vertices and a k-simplex $\sigma \in \Delta^V$, define the *contraction of* σ to be the complex S/σ given as follows: for any p-simplex $\tau \in S$ with $\sigma \subset \tau$, replace τ with the $(p-k)$-simplex given by identifying the $k+1$ vertices of σ to a single vertex v. The *tidied contraction* of σ is $S_{/\sigma} = (S/\sigma) \setminus \mathrm{St}(v)$.

A *minimal nonface* of S is a simplex $\sigma \in \Delta^V$ so that $\sigma \notin S$ but every face of $\partial\sigma$ is in S. For any minimal nonface σ of S, define the *addition of* σ to be the complex $S \cup \{\sigma\}$.

Remark 3. Note that for a graph G and an edge $e \in E(G)$, $I(G-e) = I(G) \cup \{e\}$. The definition of the tidied contraction is chosen so that $I(G/e) = I(G)_{/e}$. Then, by Proposition 2.2, we have

$$M_{G-e} = M_{I(G-e)} = M_{\mathrm{Cl}(G^c+e)} = M_{I(G)\cup\{e\}}$$

and

$$M_{G/e} = M_{I(G/e)} = M_{I(G)_{/e}}$$

Eastwood-Huggett's categorification relates M_{G-e} and $M_{G/e}$ to M_G; thus we should expect to find a relation between $M_{S\cup\{\sigma\}}$, $M_{S_{/\sigma}}$, and M_S.

3. Homology of M_S

In this section, we consider the homology and cohomology of simplicial configuration space M_S, establishing the *deletion-contraction long exact sequence* and *addition-contraction long exact sequence*, which are inspired by Eastwood-Huggett's geometric categorification of the deletion-contraction formula for the chromatic polynomial.

By Proposition 2.2, we have

Proposition 3.1. *For any graph G, $H_*(M_{(I(G)})\cong EH_*(G,M)$.*

3.1. Deletion-contraction and addition-contraction sequences. The deletion-contraction and addition-contraction long exact sequences arise from the Leray long exact sequence (sometimes also called the Thom-Gysin sequence).

Leray long exact sequences [19]**.** *Given a manifold B, and a closed submanifold A with orientable normal bundle and codimension m, there are long exact sequences (in homology and cohomology with compact supports, respectively)*

$$\cdots\to H_p(B\setminus A)\to H_p(B)\to H_{p-m}(A)\to H_{p-1}(B\setminus A)\cdots\to \tag{3}$$

$$\cdots\to H^p_c(B\setminus A)\to H^p_c(B)\to H^p_c(A)\to H^{p+1}_c(B\setminus A)\to\cdots \tag{4}$$

The map $H_p(B\setminus A)\to H_p(B)$ is induced by the inclusion of $B\setminus A\subset B$, and the map $H_p(B)\to H_{p-m}(A)$ comes from intersecting a cycle in B with $A\subset B$ to obtain a cycle in A; the map $H^p_c(B\setminus A)\to H^p_c(B)$ is the so-called shriek map *coming from the inclusion of the open submanifold $B\setminus A\subset B$, and the map $H^p_c(B)\to H^p_c(A)$ is induced by the inclusion of the closed submanifold $A\subset B$.*

Theorem 3.2. *For any simplicial complex S, any manifold M of dimension m, and any k-simplex $\sigma\in S$, we have the long exact sequences*

$$\cdots\to H_p(M_{S\setminus\sigma})\to H_p(M_S)\to H_{p-mk}(M_{S_{/\sigma}})\to H_{p-1}(M_{S\setminus\sigma},)\cdots$$

and

$$\cdots\to H^p_c(M_{S\setminus\sigma})\to H^p_c(M_S)\to H^p_c(M_{S_{/\sigma}})\to H^{p+1}_c(M_{S\setminus\sigma})\to\cdots$$

Proof. We will apply the sequences (3) and (4) with $B=M_S$, $A=B\cap D_\sigma$.

By Lemma 2.1, $D_\sigma = \bigcup_{\substack{\tau\in\Delta^V\\ \sigma\in\tau}} D_\tau$, so we can compute

$$\begin{aligned} B \setminus A &= \left(M^n \setminus \bigcup_{\rho\in\Delta^V\setminus S} D_\rho\right) \setminus D_\sigma \\ &= \left(M^n \setminus \bigcup_{\rho\in\Delta^V\setminus S} D_\rho\right) \setminus \bigcup_{\substack{\tau\in\Delta^V\\ \sigma\in\tau}} D_\tau \\ &= M^n \setminus \bigcup_{\rho\in\Delta^V\setminus(S\setminus\sigma)} D_\rho = M_{S\setminus\sigma} \end{aligned}$$

Lemma 3.3. *$A = B \cap D_\sigma$ is homeomorphic to $M_{S_{/\sigma}}$.*

Proof. Relabel the vertices so that $\sigma = [v_1 v_2 \cdots v_{k+1}]$. Then the projection $p_1 : M^n \to M^{n-k}$ onto the last $n-k$ coordinates is a homeomorphism when restricted to D_σ. Consider the simplex $\Delta = [v, v_{k+2}, \ldots, v_n]$ and note that there is a quotient map $\Delta^V \to \Delta$ is obtained by identifying $v_1, v_2, \ldots, v_{k+1}$ and labeling the result v.

Now let $x \in A$ and consider some $\rho \in \Delta \setminus (S_{/\sigma})$. Since $\rho = [v_{i_0} \cdots v_{i_\ell}]$ does not interact with v, we may consider $\tilde\rho \in \Delta^V \setminus (S \setminus \mathrm{St}(v_1, v_2, \ldots, v_{k+1}))$ given by $\tilde\rho = [v_{i_0} \cdots v_{i_\ell}]$. We claim $p_1(x) \notin D_\rho$.

If $\tilde\rho \in \Delta^V \setminus S$, then $x \notin D_{\tilde\rho}$, so $p_1(x) \notin p_1(D_{\tilde\rho}) = D_\rho$.

The other possibility is that $\tilde\rho \in \mathrm{St}(v_1, v_2, \ldots, v_{k+1}) \cap S$. There are two cases.

Suppose $\tilde\rho = [v_1 v_2 \cdots v_{k+1}\tau]$ for some τ. Now $D_{\tilde\rho} = D_\sigma \cap D_\tau$. Since $\tilde\rho \in S$, $x \notin D_{\tilde\rho}$. However, $x \in D_\sigma$. So we have $x \notin D_\tau$. Now observe that $\rho = [v\tau]$, so $D_\rho \subseteq D_\tau$, so $p_1(x) \notin D_\rho$.

Now if $\tilde\rho$ contains some (but not all) vertices of σ, say $v_{j_1}, v_{j_2}, \ldots, v_{j_s}$. Let τ be the face of σ consisting of the other vertices of $\tilde\rho$, so $\tau \notin \mathrm{St}(v_1, v_2, \ldots, v_{k+1})$, say $\tau = [v_{i_0} \cdots v_{i_\ell}]$. Then the fact that $x \notin D_{\tilde\rho}$ means that some of $x^{j_1}, x^{j_2}, \ldots, x^{j_s}, x^{i_0}, \ldots x^{i_k}$ must be distinct. But this exactly means that $p_1(x) = (x^1, x^2, \ldots, x^{n+1}) \notin D_\rho$.

All in all, we have shown that $p_1(A) \subseteq M_{S_{/\sigma}}$.

Now consider $y = (y_1, \ldots, y_n) \in M_{S_{/\sigma}}$ and $\tau \in \Delta^V \setminus S$. Then $\tau \in \Delta^V \setminus (S \setminus \mathrm{St}(v_1, v_2, \ldots, v_{k+1}))$. Now $p_1(D_\tau) = D_{\tau_{/\sigma}}$, so $y \notin D_{\tau_{/\sigma}}$ means that $p_1^{-1}(y) \notin D\tau$. This shows that $p_1^{-1}(M_{S_{/\sigma}}) \subseteq A$.

This shows that p_1 is a homeomorphism between A and $M_{S_{/\sigma}}$. □

Since $A = D_\sigma$ has codimension mk, the Leray sequence in homology reads

$$\cdots \to H_p(M_{S\setminus\sigma}) \to H_p(M_S) \to H_{p-mk}(M_{S_{/\sigma}}) \to H_{p-1}(M_{S\setminus\sigma}) \cdots \tag{5}$$

□

By replacing S with $S \cup \{\sigma\}$, we obtain the addition-contraction sequences:

Theorem 3.4. *For any simplicial complex S, any manifold M of dimension m, and any k-dimensional minimal nonface σ of S, we have long exact sequences*

$$\cdots \to H_p(M_S) \to H_p(M_{S\cup\{\sigma\}}) \to H_{p-mk}(S_{/\sigma}) \to H_{p-1}(M_S) \to \cdots$$

$$\cdots \to H_c^p(M_S) \to H_c^p(M_{S\cup\{\sigma\}}) \to H_c^p(M_{S_{/\sigma}}) \to H_c^{p+1}(M_S) \to \cdots$$

Remark 4. Observe that there is a grading shift in the homology version of each of these long exact sequences, which depends on the dimension of the face being contracted. For this reason, we prefer to work with cohomology theory for the remainder of the paper, though the reader can supply the analogous results about homology.

Remark 5. The addition-contraction operation appears as part of a *wall-crossing formula* in [1], used to compute the intersection theory of maps from n-pointed curves to an algebraic variety. We think it would be fruitful to pursue this connection in further depth.

4. Properties of the simplicial configuration space

This section provides some relations between the structure of a simplicial complex S and the geometry of simplicial configuration space M_S. Each of these, in turn, induces a relation between S and $H_c^*(M_S)$.

4.1. Functoriality in S. Now we will describe the sense in which M_S is functorial in S. To describe this functoriality, we need the following:

Lemma 4.1. *M_S is in bijection with the set of maps $f : V(S) \to M$ with the property that if $[v_{i_1} \cdots v_{i_j}] \notin S$, then $f(v_{i_1}), \ldots, f(v_{i_j})$ are not all equal.*

Proof. Each such f corresponds to the point $x_f = (f(v_1), \ldots, f(v_n)) \in M^n$. Given $x = (x^1, \ldots, x^n) \in M_S$, we assign $f_x : v_i \mapsto x^i$. □

Definition 4.1. Let S_1, S_2 be simplicial complexes with vertex sets $V(S_1)$, $V(S_2)$. For any map $\phi : V(S_1) \to V(S_2)$ and any simplex $\sigma = [v_{i_1} \cdots v_{i_j}]$ on $V(S_1)$, define $\phi(\sigma) = [\phi(v_{i_1}) \cdots \phi(v_{i_j})]$. If for each $\sigma \notin S_1$, we have $\phi(\sigma) \notin S_2$, we call ϕ a *cosimplicial map* from S_1 to S_2.

Observe that simplicial complexes together with cosimplicial maps form a category.

Proposition 4.2. *Each cosimplicial map $\phi : S_1 \to S_2$ induces a continuous map $\phi^* : M_{S_2} \to M_{S_1}$.*

Proof. Using the correspondence from Lemma 4.1, given $\phi : V(S_1) \to V(S_2)$, for each $x \in M_{S_2}$ with corresponding $f_x : V(S_2) \to M$, set $(\phi^* f_x)(v) = f_x\phi(v)$. It is straightforward to verify that because ϕ is cosimplicial, $\phi^* f_x : V(S_1) \to M$ corresponds via Lemma 4.1 to some $\phi^* x \in M_{S_1}$. □

Proposition 4.3. *If $\phi_1 : S_1 \to S_2$ and $\phi_2 : S_2 \to S_3$ are cosimplicial, then $(\phi_2 \circ \phi_1)^* = \phi_1^* \circ \phi_2^* : M_{S_3} \to M_{S_1}$.*

Corollary 4.4. *$S \mapsto M_S$ is a contravariant functor from the category of simplicial complexes with cosimplicial maps to the category of topological spaces with continuous maps.*

Composing with the contravariant functor H_c^*, we obtain:

Corollary 4.5. *$S \mapsto H_c^*(M_S)$ is a covariant functor from the category of simplicial complexes with cosimplicial maps to the category of graded abelian groups.*

4.2. Induced operations on simplicial configuration space. Several standard operations on simplicial complexes induce topological relations among the corresponding simplicial configuration spaces.

Definition 4.2. The *cone* $C(S)$ on the simplicial complex S is the complex obtained from S by replacing each simplex $\sigma \in S$ with $[v\sigma]$, where v is a new vertex.

The *join* of simplicial complexes S and T is the complex

$$S * T = \{\sigma\tau \mid \sigma \in S, \tau \in T\}$$

Theorem 4.6. *Let S and T be disjoint simplicial complexes. Then the simplicial configuration space of their join $S * T$ is the product of their simplicial configuration spaces:*

$$M_{S*T} = M_S \times M_T$$

Proof. Let k be the number of vertices of S and ℓ the number of vertices of T; label the vertices of S $v_1, \ldots, v_k$ and the vertices of T $v_{k+1}, \ldots, v_{k+\ell}$.

Set $\pi_1 : M_{S*T} \to M^k$ and $\pi_2 : M_{S*T} \to M^\ell$ be the coordinate projections.

Let $z = (z^1, \ldots, z^{k+\ell}) \in M_{S*T}$. We will show that $\pi_1(z) \in M_S$ and $\pi_2(z) \in M_T$. To this end, suppose that $1 \le i_1, \ldots, i_j \le k$ are a choice of indices so that $z^{i_1} = \cdots = z^{i_j}$. Then we know that $\sigma = [v_{i_1} \cdots v_{i_j}] \in S * T$. Since $i_1 \ldots, i_j \le k$, this means $\sigma \in S$. So $\pi_1(z) = (z^1, \ldots, z^k) \in M_S$. The proof that $\pi_2(z) \in M_T$ is similar.

Thus we have shown that if $z = (z^1, \ldots, z^k, z^{k+1}, \ldots, z^{k+\ell}) = (x, y) \in M_{S*T}$, then $x \in M_S$ and $y \in M_T$, i.e. that $M_{S*T} \subseteq M_S \times M_T$.

Now let $z = (x, y) \in M_S \times M_T$. Suppose that $1 \le i_1, \ldots, i_j \le k + \ell$ are indices so that $z^{i_1} = \cdots = z^{i_j}$. Then we know that $\sigma = [v_{i_1} \cdots v_{i_k}]$ has the property that $\sigma \cap [v_1 \cdots v_k] \in S$ and $\sigma \cap [v_{k+1} \cdots v_{k+\ell}] \in T$, which means $\sigma \in S * T$. So $z \in M_{S*T}$. □

Remark 6. Join is the coproduct in the category of simplicial complexes with cosimplicial maps; thus Theorem 4.6 says that the functor $S \mapsto M_S$ takes coproducts to products.

From Theorem 4.6 and the Künneth formula, we have:

Corollary 4.7. *Given disjoint simplicial complexes S and T cohomology of simplicial configuration spaces (M_S), (M_T) and M_{S*T} fit into the following split exact sequence*

$$0 \to \bigoplus_{i+j=k} H_c^i(M_S) \otimes H_c^j(M_T) \to H_c^k(M_{S*T}) \to \bigoplus_{i+j=k-1} \mathrm{Tor}(H_c^i(M_S), H_c^j(M_T)) \to 0$$

As a particular application, in case T is a single vertex, $S * T = C(S)$; thus we obtain

Proposition 4.8. *The simplicial configuration space of the cone of S is*

$$M_{C(S)} = M_S \times M,$$

and there is a split exact sequence

$$0 \to \bigoplus_{i+j=k} H_c^i(M_S) \otimes H_c^j(M) \to H_c^k(M_{C(S)}) \to \bigoplus_{i+j=k-1} \mathrm{Tor}(H_c^i(M_S), H_c^j(M)) \to 0$$

Another interesting operation is the suspension operation $\Sigma(S)$, which is the join of S with two points.

Proposition 4.9. *Let D denote the diagonal of $M \times M$, and $\mathring{M} = (M \times M) \setminus D$. Then $M_{\Sigma(S)} = M_S \times \mathring{M}$.*

Theorem 4.10. *Let S be a simplicial complex, and M a connected manifold of dimension at least 2. Denote by $\{pt\} \sqcup S$ the disjoint union of S with a single vertex; that is, the simplicial complex obtained by adding a single vertex and no higher-dimensional faces. Then there is an orientable fibration*

$$(M \setminus \{pt\})_S \to M_{\{pt\} \sqcup S} \to M.$$

Proof. Fix M connected manifold with dimension at least 2. Then

$$M_{\{pt\} \sqcup S} \subseteq M_{C(S)} = M \times M_S,$$

so there is a projection $\pi : M_{\{pt\} \sqcup S} \to M$. We claim that π is the projection map of a fibration.

To show that $M_{\{pt\} \sqcup S} \xrightarrow{\pi} M$ is a fibration, given any topological space W suppose we have a homotopy $h : W \times I \to M$, and an initial lift $H_0 : W \to M_{\{pt\} \sqcup S}$ so that $h(0) = \pi \circ H_0$. We need to lift the entire homotopy h to some $H : W \times I \to M_{\{pt\} \sqcup S}$.

By construction, for each $x \in M$,

$$\pi^{-1}(x) = \left\{ (x^1, \dots, x^n, x) \in M_S \middle| \forall i, x \neq x^i \right\}.$$

Set $H_0 = (H_0^1, \dots H_0^n, h(0))$. Fixing $w \in W$, choose an isotopy of M, $\alpha_w^t : M \to M$ so that α_w^0 is the identity and for each t and each i,

$\alpha_w^t(H_0^i(w)) \neq h(w,t)$. Such α_w^t exists because M is a manifold of dimension at least 2, so locally an incipient collision between $\alpha_w^t(H_0^i(w))$ and $h(w,t)$ can be avoided by perturbing $\alpha_w^t(H_0^i(w))$ slightly. Moreover we can choose α_w^t to depend continuously on w. Then set $H(w,t) = (\alpha_w^t(H_0^1(w)), \ldots, \alpha_w^t(H_0^n(w)), h(w,t))$. Because each α_w^t is a homeomorphism, $\alpha_w^t(H_0^i(w)) = \alpha_w^t(H_0^j(w))$ iff $H_0^i(w) = H_0^j(w)$. Further, by construction $\alpha_w^t(H_0^i(w)) \neq h(w,t)$. Thus $H : W \times I \to M_{\{pt\}\sqcup S}$. Clearly H lifts h. Thus $M(\{pt\} \sqcup S) \xrightarrow{\pi} M$ is a fibration.

The fiber $\pi^{-1}(x) = \left\{(x^1, \ldots, x^n, x) \in M_S \middle| \forall i, x \neq x^i\right\}$ is homeomorphic to $(M \setminus \{x\})_S$. Clearly the fibration is orientable. □

Since M is a connected manifold, it is path connected. So the Leray-Serre spectral sequence (see Chapter 5 of [21]) reads:

Corollary 4.11. *If M is connected of dimension at least 2, there is a spectral sequence abutting to $H_c^*(M_{\{pt\}\sqcup S})$ whose E_2 page is given by*

$$E_2^{ij} = H_c^i(M, H_c^j((M \setminus \{pt\})_S)$$

We call this sequence, along with its its homology counterpart, the *orphan point spectral sequences.*

Remark 7. We mention that Theorem 4.10 is a generalization of a result of Fadell and Neuwirth [14] which says that $\mathrm{Conf}_n(M)$ fibers over M with fiber $\mathrm{Conf}_{n-1}(M \setminus \{pt\})$.

5. M matters

The homology theory $H_c^*(M_S)$ can be thought of in two ways: for a fixed S, it is an invariant of the space M; for a fixed M, it is an invariant of the simplicial complex S. In this section we will show that the strength of the invariant $H_c^*(M_-)$ in fact does depend on the choice of the space M.

Consider the one-dimensional simplicial complexes (graphs) in Figure 1, denoted by G_1 and G_2, respectively. We will use these graphs to demonstrate that the strength of $H_c^*(M_S)$ depends strongly on the choice of M.

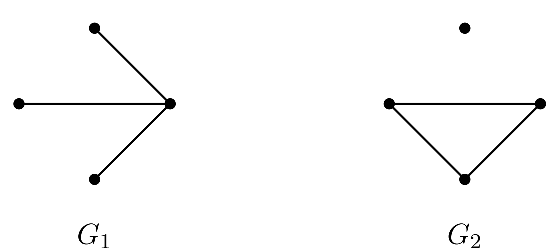

FIGURE 1. Complexes G_1 and G_2 with $H_c^*\left(\mathbb{CP}^1_{G_1}\right) \cong H_c^*\left(\mathbb{CP}^1_{G_2}\right)$ but $H_c^*\left(\mathbb{CP}^2_{G_1}\right) \not\cong H_c^*\left(\mathbb{CP}^2_{G_2}\right)$.

Proposition 5.1. *For simplicial complexes G_1 and G_2 as in Figure* 1*,*

$$H_c^*(\mathbb{CP}^1_{G_1}) \cong H_c^*(\mathbb{CP}^1_{G_2}) \cong \begin{cases} \mathbb{Z} & k = 3, 5, 6, 8 \\ 0 & \textit{otherwise.} \end{cases}$$

The following theorem shows that $H_c^*(\mathbb{CP}^1_-)$ is not stronger than $H_c^*(\mathbb{CP}^2_-)$:

Theorem 5.2. *For simplicial complexes G_1 and G_2 as in Figure* 1*, we have*

$$H_c^*(\mathbb{CP}^2_{G_1}) \not\cong H_c^*(\mathbb{CP}^2_{G_2}).$$

Proof. We will distinguish $H_c^*(\mathbb{CP}^2_{G_1})$ from $H_c^*(\mathbb{CP}^2_{G_2})$ by showing that they have different ranks in degrees 6, 7, 8, and 9. For this reason and to simplify some computations, we will work over $\mathbb{Q}$ for the remainder of this proof.

To prove Theorem 5.2, we will make use of the orphan point spectral sequence Corollary 4.11. Because our coefficients are $\mathbb{Q}$, the convergence of the orphan point spectral sequence means $H_c^k(M_{\{pt\}\sqcup S}) = \bigoplus_{i+j=k} E_\infty^{ij}$.

We have the following elementary computations.

H_c^k	0	1	2	3	4	5	6	7	8	9	10	11	12
$\mathbb{CP}^2$	$\mathbb{Q}$		$\mathbb{Q}$		$\mathbb{Q}$								
$\mathbb{CP}^2 \setminus \{\mathrm{pt}\}$			$\mathbb{Q}$		$\mathbb{Q}$								
$\left(\mathbb{CP}^2 \setminus \{\mathrm{pt}\}\right)^2$					$\mathbb{Q}$		$\mathbb{Q}^2$		$\mathbb{Q}$				
$\left(\mathbb{CP}^2 \setminus \{\mathrm{pt}\}\right)^3$							$\mathbb{Q}$		$\mathbb{Q}^3$		$\mathbb{Q}^3$		$\mathbb{Q}$

Throughout this section, we denote by $\bullet$, $\bullet\bullet$, and $\bullet\bullet\bullet$ the complexes with one, two, and three vertices, respectively, and no higher-dimensional faces.

First we apply the orphan point sequence to compute $H_c^*(\mathbb{CP}^2_{\bullet\bullet})$. The E_2 page is:

$p \diagdown q$	0	1	2	3	4
0			$\mathbb{Q}$		$\mathbb{Q}$
1					
2			$\mathbb{Q}$		$\mathbb{Q}$
3					
4			$\mathbb{Q}$		$\mathbb{Q}$

The differential of this page is $d_2 : E_2^{p,q} \to E_2^{p+2,q+1}$, which must evidently vanish. Moreover, all subsequent differentials of the spectral sequence are zero. So the sequence collapses at E_2. We have $H_c^k(\mathbb{CP}^2_{\bullet\bullet}) = \bigoplus_p E_2^{p,k-p}$, that is:

$$H_c^k(\mathbb{CP}^2_{\bullet\bullet}) = \begin{cases} \mathbb{Q} & k = 2, 8 \\ \mathbb{Q}^2 & k = 4, 6 \\ 0 & \text{otherwise.} \end{cases}.$$

Similarly, we have for $\{pt\}\sqcup\Delta^1$, the E_2 page of the Leray spectral sequence is:

$p\diagdown q$	0	1	2	3	4	5	6	7	8
0					$\mathbb{Q}$		$\mathbb{Q}^2$		$\mathbb{Q}$
1									
2					$\mathbb{Q}$		$\mathbb{Q}^2$		$\mathbb{Q}$
3									
4					$\mathbb{Q}$		$\mathbb{Q}^2$		$\mathbb{Q}$

Again the differentials d_k for $k \geq 2$ vanish, so we obtain

$$H_c^k(\mathbb{CP}^2_{\{pt\}\sqcup\Delta^1}) = \begin{cases} \mathbb{Q} & k = 4, 12 \\ \mathbb{Q}^3 & k = 6, 10 \\ \mathbb{Q}^4 & k = 8 \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we have for $\{pt\} \sqcup \Delta^2$, the E_2 page:

$p\diagdown q$	0	1	2	3	4	5	6	7	8	9	10	11	12
0							$\mathbb{Q}$		$\mathbb{Q}^3$		$\mathbb{Q}^3$		$\mathbb{Q}$
1													
2							$\mathbb{Q}$		$\mathbb{Q}^3$		$\mathbb{Q}^3$		$\mathbb{Q}$
3													
4							$\mathbb{Q}$		$\mathbb{Q}^3$		$\mathbb{Q}^3$		$\mathbb{Q}$

Yet again the differentials vanish, so we obtain

$$H_c^k(\mathbb{CP}^2_{\{pt\}\sqcup\Delta^2}) = \begin{cases} \mathbb{Q} & k = 6, 16 \\ \mathbb{Q}^4 & k = 8, 14 \\ \mathbb{Q}^7 & k = 10, 12 \\ 0 & \text{otherwise.} \end{cases}$$

Now we use the addition-contraction sequence $G_2 \to \{pt\} \sqcup \Delta^2 \to \bullet\bullet$ to obtain the long exact sequence

$$\begin{aligned} 0 &\to H_c^0(\mathbb{CP}^2_{G_2}) \to 0 \to 0 \to H_c^1(\mathbb{CP}^2_{G_2}) \to 0 \\ &\to 0 \to H_c^2(\mathbb{CP}^2_{G_2}) \to 0 \to \mathbb{Q} \to H_c^3(\mathbb{CP}^2_{G_2}) \to 0 \\ &\to 0 \to H_c^4(\mathbb{CP}^2_{G_2}) \to 0 \to \mathbb{Q}^2 \to H_c^5(\mathbb{CP}^2_{G_2}) \to 0 \\ &\to 0 \to H_c^6(\mathbb{CP}^2_{G_2}) \to \mathbb{Q} \to \mathbb{Q}^2 \to H_c^7(\mathbb{CP}^2_{G_2}) \to 0 \\ &\to 0 \to H_c^8(\mathbb{CP}^2_{G_2}) \to \mathbb{Q}^4 \to \mathbb{Q} \to H_c^9(\mathbb{CP}^2_{G_2}) \to 0 \\ &\to 0 \to H_c^{10}(\mathbb{CP}^2_{G_2}) \to \mathbb{Q}^7 \to 0 \to H_c^{11}(\mathbb{CP}^2_{G_2}) \to 0 \\ &\to 0 \to H_c^{12}(\mathbb{CP}^2_{G_2}) \to \mathbb{Q}^7 \to 0 \\ &\to H_c^{13}(\mathbb{CP}^2_{G_2}) \to 0 \to 0 \to H_c^{14}(\mathbb{CP}^2_{G_2}) \to \mathbb{Q}^4 \to 0 \\ &\to H_c^{15}(\mathbb{CP}^2_{G_2}) \to 0 \to 0 \to H_c^{16}(\mathbb{CP}^2_{G_2}) \to \mathbb{Q} \to 0 \end{aligned} \tag{6}$$

which determines all but four of the groups $H_c^*(\mathbb{CP}^2_{G_2})$. The map $\mathbb{Q} \to \mathbb{Q}^2$ is injective, and the map $\mathbb{Q}^4 \to \mathbb{Q}$ is surjective, which determines the cases $k = 6, 7, 8, 9$. We have:

$$H_c^k(\mathbb{CP}^2_{G_2}) = \begin{cases} \mathbb{Q} & k = 3, 7, 16 \\ \mathbb{Q}^2 & k = 5 \\ \mathbb{Q}^3 & k = 8 \\ \mathbb{Q}^4 & k = 14 \\ \mathbb{Q}^7 & k = 10, 12 \\ 0 & \text{otherwise.} \end{cases}$$

Now we compute the homology for G_1. First we use the addition-contraction sequence $\bullet\bullet \to \Delta^1 \to \bullet$ to compute $H_c^*((\mathbb{CP}^2 \setminus \{\text{pt}\})_{\bullet\bullet})$:

$$\begin{aligned} 0 &\to H_c^0((\mathbb{CP}^2 \setminus \{\text{pt}\})_{\bullet\bullet}) \to 0 \to 0 \to H_c^1((\mathbb{CP}^2 \setminus \{\text{pt}\})_{\bullet\bullet}) \to 0 \to 0 \\ &\to H_c^2((\mathbb{CP}^2 \setminus \{\text{pt}\})_{\bullet\bullet}) \to 0 \to \mathbb{Q} \to H_c^3((\mathbb{CP}^2 \setminus \{\text{pt}\})_{\bullet\bullet}) \to 0 \to 0 \\ &\to H_c^4((\mathbb{CP}^2 \setminus \{\text{pt}\})_{\bullet\bullet}) \to \mathbb{Q} \to \mathbb{Q} \to H_c^5((\mathbb{CP}^2 \setminus \{\text{pt}\})_{\bullet\bullet}) \to 0 \to 0 \\ &\to H_c^6((\mathbb{CP}^2 \setminus \{\text{pt}\})_{\bullet\bullet}) \to \mathbb{Q}^2 \to 0 \to H_c^7((\mathbb{CP}^2 \setminus \{\text{pt}\})_{\bullet\bullet}) \to 0 \to 0 \\ &\to H_c^8((\mathbb{CP}^2 \setminus \{\text{pt}\})_{\bullet\bullet}) \to \mathbb{Q} \to 0 \end{aligned} \tag{7}$$

which determines all but two of the groups, namely those in degrees 4 and 5. The map $H_c^4((\mathbb{CP}^2 \setminus \{pt\})^2) \to H_c^4(\mathbb{CP}^2 \setminus \{pt\})$ is induced from the inclusion of the diagonal in $(\mathbb{CP}^2 \setminus \{pt\})^2$; it is an isomorphism. Thus by exactness, $H_c^4((\mathbb{CP}^2 \setminus \{\text{pt}\})_{\bullet\bullet})$ and $H_c^5((\mathbb{CP}^2 \setminus \{\text{pt}\})_{\bullet\bullet})$ are trivial. Therefore:

$$H_c^*((\mathbb{CP}^2 \setminus \{\text{pt}\})_{\bullet\bullet}) = \begin{cases} \mathbb{Q} & k = 3, 8 \\ \mathbb{Q}^2 & k = 6 \\ 0 & k = 0, 1, 2, 4, 5, 7. \end{cases} \tag{8}$$

Now we use the spectral sequence argument to compute $H_c^*(\mathbb{CP}^2_{\bullet\bullet\bullet})$. Then $E_2^{p,q} = H_c^p(\mathbb{CP}^2, H_c^q((\mathbb{CP}^2 \setminus \{\text{pt}\})_{\bullet\bullet}))$ is:

$p \diagdown q$	0	1	2	3	4	5	6	7	8
0				$\mathbb{Q}$			$\mathbb{Q}^2$		$\mathbb{Q}$
1									
2				$\mathbb{Q}$			$\mathbb{Q}^2$		$\mathbb{Q}$
3									
4				$\mathbb{Q}$			$\mathbb{Q}^2$		$\mathbb{Q}$

Most of the differentials of this spectral sequence vanish; we have

k	0	1	2	3	4	5	6	7	8	9	10	11	12
$\text{rk}(H_c^k(\mathbb{CP}^2_{\bullet\bullet\bullet}))$	0	0	0	1	0	1	r_1	r_2	3	0	3	0	1

where $r_1 = 2$ and $r_2 = 1$ or $r_1 = 1$ and $r_2 = 0$.

Observing that $G_1 = C(\bullet\bullet\bullet)$ is the cone on three points, we get

$$H_c^*(\mathbb{CP}^2_{G_1},) = H_c^*(\mathbb{CP}^2) \otimes H_c^*(\mathbb{CP}^2_{\bullet\bullet\bullet}).$$

In particular:

$$\mathrm{rk}(H_c^k(\mathbb{CP}^2_{G_1})) = \begin{cases} 1 & k = 3, 16 \\ 2 & k = 5 \\ 2 \text{ or } 1 & k = 6, 9 \\ 3 \text{ or } 2 & k = 7 \\ 5 \text{ or } 4 & k = 8 \\ 8 \text{ or } 7 & k = 10 \\ 7 & k = 12 \\ 4 & k = 14 \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

To complete the proof of Theorem 5.2, we simply observe that

$$\begin{aligned} &\mathrm{rk}(H_c^6(\mathbb{CP}^2_{G_1})) \geq 1 > 0 = \mathrm{rk}(H_c^6(\mathbb{CP}^2_{G_2})), \\ &\mathrm{rk}(H_c^7(\mathbb{CP}^2_{G_1})) \geq 2 > 1 = \mathrm{rk}(H_c^7(\mathbb{CP}^2_{G_2})), \\ &\mathrm{rk}(H_c^8(\mathbb{CP}^2_{G_1})) \geq 4 > 3 = \mathrm{rk}(H_c^8(\mathbb{CP}^2_{G_2})), \text{ and} \\ &\mathrm{rk}(H_c^9(\mathbb{CP}^2_{G_1})) \geq 1 > 0 = \mathrm{rk}(H_c^9(\mathbb{CP}^2_{G_2})). \end{aligned}$$

□

Remark 8. Lowrance-Sazdanović showed [20] that the chromatic homology over $\mathbb{Z}[x]/_{(x^2=0)}$ is determined by the chromatic polynomial while it was known from the work of Pabiniak-Przytycki-Sazdanović [22] that the chromatic homology over $\mathbb{Z}[x]/_{(x^3=0)}$ is more discriminative than the chromatic polynomial. Since $H_c^*(\mathbb{CP}^1) \cong \mathbb{Z}[x]/_{(x^2=0)}$ and $H_c^*(\mathbb{CP}^2) \cong \mathbb{Z}[x]/_{(x^3=0)}$, the Baranovsky-Sazdanović spectral sequence [4] implies that $EH_*(G, \mathbb{CP}^1)$ is also determined by the chromatic polynomial. Proposition 5.1 and Theorem 5.2 imply that a similar phenomenon might hold in the simplicial setting: perhaps $H_c^*(\mathbb{CP}^1_S)$ is determined by the simplicial chromatic polynomial χ_c discussed in the next section.

6. The simplicial chromatic polynomial

The starting point in the work of Eastwood and Huggett [12] as well as Helme-Guizon and Rong [15] is the chromatic polynomial which is lifted to a homology theory via a process called categorification. In this section we reverse this process, i.e. we decategorify our homology theory to a polynomial. More precisely, first we take the Euler characteristic of $H_c^*(M_S)$ to obtain a numerical invariant which depends on S and M: denote by $\chi_c(S, M)$ the

Euler characteristic

$$\chi_c(S, M) = \sum (-1)^k \operatorname{rk} H_c^k(M_S)$$

The following statements about this Euler characteristic are obtained by applying the Euler characteristic to the homological results in §§3,4:

Corollary 6.1 (of Theorem 3.4)**.** *For any simplicial complex S, any manifold M, and any minimal nonface σ of S,*

$$0 = \chi_c(S, M) - \chi_c(S \cup \{\sigma\}, M) + \chi_c(S_{/\sigma}, M).$$

Corollary 6.2 (of Proposition 2.4)**.** *For any M, $\chi_c(\Delta^{n-1}, M) = \chi(H_c^*(M))^n$.*

Proposition 6.3. *For any simplicial complex S with n vertices, $\chi_c(S, M)$ is a monic polynomial in $\chi(H_c^*(M))$, of degree n.*

Proof. We proceed by induction on n. If $n = 1$, then $M_S = M$, so we are done.

For the induction, observe that if we choose an edge $e \notin S$, we may apply Corollary 6.1 to add/contract e. The term $\chi_c(S_{/e}, M)$ is, by hypothesis, a monic polynomial of degree $n - 1$. We may continue adding/contracting edges until we obtain S' whose 1-skeleton is a complete graph, so that

$$\chi_c(S, M) = \chi_c(S', M) + \text{ polynomial of degree at most } n - 1 \tag{10}$$

Then we may apply Corollary 6.1 to add 2-simplices, observing that the contraction term will be $\chi_c(T, M)$ for T with $n - 2$ simplices, which is by inductive assumption a monic polynomial of degree $n - 2$.

Continuing in this way, we will obtain

$$\chi_c(S, M) = \chi_c(\Delta^V, M) + \sum (-1)^\varepsilon \chi_c(T, M) \tag{11}$$

where each T is a simplicial complex on at most $n-1$ vertices. By Corollary 6.2, this shows the leading term of $\chi_c(S, M)$ is $\chi(H_c^*(M))^n$. □

Observe that $\chi_c(S, M)$ depends only on $\chi(H_c^*(M))$, so we may choose any space M with $\chi_c(M) = t$ — we may as well use $M = \mathbb{CP}^{t-1}$ — to define

Definition 6.1. The *simplicial chromatic polynomial* is the polynomial determined by the assignment $\chi_c(S) : t \mapsto \chi_c(S, \mathbb{CP}^{t-1})$.

Proposition 6.4. *The normalization $\chi_c(\Delta^{n-1}, t) = t^n$ and the addition-contraction formula*

$$0 = \chi_c(S, t) - \chi_c(S \cup \{\sigma\}, t) + \chi_c(S_{/\sigma}, t)$$

determine a unique polynomial invariant of simplicial complexes.

Corollary 6.5 (of Proposition 2.2)**.** *For any graph G, $\chi_c(I(G), t) = P_G(t)$.*

The properties of the homology discussed in §4.2 descend to χ_c:

Corollary 6.6 (of Corollary 4.7)**.** *For any simplicial complexes S, T,*

$$\chi_c(S * T, t) = \chi_c(S, t) \cdot \chi_c(T, t).$$

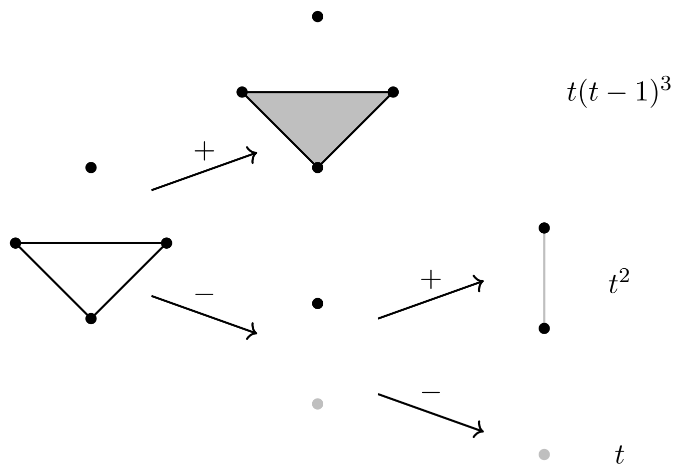

FIGURE 2. An addition-contraction computation for G_2. Added minimal nonfaces are shaded in gray. We have $\chi_c(G_2,t) = t(t-1)^3 - t^2 + t = t^4 - 3t^3 + 2t^2$.

Corollary 6.7 (of Theorem 4.10)**.** *For any simplicial complex S,*

$$\chi_c(\{pt\} \sqcup S, t) = t \cdot \chi_c(S, t-1).$$

Proof. Besides Theorem 4.10, the only additional observation required is that

$$\chi(H_c^*(\mathbb{CP}^{t-1} \setminus \{pt\})) = t - 1.$$

□

Applying these corollaries, one computes as in Figure 2, that, for example, for G_1 and G_2 as in §5,

$$\chi_c(G_1, t) = t^4 - 3t^3 + 2t^2 = \chi_c(G_2, t).$$

In particular, we see that G_1 and G_2 have the property that, for any M, the homology theories $H_c^*(M_{G_1})$ and $H_c^*(M_{G_2})$ have the same Euler characteristic. Then Theorem 5.2 demonstrates that $H_c^*(M_S)$, even for a single choice of M, may have strictly more discriminative power than the polynomial $\chi_c(S)$.

Corollary 6.8 (of Proposition 4.9 and Corollary 6.7)**.** *For any simplicial complex S,*

$$\chi_c(\Sigma(S), t) = t \cdot (t-1) \cdot \chi_c(S, t).$$

Proof. By Corollary 6.7, $\chi_c(\bullet\bullet, t) = t \cdot (t-1)$. Since $\Sigma(S) = S * (\bullet\bullet)$, the result follows. □

Proposition 6.9. *Denote by S_n the standard triangulation of the n-sphere. Then*

$$\chi_c(S_n, t) = t^{n+1}(t-1)^{n+1}.$$

Proof. The standard triangulation S_n is defined recursively by $S_0 = \bullet\bullet$ and $S_n = \Sigma(S_{n-1})$. The result then follows by applying Corollary 6.8. □

7. Comparison to other invariants

In this section, we note that the polynomial χ_c, hence the homology H_c^*, is novel in the sense that it is not a specialization of known polynomial invariants of graphs, in the case S is a graph, or simplicial complexes.

7.1. Bott-Whitney polynomial. Bott [8] gave a family of polynomial invariants for polyhedra, which captures both combinatorial and also some topological information [25]. In particular, the R-polynomial is defined as follows:

Definition 7.1. For a finite cell complex π of dimension D, we define

$$R(\pi, t) = \sum_{s \subseteq I_N} (-1)^{|s|} t^{\beta_D(\pi \setminus s)}$$

where N is the number of D-dimensional cells in π, $I_N = \{1, \dots, D\}$ is a set indexing those cells, β_D is the D^{th} Betti number, and $\pi \setminus s$ means the complex obtained by omitting cells with labels in $s \subseteq I_N$.

We have, for G_1 and G_2 as in Figure 1, that $R(G_1, t) = 0$ and $R(G_2, t) = t - 1$; this shows that having the same χ_c polynomial does not imply having the same R-polynomial.

The reverse is also true; that is, $R(S_1, t) = R(S_2, t)$ does not imply $\chi_c(S_1, t) = \chi_c(S_2, t)$. To see this, consider

Example 4.1 of [25]**.** *If π is a subdivision of an n-cell, then $R(\pi, t) = 0$.*

On the other hand, if we set B to be the barycentric subdivision of Δ^2, $\chi_c(\Delta^2, t) = t^3$ whereas $\chi_c(B, t)$ is a polynomial of degree 7.

In particular, Wang's Example 4.1 implies the R-polynomial is trivial on trees; in forthcoming [9], we show examples of trees which are distinguished by χ_c.

7.2. Chromatic and Tutte polynomials for graphs. Consider the graphs G_3 and G_4 as in Figure 3, which are outerplanar and are composed of two triangles and one square, glued along edges, hence have the same chromatic polynomial [24]:

We have

$$\chi_c(G_3, t) = t^6 - 7t^5 + 19t^4 - 27t^3 + 22t^2 - 8t$$

but

$$\chi_c(G_4, t) = t^6 - 7t^5 + 18t^4 - 20t^3 + 8t^2$$

so χ_c, hence $H_c^*(M_-)$, distinguishes some graphs which the chromatic polynomial does not.

Similarly, the graphs G_5 and G_6 in Figure 4 are related by a Whitney flip, hence have the same Tutte polynomial:

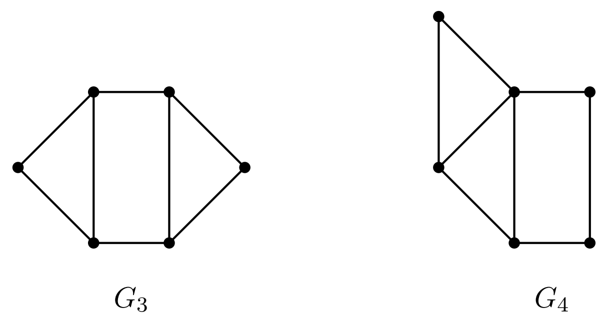

FIGURE 3. Two cochromatic graphs G_3 and G_4, which are distinguished by χ_c.

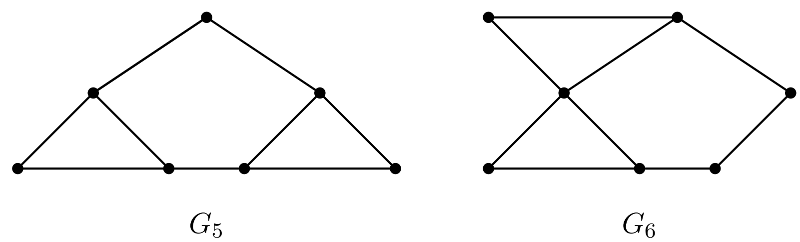

FIGURE 4. Two graphs G_5 and G_6 which are co-Tutte but distinguished by χ_c.

however

$$\chi_c(G_5, t) = t^7 - 12t^6 + 60t^5 - 161t^4 + 245t^3 - 199t^2 + 66t$$

and

$$\chi_c(G_6, t) = t^7 - 12t^6 + 61t^5 - 171t^4 + 280t^3 - 249t^2 + 90t$$

so χ_c distinguishes some graphs which the Tutte polynomial does not.

7.3. Tutte-Krushkal-Renardy polynomial. Krushkal-Renardy [17] gave a family of two-variable polynomials $T_{n,K}(x, y)$ associated to a cellular complex K. We have

$$T_{2,C(G_1)}(x, y) = 1 + 3x + 3x^2 + x^3$$

and

$$T_{2,C(G_2)}(x, y) = 4 + y + 6x + 4x^2 + x^3.$$

On the other hand,

$$\chi_c(C(G_1), t) = t \cdot \chi_c(G_1, t) = t \cdot \chi_c(G_2, t) = \chi_c(C(G_2), t)$$

showing that the Tutte-Renardy-Krushkal polynomial distinguishes some complexes which χ_c does not.

Conversely, consider S, the simplicial complex on 4 vertices with exactly 3 two-dimensional faces. We have $T_{2,S}(x, y) = 1 + 3x + 3x^2 + x^3 =$

$T_{2,C(G_1)}(x,y)$, but S has 4 vertices and $C(G_1)$ has five vertices, so $\chi_c(S,t)$ and $\chi_c(C(G_1),t)$ have different degrees.

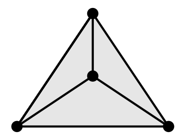

FIGURE 5. A simplicial complex which is co-Tutte-Renardy-Krushkal to $C(G_1)$, which χ_c distinguishes from $C(G_1)$.

8. Questions and future directions

8.1. The chromatic polynomial for simplicial complexes. A forthcoming paper [9] describes some further properties of the polynomial χ_c, including some applications to graph theory. Observe that our construction is based on the deletion-contraction rule, but is not obviously 'chromatic' in the sense of enumerating colorings. Describing a notion of 'colorings' of a simplicial complex which χ_c counts is another natural direction for future work.

8.2. A two-variable polynomial. Krushkal-Renardy have defined a family of Tutte polynomials for cell complexes, of which the Bott polynomial is a specialization. We expect that a two-variable generalization of χ_c should exist, and satisfy interesting duality properties.

8.3. An algebraic categorification of χ_c. Helme-Guizon and Rong's chromatic homology [15] is a Khovanov-type homology theory: a categorification of the chromatic polynomial for graphs whose additional input is a certain graded algebra. [11] describes the corresponding construction for χ_c as well as a spectral sequence relating the algebraic and geometric constructions as in [4].

8.4. An algebraic description. Arnol'd (for $M = \mathbb{C}$ [2]), followed by Križ [16] and independently Totaro [23], gave a description of $H^*(\mathrm{Conf}_n M;\mathbb{Q})$ as the homology of a differential graded algebra over $H^*(M^n;\mathbb{Q})$. We suspect that the cohomology of M_S should have a similar description; a forthcoming paper addresses the case $S = I(G)$ is the independence complex of a graph [10].

8.5. Functoriality in M. We have shown that M_S and $H^*(M_S)$ are functorial in S, and exploited this to study S. A natural question to ask is: in what sense are the constructions functorial in M? What information about M can be read from M_S or $H^*_c(M_S)$ as S varies?

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This paper is available via http://nyjm.albany.edu/j/2019/25-34.html.