

## Boundaries, bundles, and trace algebras

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*To the memory of Bill Arveson*

ABSTRACT. We describe how noncommutative function algebras built from noncommutative functions in the sense of Dmitry S. Kaliuzhnyi-Verbovetskyi and Victor Vinnikov may be studied as subalgebras of homogeneous  $C^*$ -algebras.

### CONTENTS

1. Introduction	136
2. The concomitants and cross sections	144
3. Function theory in bundles	147
4. Boundary representations	148
5. Azumaya algebras	149
6. Concluding remarks	150
Acknowledgement	152
References	152

### 1. Introduction

This note grew out of efforts to apply Arveson’s boundary theory [2, 4, 6, 5] to operator algebras that arise naturally in free analysis. They are built from the representation theory of free algebras, but our point of view was inspired to a great extent by the recent book and perspective of D. Kaliuzhnyi-Verbovetskyi and V. Vinnikov [16]. In a sense, our purpose is to present “a proof of concept”. The problem which drew us to the topics discussed here remains unsolved. We will discuss it in the final section, Section 6. Our efforts to solve this problem led us to methods from algebraic geometry, geometric invariant theory and polynomial identity algebras - subjects largely unfamiliar to us. Nevertheless, we hope to show that these subjects carry useful information for free analysis and its associated operator algebras. We have not striven for maximal generality in the theorems and proofs

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presented in this paper. Rather, we have tried to present a story whose purpose is to stimulate interest among the operator algebra community in the algebras described here and to stimulate future research. Consequently, the Introduction is the bulk of the paper. It carries most of the narrative and the statements of the main theorems. Most proofs and details are relegated to subsequent (shorter) sections.

The fundamental feature of the functions that we want to exploit is that they are (*holomorphic matrix concomitants*). Various algebras they generate will be identified as subalgebras of homogeneous  $C^*$ -algebras. To describe the functions and algebras, we need to develop notation and provide background information. Throughout this note  $G$  will denote the projective linear group,  $PGL(n, \mathbb{C})$ , which will be viewed as the group of *automorphisms* of the full algebra of complex  $n \times n$  matrices,  $M_n(\mathbb{C})$ . The subgroup of  $G$  that preserves the usual  $*$ -structure on  $M_n(\mathbb{C})$  is the projective unitary group,  $PU(n, \mathbb{C})$ . It will be denoted by  $K$ . We frequently identify  $G$  with  $GL(n, \mathbb{C})$  and write  $s^{-1}as$ ,  $a \in M_n(\mathbb{C})$ ,  $s \in G$ , for what should be written as  $a \cdot s$  or  $s^{-1} \cdot a$ . This should cause no confusion since when  $GL(n, \mathbb{C})$  appears in this note, it always acts through conjugation of matrices. We study actions of  $G$  on  $d$ -tuples of  $n \times n$  matrices,  $M_n(\mathbb{C})^d$ , via the “diagonal” action. That is, we write elements of  $M_n(\mathbb{C})^d$  as  $\mathfrak{z} = (Z_1, Z_2, \dots, Z_d)$ , with  $Z_i \in M_n(\mathbb{C})$ , and we write  $\mathfrak{z} \cdot s = s^{-1}\mathfrak{z}s$  for  $(s^{-1}Z_1s, s^{-1}Z_2s, \dots, s^{-1}Z_ds)$ ,  $s \in G$ . We are interested in domains  $\mathcal{D} \subseteq M_n(\mathbb{C})^d$  that are invariant under this action of  $G$ . A function  $f$  defined on such a domain  $\mathcal{D}$  and mapping to  $M_n(\mathbb{C})$  is called a *matrix concomitant* if  $f$  satisfies the equation

$$(1.1) \quad f(s^{-1}\mathfrak{z}s) = s^{-1}f(\mathfrak{z})s,$$

for all  $s \in G$  and all  $\mathfrak{z} \in \mathcal{D}$ . The collection of all holomorphic matrix concomitants defined on a domain  $\mathcal{D}$  will be denoted  $Hol(\mathcal{D}, M_n(\mathbb{C}))^G$ . These are the principal objects of study in this note. Unless explicitly stated otherwise  $d$  and  $n$  will be assumed to be at least 2 when discussing  $d$ -tuples of  $n \times n$  matrices.

Examples of holomorphic matrix concomitants are easy to come by. For  $i = 1, 2, \dots, d$ , we let  $\mathcal{Z}_i$  denote the function on  $M_n(\mathbb{C})^d$  defined by

$$\mathcal{Z}_i(\mathfrak{z}) := Z_i, \quad \mathfrak{z} = (Z_1, Z_2, \dots, Z_d).$$

That is, the  $\mathcal{Z}_i$  are just the *matrix coordinate functions* defined on  $M_n(\mathbb{C})^d$ . Clearly, each  $\mathcal{Z}_i$  is a holomorphic matrix concomitant. Since matrix concomitants form an algebra under pointwise sums and products, the algebra generated by the  $\mathcal{Z}_i$  consists of holomorphic matrix concomitants. This algebra is denoted  $\mathbb{G}_0(d, n)$  and is called the *algebra of  $d$  generic  $n \times n$  matrices*. Evidently, it is the image of the free algebra on  $d$  variables,  $\mathbb{C}\langle X_1, X_2, \dots, X_d \rangle$  under the map that takes  $X_i$  to  $\mathcal{Z}_i$ ,  $i = 1, 2, \dots, d$ . Another important algebra of holomorphic matrix concomitants is built from the algebra polynomial matrix *invariants*,  $\mathbb{I}_0(d, n)$ , which is the set of all polynomial functions  $p : M_n(\mathbb{C})^d \rightarrow \mathbb{C}$  such that  $p(s^{-1}\mathfrak{z}s) = p(\mathfrak{z})$ ,  $s \in G$ ,

$\mathfrak{z} \in M_n(\mathbb{C})^d$ . We identify  $p \in \mathbb{I}_0(d, n)$  with the matrix-valued function  $\mathfrak{z} \rightarrow p(\mathfrak{z})I_n$ , obtaining a polynomial matrix concomitant. The algebra generated by  $\mathbb{G}_0(d, n)$  and  $\mathbb{I}_0(d, n)$  is denoted  $\mathbb{S}_0(d, n)$  and is called the *trace algebra* of the generic matrices. In [24, Theorem 2.1], Procesi proved that  $\mathbb{S}_0(d, n)$  is precisely the set of all *polynomial* matrix concomitants. That is,  $\mathbb{S}_0(d, n)$  consists of all the matrix concomitants whose entries are polynomial functions of  $dn^2$  variables, organized as  $d$ -tuples of  $n \times n$  matrices.

**Lemma 1.1.**  *$Hol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^G$  is the closure of  $\mathbb{S}_0(d, n)$  in topology of uniform convergence on compact subsets of  $M_n(\mathbb{C})^d$ .*

**Proof.** This is an easy application of Weyl’s unitarian trick, which is often regarded as the assertion that the maximal compact subgroup of a reductive algebraic group is Zariski dense in the algebraic group [25, Page 224 ff]. In our situation, it means that any polynomial function on  $M_n(\mathbb{C})^d$  that is invariant under the action of  $K$  is automatically invariant under the action of  $G$ . Given  $f \in Hol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^G$  choose a sequence  $\{p_l\}_{l \geq 1}$  of  $n \times n$  matrices, whose entries are polynomial functions on  $M_n(\mathbb{C})^d$ , that converges to  $f$  uniformly on compact subsets of  $M_n(\mathbb{C})^d$ , and define

$$\tilde{p}_l(\mathfrak{z}) = \int_K kp_l(k^{-1}\mathfrak{z}k)k^{-1} dk,$$

where “ $dk$ ” denotes Haar measure on  $K$ . Then easy estimates show that the  $\tilde{p}_l$  converge to  $f$  uniformly on compact subsets of  $M_n(\mathbb{C})^d$ . The  $\tilde{p}_l$  satisfy the equation  $\tilde{p}_l(k^{-1}\mathfrak{z}k) = k^{-1}\tilde{p}_l(\mathfrak{z})k$  for all  $k \in K$ . Since the  $\tilde{p}_l$  are all polynomials, they are matrix concomitants by Weyl’s unitarian trick, and Procesi’s theorem (loc. cit.) completes the proof.  $\square$

Procesi also proved in [24, Theorem 3.4a] that  $\mathbb{I}_0(d, n)$  is generated by the traces  $tr(Z_{i_1}Z_{i_2}\cdots, Z_{i_s})$ , where  $s \leq 2^n - 1$ . Thus  $\mathbb{I}_0(d, n)$  is finitely generated. We may therefore consider the spectrum of  $\mathbb{I}_0(d, n)$ ,  $Q(d, n)$ , as an abstract affine algebraic variety defined over  $\mathbb{C}$ . The inclusion of  $\mathbb{I}_0(d, n)$  in the polynomial functions mapping  $M_n(\mathbb{C})^d$  to  $\mathbb{C}$  induces, by way of duality, a (regular) map  $\pi_0$  from  $M_n(\mathbb{C})^d$  onto  $Q(d, n)$ .

If  $\mathcal{V}(d, n)$  denotes the set of all  $\mathfrak{z} = (Z_1, Z_2, \dots, Z_d) \in M_n(\mathbb{C})^d$  such that  $Z_1, Z_2, \dots, Z_d$  generate  $M_n(\mathbb{C})$  as an algebra over  $\mathbb{C}$ , then  $\mathcal{V}(d, n)$  is a  $G$ -invariant, Zariski-open subset of  $M_n(\mathbb{C})^d$ , which we call the set of *irreducible points* of  $M_n(\mathbb{C})^d$ . Another fundamental theorem of Procesi [23, Theorem 5.10] asserts that the image of  $\mathcal{V}(d, n)$  under  $\pi_0$ , which we denote by  $Q_0(d, n)$ , is an open subset of the smooth points of  $Q(d, n)$  and that  $(\mathcal{V}(d, n), \pi_0, Q_0(d, n))$  has the structure of a holomorphic principal  $G$ -bundle, denoted here by  $\mathfrak{B}(d, n)$ .

We write  $\mathfrak{M}(d, n)$  for the associated fibre bundle with fibre  $M_n(\mathbb{C})$ , i.e., the bundle space of  $\mathfrak{M}(d, n)$  is  $\mathcal{V}(d, n) \times_G M_n(\mathbb{C})$ , where  $G$  acts on  $\mathcal{V}(d, n) \times M_n(\mathbb{C})$  via the formula  $(\mathfrak{z}, A) \cdot s = (s^{-1}\mathfrak{z}s, s^{-1}As) = (\mathfrak{z} \cdot s, s^{-1} \cdot A)$ . The projection  $\pi : \mathcal{V}(d, n) \times_G M_n(\mathbb{C}) \rightarrow Q_0(d, n)$  is given by formula  $\pi([\mathfrak{z}, A]) =$

$[\mathfrak{z}]$ , in which we adopt the convention that when  $G$  acts on a set, say,  $X$ , then the orbit of a point  $x \in X$  is written  $[x]$ , i.e.,  $[x] := \{x \cdot g \mid g \in G\}$ . Thus, in particular,  $\pi([\mathfrak{z}, A]) = \pi_0(\mathfrak{z})$ .

Our first result identifies the holomorphic cross sections of  $\mathfrak{M}_n(d, n)$ ,  $\Gamma_h(Q_0(d, n), \mathfrak{M}(d, n))$ , with the holomorphic matrix concomitants on  $\mathcal{V}(d, n)$ . While the proof will be presented in Section 2, it will be helpful to reflect here on the connection between cross sections and concomitants. Everything boils down to parsing this equation:

$$(1.2) \quad \sigma([\mathfrak{z}]) = [\mathfrak{z}, \phi(\mathfrak{z})],$$

$\mathfrak{z} \in \mathcal{V}(d, n)$ , where  $\sigma$  is a cross section of  $\mathfrak{M}(d, n)$  and  $\phi$  is a matrix concomitant. The key for this is to note that if we are given  $\mathbf{u} \in Q_0(d, n)$  and  $\mathbf{a} \in \mathfrak{M}(d, n)$  such that  $\pi(\mathbf{a}) = \mathbf{u}$ , then once  $\mathfrak{z} \in \mathcal{V}(d, n)$  is chosen so that  $\pi_0(\mathfrak{z}) = \mathbf{u}$ , there is one and only one  $A \in M_n(\mathbb{C})$  such that  $\mathbf{a} = [\mathfrak{z}, A]$ . Now let's read (1.2) from left to right and suppose  $\sigma$  is a cross section of  $\mathfrak{M}(d, n)$ . If  $\mathbf{u} \in Q_0(d, n)$ , then for  $\mathfrak{z} \in \pi_0^{-1}(\mathbf{u})$ , there is one and only one matrix  $\phi(\mathfrak{z}) \in M_n(\mathbb{C})$  such that  $[\mathfrak{z}, \phi(\mathfrak{z})] = \sigma(\mathbf{u})$ . This defines  $\phi$  on  $\pi_0^{-1}(\mathbf{u})$  for each  $\mathbf{u} \in Q_0(d, n)$ , and so the  $M_n(\mathbb{C})$ -valued function,  $\phi$ , is well defined on all of  $\mathcal{V}(d, n)$ . On the other hand,  $\pi_0(\mathfrak{z} \cdot s) = \mathbf{u}$  for any  $s \in G$ . So  $\pi([\mathfrak{z} \cdot s, \phi(\mathfrak{z} \cdot s)]) = \mathbf{u}$ , too. But by definition of the action of  $G$  on  $\mathcal{V}(d, n) \times M_n(\mathbb{C})$ ,  $[\mathfrak{z} \cdot s, \phi(\mathfrak{z} \cdot s)] = [\mathfrak{z}, s \cdot \phi(\mathfrak{z} \cdot s)]$ , which shows that  $s \cdot \phi(\mathfrak{z} \cdot s) = \phi(\mathfrak{z})$ , i.e.,  $\phi(\mathfrak{z} \cdot s) = s^{-1}\phi(\mathfrak{z})s$ . Reading (1.2) from right to left, suppose  $\phi$  is a matrix concomitant on  $\mathcal{V}(d, n)$ . Then  $[\mathfrak{z}, \phi(\mathfrak{z})]$  is an element in  $\mathfrak{M}(d, n)$  such that  $\pi([\mathfrak{z}, \phi(\mathfrak{z})]) = \pi_0(\mathfrak{z}) = [\mathfrak{z}]$ . But for each  $s \in G$ ,  $\pi([\mathfrak{z} \cdot s, \phi(\mathfrak{z} \cdot s)]) = \pi_0(\mathfrak{z} \cdot s) = [\mathfrak{z}]$ , too, and  $[\mathfrak{z} \cdot s, \phi(\mathfrak{z} \cdot s)] = [\mathfrak{z}, s \cdot \phi(\mathfrak{z} \cdot s)] = [\mathfrak{z}, \phi(\mathfrak{z})]$  because  $\phi$  is a concomitant. Therefore, if we set  $\sigma([\mathfrak{z}]) = [\mathfrak{z}, \phi(\mathfrak{z})]$ , then  $\sigma$  is well defined.

Henceforth, then, given a matrix concomitant  $\phi$ , we shall write  $\sigma_\phi$  for the cross section of  $\mathfrak{M}(d, n)$  determined by  $\phi$  via (1.2) and conversely, given a cross section  $\sigma$  of  $\mathfrak{M}(d, n)$ , we shall write  $\phi_\sigma$  for the matrix concomitant defined through (1.2).

**Theorem 1.2.** *For  $d \geq 2$  and  $n \geq 2$ , the correspondence  $\phi \rightarrow \sigma_\phi$  defines an algebra isomorphism  $\Psi : \text{Hol}(\mathcal{V}(d, n), M_n(\mathbb{C}))^G \rightarrow \Gamma_h(Q_0(d, n), \mathfrak{M}(d, n))$ , with inverse given by  $\sigma \rightarrow \phi_\sigma$ . If, in addition,  $d$  or  $n$  is greater than 2, then every concomitant in  $\text{Hol}(\mathcal{V}(d, n), M_n(\mathbb{C}))^G$  admits a unique extension to a concomitant in  $\text{Hol}(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^G$ . The domain  $\mathcal{V}(2, 2)$ , on the other hand, is a domain of holomorphy and there are concomitants in  $\text{Hol}(\mathcal{V}(2, 2), M_2(\mathbb{C}))^G$  that do not extend to  $M_2(\mathbb{C})^2$ .*

Theorem 1.2 gives a faithful representation of  $\text{Hol}(\mathcal{V}(d, n), M_n(\mathbb{C}))^G$  as a space of functions on the space of similarity classes of its irreducible matrix representations. It has the following immediate corollary.

**Corollary 1.3.** *The bundle  $\mathfrak{M}(d, n)$  is not trivial when  $(d, n) \neq (2, 2)$ .*

**Proof.** By [19, Proposition 4.4],  $Hol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^G$  has no zero divisors. Since  $Hol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^G$  and  $\Gamma_h(Q_0(d, n), \mathfrak{M}(d, n))$  are isomorphic when  $(d, n) \neq (2, 2)$ , neither does  $\Gamma_h(Q_0(d, n), \mathfrak{M}(d, n))$  in this case. However, if  $\mathfrak{M}(d, n)$  were trivial,  $\Gamma_h(Q_0(d, n), \mathfrak{M}(d, n))$  would be isomorphic to the  $n \times n$  matrices over the space of holomorphic functions on  $Q_0(d, n)$ , which has plenty of zero divisors.  $\square$

The bundle  $\mathfrak{M}(2, 2)$  is also nontrivial, but a different proof seems to be required. We will present one in Proposition 2.2.

Our focus then turns to domains  $\mathcal{D}$  such that  $\overline{\mathcal{D}}$  is a compact subset of  $Q_0(d, n)$ . Since  $Q_0(d, n)$  is the spectrum of  $\mathbb{I}_0(d, n)$ , the image of  $\mathbb{I}_0(d, n)$  under  $\Psi$  coincides with the algebra of regular  $\mathbb{C}$ -valued functions on  $Q_0(d, n)$ . That is, if  $\mathfrak{w} \in Q_0(d, n)$  and if  $\mathfrak{z} \in M_n(\mathbb{C})^d$  is such that  $\pi_0(\mathfrak{z}) = \mathfrak{w}$ , then for  $f \in \mathbb{I}_0(d, n)$ , we get  $\Psi(f)(\mathfrak{w}) = f(\mathfrak{z})$ , identified with the cross section of  $\mathfrak{M}(d, n)$  that  $f$  determines. That is,  $\Psi(f)([\mathfrak{z}]) = [\mathfrak{z}, f(\mathfrak{z})]$ . We let  $\mathbb{I}(\mathcal{D}; d, n)$  denote the closure of  $\{\Psi(f) \mid f \in \mathbb{I}_0(d, n)\}$  in the space of continuous  $\mathbb{C}$ -valued functions on  $\overline{\mathcal{D}}$ ,  $C(\overline{\mathcal{D}})$ . Since  $\mathbb{I}_0(d, n)$  contains the constant functions and separates the points of  $Q_0(d, n)$ ,  $\mathbb{I}(\mathcal{D}; d, n)$  is a function algebra on  $\overline{\mathcal{D}}$ , consisting of functions that are continuous on  $\overline{\mathcal{D}}$  and holomorphic on  $\mathcal{D}$ . Although  $\overline{\mathcal{D}}$  need not be the maximal ideal space of  $\mathbb{I}(\mathcal{D}; d, n)$ ,  $\overline{\mathcal{D}}$  contains the Shilov boundary of the maximal ideal space, which we denote by  $\partial\mathcal{D}$ . (This is the case simply because  $\mathbb{I}(\mathcal{D}; d, n)$  is a function algebra on  $\overline{\mathcal{D}}$ .) The extreme boundary, or Choquet boundary of  $\mathcal{D}$ , will be denoted  $\partial_e\mathcal{D}$ . It is a dense subset of  $\partial\mathcal{D}$  that consists of all points in  $\overline{\mathcal{D}}$  that have unique representing measures for  $\mathbb{I}(\mathcal{D}; d, n)$  supported in  $\overline{\mathcal{D}}$ .

We are interested both in the holomorphic cross sections of  $\mathfrak{M}(d, n)$  and in its continuous cross sections,  $\Gamma_c(Q_0(d, n), \mathfrak{M}(d, n))$ . The problem we face is that there is no *evident* natural involution on  $\mathfrak{M}(d, n)$  with respect to which  $\Gamma_c(X, \mathfrak{M}(d, n))$  is a  $C^*$ -algebra for every compact subset  $X \subseteq Q_0(d, n)$ . This is because  $\mathcal{V}(d, n)$  is a principal  $G$ -bundle and so in a coordinate representation of  $\mathcal{V}(d, n)$  the transition functions need not take their values in  $K$ . In fact,  $\Gamma_c(X, \mathfrak{M}(d, n))$  does not carry a *canonical* Banach algebra structure. Nevertheless, there are many *ad hoc* Banach algebra structures on  $\Gamma_c(X, \mathfrak{M}(d, n))$ , which may be constructed as follows. Take a locally finite open cover  $\mathcal{U}$  of  $Q_0(d, n)$  with an associated set of transition functions  $\{g_{UV}\}_{U, V \in \mathcal{U}}$  that define  $\mathcal{V}(d, n)$  as a principal bundle. Then take isomorphisms  $F_U : \mathfrak{M}(d, n)|_U \rightarrow U \times M_n(\mathbb{C})$  that allow one to identify continuous cross sections of  $\mathfrak{M}(d, n)$  over  $U$  with continuous  $M_n(\mathbb{C})$ -valued functions  $f_U$  on  $U$  that satisfy  $f_U(\mathbf{u}) = g_{UV}(\mathbf{u}) \circ f_V(\mathbf{u})$  on  $U \cap V$ . For a given compact subset  $X \subseteq Q_0(d, n)$  one can then define a Banach algebra norm on  $\Gamma_c(X, \mathfrak{M}(d, n))$  by setting

$$(1.3) \quad \|\sigma\|_{\mathcal{U}} := \sup_{x \in X} \sup_{x \in U} \|F_U(\sigma)(x)\|, \quad \sigma \in \Gamma_c(X, \mathfrak{M}(d, n)).$$

Here the norm  $\|F_U(\sigma)(x)\|$  refers to the Hilbert space operator norm one obtains by viewing  $M_n(\mathbb{C})$  as operators on  $\mathbb{C}^n$  in the usual way. Different systems of data  $(\mathcal{U}, \{g_{UV}\}_{U,V \in \mathcal{U}}, \{F_U\}_{U \in \mathcal{U}})$  give different norms, but the norms are all equivalent, i.e., the Banach algebras constructed are mutually isomorphic, and they all yield the compact-open topology on  $\Gamma_c(X, \mathfrak{M}(d, n))$  for any compact set  $X \subseteq Q_0(d, n)$ .

It may come as a pleasant surprise, therefore, to learn that there *is* a way to put a  $C^*$ -algebra structure on  $\Gamma_c(X, \mathfrak{M}(d, n))$  for each compact set  $X \subseteq Q_0(d, n)$ . In fact, any two  $C^*$ -algebra structures on  $\Gamma_c(X, \mathfrak{M}(d, n))$  are  $*$ -isomorphic. We must emphasize the difference between ‘isomorphic’ and ‘equal’ here because the isomorphisms involved almost always map some holomorphic sections to non-holomorphic sections. Each  $C^*$ -structure on  $\Gamma_c(X, \mathfrak{M}(d, n))$  is obtained from a *reduction*  $\mathfrak{P}$  of  $\mathcal{V}(d, n)$  to a principal  $K$ -bundle over  $X$ <sup>1</sup>. For our purposes, this means that  $\mathfrak{P}$  is a principal  $K$  bundle obtained from a  $K$ -invariant subset  $\mathcal{P}$  of  $\mathcal{V}(d, n)$  that  $\pi$  maps onto  $X$  in such a way that  $\pi$  induces a homeomorphism of  $\mathcal{P}/K$  with  $X$ . From a coordinate point of view, the transition functions defining  $\mathfrak{P}$  take their values in  $K$  and so the associated  $M_n(\mathbb{C})$ -fibre bundle, which we denote by  $\mathfrak{M}^*(\mathfrak{P}; d, n)$ , has a natural, fibre-wise-defined involution. The bundles  $\mathfrak{M}(d, n)$  and  $\mathfrak{M}^*(\mathfrak{P}; d, n)$  are isomorphic as topological bundles [15, Theorem 6.3.1]. Therefore for any compact subset  $X$  of  $Q_0(d, n)$ ,  $\Gamma_c(X, \mathfrak{M}^*(\mathfrak{P}; d, n))$  and  $\Gamma_c(X, \mathfrak{M}(d, n))$  are isomorphic Banach algebras, where  $\Gamma_c(X, \mathfrak{M}(d, n))$  is given any of the norms  $\|\cdot\|_{\mathcal{U}}$  defined in (1.3) using a choice of the data  $(\mathcal{U}, \{g_{UV}\}_{U,V \in \mathcal{U}}, \{F_U\}_{U \in \mathcal{U}})$ .

In the norm on  $\Gamma_c(\overline{\mathcal{D}}, \mathfrak{M}(d, n))$ , elements in  $\Gamma_h(\overline{\mathcal{D}}, \mathfrak{M}(d, n))$  achieve their maximums on  $\partial\mathcal{D}$ . However, it is easy to construct examples of reductions  $\mathfrak{P}$  of  $\mathcal{V}(d, n)$  such that the image of an element from  $\Gamma_h(\overline{\mathcal{D}}, \mathfrak{M}(d, n))$  in  $\Gamma_c(\overline{\mathcal{D}}, \mathfrak{M}^*(\mathfrak{P}; d, n))$  need not take its maximum norm on  $\partial\mathcal{D}$ . For this reason, we adjust our focus and concentrate *directly* on  $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$ .

**Definition 1.4.** The closure of  $\Psi(\mathbb{S}_0(d, n))$  in  $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$  will be denoted  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  and will be called the *tracial function algebra* of  $\mathcal{D}$  determined by  $\mathfrak{P}$  and  $\mathbb{S}_0(d, n)$ .

Observe that when  $n = 1$ ,  $G = K$  is the trivial group;  $\mathcal{V}(d, n)$ ,  $\mathcal{P}$ , and  $Q_0(d, n)$  become identified with  $\mathbb{C}^d \setminus \{0\}$ ;  $\mathfrak{M}(d, n) = \mathfrak{M}^*(\mathfrak{P}; d, n)$  is the trivial line bundle on  $\mathbb{C}^d$ ; and the algebras  $\mathbb{I}(\mathcal{D}; d, n)$  and  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  are identified with  $\mathcal{P}(\overline{\mathcal{D}})$ , the sup-norm closure of the polynomial functions on  $\mathbb{C}^d$  in the continuous functions on  $\overline{\mathcal{D}}$ . Of course,  $\mathcal{P}(\overline{\mathcal{D}})$  is a much studied algebra in complex analysis (see, e.g. [32]), but there does not seem to be a universally accepted term for it. Our current thinking is that  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  is the natural generalization of  $\mathcal{P}(\overline{\mathcal{D}})$ .

We note that the center of  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  may be identified in a natural fashion with  $\mathbb{I}(\mathcal{D}; d, n)$ , no matter what reduction is chosen. We shall give

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<sup>1</sup>We follow Steenrod [30] in the use of the term “reduction”. Husemoller uses the term “restriction”.

a proof of this fact in Section 3. The reason the assertion is true is that elements of  $\mathbb{I}(\mathcal{D}; d, n)$  are identified with sections whose values are scalar multiples of the identity and these are unaffected by the transition functions that describe the bundles. The fact that the center of  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  is  $\mathbb{I}(\mathcal{D}; d, n)$  shows in particular that  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  is a *proper* subalgebra of  $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$ . This is not evident, *a priori*.

The  $C^*$ -algebra  $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$  is an  $n$ -homogeneous  $C^*$ -algebra [33, Theorem 8] and each irreducible representation of it is given, essentially, by evaluation at a unique point of  $\partial\mathcal{D}$ . In more detail, note that for  $\mathbf{u} \in Q_0(d, n)$ ,  $\pi^{-1}(\mathbf{u}) = \{\mathfrak{z}, A\} \in \mathcal{P} \times_K M_n(\mathbb{C}) \mid \pi_0(\mathfrak{z}) = \mathbf{u}, A \in M_n(\mathbb{C})\}$ . So, once  $\mathfrak{z}$  is chosen so that  $\pi_0(\mathfrak{z}) = \mathbf{u}$  the map  $A \rightarrow [\mathfrak{z}, A]$  is a unital  $*$ -homomorphism  $\rho$  of  $M_n(\mathbb{C})$  into  $\pi^{-1}(\mathbf{u})$ . Since  $M_n(\mathbb{C})$  is simple, the map is injective. It is surjective because if  $[\mathfrak{w}, B]$  lies in  $\pi^{-1}(\mathbf{u})$ , then there is a unique  $s \in K$  such that  $\mathfrak{w} = \mathfrak{z} \cdot s$  and we may write:  $[\mathfrak{w}, B] = [\mathfrak{z} \cdot s, B] = [\mathfrak{z}, s \cdot B]$ , which is in the image of  $\rho$ . Thus, if for each  $\mathbf{u} \in \partial\mathcal{D}$ , we write  $ev_{\mathbf{u}}$  for the  $*$ -homomorphism from  $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$  into  $\pi^{-1}(\mathbf{u})$  defined by evaluating a section in  $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$  at  $\mathbf{u}$ , then  $\rho^{-1} \circ ev_{\mathbf{u}}$  is an irreducible representation of  $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$  and every irreducible representation of  $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$  is unitarily equivalent to  $\rho^{-1} \circ ev_{\mathbf{u}}$  for a unique  $\mathbf{u} \in \partial\mathcal{D}$  by [11, Corollary 10.4.4].

The two principal theorems of this note are Theorems 1.6 and 1.9, below. For the first, and its corollary, Corollary 1.7, we need to recall Arveson's definition of a boundary representation, and related ideas.

**Definition 1.5.** [2, Definition 2.1.1] If  $B$  is a unital  $C^*$ -algebra and if  $A$  is a norm-closed subalgebra of  $B$  that contains the unit of  $B$  and generates  $B$  as a  $C^*$ -algebra, then an irreducible representation  $\pi : B \rightarrow B(H_\pi)$  is a *boundary representation* for  $A$  in case  $\pi$  is the only unital completely positive map  $\omega : B \rightarrow B(H_\pi)$  such that  $\pi|_A = \omega|_A$ .

**Theorem 1.6.** *For every  $\mathbf{u} \in \partial_e\mathcal{D}$ , the evaluation representation  $ev_{\mathbf{u}}$  is a boundary representation of  $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$  for  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$ .*

In the setting of Definition 1.5, an ideal  $\mathfrak{I}$  in  $B$  is called a *boundary ideal* in case the restriction to  $A$  of the quotient map  $q : B \rightarrow B/\mathfrak{I}$  is completely isometric. The intersection of the kernels of the boundary representations of  $B$  for  $A$  is the largest boundary ideal, which is called the *Shilov boundary ideal* of  $B$  for  $A$ <sup>2</sup>. The quotient of  $B$  by the Shilov boundary ideal is unique up to  $C^*$ -isomorphism in a very strong sense [2, Theorem 2.2.6]. The quotient is called the  *$C^*$ -envelope of  $A$* .

**Corollary 1.7.** *For each reduction  $\mathfrak{P}$  of  $\mathcal{V}(d, n)$  and for each domain  $\mathcal{D}$  with  $\overline{\mathcal{D}}$  contained in  $Q_0(d, n)$ , the Shilov boundary ideal of  $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$*

<sup>2</sup>When [2] was written, it was not known if boundary representations always exist and the Shilov boundary ideal was defined differently; the existence of the Shilov boundary ideal was problematic. Today, thanks to [4] and [8], it is known that in every setting there are sufficiently many boundary representations to determine the Shilov boundary ideal.

for  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  vanishes. Consequently,  $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$  is the  $C^*$ -envelope of  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$ .

The pair  $(Q(d, n), \mathbb{I}_0(d, n))$  is an example of what Rickart calls a natural function algebra [28], where  $Q(d, n)$  is considered with its analytic topology. If  $X \subseteq Q(d, n)$  is a compact subset, then the  $\mathbb{I}_0(d, n)$ -convex hull of  $X$ ,  $\widehat{X}$ , is defined to be  $\{\mathfrak{z} \in Q(d, n) \mid |f(\mathfrak{z})| \leq \|f\|_X, f \in \mathbb{I}_0(d, n)\}$ , where  $\|f\|_X := \sup_{\mathfrak{z} \in X} |f(\mathfrak{z})|$ . If  $X = \widehat{X}$ , then  $X$  is called  $\mathbb{I}_0(d, n)$ -convex. The maximal ideal space of the closure of  $\mathbb{I}_0(d, n)$  in  $C(X)$  is  $\widehat{X}$ .

We note in passing that when  $d = n = 2$ ,  $\mathbb{I}_0(d, n)$  is isomorphic to the polynomial algebra in five variables; so  $Q(2, 2)$  may be identified with  $\mathbb{C}^5$  (see, e.g., [18, P. 14 ff]). Thus, in this case, the  $\mathbb{I}_0(2, 2)$ -convex hull of a compact set  $X$  coincides with its polynomially convex hull. In general, however,  $\mathbb{I}_0(d, n)$  is more complicated and still largely mysterious. It is worth noting that when  $d = n = 2$ , the identification of  $Q(2, 2)$  with  $\mathbb{C}^5$  is through the map

$$(Z_1, Z_2) \rightarrow (\text{tr}(Z_1), \text{tr}(Z_2), \det(Z_1), \det(Z_2), \text{tr}(Z_1 Z_2)).$$

So even in this setting, the interaction of the map with the norms involved is unclear. The situation is further complicated by the fact that generators of  $\mathbb{I}_0(2, 2)$  are not uniquely determined and it is not at all clear which ones are best for, or even well adapted to, analysis.

**Definition 1.8.** A unital algebra  $\mathfrak{A}$  with center  $\mathfrak{Z}$  is called an *Azumaya algebra* in case

- (1) As a right module over  $\mathfrak{Z}$ ,  $\mathfrak{A}$  is projective, and
- (2) The map from  $\mathfrak{A} \otimes_{\mathfrak{Z}} \mathfrak{A}^{op}$  to  $\text{End}(\mathfrak{A}_{\mathfrak{Z}})$  defined by identifying  $a \otimes b$  with the endomorphism

$$a \otimes b(c) := acb, \quad c \in \mathfrak{A},$$

is an isomorphism.

This is one of many equivalent definitions. For further background on such algebras, see [9]. The importance of these algebras for us is that they are algebraic versions of  $n$ -homogeneous  $C^*$ -algebras by [1, Theorem 8.3]. Specifically, Artin proved in his Theorem 8.3 (specialized to algebras over  $\mathbb{C}$ ) that if  $A$  is a unital  $\mathbb{C}$ -algebra, then  $A$  is an Azumaya algebra of rank  $n^2$  over its center if and only if  $A$  satisfies the identities of the  $n \times n$  matrices and  $A$  has no (unital) representations in  $M_r(\mathbb{C})$  for  $r \leq n$ . (To say in this setting that  $A$  has rank  $n^2$  over its center means that for each maximal 2-sided ideal  $\mathfrak{m}$  of  $A$ ,  $A/\mathfrak{m} \simeq M_n(\mathbb{C})$ .) Equivalently, under the hypothesis that  $A$  satisfies the identities of the  $n \times n$  matrices, the theorem asserts that  $A$  is an Azumaya algebra if and only if each (algebraically) irreducible representation of  $A$  is  $n$ -dimensional. Artin was inspired, in part, by Tomiyama and Takesaki’s representation of an  $n$ -homogeneous  $C^*$ -algebra as the continuous cross sections of a matrix bundle in [33]. Thus, in one sense, the following



theorem may easily be anticipated, given that the algebra in question is a subalgebra of an  $n$  homogeneous  $C^*$ -algebra. However, the proof may not seem immediate. Further, the theorem has consequences that appear difficult to establish without it, e.g., Corollary 1.10.

**Theorem 1.9.** *If  $\overline{\mathcal{D}}$  is  $\mathbb{I}_0(d, n)$ -convex, then the algebra  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  is a rank  $n^2$  Azumaya algebra over  $\mathbb{I}(\mathcal{D}; d, n)$ .*

**Corollary 1.10.** *If  $\overline{\mathcal{D}}$  is  $\mathbb{I}_0(d, n)$ -convex, then there is a bijective correspondence between ideals  $\mathfrak{a}$  of  $\mathbb{I}(\mathcal{D}; d, n)$  and ideals  $\mathfrak{A}$  of  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  given by  $\mathfrak{a} \rightarrow \mathfrak{a}\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  and  $\mathfrak{A} \rightarrow \mathfrak{A} \cap \mathbb{I}(\mathcal{D}; d, n)$ .*

**Proof.** This is an application of Corollary II.3.7 of [9], which is valid for any Azumaya algebra.  $\square$

## 2. The concomitants and cross sections

The map we call  $\Psi$  in Theorem 1.2 is a special case of the bijection described in [15, Theorem 4.8.1]. There, Husemoller deals with general fibre bundles associated to principal bundles. However, when specialized to our setting it is clear that  $\Psi$  is a bijection that takes continuous concomitants to continuous cross sections. It also clearly preserves the algebraic structures involved. So to prove Theorem 1.2, it suffices to show that  $\Psi$  maps holomorphic concomitants to holomorphic cross sections and that  $\Psi^{-1}$  maps holomorphic cross sections to holomorphic concomitants.

Since the property of being holomorphic is a local property, we may restrict our attention to an open subset  $U \subseteq Q_0(d, n)$  over which  $\mathcal{V}(d, n)$  is trivial. We let  $\mathcal{V}_0 = \pi_0^{-1}(U)$ , so  $\mathcal{V}_0$  is an open,  $G$ -invariant subset of  $\mathcal{V}(d, n)$ , and we fix a biholomorphic bundle isomorphism  $F : U \times G \rightarrow \mathcal{V}_0$ . Thus  $F$  is  $G$ -equivariant and  $\pi_0 \circ F = \pi_1$ , where  $\pi_1$  is the projection of  $U \times G$  onto the first factor. (This implies that  $\mathbf{u} \rightarrow F(\mathbf{u}, e)$  is a holomorphic section of  $\mathfrak{V}(d, n)|_U$ , and conversely, each holomorphic section  $f$  of  $\mathfrak{V}(d, n)|_U$  determines a biholomorphic bundle isomorphism from  $U \times G$  onto  $\mathcal{V}_0$  via the formula  $F(\mathbf{u}, g) = f(\mathbf{u})g$ .) The isomorphism  $F$ , in turn, induces a biholomorphic bundle isomorphism  $\widehat{F} : U \times M_n(\mathbb{C}) \rightarrow \mathcal{V}_0 \times_G M_n(\mathbb{C})$  via the formula  $\widehat{F}(\mathbf{u}, A) = [F(\mathbf{u}, e), A]$ .

Suppose that  $\phi : \mathcal{V}(d, n) \rightarrow M_n(\mathbb{C})$  is a holomorphic matrix concomitant. Then the restriction to  $U$  of the section  $\sigma_\phi$  defined above is given by the formula

$$\sigma_\phi(\mathbf{u}) = [\mathfrak{z}, \phi(\mathfrak{z})], \quad \mathbf{u} \in U,$$

where  $\mathfrak{z} \in \mathcal{V}_0$  is any point such that  $\pi_0(\mathfrak{z}) = \mathbf{u}$ . To show  $\sigma_\phi$  is holomorphic on  $U$ , it suffices to show that  $\widehat{F}^{-1} \circ \sigma_\phi$  is holomorphic on  $U$ . To get a formula for  $\widehat{F}^{-1} \circ \sigma_\phi$ , fix both  $\mathbf{u} \in U$  and  $\mathfrak{z} \in \mathcal{V}_0$  such that  $\pi_0(\mathfrak{z}) = \mathbf{u}$ . Then there is a unique  $g \in G$  such that  $F(\mathbf{u}, g) = \mathfrak{z}$ . Since we also have  $F(\mathbf{u}, g) = F(\mathbf{u}, e)g$ ,

we arrive at the following equation,

$$\begin{aligned}\widehat{F}^{-1} \circ \sigma_\phi(\mathbf{u}) &= \widehat{F}^{-1}([\mathfrak{z}, \phi(\mathfrak{z})]) = \widehat{F}^{-1}([F(\mathbf{u}, g), \phi(\mathfrak{z})]) \\ &= \widehat{F}^{-1}([F(\mathbf{u}, e), g \cdot \phi(\mathfrak{z})]) = \widehat{F}^{-1}(\widehat{F}(\mathbf{u}, \phi(\mathfrak{z} \cdot g^{-1}))) \\ &= (\mathbf{u}, \phi(\mathfrak{z} \cdot g^{-1})) = (\mathbf{u}, \phi \circ F(\mathbf{u}, e)),\end{aligned}$$

which shows that  $\widehat{F}^{-1} \circ \sigma_\phi$  is holomorphic on  $U$ , since  $\mathbf{u} \rightarrow (\mathbf{u}, \phi \circ F(\mathbf{u}, e))$  is certainly holomorphic.

If  $\sigma$  is a holomorphic section of  $\mathfrak{M}(d, n)$ , then to show that  $\phi_\sigma$  is holomorphic, it suffices to show that the restriction of  $\phi_\sigma$  to  $\mathcal{V}_0$  is holomorphic; and for this, it suffices to show that  $\phi_\sigma \circ F$  is holomorphic on  $U \times G$ . Since  $\mathfrak{M}(d, n)$  is trivial over  $U$  and  $\sigma|_U$  is a section of  $\mathfrak{M}(d, n)|_U$ ,  $\widehat{F}^{-1} \circ \sigma|_U$  a section of the product bundle  $U \times M_n(\mathbb{C})$  over  $U$ . Consequently, there is a function  $f : U \rightarrow M_n(\mathbb{C})$  such that  $\widehat{F}^{-1} \circ \sigma(\mathbf{u}) = (\mathbf{u}, f(\mathbf{u}))$ . The assumption that  $\sigma$  is holomorphic guarantees that  $f$  is holomorphic, too. On the other hand, the matrix concomitant  $\phi_\sigma$  determined by  $\sigma$  satisfies (1.2). Therefore,  $(\mathbf{u}, f(\mathbf{u})) = \widehat{F}^{-1} \circ \sigma(\mathbf{u}) = \widehat{F}^{-1}([\mathfrak{z}, \phi_\sigma(\mathfrak{z})])$  for any  $\mathfrak{z}$  such that  $\pi_0(\mathfrak{z}) = \mathbf{u}$ . So,

$$\widehat{F}(\mathbf{u}, f(\mathbf{u})) = [\mathfrak{z}, \phi_\sigma(\mathfrak{z})].$$

However, by definition of  $\widehat{F}$  in terms of  $F$ , we may rewrite the left-hand side of this equation as

$$\begin{aligned}\widehat{F}(\mathbf{u}, f(\mathbf{u})) &= [F(\mathbf{u}, e), f(\mathbf{u})] \\ &= [F(\mathbf{u}, e) \cdot g, g^{-1} \cdot f(\mathbf{u})] = [F(\mathbf{u}, g), g^{-1} \cdot f(\mathbf{u})].\end{aligned}$$

If we write  $\mathfrak{z} = F(\mathbf{u}, g)$ , these two equations yield

$$[F(\mathbf{u}, g), g^{-1} \cdot f(\mathbf{u})] = \widehat{F}(\mathbf{u}, f(\mathbf{u})) = [F(\mathbf{u}, g), \phi_\sigma(F(\mathbf{u}, g))].$$

Hence, there is an  $h \in G$  such  $F(\mathbf{u}, g) \cdot h = F(\mathbf{u}, g)$  and  $h^{-1} \cdot g^{-1} \cdot f(\mathbf{u}) = \phi_\sigma(F(\mathbf{u}, g))$ . However, since  $G$  acts freely on  $\mathcal{V}(d, n)$ , we conclude that  $h = e$ , proving that

$$\phi_\sigma \circ F(\mathbf{u}, g) = g^{-1} \cdot f(\mathbf{u}).$$

Since  $f$  is holomorphic on  $U$  and the action of  $G$  on  $M_n(\mathbb{C})$  is holomorphic, we see that  $\phi_\sigma \circ F$  is holomorphic on  $U \times G$ , as required. This completes the proof of the first assertion in Theorem 1.2.

Turning to the second, we begin with the following theorem. It, or something akin to it, seems to have been known to Luminet [19, Remark 4.14]. However, no proof or reference was given. We are grateful to Zinovy Reichstein for the formulation of the theorem and for allowing us to include his proof here.

**Theorem 2.1.** *Suppose  $d, n \geq 2$  and for  $k = 1, 2, \dots, n-1$ , let  $X_k$  be the set of all  $(A_1, A_2, \dots, A_d) \in M_n(\mathbb{C})^d$  such that the  $A_i$  have a common  $k$ -dimensional invariant subspace. Then  $X_k$  is an irreducible algebraic variety*

of dimension  $dn^2 - (d-1)k(n-k)$ , and

$$\cup_{k=1}^{n-1} X_k = M_n(\mathbb{C})^d \setminus \mathcal{V}(d, n).$$

**Proof.** Evidently, the union of the  $X_k$  is  $M_n(\mathbb{C})^d \setminus \mathcal{V}(d, n)$ . Let  $\text{Gr}(k, n)$  denote the Grassmannian consisting of all  $k$ -dimensional subspaces of  $\mathbb{C}^n$  and let

$$Y_k = \{(A_1, A_2, \dots, A_d; W) \in M_n(\mathbb{C})^d \times \text{Gr}(k, n) \mid A_i W \subseteq W, 1 \leq i \leq d\}.$$

Clearly,  $Y_k$  is an algebraic subvariety of  $M_n(\mathbb{C})^d \times \text{Gr}(k, n)$ . Let  $\pi_{2k} : Y_k \rightarrow \text{Gr}(k, n)$  be the projection onto the last component. Then  $\pi_{2k}$  is surjective, and its fibres are vector spaces of block-upper triangular matrices (in appropriate bases), with blocks of size  $k$  and  $n-k$ . So the fibres are irreducible varieties of the same dimension, viz.,  $d(n^2 - k(n-k))$ . By the fibre dimension theorem [29, Theorem I.6.7, p.76], the  $Y_k$  are irreducible and

$$\dim Y_k = d(n^2 - k(n-k)) + \dim \text{Gr}(k, n) = dn^2 - (d-1)k(n-k).$$

Consider the map  $\pi_{1k} : Y_k \rightarrow M_n(\mathbb{C})^d$  which projects onto the first  $d$  components. The image of  $\pi_{1k}$  is  $X_k$ . Therefore,  $X_k$  is irreducible. Further, the set of  $(A_1, A_2, \dots, A_d) \in X_k$  such that  $A_1$  has distinct eigenvalues is a Zariski open subset of  $X_k$  and so  $\dim X_k = \dim Y_k = dn^2 - (d-1)k(n-k)$ , as claimed.  $\square$

If  $(d, n) \neq (2, 2)$ , the complement of  $\mathcal{V}(d, n)$  in  $M_n(\mathbb{C})^d$  is the finite union of algebraic varieties of codimension  $\geq 2$  by Theorem (2.1). Consequently, by [14, Theorem K.1] every function that is holomorphic on  $\mathcal{V}(d, n)$  extends uniquely to a function that is holomorphic on all of  $M_n(\mathbb{C})^d$ .

Suppose, finally,  $(d, n) = (2, 2)$ , and consider the commutator  $[Z_1, Z_2]$  in  $\mathbb{G}_0(2, 2)$ . It is well known in some circles that  $\mathcal{V}(2, 2) = \{\mathfrak{z} = (Z_1, Z_2) \mid [Z_1, Z_2] \text{ is invertible}\}$ . Since we don't have an explicit reference for this, here is a simple proof: One may assume, without loss of generality, that  $Z_1$  is in Jordan canonical form and that  $Z_1$  either has distinct eigenvalues or is the Jordan cell,  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ . If  $Z_1$  has distinct eigenvalues, say  $a$  and  $c$ , then we may write

$$[Z_1, Z_2] = \left[ \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}, \begin{bmatrix} w & x \\ y & z \end{bmatrix} \right] = \begin{bmatrix} 0 & (a-c)x \\ (c-a)y & 0 \end{bmatrix}.$$

If  $Z_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , then

$$[Z_1, Z_2] = \left[ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} w & x \\ y & z \end{bmatrix} \right] = \begin{bmatrix} -y & w-z \\ 0 & y \end{bmatrix}.$$

In either case, it is clear that  $[Z_1, Z_2]$  is invertible if and only if  $Z_1$  and  $Z_2$  have no common invariant subspace. Thus  $\det[Z_1, Z_2]$  is a polynomial in  $\mathbb{I}(2, 2)$  whose zero set is  $M_2(\mathbb{C})^2 \setminus \mathcal{V}(2, 2)$ . Thus  $f(\mathfrak{z}) := (\det[Z_1, Z_2])^{-1}$  is a holomorphic matrix concomitant on  $\mathcal{V}(2, 2)$  that cannot be analytically

extended beyond  $\mathcal{V}(2, 2)$ . Thus  $\mathcal{V}(2, 2)$  is a domain of holomorphy in  $M_2(\mathbb{C})^2$  and the proof of Theorem 1.2 is complete.

As a corollary of the calculation in the previous paragraph, we may formulate the following proposition whose proof is a slight modification of an argument shown to us by Ben Williams.

**Proposition 2.2.** The bundles  $\mathfrak{V}(2, 2)$  and  $\mathfrak{M}(2, 2)$  are nontrivial both as topological bundles and as holomorphic bundles.

**Proof.** It suffices to focus on  $\mathfrak{V}(2, 2)$ . Recall that the bundle space of  $\mathfrak{V}(2, 2)$  is  $\mathcal{V}(2, 2) = \{\mathfrak{z} = (Z_1, Z_2) \mid [Z_1, Z_2] \text{ is invertible}\}$  and note that the complement of  $\mathcal{V}(2, 2)$  is the hypersurface in  $M_2(\mathbb{C})^2 \simeq \mathbb{C}^8$  given by the equation  $\det[Z_1, Z_2] = 0$ . By [10, Corollary 1.4 on page 103], then, the first integral homology group of  $\mathcal{V}(2, 2)$  is  $\mathbb{Z}^k$ , where  $k$  is the number of irreducible components of its complement. (The hypersurface is actually irreducible, but we don't need that here.) On the other hand, if  $\mathfrak{V}(2, 2)$  were trivial, the first integral homology of  $\mathcal{V}(2, 2)$  would contain the first integral homology of  $PGL(2, \mathbb{C})$  as a direct summand. But this homology group is  $\mathbb{Z}/2\mathbb{Z}$ , and so we would contradict the fact that the first integral homology group of  $\mathcal{V}(2, 2)$  is  $\mathbb{Z}^k$ . The fact that the homology of  $PGL(2, \mathbb{C})$  is  $\mathbb{Z}/2\mathbb{Z}$  seems to be well known in some circles. However, we lack an explicit reference in the literature. For completeness, we note that in general  $PGL(n, \mathbb{C})$  is contractible to  $PU(n, \mathbb{C})$ . Then, as Tomiyama and Takesaki argue in the paragraph before the Remark on page 517 of [33], the fundamental group of  $PU(n, \mathbb{C})$  is  $\mathbb{Z}/n\mathbb{Z}$ . Since  $PU(n, \mathbb{C})$  is connected and its fundamental group is abelian, the first integral homology group of  $PU(n, \mathbb{C})$  is the same as its fundamental group, i.e.,  $\mathbb{Z}/n\mathbb{Z}$ .  $\square$

### 3. Function theory in bundles

Our first objective is to show that the center of  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  is  $\mathbb{I}(\mathcal{D}; d, n)$  independent of the reduction  $\mathfrak{P}$ . Of course,  $\mathbb{I}(\mathcal{D}; d, n)$  is contained in the center. The problem is the reverse inclusion. It is easy to see that every element in the center of  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  is the restriction to  $\partial\mathcal{D}$  of a continuous function on  $\overline{\mathcal{D}}$  that is holomorphic on  $\mathcal{D}$ , but why must it be in  $\mathbb{I}(\mathcal{D}; d, n)$ ? The reason is due, really, to Procesi who shows that the center of  $\mathbb{S}_0(d, n)$  is  $\mathbb{I}_0(d, n)$  [22, Page 94].

First, note that the cross section  $\varepsilon$  in  $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$  defined by the formula  $\varepsilon([\mathfrak{z}]) := [\mathfrak{z}, I_n]$ , where  $I_n$  is the identity  $n \times n$  matrix, is the identity of  $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$ . Further, the center of  $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$ ,  $\mathfrak{Z}\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$ , is the set of all cross sections  $\sigma$  of the form  $\sigma([\mathfrak{z}]) = [\mathfrak{z}, c([\mathfrak{z}])I_n]$ , where  $c : \partial\mathcal{D} \rightarrow \mathbb{C}$  is a continuous complex-valued function. We shall usually write such a section as  $c \cdot \varepsilon$ , and we shall identify the center,  $\mathfrak{Z}\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$ , with  $C(\partial\mathcal{D})$  through the isomorphism  $c \rightarrow c \cdot \varepsilon$ . We shall write  $\tau_0$  for the normalized trace on  $M_n(\mathbb{C})$ , i.e.,  $\tau_0(I_n) = 1$ , and we

shall define  $\tau : \mathfrak{M}^*(\mathfrak{P}; d, n) \rightarrow \mathbb{C}$  by  $\tau([\mathfrak{z}, A]) := \tau_0(A)$ . Then  $\tau$  is a well-defined continuous function on  $\mathfrak{M}^*(\mathfrak{P}; d, n)$ . We now define

$$T : \Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n)) \rightarrow \mathfrak{Z}\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$$

by the formula,

$$T(\sigma) := \tau \circ \sigma \cdot \varepsilon, \quad \sigma \in \Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n)).$$

Then it is straightforward to verify that  $T$  is a conditional expectation from  $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$  onto  $\mathfrak{Z}\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$  that also satisfies the equation

$$T(\Psi(\phi))([\mathfrak{z}]) = \tau_0(\phi(\mathfrak{z}))\varepsilon([\mathfrak{z}]), \quad \phi \in \mathbb{S}_0(d, n), \mathfrak{z} \in \mathcal{V}(d, n).$$

**Theorem 3.1.**  *$T$  maps  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  onto  $\mathbb{I}(\mathcal{D}; d, n)$  and  $\mathbb{I}(\mathcal{D}; d, n)$  is the center of  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$ .*

**Proof.** Since  $T(\Psi(\phi))([\mathfrak{z}]) = \tau_0(\phi(\mathfrak{z}))\varepsilon([\mathfrak{z}])$  for every  $\phi \in \mathbb{S}_0(d, n)$  and since  $\mathfrak{z} \rightarrow \tau_0(\phi(\mathfrak{z}))$  is a  $G$ -invariant polynomial function, the image of  $T$  restricted to  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  is contained in  $\mathbb{I}(\mathcal{D}; d, n)$ . If  $\sigma$  is a section in the center of  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$ , then  $\sigma([\mathfrak{z}])$  lies in the center of the fibre of  $\mathfrak{M}^*(\mathfrak{P}; d, n)$  over  $[\mathfrak{z}]$ ,  $\pi^{-1}([\mathfrak{z}])$ , which is (isomorphic to)  $M_n(\mathbb{C})$ . Thus  $\sigma([\mathfrak{z}])$  is a multiple of  $[\mathfrak{z}, I_n]$ . Hence,  $\sigma \in \mathfrak{Z}\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$ . Since  $\sigma \in \mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$ , there is a sequence  $\{\phi_n\}_{n \geq 1}$  in  $\mathbb{S}_0(d, n)$  such that  $\Psi(\phi_n) \rightarrow \sigma$  in  $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$ , by definition of  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$ . But then  $T(\Psi(\phi_n)) \rightarrow T(\sigma) = \sigma$  and each  $T(\Psi(\phi_n)) \in \mathbb{I}(\mathcal{D}; d, n)$ . Thus  $\sigma \in \mathbb{I}(\mathcal{D}; d, n)$ .  $\square$

**Corollary 3.2.**  *$\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  is a proper subalgebra of  $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$ .*

## 4. Boundary representations

In this section, we prove Theorem 1.6. It rests on a simple observation of Kleski [17, Remark 3.4], which is a corollary of his deep Theorem 3.1. Recall Arveson's definition of a peaking representation.

**Definition 4.1.** [6, Definition 7.1] Suppose  $A$  is a norm closed subalgebra of a unital  $C^*$ -algebra  $B$  that generates  $B$  as a  $C^*$ -algebra and contains the unit of  $B$ . An irreducible  $C^*$ -representation  $\pi : B \rightarrow B(H_\pi)$  is called a *peaking representation for  $A$*  if there is an integer  $n \geq 1$  and an  $n \times n$  matrix  $(a_{ij}) \in M_n(A)$  such that

$$\|(\pi(a_{ij}))\| \geq \|(\sigma(a_{ij}))\|$$

for every irreducible representation  $\sigma$  for  $B$  that is not unitarily equivalent to  $\pi$ . We also say that  $\pi$  peaks at  $(a_{ij})$ .

Arveson defines the notion of a peaking representation in the context of operator systems, i.e., unital, closed, and self-adjoint subspaces of  $C^*$ -algebras. However, thanks to [2, Proposition 1.2.8], if a representation is peaking in the sense of our Definition 4.1 it is a peaking representation with

respect to the operator system generated by  $A$ , i.e., the norm-closure of  $A + A^*$ .

In [17, Theorem 3.1], Kleski proves that if  $(a_{ij})$  is any element in  $M_n(A)$  then there is a boundary representation  $\pi_0$  of  $B$  for  $A$  such that

$$(4.1) \quad \|(a_{ij})\| = \|(\pi_0(a_{ij}))\|.$$

As Kleski observes in [17, Remark 3.4], this implies that a peaking representation is a boundary representation. Indeed, if  $\pi$  is an irreducible representation of  $B$  that peaks at  $(a_{ij})$ , then we would have  $\|(a_{ij})\| \geq \|(\pi(a_{ij}))\| \geq \|(\pi_0(a_{ij}))\|$  if  $\pi_0 \approx \pi$ , which would contradict (4.1). Thus  $\pi \sim \pi_0$  and therefore  $\pi$  is a boundary representation.

**Proof of Theorem 1.6.** To apply these remarks to the situation of Theorem 1.6 is very easy. Our  $B$  is  $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$  and our  $A$  is  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$ . Our hypothesis is that  $\mathbf{u} \in \partial_e\mathcal{D}$  - the extreme boundary of  $\mathcal{D}$ . Since  $Q_0(d, n)$  is metrizable, so is  $\overline{\mathcal{D}}$ . Therefore  $\mathbf{u}$  is a peak point in the function algebra sense [31, Theorem 1.7.26], i.e., there is a function  $f \in \mathbb{I}(\mathcal{D}; d, n)$  such that  $f(\mathbf{u}) = 1$ , but  $|f(\mathbf{v})| \leq 1$  for all  $\mathbf{v} \neq \mathbf{u}$ . But then, we may simply view  $f$  as a  $1 \times 1$  matrix over  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  and conclude that  $ev_{\mathbf{u}}$  peaks at  $f$ . Hence  $ev_{\mathbf{u}}$  is a boundary representation of  $\Gamma_c(\partial\mathcal{D}, \mathfrak{P}; d, n)$  for  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$ .  $\square$

**Proof of Corollary 1.7.** Any section  $\sigma \in \Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$  in the kernel of  $ev_{\mathbf{u}}$  vanishes at  $\mathbf{u}$ . So any section in  $\bigcap_{\mathbf{u} \in \partial_e\mathcal{D}} \ker(ev_{\mathbf{u}})$  vanishes on  $\partial_e\mathcal{D}$ . Since  $\partial_e\mathcal{D}$  is dense in  $\partial\mathcal{D}$  [31, Theorem I.7.24], any such section is the zero section. Therefore, by Theorem 1.6, the intersection of the kernels of the boundary representations, which is the Shilov boundary ideal, must be zero, i.e.,  $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\mathfrak{P}; d, n))$  is the  $C^*$ -envelope of  $\mathbb{S}(\partial\mathcal{D}, \mathfrak{P}; d, n)$ .  $\square$

### 5. Azumaya algebras

The proof of Theorem 1.9 is an application of Procesi’s extension [22, Theorem VIII.2.1] of Artin’s theorem that was discussed earlier. A  $d$ -variable *central polynomial* for the  $n \times n$  matrices is a nonzero polynomial  $p$  in the center of  $\mathbb{G}_0(d, n)$  that is without constant term. It is not evident, *a priori*, that such polynomials exist. However, they do - for every  $d$  - thanks to the work of Formanek [13] and Razmyslov [26]. Procesi’s theorem asserts (among many things) that if  $R$  is a ring satisfying the identities of the  $n \times n$  matrices then  $R$  is an Azumaya algebra if and only if  $R = F(R)R$  - the ideal generated by the Formanek center,  $F(R)$ . The Formanek center, in turn, is the collection of elements in  $R$  obtained by evaluating all the central polynomials for the  $d$  generic  $n \times n$  matrices for all  $d$  at all  $d$ -tuples of elements of  $R$ . Here, of course, when forming  $F(R)$ , we are viewing a  $d$ -variable central polynomial  $p$  as an element  $\mathbb{C}\langle X_1, X_2, \dots, X_d \rangle$ . Notice, too, that when  $R = \mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$ , then a  $p \in \mathbb{G}_0(d, n) \subseteq \mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  may be identified with its evaluation at the  $d$  coordinate functions  $\mathcal{Z}_i$ , i.e.,  $p = p(\mathcal{Z}_1, \mathcal{Z}_2, \dots, \mathcal{Z}_d)$ ,

where, recall,  $\mathcal{Z}_i(\mathfrak{z}) = Z_i$ , if  $\mathfrak{z} = (Z_1, Z_2, \dots, Z_d)$ . We require the following special case of a lemma of Reichstein and Vonesen [27, Lemma 2.10]: For every  $\mathfrak{z} \in \mathcal{V}(d, n)$  there is a  $d$ -variable central polynomial  $p$  such that  $p(\mathfrak{z}) = I_n$ .

**Proof of Theorem 1.9.** Now  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  certainly satisfies the identities of the  $n \times n$  matrices and our hypothesis on  $\mathcal{D}$  is that  $\overline{\mathcal{D}}$  is the maximal ideal space of  $\mathbb{I}(\mathcal{D}; d, n)$ . Also, our Theorem 3.1 tells us that  $\mathbb{I}(\mathcal{D}; d, n)$  is the center of  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$ . So given any point  $\mathbf{u} \in \overline{\mathcal{D}}$ , we choose a  $\mathfrak{z}$  in the bundle space  $\mathcal{P}$  of  $\mathfrak{P}$  such that  $\pi(\mathfrak{z}) = \mathbf{u}$ . Then, using the Reichstein-Vonesen lemma, we choose a  $d$ -variable central polynomial  $p$  such that  $p(\mathfrak{z}) = I_n$ . Since a central polynomial certainly is invariant, we may view  $p$  as a function on  $Q(d, n)$  that is 1 at  $\mathbf{u}$ . So, by the compactness of  $\overline{\mathcal{D}}$  we may choose a finite number of central polynomials,  $p_1, p_2, \dots, p_N$ , that have no common zero on  $\overline{\mathcal{D}}$ . It follows that  $p_1\mathbb{I}(\mathcal{D}; d, n) + p_2\mathbb{I}(\mathcal{D}; d, n) + \dots + p_N\mathbb{I}(\mathcal{D}; d, n) = \mathbb{I}(\mathcal{D}; d, n)$  and, *a fortiori*, that  $F(\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n))\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n) = \mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$ . Thus,  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  is an Azumaya algebra.  $\square$

## 6. Concluding remarks

One may wonder about the extent of our results. How comprehensive are the examples they cover? While we have formulated our analysis in the context of the trace algebra of the algebra of generic matrices, everything we have written goes over without significant changes to the more general situation of what Reichstein and Vonesen call  $n$ -varieties.

**Definition 6.1.** [27, Definition 3.1] An  $n$ -variety is a  $G$ -invariant subset  $X$  of  $\mathcal{V}(d, n)$ , for some  $d \geq 2$ , with the property that  $X = \overline{X} \cap \mathcal{V}(d, n)$  where  $\overline{X}$  denotes the Zariski closure of  $X$  in  $M_n(\mathbb{C})^d$ .

When passing from  $\mathcal{V}(d, n)$  to an  $n$ -variety, one replaces  $\mathbb{G}_0(d, n)$  by  $\mathbb{G}_0(d, n)/\mathcal{I}(X)$ , where  $\mathcal{I}(X) := \{p \in \mathbb{G}_0(d, n) \mid p(\mathfrak{z}) = 0, \mathfrak{z} \in X\}$ . The quotient  $\mathbb{G}_0(d, n)/\mathcal{I}(X)$  is a noncommutative analogue of the coordinate ring of an algebraic variety and the thrust of [27] is that noncommutative algebraic geometry should take place in the context of  $n$ -varieties, their coordinate rings, and associated noncommutative function fields. These latter are central simple algebras and each can be written as the algebra of rational matrix concomitants mapping  $X$  into  $M_n(\mathbb{C})$ . Further, by [27, Lemma 8.1], every irreducible algebraic variety on which  $G$  acts freely on a Zariski open set is birational to an irreducible  $n$ -variety. Thus, with technical adjustments, the results we have discussed make sense at this level.

In another direction, which we are currently investigating, the results of [21] suggest how to replace  $PGL(n, \mathbb{C})$  with certain more complicated reductive groups and formulate a function theory on quiver varieties and other structures that can be built from  $C^*$ -correspondences.

The work of Craw, Raeburn and Taylor [7] was also a source of inspiration for us. They introduced the notion of a Banach Azumaya algebra over

a commutative Banach algebra. Their purpose was to use the theory of Azumaya algebras to illuminate the topological properties of the maximal ideal space of the commutative Banach algebra. However, it seems difficult to identify naturally occurring Azumaya Banach algebras “in the wild”. Our results, coupled with their Proposition 2.6, show that such algebras arise quite naturally and quite frequently.

The specific problem which led us to the results we have presented here stems from [20] and [21]. In [20] we identified the completely contractive representations of the tensor algebra of a  $C^*$ -correspondence. When the correspondence is specialized to complex  $d$ -space  $\mathbb{C}^d$ , one finds that the completely contractive  $n$ -dimensional representations of the tensor algebra,  $\mathcal{T}_+(\mathbb{C}^d)$ , are parametrized by the closed “disc”,  $\overline{\mathbb{D}(d, n)}$ , where  $\mathbb{D}(d, n) = \{\mathfrak{z} \in M_n(\mathbb{C})^d \mid \|\mathfrak{z}\mathfrak{z}^*\| < 1\}$ . When viewed simply as a subset of the complex space  $\mathbb{C}^{dn^2}$ ,  $\mathbb{D}(d, n)$  is a classical symmetric domain. If  $\mathbb{G}(d, n)$  (resp.  $\mathbb{S}(d, n)$ ) is the closure of  $\mathbb{G}_0(d, n)$  (resp.  $\mathbb{S}_0(d, n)$ ) in  $C(\overline{\mathbb{D}(d, n)}, M_n(\mathbb{C}))$ , then  $\mathbb{G}(d, n)$  is precisely the sup-norm closure of the algebra of functions on  $\overline{\mathbb{D}(d, n)}$  that one obtains from evaluating the elements of  $\mathcal{T}_+(\mathbb{C}^d)$  on  $\overline{\mathbb{D}(d, n)}$ . (Note that Arveson [3] showed that in general  $\mathbb{G}(d, n)$  is strictly larger than the algebra of evaluations from  $\mathcal{T}_+(\mathbb{C}^d)$ .) The elements of  $\mathbb{G}(d, n)$  and  $\mathbb{S}(d, n)$  are continuous  $M_n(\mathbb{C})$ -valued functions on  $\overline{\mathbb{D}(d, n)}$  that are analytic on  $\mathbb{D}(d, n)$  and for each  $f \in \mathbb{S}(d, n)$ , the maximum of  $\|f(\mathfrak{z})\|$ , for  $\mathfrak{z} \in \overline{\mathbb{D}(d, n)}$ , is taken on the Shilov boundary of  $\mathbb{D}(d, n)$ ,  $\partial_e \mathbb{D}(d, n)$ . The question which motivated this paper is “What are the boundary representations for  $\mathbb{G}(d, n)$  and  $\mathbb{S}(d, n)$  and what are the  $C^*$ -envelopes of these algebras?” For this, we need to know how to describe the  $C^*$ -algebras they generate.

The functions in  $\mathbb{S}(d, n)$  are  $K$ -concomitants, i.e.,  $f(k^{-1}\mathfrak{z}k) = k^{-1}f(\mathfrak{z})k$  for all  $\mathfrak{z} \in \overline{\mathbb{D}(d, n)}$  and all  $k \in K$ . Therefore, the natural place to study them is on the quotient space  $\overline{\mathbb{D}(d, n)}/K$ , which is a compact Hausdorff space on which all the continuous  $K$ -concomitants,  $C(\overline{\mathbb{D}(d, n)}, M_n(\mathbb{C}))^K$ , naturally live. This algebra, in turn, is naturally isomorphic to the  $C^*$ -algebra of continuous cross sections of a certain  $C^*$ -bundle of finite dimensional  $C^*$ -algebras over  $\overline{\mathbb{D}(d, n)}/K$ , by [12, Lemma 2.2]. However, it is *not* a homogeneous  $C^*$ -algebra because  $K$  does not act freely on  $\overline{\mathbb{D}(d, n)}$ . There are some obvious candidates for the boundary representations of  $C(\overline{\mathbb{D}(d, n)}, M_n(\mathbb{C}))^K$  for  $\mathbb{S}(d, n)$ , but we do not yet know how to check them. Problems with isotropy prevent us from applying the ideas that we have presented above. Nevertheless, the algebra  $\mathbb{S}(d, n)$  seems to have a lot in common with the algebras  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  that we have discussed here. We focused on these first because we could avoid difficulty with isotropy. The algebras  $\mathbb{S}(\mathcal{D}, \mathfrak{P}; d, n)$  turn out to be quite interesting in their own right, however, and they deserve further exploration.



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