

# Interactions and dynamical systems of type $(n, m)$ - a case study

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ABSTRACT. In this paper we prove that the  $C^*$ -algebra of the universal  $(n, m)$ -dynamical system may be obtained, up to Morita-Rieffel equivalence, as the crossed-product relative to an interaction on a commutative  $C^*$ -algebra. The interaction involved is shown not to be part of an interaction group.

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## 1. Introduction.

The notion of interactions was introduced in [11] in order to provide a common generalization for endomorphisms of  $C^*$ -algebras and their transfer operators. One of the main results in [11], namely Theorem 6.3, is the proof of the existence of a *covariant representation* for any given interaction, but no consistent notion of *crossed product* was introduced.

In reality, in the last section of [11], an admittedly experimental attempt was made to provide some sort of crossed product in terms of a certain generalization of the Cuntz-Pimsner algebra to a context in which the correspondence is replaced by a *generalized correspondence* [11, Definition 7.1]. However, no nontrivial examples were provided so the theory was not put through any significant test.

The notion of interactions was later given a (non-equivalent) alternative form in [12] (see also [13]), the catch-word being *interaction groups*, and a well developed notion of crossed product was introduced. Several examples were later exhibited in [15], including the case of the *multi-valued map*  $z \mapsto$

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$z^{2/3}$  on the circle, which has not received a lot of attention in the literature, except for the paper [3] by Arzumanian and Renault (which was in fact slightly rectified by [15]) and some recent work by Arzumanian [2] and by Cuntz and Vershik [5].

In a very rough sense, interactions are related to partial isometries, while interaction groups are related to power partial isometries, meaning partial isometries whose powers are still partial isometries. Since partial isometries not satisfying the latter property are hard to study objects, I eventually developed the impression that interactions should be likewise considered.

Roughly five years after the appearance of [11], I was involved in a seemingly unrelated joint project with P. Ara and T. Katsura [1], where we introduced the notion of  $(n, m)$ -dynamical systems and their accompanying  $C^*$ -algebras, denoted  $\mathcal{O}_{n,m}$ , which turned out to be a generalization of the Cuntz algebras. The method used to study  $\mathcal{O}_{n,m}$  was based on partial actions and in no moment did it occur to us to study it from the point of view of interactions.

By an  $(n, m)$ -dynamical system we mean two compact spaces  $X$  and  $Y$ , with maps

$$h_1, \dots, h_n, v_1, \dots, v_m : Y \rightarrow X,$$

which are homeomorphisms onto their ranges, and such that

$$X = \bigcup_{i=1}^n h_i(Y) = \bigcup_{j=1}^m v_j(Y),$$

both unions being disjoint unions. Given such a system, one may consider a local homeomorphism  $\alpha : X \rightarrow Y$ , defined to coincide with  $h_i^{-1}$  on the range of each  $h_i$ . One might consider  $\alpha$  as some version of Bernoulli's shift, for which the  $h_i$ 's are the inverse branches.

Replacing the  $h_i$  by the  $v_j$ , one may similarly define another local homeomorphism, say  $\beta : X \rightarrow Y$ , having  $v_j$  as inverse branches.

Evidently neither  $\alpha$  nor  $\beta$  are invertible (unless  $n$  or  $m = 1$ ), but we may view the *multi-valued* map

$$L : y \mapsto \{h_1(y), \dots, h_n(y)\},$$

as playing the role of the inverse of  $\alpha$ . Likewise

$$M : y \mapsto \{v_1(y), \dots, v_m(y)\}$$

may be considered as some sort of inverse for  $\beta$ . Playing in a totally careless way with these maps, one may define

$$\mathcal{V}, \mathcal{H} : X \rightarrow X,$$

by  $\mathcal{V} = M\alpha$ , and  $\mathcal{H} = L\beta$ , and argue that

$$\mathcal{V}^{-1} = \alpha^{-1}M^{-1} = L\beta = \mathcal{H}.$$

Evidently all of this is nonsense, but the notion of interactions may give it a precise and meaningful treatment. The main idea is that, when a map

is multivalued, it defines a (singly valued) map on the algebra of continuous functions by averaging over the multiple values. In the present case, this leads us to defining maps  $\mathcal{V}$  and  $\mathcal{H}$  on the algebra  $C(X)$  by

$$(\dagger) \quad \mathcal{V}(f)|_x = \frac{1}{m} \sum_{j=1}^m f(v_j(\alpha(x))), \quad \text{and} \quad \mathcal{H}(f)|_x = \frac{1}{n} \sum_{i=1}^n f(h_i(\beta(x)))$$

for all  $f \in C(X)$ , and all  $x \in X$ .

The interesting fact is that the pair  $(\mathcal{V}, \mathcal{H})$  turns out to be an interaction and, moreover, the experimental notion of crossed product introduced in [11] fits like a glove in the present situation, producing the expected result, namely the full hereditary subalgebra of  $\mathcal{O}_{n,m}$  associated to the characteristic function on  $X$ .

Besides briefly recalling the necessary background, the content of this paper is precisely to prove the isomorphism of the crossed product  $C(X) \rtimes_{\mathcal{V}, \mathcal{H}} \mathbb{N}$  with the hereditary subalgebra of  $\mathcal{O}_{n,m}$  mentioned above.

Another question that we discuss is the possibility of fitting the theory of interaction groups to  $\mathcal{O}_{n,m}$  but we unfortunately find in (3.11) that this is not possible.

Before we actually begin, we should say that the description of  $\mathcal{V}$  and  $\mathcal{H}$  given in  $(\dagger)$ , above, is not quite the one we use below, as we have chosen to emphasize the algebraic aspects of  $\mathcal{O}_{n,m}$  over its dynamical picture. However, without too much effort, the reader may use the results in [1] to show that  $(\dagger)$  agrees with the definitions of  $\mathcal{V}$  and  $\mathcal{H}$  given in (3.7), below.

## 2. Interactions.

In this section we will give a brief overview of the notions of interactions and the corresponding crossed product. For more information the reader is referred to [11].

► From now on we will let  $A$  be a fixed unital  $C^*$ -algebra.

**Definition 2.1.** [11, Definition 3.1] *A pair  $(\mathcal{V}, \mathcal{H})$  of maps*

$$\mathcal{V}, \mathcal{H} : A \rightarrow A$$

*will be called an interaction over  $A$ , if*

- (i)  $\mathcal{V}$  and  $\mathcal{H}$  are positive, bounded, unital linear maps,
- (ii)  $\mathcal{V}\mathcal{H}\mathcal{V} = \mathcal{V}$ ,
- (iii)  $\mathcal{H}\mathcal{V}\mathcal{H} = \mathcal{H}$ ,
- (iv)  $\mathcal{V}(xy) = \mathcal{V}(x)\mathcal{V}(y)$ , if either  $x$  or  $y$  belong to  $\mathcal{H}(A)$ ,
- (v)  $\mathcal{H}(xy) = \mathcal{H}(x)\mathcal{H}(y)$ , if either  $x$  or  $y$  belong to  $\mathcal{V}(A)$ .

► Let us assume, for the remainder of this section, that  $(\mathcal{V}, \mathcal{H})$  is a fixed interaction over  $A$ .

**Definition 2.2.** [11, Definition 3.5] *A covariant representation of  $(\mathcal{V}, \mathcal{H})$  in a given unital  $C^*$ -algebra  $B$  is a pair  $(\pi, s)$ , where  $\pi$  is a unital  $*$ -homomorphism of  $A$  into  $B$ , and  $s$  is a partial isometry in  $B$  such that*

- (i)  $s\pi(a)s^* = \pi(\mathcal{V}(a))ss^*$ , and
- (ii)  $s^*\pi(a)s = \pi(\mathcal{H}(a))s^*s$ ,

for every  $a$  in  $A$ .

**Definition 2.3.** We will denote by  $\mathcal{T}(A, \mathcal{V}, \mathcal{H})$  the universal unital<sup>1</sup>  $C^*$ -algebra generated by a copy of  $A$  and a partial isometry  $\check{s}$ , subject to the relations

- (i)  $\check{s}a\check{s}^* = \mathcal{V}(a)\check{s}\check{s}^*$ , and
- (ii)  $\check{s}^*a\check{s} = \mathcal{H}(a)\check{s}^*\check{s}$ ,

for every  $a$  in  $A$ . The canonical mapping from  $A$  to  $\mathcal{T}(A, \mathcal{V}, \mathcal{H})$  will be denoted by  $\tilde{\pi}$ .

It is readily seen that  $(\tilde{\pi}, \check{s})$  is a covariant representation of  $(\mathcal{V}, \mathcal{H})$  in  $\mathcal{T}(A, \mathcal{V}, \mathcal{H})$ . In addition,  $\mathcal{T}(A, \mathcal{V}, \mathcal{H})$  is clearly the universal  $C^*$ -algebra for covariant representations of  $(\mathcal{V}, \mathcal{H})$  in the sense that any covariant representation factors through  $\mathcal{T}(A, \mathcal{V}, \mathcal{H})$ .

We should remark that, as we are working in the category of unital  $C^*$ -algebras and morphisms, the natural inclusion  $\tilde{\pi}$  of  $A$  in  $\mathcal{T}(A, \mathcal{V}, \mathcal{H})$  is a unital map and, in particular,

$$(2.4) \quad \tilde{\pi}(1)\check{s} = \check{s}\tilde{\pi}(1) = \check{s}.$$

Quite likely is also possible to develop a similar theory for non-unital algebras but, given the examples we have in mind, we have decided to concentrate on the unital case here.

**Proposition 2.5.** [11] *The closed linear span of  $\tilde{\pi}(A)\check{s}\tilde{\pi}(A)$ , henceforth denoted by  $\mathcal{X}$ , is a ternary ring of operators [16], meaning that it satisfies*

$$\mathcal{X}\mathcal{X}^*\mathcal{X} \subseteq \mathcal{X}.$$

**Proof.** For all  $a, b, c, d, e, f \in A$ , we have

$$\begin{aligned} & (\tilde{\pi}(a)\check{s}\tilde{\pi}(b))(\tilde{\pi}(c)\check{s}\tilde{\pi}(d))^*(\tilde{\pi}(e)\check{s}\tilde{\pi}(f)) = \tilde{\pi}(a)\check{s}\tilde{\pi}(bd^*)\check{s}^*\tilde{\pi}(c^*e)\check{s}\tilde{\pi}(f) \\ & = \tilde{\pi}(a)\tilde{\pi}(\mathcal{V}(bd^*))\check{s}^*\check{s}\tilde{\pi}(\mathcal{H}(c^*e))\tilde{\pi}(f) = \tilde{\pi}(a\mathcal{V}(bd^*))\check{s}\tilde{\pi}(\mathcal{H}(c^*e)f) \in \mathcal{X}. \end{aligned}$$

□

From the above result it follows that

$$\mathcal{K}_{\mathcal{V}} := \overline{\text{span}}\mathcal{X}\mathcal{X}^*$$

as well as

$$\mathcal{K}_{\mathcal{H}} := \overline{\text{span}}\mathcal{X}^*\mathcal{X}$$

are closed  $*$ -subalgebras of  $\mathcal{T}(A, \mathcal{V}, \mathcal{H})$ . It also follows that

$$\mathcal{K}_{\mathcal{V}}\mathcal{X} \subseteq \mathcal{X}, \quad \text{and} \quad \mathcal{X}\mathcal{K}_{\mathcal{H}} \subseteq \mathcal{X},$$

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<sup>1</sup>When we say “universal unital” we mean that we are working in the category of unital  $C^*$ -algebras and hence all algebras and morphisms involved in its universal properties are supposed to be unital.

and hence that  $\mathcal{X}$  is a  $\mathcal{K}_{\mathcal{V}} - \mathcal{K}_{\mathcal{H}}$ -bimodule. On the other hand it is easily seen that  $\mathcal{X}$  is an  $A - A$ -bimodule.

**Definition 2.6.** [11]

- (a) *A left [resp. right-] redundancy is a pair  $(a, k)$  in  $A \times \mathcal{K}_{\mathcal{V}}$  [resp.  $A \times \mathcal{K}_{\mathcal{H}}$ ] such that*

$$\tilde{\pi}(a)x = kx \text{ [resp. } x\tilde{\pi}(a) = xk], \quad \forall x \in \mathcal{X}.$$

- (b) *The redundancy ideal is the closed two-sided ideal of  $A$  generated by the set*

$$\begin{aligned} & \{ \tilde{\pi}(a) - k : (a, k) \text{ is a left-redundancy} \} \\ & \cup \{ \tilde{\pi}(a) - k : (a, k) \text{ is a right-redundancy} \}. \end{aligned}$$

- (c) *The quotient of  $\mathcal{T}(A, \mathcal{V}, \mathcal{H})$  by the redundancy ideal will be called the covariance algebra or the crossed product for the interaction  $(\mathcal{V}, \mathcal{H})$ , and will be denoted by  $A \rtimes_{\mathcal{V}, \mathcal{H}} \mathbb{N}$ .*
- (d) *Letting*

$$q : \mathcal{T}(A, \mathcal{V}, \mathcal{H}) \rightarrow A \rtimes_{\mathcal{V}, \mathcal{H}} \mathbb{N}$$

*be the quotient map, we will let  $\hat{\pi} = q \circ \tilde{\pi}$ , and  $\hat{s} = q(\check{s})$ .*

Again we have that  $(\hat{\pi}, \hat{s})$  is a covariant representation of  $(\mathcal{V}, \mathcal{H})$  in  $A \rtimes_{\mathcal{V}, \mathcal{H}} \mathbb{N}$ .

The following is an elementary result which slightly simplifies some computations involving redundancies:

**Proposition 2.7.** *A pair  $(a, k)$  in  $A \times \mathcal{K}_{\mathcal{V}}$  [resp.  $A \times \mathcal{K}_{\mathcal{H}}$ ] is a left- [resp. right-] redundancy if and only if*

$$\tilde{\pi}(a)\tilde{\pi}(b)\check{s} = k\tilde{\pi}(b)\check{s} \text{ [resp. } \check{s}\tilde{\pi}(b)\tilde{\pi}(a) = \check{s}\tilde{\pi}(b)k], \quad \forall a \in A.$$

**Proof.** This follows immediately from 2.4 and the density of  $\tilde{\pi}(A)\check{s}\tilde{\pi}(A)$  in  $\mathcal{X}$ .  $\square$

### 3. Brief description of $\mathcal{O}_{n,m}$ .

Let us now introduce the algebra  $\mathcal{O}_{n,m}$  which will play a prominent role in our main result below. For further details on the properties and structure of  $\mathcal{O}_{n,m}$  the reader is referred to [1].

**Definition 3.1.** Given integers  $n, m \geq 1$ , the Leavitt  $C^*$ -algebra of type  $(n, m)$ , henceforth denoted by  $L_{n,m}$ , is the universal unital  $C^*$ -algebra generated by partial isometries  $s_1, \dots, s_n, t_1, \dots, t_m$  satisfying the relations

$$\begin{aligned} s_i^* s_k &= 0, \text{ for } i \neq k, \\ t_j^* t_l &= 0, \text{ for } j \neq l, \\ s_i^* s_i &= t_j^* t_j =: q, \\ \sum_{i=1}^n s_i s_i^* &= \sum_{j=1}^m t_j t_j^* =: p, \\ pq &= 0, \quad p + q = 1. \end{aligned}$$

As observed in [1, Section 2], when  $n, m > 1$ , the partial isometries  $s_i$  and  $t_j$  in  $L_{n,m}$  do not form a tame set, in the sense that the multiplicative subsemigroup of  $L_{n,m}$  generated by

$$\{s_1, \dots, s_n, s_1^*, \dots, s_n^*, t_1, \dots, t_m, t_1^*, \dots, t_m^*\}$$

does not consist of partial isometries. In order to fix this, we consider the ideal  $J \trianglelefteq L_{n,m}$  generated by all elements of the form  $xx^*x - x$ , where  $x$  runs in the above mentioned semigroup.

**Definition 3.2.**  $\mathcal{O}_{n,m}$  is the quotient of  $L_{n,m}$  by the ideal  $J$  described above.

From now on we will concentrate our attention on  $\mathcal{O}_{n,m}$ , whereas  $L_{n,m}$  will not play any further role in this work. We will therefore not bother to introduce any new notation for the images of the  $s_i$  and  $t_j$  in  $\mathcal{O}_{n,m}$ , denoting them again by  $s_i$  and  $t_j$ , as no confusion will arise.

**Definition 3.3.** The multiplicative subsemigroup of  $\mathcal{O}_{n,m}$  generated by all the  $s_i$ , all the  $t_j$ , as well as their adjoints, will be denoted by  $\mathcal{S}_{n,m}$ .

It is then clear that  $\mathcal{S}_{n,m}$  is formed by partial isometries, and hence it is an *inverse semigroup*. Its *idempotent semi-lattice*, namely

$$E(\mathcal{S}_{n,m}) = \{ss^* : s \in \mathcal{S}_{n,m}\} = \{e \in \mathcal{S}_{n,m} : e^2 = e\}$$

is a set of commuting projections and therefore generates an abelian sub- $C^*$ -algebra of  $\mathcal{O}_{n,m}$ , which we will denote by  $A$ . Sections (2) and (4) of [1] give two different descriptions of the spectrum of  $A$ .

Evidently  $p$  and  $q$  are complementary (central) projections in  $A$ , so  $A$  admits a decomposition as a direct sum of two ideals:

$$A = A_p \oplus A_q,$$

where  $A_p = pA$ , and  $A_q = qA$ . Observing that

$$(3.4) \quad s_i = ps_iq, \quad \text{and} \quad t_j = pt_jq,$$

for all  $i$  and  $j$ , we see that the ideals generated by either  $p$  or  $q$  in  $\mathcal{O}_{n,m}$  coincide with the whole of  $\mathcal{O}_{n,m}$ , which is to say that  $p$  and  $q$  are full projections. Consequently the subalgebras of  $\mathcal{O}_{n,m}$  given by

$$\mathcal{O}_{n,m}^p := p(\mathcal{O}_{n,m})p, \quad \text{and} \quad \mathcal{O}_{n,m}^q := q(\mathcal{O}_{n,m})q$$

are both *full corners*, and hence Morita-Rieffel equivalent [4, Theorem 1.1] to  $\mathcal{O}_{n,m}$ .

**Proposition 3.5.** *For every  $i \leq n$ , and  $j \leq m$ , the correspondences*

$$\alpha_i : a \mapsto s_i a s_i^*, \quad \text{and} \quad \beta_j : a \mapsto t_j a t_j^*$$

*give well defined \*-homomorphisms from  $A_q$  to  $A_p$ , and the same is true with respect to*

$$\alpha := \sum_{i=1}^n \alpha_i, \quad \text{and} \quad \beta := \sum_{j=1}^m \beta_j.$$

*Moreover  $\alpha$  and  $\beta$  are unital.*

**Proof.** Left to the reader.  $\square$

The \*-homomorphisms  $\alpha_i$  and  $\beta_j$  above are closely related to a *partial action* of the free group  $\mathbb{F}_{n+m}$  on  $A$  which contains enough information to reconstruct  $\mathcal{O}_{n,m}$  in the sense that  $\mathcal{O}_{n,m}$  is isomorphic to the crossed product  $A \rtimes \mathbb{F}_{n+m}$ . See [1] for more information on this.

Another easy consequence of the relations defining  $\mathcal{O}_{n,m}$  above is in order.

**Proposition 3.6.** *Define maps  $L, M : A_p \rightarrow A_q$  by*

$$L(f) = \frac{1}{n} \sum_{i=1}^n s_i^* f s_i, \quad \text{and} \quad M(f) = \frac{1}{m} \sum_{j=1}^m t_j^* f t_j.$$

*Then  $L$  and  $M$  are unital positive linear maps and moreover*

- (i)  $L\alpha$  and  $M\beta$  coincide with the identity of  $A_q$ .
- (ii)  $L(\alpha(g)f) = gL(f)$ , for all  $g \in A_q$ , and  $f \in A_p$ .
- (iii)  $M(\beta(g)f) = gM(f)$ , for all  $g \in A_q$ , and  $f \in A_p$ .

**Proof.** It is clear that  $L$  and  $M$  are positive linear maps. Observing that  $p$  is the unit of  $A_p$ , we have that

$$L(p) = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n s_i^* s_j s_j^* s_i = \frac{1}{n} \sum_{i=1}^n s_i^* s_i = q,$$

which is the unit of  $A_q$ . Therefore  $L$  is indeed a unital map, and a similar argument applies to prove that  $M$  is also unital. In order to prove (ii), let  $g \in A_q$ , and  $f \in A_p$ . Then

$$L(\alpha(g)f) = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n s_i^* s_j g s_j^* f s_i = \frac{1}{n} \sum_{i=1}^n s_i^* s_i g s_i^* f s_i = \frac{1}{n} \sum_{i=1}^n g s_i^* f s_i = gL(f).$$

The proof of (iii) follows along similar lines and, finally, (i) follows from (ii) and (iii) upon plugging  $f = 1$ .  $\square$

Notice that equations (3.6.ii-iii) bear a close similarity with the axioms defining transfer operators in [10].

**Proposition 3.7.** *Let  $\mathcal{V}$  and  $\mathcal{H}$  be the linear operators on  $A_p$  defined by*

$$\mathcal{V} = \alpha M, \quad \text{and} \quad \mathcal{H} = \beta L.$$

*Then  $(\mathcal{V}, \mathcal{H})$  is an interaction over  $A_p$ .*

**Proof.** It is clear that  $\mathcal{V}$  and  $\mathcal{H}$  are bounded positive linear maps. In order to prove (2.1.ii), we have by (3.6) that

$$\mathcal{V}\mathcal{H}\mathcal{V} = \alpha M \beta L \alpha M = \alpha M = \mathcal{V}.$$

The proof of (2.1.iii) is similar. As for (2.1.v), let  $f_1, f_2 \in A_p$ , with  $f_1 \in \mathcal{V}(A_p)$ . Then there is  $k \in A_p$  such that

$$f_1 = \mathcal{V}(k) = \alpha(M(k)) = \alpha(g),$$

where  $g = M(k) \in A_q$ . We then have by (3.6) that

$$\begin{aligned} \mathcal{H}(f_1 f_2) &= \beta(L(\alpha(g) f_2)) = \beta(gL(f_2)) \\ &= \beta(g) \beta(L(f_2)) = \beta(M(k)) \mathcal{H}(f_2) = \dots \end{aligned}$$

Noticing that  $\mathcal{H}(f_1) = \beta L \alpha M(k) = \beta(M(k))$ , the above equals

$$\dots = \mathcal{H}(f_1) \mathcal{H}(f_2),$$

proving (2.1.v). The proof of (2.1.iv) is similar.  $\square$

Our next goal will be to produce a covariant representation of  $(\mathcal{V}, \mathcal{H})$  in  $\mathcal{O}_{n,m}^p$ . The partial isometry involved will actually be produced in terms of two other partial isometries, as follows:

**Proposition 3.8.** *Let*

$$S = \frac{1}{\sqrt{n}} \sum_{i=1}^n s_i, \quad \text{and} \quad T = \frac{1}{\sqrt{m}} \sum_{j=1}^m t_j.$$

*Then  $S^*S = T^*T = q$ . Consequently  $S$  and  $T$  are partial isometries. In addition*

$$R := ST^*$$

*is a partial isometry belonging to  $\mathcal{O}_{n,m}^p$ , which satisfies  $RR^* = SS^*$  and  $R^*R = TT^*$ .*

**Proof.** We have

$$S^*S = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n s_i^* s_j = \frac{1}{n} \sum_{i=1}^n s_i^* s_i = q,$$



and similarly  $T^*T = q$ . This shows that  $S$  and  $T$  are partial isometries. In order to show that  $R$  is also a partial isometry we compute

$$RR^*R = ST^*TS^*ST^* = SqT^* = SqT^* = ST^* = R.$$

By 3.4 we have that  $S = pS$ , and  $T = pT$ , so

$$R = ST^* = pST^*p \in \mathcal{O}_{n,m}^p.$$

□

The partial isometries  $S$  and  $T$  above have a close relationship with the maps  $\alpha$ ,  $\beta$ ,  $L$  and  $M$  studied above, as we shall now see.

**Proposition 3.9.** *For every  $f \in A_p$ , and for every  $g \in A_q$ , one has that*

- (i)  $S^*fS = L(f)$ ,
- (ii)  $T^*fT = M(f)$ ,
- (iii)  $Sg = \alpha(g)S$ ,
- (iv)  $Tg = \beta(g)T$ .

**Proof.** For  $f \in A_p$ , and  $i \neq j$ , one has that

$$s_i^*fs_j = s_i^*s_i s_i^*fs_j = s_i^*fs_i s_i^*s_j = 0.$$

Therefore

$$S^*fS = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n s_i^*fs_j = \frac{1}{n} \sum_{i=1}^n s_i^*fs_i = L(f),$$

proving (i). A similar argument proves (ii). Given  $g \in A_q$ , we have that

$$\alpha(g)S = \sum_{i=1}^n s_i g s_i^* \frac{1}{\sqrt{n}} \sum_{j=1}^n s_j = \frac{1}{\sqrt{n}} \sum_{i=1}^n \sum_{j=1}^n s_i g s_i^* s_j = \frac{1}{\sqrt{n}} \sum_{i=1}^n s_i g q = Sg.$$

proving (iii). A similar computation proves (iv). □

The similarity of (3.9.i-iv) with the axioms defining covariant representations in the context of endomorphisms and transfer operators [10] should again be noticed.

The covariant representation announced above may now be presented.

**Proposition 3.10.** *Let  $\iota$  denote the inclusion of  $A_p$  into  $\mathcal{O}_{n,m}^p$ . Then  $(\iota, R)$  is a covariant representation of the interaction  $(\mathcal{V}, \mathcal{H})$  in  $\mathcal{O}_{n,m}^p$ . Therefore there is a  $*$ -homomorphism*

$$\Phi : \mathcal{T}(A_p, \mathcal{V}, \mathcal{H}) \rightarrow \mathcal{O}_{n,m}^p$$

satisfying  $\Phi(\tilde{\pi}(a)) = a$ , for all  $a$  in  $A_p$ , and such that  $\Phi(\check{s}) = R$ .

**Proof.** Given  $f \in A_p$ , we have by (3.9) that

$$RfR^* = ST^*fTS^* = SM(f)S^* = \alpha(M(f))SS^* = \mathcal{V}(f)RR^*.$$

while

$$R^*fR = TS^*fST^* = TL(f)T^* = \beta(L(f))TT^* = \mathcal{H}(f)R^*R.$$

This shows that  $(\iota, R)$  is indeed a covariant representation. The last sentence of the statement now follows immediately from the universal property of  $\mathcal{T}(A_p, \mathcal{V}, \mathcal{H})$ .  $\square$

Let us make a short pause to compare the representation above with the representations arising from the theory of interaction groups [12]. Observe that the partial isometries  $v_g$  of [12, Definition 4.1] lie in the range of a \*-partial representation. Moreover, by [8, Proposition 2.4.iii], any two partial isometries belonging to the range of the same partial representation must have commuting range projections. In particular, every such partial isometry is necessarily a *power partial isometry*, meaning that its powers are still partial isometries.

**Proposition 3.11.** *If  $n$  and  $m$  are both greater or equal to 2, the partial isometry  $R$  introduced in (3.8) is not a power partial isometry. More precisely,  $R^2$  is not a partial isometry.*

**Proof.** It is well known (see e.g. [9, Lemma 5.3]) that the product of two partial isometries  $u$  and  $v$  is a partial isometry if and only if the source projection of  $u$  commutes with the range projection of  $v$ . Thus,  $R^2$  is a partial isometry if and only if  $R^*R$  commutes with  $RR^*$ . In view of the last sentence of (3.8), we must check whether or not  $SS^*$  commutes with  $TT^*$ . We have

$$SS^*TT^* = \frac{1}{nm} \sum_{i,k=1}^n \sum_{j,l=1}^m s_i s_k^* t_j t_l^*,$$

while

$$TT^*SS^* = \frac{1}{nm} \sum_{j,l=1}^m \sum_{i,k=1}^n t_j t_l^* s_i s_k^*.$$

Using the description of  $\mathcal{O}_{n,m}$  as a partial crossed product [1, 2.5], and also the fact that the crossed product may be defined [14, Section 2] as the cross sectional  $C^*$ -algebra of the semi-direct product Fell bundle [6, Definition 2.8], we deduce that  $\mathcal{O}_{n,m}$  is a cross sectional algebra for a Fell bundle over the free group  $\mathbb{F}_{n+m}$ .

Moreover, if the generators of  $\mathbb{F}_{n+m}$  are denoted  $a_1, \dots, a_n, b_1, \dots, b_m$ , each summand  $s_i s_k^* t_j t_l^*$  in the expression for  $SS^*TT^*$  above lie in the homogeneous space associated to the group element  $a_i a_k^{-1} b_j b_l^{-1}$ , and a similar fact holds for the terms  $t_j t_l^* s_i s_k^*$  in the expression for  $TT^*SS^*$ . Should  $SS^*$  commute with  $TT^*$ , the *Fourier coefficient* [7, Definition 2.7] of  $SS^*TT^*$  relative to the group element  $a_1 a_2^{-1} b_1 b^{-1}$  would be zero, as this is clearly the case for  $TT^*SS^*$ . This means that  $s_1 s_2^* t_1 t_2^* = 0$ , a contradiction.  $\square$

As already discussed before the statement of the Proposition above, the fact that  $R$  is not a power partial isometry says that it is impossible to view the covariant representation given by (3.10) as part of some interaction group.

Our next task will be to show that  $\Phi$  vanishes on the redundancy ideal of  $\mathcal{T}(A_p, \mathcal{V}, \mathcal{H})$ . The following technical result initiates our preparations for this.

**Proposition 3.12.** *Let  $y \in \mathcal{O}_{n,m}^p$ .*

- (i) *If  $yA_pR = \{0\}$ , then  $y = 0$ .*
- (ii) *If  $yA_pR^* = \{0\}$ , then  $y = 0$ .*

**Proof.** (i) For any given  $k \leq n$ , notice that  $s_k s_k^* \in A_p$ , so

$$\begin{aligned} 0 &= y s_k s_k^* R = y s_k s_k^* S T^* = y s_k s_k^* \left( \frac{1}{\sqrt{nm}} \sum_{i=1}^n \sum_{j=1}^m s_i t_j^* \right) \\ &= \frac{1}{\sqrt{nm}} \sum_{i=1}^n \sum_{j=1}^m y s_k s_k^* s_i t_j^* = \frac{1}{\sqrt{nm}} \sum_{j=1}^m y s_k t_j^*. \end{aligned}$$

Multiplying this on the right by  $t_1$ , we deduce that

$$0 = \sum_{j=1}^m y s_k t_j^* t_1 = y s_k t_1^* t_1 = y s_k q = y s_k.$$

Therefore

$$yp = \sum_{k=1}^n y s_k s_k^* = 0.$$

Since  $y \in \mathcal{O}_{n,m}^p$  by hypothesis, we have that  $y = yp = 0$ . The proof of (ii) is similar.  $\square$

We may now show the existence of natural a \*-homomorphism from  $A_p \rtimes_{\mathcal{V}, \mathcal{H}} \mathbb{N}$  to  $\mathcal{O}_{n,m}^p$ .

**Proposition 3.13.** *The map  $\Phi$  of (3.10) vanishes on the redundancy ideal of  $\mathcal{T}(A_p, \mathcal{V}, \mathcal{H})$ . Consequently there exists a \*-homomorphism*

$$\Psi : A_p \rtimes_{\mathcal{V}, \mathcal{H}} \mathbb{N} \rightarrow \mathcal{O}_{n,m}^p,$$

*such that  $\Psi(\hat{\pi}(a)) = a$ , for all  $a \in A_p$ , and  $\Psi(\hat{s}) = R$ .*

**Proof.** Let  $(a, k) \in A_p \times \mathcal{K}_{\mathcal{V}}$  be a left-redundancy. Then, taking (2.7) into account, for every  $b \in A_p$ , we have that

$$0 = (\tilde{\pi}(a) - k) \tilde{\pi}(b) \check{s}.$$

Applying  $\Phi$  to this leads to

$$0 = (a - \Phi(k)) b R.$$

In other words, we have that  $(a - \Phi(k)) A_p R = 0$ , and hence by (3.12) we conclude that

$$0 = a - \Phi(k) = \Phi(\tilde{\pi}(a) - k).$$

In the same way we may prove that  $(\dagger)$  holds for right-redundancies, hence concluding the proof.  $\square$

Recall from (3.3) that  $\mathcal{S}_{n,m}$  is the multiplicative subsemigroup of  $\mathcal{O}_{n,m}$  generated by all the  $s_i$ , all the  $t_j$ , as well as their adjoints. In addition to  $\mathcal{S}_{n,m}$ , we wish to introduce the following subsets of  $\mathcal{S}_{n,m}$ :

**Definition 3.14.**

(a) We shall denote by  $\mathcal{G}$  the subset of  $\mathcal{S}_{n,m}$  given by

$$\mathcal{G} = \{s_1, \dots, s_n, t_1, \dots, t_m\}.$$

(b) We shall denote by  $\mathcal{F}$  the subset of  $\mathcal{S}_{n,m}$  given by

$$\mathcal{F} = \{s_i t_j^* : i \leq n, j \leq m\}.$$

(c) The subsemigroup of  $\mathcal{S}_{n,m}$  generated by  $\mathcal{F} \cup \mathcal{F}^*$  will be denoted by  $\mathcal{R}_{n,m}$ .

Observe that, since  $\mathcal{S}_{n,m}$  is an inverse semigroup and since the generating set of  $\mathcal{R}_{n,m}$  is self-adjoint, one has that  $\mathcal{R}_{n,m}$  is itself an inverse semigroup.

**Proposition 3.15.**

- (i)  $\mathcal{O}_{n,m}^p$  is generated as a  $C^*$ -algebra by  $\mathcal{F}$ .
- (ii)  $A_p$  is generated as a  $C^*$ -algebra by  $E(\mathcal{R}_{n,m})$ , the idempotent semi-lattice of  $\mathcal{R}_{n,m}$ .

**Proof.** In order to prove (i), let us temporarily denote by  $B$  the closed  $*$ -subalgebra of  $\mathcal{O}_{n,m}$  generated by  $\mathcal{F}$ . By (3.4) we have that

$$(3.16) \quad s_i t_j^* = p s_i t_j^* p \in \mathcal{O}_{n,m}^p,$$

and hence  $B \subseteq \mathcal{O}_{n,m}^p$ . In order to prove the reverse inclusion it is clearly enough to prove that

$$z := p x_1 \dots x_r p \in B,$$

whenever  $x_k \in \mathcal{G} \cup \mathcal{G}^*$ , for every  $k \leq r$ . If  $r = 0$ , that is, if  $z = p$ , then

$$z = p = \sum_{i=1}^n s_i s_i^* = \sum_{i=1}^n s_i q s_i^* = \sum_{i=1}^n s_i t_1^* t_1 s_i^* = \sum_{i=1}^n s_i t_1^* (s_i t_1^*)^* \in B.$$

In case  $r > 0$ , we claim that, unless  $z = 0$ , the  $x_k$ 's above must:

- (a) start with an element from  $\mathcal{G}$ ,
- (b) end in an element from  $\mathcal{G}^*$ , and
- (c) alternate elements from  $\mathcal{G}$  and  $\mathcal{G}^*$ .

In order to prove (a), suppose by contradiction that  $x_1 \in \mathcal{G}^*$ . Then  $x_1^* = p x_1^* q$ , by (3.4), so  $p x_1 = p q x_1 p = 0$ , and we would have that  $z = 0$ . A similar reasoning proves (c). As for (b), if two consecutive terms, say  $x_k$  and  $x_{k+i}$ , both lie in  $\mathcal{G}$ , then, again by (3.4), we would have that  $x_k x_{k+1} = p x_k q p x_{k+1} q = 0$ , and again  $z = 0$ .

This said, we may rewrite  $z$  as

$$z = p u_1 v_1^* \dots u_l v_l^* p,$$

where  $u_k, v_k \in \mathcal{G}$  (as opposed to  $\mathcal{G} \cup \mathcal{G}^*$ ). Therefore it suffices to prove that each  $u_k v_k^* \in B$ .

Since  $u_k$  and  $v_k$  may be chosen among the  $s_i$ 's or the  $t_j$ 's, we are left with the task of proving that

$$s_i t_j^*, t_j s_i^*, s_i s_j^*, t_i t_j^* \in B.$$

It is evident that the first two terms above do lie in  $B$ , while

$$s_i s_j^* = s_i q s_j^* = s_i t_1^* t_1 s_j^* = s_i t_1^* (s_j t_1^*)^* \in B.$$

A similar argument proves that  $t_i t_j^* \in B$ .

Focusing now on (ii), let  $C$  be the closed  $*$ -subalgebra of  $\mathcal{O}_{n,m}$  generated by  $E(\mathcal{R}_{n,m})$ . Given any element  $e \in E(\mathcal{R}_{n,m})$ , choose  $z \in \mathcal{R}_{n,m}$ , such that  $e = zz^*$ . Then clearly  $e \in E(\mathcal{S}_{n,m})$ , and hence  $a \in A$ . Since  $z \in \mathcal{R}_{n,m}$ , equation 3.16 implies that  $z = pz$ , so  $e = pe$ , and hence  $e \in pA = A_p$ . This shows that  $C \subseteq A_p$ .

Recall that  $A$  is generated by  $E(\mathcal{S}_{n,m})$ , and consequently  $A_p (= pA)$  is generated by  $pE(\mathcal{S}_{n,m})$ . In order to prove that  $A_p \subseteq C$ , it is therefore enough to prove that

$$pe \in C, \forall e \in E(\mathcal{S}_{n,m}).$$

Write  $e = zz^*$ , for some  $z \in \mathcal{S}_{n,m}$ , and further write  $z = x_1 \dots x_r$ , where  $x_k \in \mathcal{G} \cup \mathcal{G}^*$ , for every  $k \leq r$ . Summarizing, we must prove that

$$f := px_1 \dots x_r x_r^* \dots x_1^* p \in C,$$

observing that the extra  $p$  on the right-hand side above may be added because  $E(S)$  is commutative. Excluding the trivial case in which  $f = 0$ , we have already seen that (a)–(c) above must hold. Depending on whether  $r$  is even or odd, we therefore have two alternatives:

$$(3.17) \quad f = p u_1 v_1^* \dots u_{l-1} v_{l-1}^* u_l v_l^* v_l u_l^* v_{l-1} u_{l-1}^* \dots v_1 u_1^* p,$$

or

$$(3.18) \quad f = p u_1 v_1^* \dots u_{l-1} v_{l-1}^* u_l u_l^* v_{l-1} u_{l-1}^* \dots v_1 u_1^* p,$$

where  $u_k, v_k \in \mathcal{G}$ . However (3.18) may easily be reduced to (3.17), by plugging  $v_l = u_l$ , so we may assume (3.17). In order to conclude the proof, it is now enough to show that  $u_k v_k^* \in \mathcal{R}_{n,m}$ , for all  $k$ , which we do by observing that

$$\begin{aligned} s_i t_j^* &\in \mathcal{R}_{n,m}, \\ t_j s_i^* &\in \mathcal{R}_{n,m}, \\ s_i s_j^* = s_i t_1^* t_1 s_j^* &\in \mathcal{R}_{n,m}, \\ t_i t_j^* = t_i s_1^* s_1 t_j^* &\in \mathcal{R}_{n,m}, \end{aligned}$$

and the proof is concluded.  $\square$

**Proposition 3.19.** *For every  $i \leq n$ , and  $j \leq m$ , let*

$$p_i = s_i s_i^*, q_j = t_j t_j^*, \hat{p}_i = \hat{\pi}(p_i), \hat{q}_j = \hat{\pi}(q_j), \text{ and } r_{i,j} = \sqrt{nm} \hat{p}_i \hat{s} \hat{q}_j.$$

Then  $\Psi(r_{i,j}) = s_i t_j^*$ . Consequently  $\Psi$  is surjective.

**Proof.** Given  $i$  and  $j$ , we have

$$\Psi(r_{i,j}) = \sqrt{nm} \psi(\hat{\pi}(p_i) \hat{s} \hat{\pi}(q_j)) = \sqrt{nm} p_i S T^* q_j = \sum_{k=1}^n \sum_{l=1}^m p_i s_k t_l^* q_j = s_i t_j^*.$$

The last sentence in the statement then follows from (3.15.i).  $\square$

#### 4. Mapping $\mathcal{O}_{n,m}^p$ into $A_p \rtimes_{\mathcal{V}, \mathcal{H}} \mathbb{N}$

Our main goal is to prove that  $\Psi$  is in fact an isomorphism. In order to accomplish this we will find a representation of the generators and relations defining  $\mathcal{O}_{n,m}$  within the algebra of  $2 \times 2$  matrices over  $A_p \rtimes_{\mathcal{V}, \mathcal{H}} \mathbb{N}$ , and then we will employ the universal property of  $\mathcal{O}_{n,m}$  to construct an inverse for  $\Psi$ . Let us begin by proving some useful algebraic relations.

**Proposition 4.1.** *For every  $i \leq n$ ,  $j \leq m$ , and  $f \in A_p$ , one has*

- (i)  $M(q_j) = \frac{1}{m} q$ ,
- (ii)  $L(p_i) = \frac{1}{n} q$ ,
- (iii)  $\mathcal{V}(q_j) = \frac{1}{m} p$ ,
- (iv)  $\mathcal{H}(p_i) = \frac{1}{n} p$ ,
- (v)  $p_i \mathcal{V}(\mathcal{H}(p_i f)) = \frac{1}{n} p_i f$ ,
- (vi)  $q_j \mathcal{H}(\mathcal{V}(q_j f)) = \frac{1}{m} q_j f$ ,
- (vii)  $\hat{p}_i \hat{s} \hat{s}^* \hat{p}_i = \frac{1}{n} \hat{p}_i$ ,
- (viii)  $\hat{q}_j \hat{s} \hat{s}^* \hat{q}_j = \frac{1}{m} \hat{q}_j$ .

**Proof.** In order to prove (i), we compute:

$$M(q_j) = \frac{1}{m} \sum_{k=1}^m t_k^* q_j t_k = \frac{1}{m} \sum_{k=1}^m t_k^* t_j t_k^* t_k = \frac{1}{m} t_j^* t_j = \frac{1}{m} q,$$

while (ii) follows similarly. As for (iii), we have

$$\mathcal{V}(q_j) = \alpha(M(q_j)) \stackrel{(i)}{=} \frac{1}{m} \alpha(q) = \frac{1}{m} \sum_{i=1}^n s_i q s_i^* = \frac{1}{m} \sum_{i=1}^n s_i s_i^* = \frac{1}{m} p,$$

proving (iii), and a similar argument proves (iv). As for (v), we have

$$\mathcal{V}(\mathcal{H}(p_i f)) = \alpha M \beta L(p_i f) = \alpha L(p_i f).$$

Notice that

$$L(p_i f) = \frac{1}{n} \sum_{k=1}^n s_k^* p_i f s_k = \frac{1}{n} s_i^* s_i s_i^* f s_i = \frac{1}{n} s_i^* f s_i.$$

So

$$\alpha(L(p_i f)) = \frac{1}{n} \alpha(s_i^* f s_i) = \frac{1}{n} \sum_{k=1}^n s_k s_i^* f s_i s_k^*,$$

and consequently

$$p_i \alpha(L(p_i f)) = \frac{1}{n} \sum_{k=1}^n p_i s_k s_i^* f s_i s_k^* = \frac{1}{n} s_i s_i^* s_i s_i^* f s_i s_i^* = \frac{1}{n} p_i f,$$

thus proving (v), while (vi) follows from a similar argument.

Focusing on (vii), and letting  $\check{p}_i = \tilde{\pi}(p_i)$ , we claim that  $(\frac{1}{n} p_i, \check{p}_i \check{s} \check{s}^* \check{p}_i)$  is a left-redundancy. To see this, again taking (2.7) into account, pick  $f \in A_p$ . Then

$$\begin{aligned} \check{p}_i \check{s} \check{s}^* \check{p}_i \tilde{\pi}(f) \check{s} &= \tilde{\pi}(p_i) \check{s} \check{s}^* \tilde{\pi}(p_i f) \check{s} = \tilde{\pi}(p_i) \tilde{\pi}(\mathcal{V}(\mathcal{H}(p_i f))) \check{s} \\ &= \tilde{\pi}(p_i \mathcal{V}(\mathcal{H}(p_i f))) \check{s} \stackrel{(v)}{=} \frac{1}{n} \tilde{\pi}(p_i f) \check{s} = \tilde{\pi}\left(\frac{1}{n} p_i\right) \tilde{\pi}(f) \check{s}, \end{aligned}$$

proving the claim, and hence that  $\frac{1}{n} \check{p}_i = \check{p}_i \check{s} \check{s}^* \check{p}_i$ , in  $A_p \rtimes_{\mathcal{V}, \mathcal{H}} \mathbb{N}$ . The last point is proved similarly.  $\square$

We next present some important algebraic relations among the elements  $r_{i,j}$  introduced in (3.19).

**Lemma 4.2.** *For every  $i, k \leq n$ , and every  $j, l \leq m$ , one has that*

- (i)  $r_{i,j} r_{k,l}^* = 0$ , if  $j \neq l$ ,
- (ii)  $r_{i,j}^* r_{k,l} = 0$ , if  $i \neq k$ ,
- (iii)  $r_{i,j} r_{i,j}^* = \hat{p}_i$ ,
- (iv)  $r_{i,j}^* r_{i,j} = \hat{q}_j$ ,

**Proof.** Point (i) follows from the fact that the  $\hat{q}_j$  are pairwise orthogonal projections, while (ii) follows from a similar assertion about the  $\hat{p}_i$ . As for (iii), notice that

$$\begin{aligned} r_{i,j} r_{i,j}^* &= n m \hat{p}_i \hat{s} \hat{q}_j \hat{s}^* \hat{p}_i = n m \hat{p}_i \hat{s} \hat{\pi}(q_j) \hat{s}^* \hat{p}_i = \\ &= n m \hat{p}_i \hat{\pi}(\mathcal{V}(q_j)) \hat{s} \hat{s}^* \hat{p}_i \stackrel{(4.1 \text{ vii})}{=} n \hat{p}_i \hat{\pi}(p) \hat{s} \hat{s}^* \hat{p}_i = n \hat{p}_i \hat{s} \hat{s}^* \hat{p}_i \stackrel{(4.1 \text{ vii})}{=} \hat{p}_i, \end{aligned}$$

proving (iii). The proof of (iv) is similar.  $\square$

We will now describe a representation of the generators and relations defining  $\mathcal{O}_{n,m}$  within the algebra of  $2 \times 2$  matrices over  $A_p \rtimes_{\mathcal{V}, \mathcal{H}} \mathbb{N}$ .

**Proposition 4.3.** *For every  $i \leq n$ , and  $j \leq m$ , consider the following elements of  $M_2(A_p \rtimes_{\mathcal{V}, \mathcal{H}} \mathbb{N})$ .*

$$\sigma_i = \begin{pmatrix} 0 & 0 \\ r_{i,1} r_{1,1}^* & 0 \end{pmatrix} = r_{i,1} r_{1,1}^* \otimes e_{2,1}$$

and

$$\tau_j = \begin{pmatrix} 0 & 0 \\ r_{1,j}^* & 0 \end{pmatrix} = r_{1,j}^* \otimes e_{2,1}.$$

Then, using brackets to denote Boolean value, we have

- (i)  $\sigma_i^* \sigma_j = [i = j] \hat{p}_1 \otimes e_{1,1}$ , for all  $i, j \leq n$ ,
- (ii)  $\tau_i^* \tau_j = [i = j] \hat{p}_1 \otimes e_{1,1}$ , for all  $i, j \leq m$ ,
- (iii)  $\sigma_i \sigma_i^* = \hat{p}_i \otimes e_{2,2}$ , for all  $i \leq n$ ,
- (iv)  $\tau_j \tau_j^* = \hat{q}_j \otimes e_{2,2}$ , for all  $j \leq m$ .

**Proof.**

(i)

$$\begin{aligned} \sigma_i^* \sigma_j &= r_{1,1} r_{i,1}^* r_{j,1} r_{1,1}^* \otimes e_{1,1} \stackrel{(4.2.ii)}{=} [i = j] r_{1,1} r_{i,1}^* r_{i,1} r_{1,1}^* \otimes e_{1,1} \\ &\stackrel{(4.2.iv)}{=} [i = j] r_{1,1} \hat{q}_1 r_{1,1}^* \otimes e_{1,1} = [i = j] r_{1,1} r_{1,1}^* \otimes e_{1,1} \stackrel{(4.2.iii)}{=} [i = j] \hat{p}_1 \otimes e_{1,1}. \end{aligned}$$

(ii)

$$\tau_i^* \tau_j = r_{1,i} r_{1,j}^* \otimes e_{1,1} \stackrel{(4.2.i)}{=} [i = j] r_{1,i} r_{1,i}^* \otimes e_{1,1} \stackrel{(4.2.iii)}{=} [i = j] \hat{p}_1 \otimes e_{1,1}.$$

(iii)

$$\sigma_i \sigma_i^* = r_{i,1} r_{1,1}^* r_{1,1} r_{i,1}^* \otimes e_{2,2} \stackrel{(4.2.iv)}{=} r_{i,1} \hat{q}_1 r_{i,1}^* \otimes e_{2,2} = r_{i,1} r_{i,1}^* \otimes e_{2,2} \stackrel{(4.2.iii)}{=} \hat{p}_i \otimes e_{2,2}.$$

(iv)

$$\tau_j \tau_j^* = r_{1,j}^* r_{1,j} \otimes e_{2,2} \stackrel{(4.2.iv)}{=} \hat{q}_j \otimes e_{2,2}.$$

□

As a consequence we see that the  $\sigma_i$  and the  $\tau_j$  satisfy the relations in (3.1), with the role of  $q$  and  $p$  being played, respectively, by  $\hat{p}_i \otimes e_{1,1}$ , and  $\hat{p} \otimes e_{2,2}$ , where

$$\hat{p} := \sum_{i=1}^n \hat{p}_i = \sum_{j=1}^m \hat{q}_j = \hat{\pi}(p).$$

We should remark that the validity of the equation “ $p+q=1$ ”, appearing in (3.1), is guaranteed by the fact that the  $\sigma_i$  and the  $\tau_j$  lie in the corner of  $M_2(A_p \rtimes_{\mathcal{V}, \mathcal{H}} \mathbb{N})$  determined by the projection

$$\hat{p}_1 \otimes e_{1,1} + \hat{p} \otimes e_{2,2} = \begin{pmatrix} \hat{p}_1 \otimes e_{1,1} & 0 \\ 0 & \hat{p} \otimes e_{2,2} \end{pmatrix}.$$

The universal property of  $\mathcal{O}_{n,m}$  therefore yields:

**Corollary 4.4.** *There exists a (not necessarily unital) \*-homomorphism*

$$\Gamma : \mathcal{O}_{n,m} \rightarrow M_2(A_p \rtimes_{\mathcal{V}, \mathcal{H}} \mathbb{N})$$

such that

$$(4.5) \quad \Gamma(s_i) = \sigma_i, \quad \text{and} \quad \Gamma(t_j) = \tau_j,$$

for all  $i \leq n$ , and all  $j \leq m$ .



Since we are mostly interested in the subalgebra  $\mathcal{O}_{n,m}^p$  of  $\mathcal{O}_{n,m}$ , it is useful to understand the behavior of  $\Gamma$  on this subalgebra.

**Proposition 4.6.** *For all  $i \leq n$ , and all  $j \leq m$ , one has that*

$$\Gamma(s_i t_j^*) = r_{i,j} \otimes e_{2,2}.$$

Consequently the image of  $\mathcal{O}_{n,m}^p$  under  $\Gamma$  is contained in the corner of  $M_2(A_p \rtimes_{\mathcal{V}, \mathcal{H}} \mathbb{N})$  determined by  $e_{2,2}$ .

**Proof.** By equation 4.5, we have that

$$\Gamma(s_i t_j^*) = \sigma_i \tau_j^* = r_{i,1} r_{1,1}^* r_{1,j} \otimes e_{2,2}.$$

We must therefore compute

$$\begin{aligned} r_{i,1} r_{1,1}^* r_{1,j} &= (nm)^{3/2} \hat{p}_i \hat{s} \hat{q}_1 \hat{s}^* \hat{p}_1 \hat{s} \hat{q}_j = (nm)^{3/2} \hat{\pi}(p_i) \hat{s} \hat{\pi}(q_1) \hat{s}^* \hat{\pi}(p_1) \hat{s} \hat{\pi}(q_j) \\ &= (nm)^{3/2} \hat{\pi}(p_i \mathcal{V}(q_1)) \hat{s} \hat{\pi}(\mathcal{H}(p_1) q_j) \\ &\stackrel{(4.1.iii \& iv)}{=} (nm)^{3/2-1} \hat{\pi}(p_i) \hat{s} \hat{\pi}(q_j) = \sqrt{nm} \hat{p}_i \hat{s} \hat{q}_j = r_{i,j}, \end{aligned}$$

concluding the calculation of  $\Gamma(s_i t_j^*)$ . The last assertion in the statement now follows from (3.15.i). □

Observing that the corner of  $M_2(A_p \rtimes_{\mathcal{V}, \mathcal{H}} \mathbb{N})$  determined by  $e_{2,2}$  is naturally isomorphic to  $A_p \rtimes_{\mathcal{V}, \mathcal{H}} \mathbb{N}$ , we deduce from the above that:

**Corollary 4.7.** *There exists a \*-homomorphism*

$$\Lambda : \mathcal{O}_{n,m}^p \rightarrow A_p \rtimes_{\mathcal{V}, \mathcal{H}} \mathbb{N},$$

such that  $\Lambda(s_i t_j^*) = r_{i,j}$ . Moreover,

$$\Gamma(a) = \Lambda(a) \otimes e_{2,2}, \quad \forall a \in \mathcal{O}_{n,m}^p.$$

We are now ready for our main result.

**Theorem 4.8.** *The homomorphism  $\Lambda$  of the Corollary above is the inverse of the homomorphisms  $\Psi$  of (3.13), and hence  $\mathcal{O}_{n,m}^p$  is \*-isomorphic to  $A_p \rtimes_{\mathcal{V}, \mathcal{H}} \mathbb{N}$ .*

**Proof.** By (3.19) we have that  $\Psi(r_{i,j}) = s_i t_j^*$ , and, as seen above,  $\Lambda(s_i t_j^*) = r_{i,j}$ . Therefore  $\Psi \circ \Lambda$  acts like the identity on the  $s_i t_j^*$ , and hence

$$(4.9) \quad \Psi \circ \Lambda = id_{\mathcal{O}_{n,m}^p},$$

by (3.15.i).

The proof will then be concluded once we prove that  $\Lambda$  is surjective. With this goal in mind, we first claim that the  $r_{i,j}$  normalize  $\hat{A}_p := \hat{\pi}(A_p)$ , in the sense that

$$r_{i,j} \hat{A}_p r_{i,j}^* \subseteq \hat{A}_p, \quad \text{and} \quad r_{i,j}^* \hat{A}_p r_{i,j} \subseteq \hat{A}_p.$$

To see this, let  $f \in A_p$  and observe that

$$\begin{aligned} r_{i,j} \hat{\pi}(f) r_{i,j}^* &= nm \hat{p}_i \hat{s} \hat{q}_j \hat{\pi}(f) \hat{q}_j \hat{s}^* \hat{p}_i = nm \hat{p}_i \hat{\pi}(\mathcal{V}(q_j f q_j)) \hat{s} \hat{s}^* \hat{p}_i = \\ &= nm \hat{\pi}(\mathcal{V}(q_j f)) \hat{p}_i \hat{s} \hat{s}^* \hat{p}_i \stackrel{(4.1.vii)}{=} m \hat{\pi}(\mathcal{V}(q_j f)) \hat{p}_i \in \hat{A}_p, \end{aligned}$$

and similarly that  $r_{i,j}^* \hat{\pi}(f) r_{i,j} \in \hat{A}_p$ . As a consequence we deduce that any element of  $A_p \rtimes_{\mathcal{V}, \mathcal{H}} \mathbb{N}$ , which is a product of some  $r_{i,j}$  and their adjoints, also normalizes  $\hat{A}_p$ . For any  $z$  in the semigroup  $\mathcal{R}_{n,m}$  introduced in (3.14.c), we have that  $\Lambda(z)$  is such a product, so it follows that

$$\Lambda(z) \hat{A}_p \Lambda(z)^* \subseteq \hat{A}_p,$$

and, in particular, that

$$\Lambda(z) \Lambda(z)^* = \Lambda(z z^*) \in \hat{A}_p.$$

Consequently, the idempotent semi-lattice of  $\mathcal{R}_{n,m}$  is mapped into  $\hat{A}_p$  by  $\Lambda$ . By (3.15.ii), we then conclude that

$$(4.10) \quad \Lambda(A_p) \subseteq \hat{A}_p.$$

By (3.13), we have that  $\Psi(\hat{\pi}(a)) = a$ , for all  $a \in A_p$ , and hence  $\Psi(\hat{A}_p) \subseteq A_p$ . Suitably restricted, we may therefore view  $\Psi$  and  $\Lambda$  as maps

$$\Lambda|_{A_p} : A_p \rightarrow \hat{A}_p, \quad \text{and} \quad \Psi|_{\hat{A}_p} : \hat{A}_p \rightarrow A_p.$$

By (4.9) it is clear that

$$(4.11) \quad \Psi|_{A_p} \circ \Lambda|_{A_p} = id_{A_p}.$$

On the other hand, again by (3.13), we have that

$$(4.12) \quad \Psi|_{A_p} \circ \hat{\pi} = id_{A_p}.$$

Viewing  $\hat{\pi}$  as a map

$$\hat{\pi} : A_p \rightarrow \hat{A}_p,$$

notice that (4.12) implies that  $\hat{\pi}$  is injective, but since it is also clearly surjective, we deduce that  $\hat{\pi}$  is an isomorphism from  $A_p$  onto  $\hat{A}_p$ . Once more employing (4.12), we conclude that  $\Psi|_{\hat{A}_p}$  is the inverse of  $\hat{\pi}$ , and hence it is also an isomorphism. Using (4.11) then implies that  $\Lambda|_{A_p}$  is an isomorphism as well and, in particular, that (4.10) is in fact an equality of sets.

This said we therefore see that  $\hat{A}_p$  is contained in the range of  $\Lambda$ . Since  $A_p \rtimes_{\mathcal{V}, \mathcal{H}} \mathbb{N}$  is generated by  $\hat{A}_p$  and  $\hat{s}$ , in order to prove our stated goal that  $\Lambda$  is surjective, it now suffices to check that  $\hat{s}$  lies in the range of  $\Lambda$ . But this follows easily from the fact that  $p$  is the unit of  $A_p$  and hence that

$$\begin{aligned} \hat{s} &= \hat{\pi}(p) \hat{s} \hat{\pi}(p) = \sum_{i=1}^n \sum_{j=1}^m \hat{p}_i \hat{s} \hat{q}_j = \frac{1}{\sqrt{nm}} \sum_{i=1}^n \sum_{j=1}^m r_{i,j} \\ &= \frac{1}{\sqrt{nm}} \sum_{i=1}^n \sum_{j=1}^m \Lambda(s_i t_j^*) \in \Lambda(\mathcal{O}_{n,m}^p). \end{aligned}$$

□

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