

# A double commutant theorem for the corona algebra of a Razak algebra

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ABSTRACT. In this short note, we prove a number of generalizations of the Voiculescu Double Commutant Theorem, in the case where the canonical ideal is stably finite.

Among other things, we have the following result:

Say that  $\mathcal{B}$  is a stable Razak algebra, and say that  $\mathcal{A} \subseteq \mathcal{M}(\mathcal{B})/\mathcal{B}$  is a separable simple nuclear unital  $C^*$ -subalgebra. Then  $\mathcal{A}'' = \mathcal{A}$ .

## CONTENTS

1. Introduction	157
2. Main results	158
References	164

## 1. Introduction

A basic result in von Neumann algebra theory is that if  $\mathcal{A}$  is a unital  $C^*$ -subalgebra of  $\mathbb{B}(\mathcal{H})$ , then the double commutant of  $\mathcal{A}$  is equal to the weak operator closure of  $\mathcal{A}$  and is equal to the strong operator closure of  $\mathcal{A}$ . In Voiculescu’s groundbreaking work on the noncommutative Weyl–von Neumann theorem, he proved an interesting analogue for the Calkin algebra ([13], [1]). Specifically, Voiculescu showed that if  $\mathcal{A} \subseteq \mathbb{B}(l_2)/\mathcal{K}$  is a separable unital  $C^*$ -subalgebra, then the relative double commutant  $\mathcal{A}'' = \mathcal{A}$ .

It is natural to search for generalizations of such a result (and indeed, Pedersen raised the question about generalizations in [11]). Some generalizations have been made by Kucerovsky and Elliott–Kucerovsky for singly-generated hereditary  $C^*$ -subalgebras ([7], [4]) and Farah for certain ultraproducts of  $C^*$ -algebras ([5]).

Perhaps one reason for the smoothness of the proof of the original Voiculescu Double Commutant Theorem is that this is a context with the “nicest possible extension theory”. Among other things, we have the BDF–Voiculescu Theorem for the Calkin algebra which, roughly speaking, says that all essential extensions are absorbing (see, e.g., [13], [6]). Based on

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this idea, we have generalized Voiculescu's Double Commutant Theorem to the case of the corona algebra of a simple stable purely infinite  $C^*$ -algebra, which is the only other context which has this "nicest possible extension theory" ([6]), thus naturally completing this beautiful circle of ideas from the theory of absorbing extensions.

It becomes of interest to search for double commutant theorems for the corona algebras of stable nonelementary stably finite  $C^*$ -algebras. The proofs are necessarily not as elegant (since the extension theory is not as nice and well-understood). In this short note, we prove some interesting results in this direction, making some progress in certain cases where the extension theory and the KK-theory can be controlled.

For basic references to the relevant extension theory and KK theory, we refer the reader to [2], [3], [8], and [9].

## 2. Main results

We begin with some notation. For any elements  $a, b$  of a  $C^*$ -algebra, and for every  $\epsilon > 0$ , " $a \approx_\epsilon b$ " means that  $a$  is norm within  $\epsilon$  of  $b$ , i.e.,  $\|a - b\| < \epsilon$ , where  $\|\cdot\|$  is the  $C^*$ -norm.

Let  $\mathcal{D}$  be a  $C^*$ -algebra and  $\mathcal{J} \subseteq \mathcal{D}$  an ideal. Let  $S \subseteq \mathcal{D}$ . Recall that an approximate unit  $\{e_\alpha\}$  for  $\mathcal{J}$  is said to *quasicentralize*  $S$  if for all  $x \in S$ ,  $\|e_\alpha x - x e_\alpha\| \rightarrow 0$ .

**Lemma 1.** *Let  $\mathcal{B}$  be a  $\sigma$ -unital nonunital simple  $C^*$ -algebra and  $A \in \mathcal{M}(\mathcal{B})_+$  be an element such that  $0 \in \text{sp}(\pi(A))$ . Suppose that  $\{e_k\}$  is an approximate unit for  $\mathcal{B}$  which quasicentralizes  $A$  and such that*

$$e_{k+1}e_k = e_k$$

for all  $k$ .

Then for all  $n \geq 1$ , for every  $\epsilon > 0$ , there exists an  $a \in \mathcal{B}_+ - \{0\}$  and  $m > n$  with

$$ae_n = 0 \text{ and } e_m a = a$$

such that for all  $b \in \text{her}(a)_+$  with  $\|b\| = 1$ ,

$$bA \approx_\epsilon 0.$$

**Proof.** We may assume that  $\|A\| = 1$ .

Since  $0 \in \text{sp}(\pi(A))$ , there exists  $B \in \mathcal{M}(\mathcal{B})_+ - \{0\}$  with  $\|B\| = \|\pi(B)\| = 1$  and  $\pi(B)\pi(A) \approx_{\epsilon/100} 0$ . Hence,

$$\limsup_{k \rightarrow \infty} \|(1 - e_k)BA(1 - e_k)\| < \epsilon/100.$$

Hence, since  $\{e_k\}$  quasicentralizes  $A$ , we must have that

$$\limsup_{k \rightarrow \infty} \|(1 - e_k)B(1 - e_k)A\| < \epsilon/100.$$

Choose  $n' > n + 1$  for which

$$\|(1 - e_{n'})B(1 - e_{n'})A\| < \epsilon/100.$$

Since  $\|(1 - e_{n'})B(1 - e_{n'})\| = 1$ , we can find  $a' \in (1 - e_{n'})\mathcal{B}_+(1 - e_{n'}) - \{0\}$  with  $\|a'\| = 1$  such that

$$(1 - e_{n'})B(1 - e_{n'})a' \approx_{\epsilon/100} a'.$$

Choose  $m' > n' + 1$  big enough so that

$$a'' =_{df} e_{m'}a'e_{m'} \approx_{\epsilon/100} a'.$$

Note that  $\|a''\| > 1 - \epsilon/100$ .

Hence,

$$(1 - e_{n'})B(1 - e_{n'})a'' \approx_{3\epsilon/100} a''.$$

By the continuous functional calculus, we can find  $a \in \text{her}(a'')_+ - \{0\}$  such that for all  $b \in \text{her}(a)_+$  with  $\|b\| = 1$ ,

$$a''b \approx_{\epsilon/100} b.$$

Therefore, for all  $b \in \text{her}(a)_+$  with  $\|b\| = 1$ ,

$$\begin{aligned} (1 - e_{n'})B(1 - e_{n'})b &\approx_{\epsilon/100} (1 - e_{n'})B(1 - e_{n'})a''b \\ &\approx_{3\epsilon/100} a''b \\ &\approx_{\epsilon/100} b. \end{aligned}$$

Hence,

$$(1 - e_{n'})B(1 - e_{n'})b \approx_{5\epsilon/100} b.$$

Hence, for all  $b \in \text{her}(a)_+$  with  $\|b\| = 1$ ,

$$\|Ab\| \approx_{5\epsilon/100} \|A(1 - e_{n'})B(1 - e_{n'})b\| \approx_{\epsilon/100} 0.$$

So

$$\|Ab\| \approx_{\epsilon/10} 0.$$

If we define  $m =_{df} m' + 1$  then we are done.  $\square$

**Lemma 2.** *Let  $\mathcal{B}$  be a nonunital  $\sigma$ -unital simple  $C^*$ -algebra and*

$$A \in \mathcal{M}(\mathcal{B})_+ - \mathcal{B}.$$

*Say that  $\{e_k\}$  is an approximate unit for  $\mathcal{B}$  which quasentralizes  $A$  and such that*

$$e_{k+1}e_k = e_k$$

*for all  $k$ .*

*Then for all  $n \geq 1$ , for every  $\epsilon > 0$ , there exists  $m > n$  and  $a \in \mathcal{B}_+ - \{0\}$  with*

$$ae_n = 0 \text{ and } e_ma = a$$

*such that for all  $b \in \text{her}(a)_+$  with  $\|b\| = 1$ ,*

$$bA \approx_{\epsilon} \|\pi(A)\|b.$$

**Proof.** We may assume that  $\|\pi(A)\| = 1$ . We may also assume that  $\epsilon < 1$ .

Replacing  $A$  with  $(1 - e_l)A(1 - e_l)$  for large enough  $l$  if necessary, we may assume that  $\|A\| < 1 + \epsilon/100 < 2$ .

We can find  $B \in \mathcal{M}(\mathcal{B})_+$  with  $\|B\| = \|\pi(B)\| = 1$  such that

$$\pi(B)\pi(A) \approx_{\epsilon/100} \pi(B).$$

So

$$\limsup_{k \rightarrow \infty} \|(1 - e_k)BA(1 - e_k) - (1 - e_k)B(1 - e_k)\| < \epsilon/100.$$

So since  $\{e_k\}$  quasentralizes  $A$ ,

$$\limsup_{k \rightarrow \infty} \|(1 - e_k)B(1 - e_k)A - (1 - e_k)B(1 - e_k)\| < \epsilon/100.$$

Choose  $n' > n + 1$  so that

$$(1 - e_{n'})B(1 - e_{n'})A \approx_{\epsilon/100} (1 - e_{n'})B(1 - e_{n'}).$$

Since  $\|(1 - e_{n'})B(1 - e_{n'})\| = 1$ , find  $a' \in (1 - e_{n'})\mathcal{A}_+(1 - e_{n'})$  with  $\|a'\| = 1$  so that

$$a'(1 - e_{n'})B(1 - e_{n'}) \approx_{\epsilon/100} a'.$$

Find  $m' > n' + 1$  so that

$$a'' =_{df} e_{m'}a'e_{m'} \approx_{\epsilon/100} a'.$$

Note that this implies that  $\|a''\| > 1 - \epsilon/100$ . So

$$a''(1 - e_{n'})B(1 - e_{n'}) \approx_{3\epsilon/100} a''.$$

We can find  $a \in \text{her}(a'')_+ - \{0\}$  such that for all  $b \in \text{her}(a)_+$  with  $\|b\| = 1$ ,

$$ba'' \approx_{\epsilon/100} b.$$

Therefore, for all  $b \in \text{her}(a)_+$  with  $\|b\| = 1$ ,

$$\begin{aligned} bA &\approx_{\epsilon/50} ba''A \\ &\approx_{3\epsilon/50} ba''(1 - e_{n'})B(1 - e_{n'})A \\ &\approx_{\epsilon/100} ba''(1 - e_{n'})B(1 - e_{n'}) \\ &\approx_{3\epsilon/100} ba'' \\ &\approx_{\epsilon/100} b. \end{aligned}$$

If we choose  $m =_{df} m' + 1$  then we would be done.  $\square$

**Lemma 3.** *Let  $\mathcal{B}$  be a nonunital simple  $\sigma$ -unital  $C^*$ -algebra.*

*Then the centre of  $\mathcal{M}(\mathcal{B})/\mathcal{B}$  is  $\mathbb{C}1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}$ .*

**Proof.** Say that  $A \in \mathcal{M}(\mathcal{B})_+ - \mathcal{B}$  is such that  $\pi(A)$  is an element of the centre of  $\mathcal{M}(\mathcal{B})/\mathcal{B}$ . We may assume that  $\|A\| \leq 1$ .

Suppose, to the contrary, that  $\pi(A)$  is not scalar, i.e., suppose that  $\pi(A) \notin \mathbb{C}1_{\pi(\mathcal{M}(\mathcal{B}))}$ .

By replacing  $A$  with  $(A - \delta 1)_+$  for appropriate  $\delta > 0$  if necessary, we may assume that  $0 \in \text{sp}(\pi(A))$ .

Choose  $\epsilon < \|\pi(A)\|/100$ .

Let  $\{e_n\}$  be an approximate unit for  $\mathcal{B}$  which quasicalentralizes  $A$  and such that  $e_{n+1}e_n = e_n$  for all  $n$ . By Lemmas 1 and 2, let  $\{m_k\}, \{m'_k\}, \{n_k\}, \{n'_k\}$  be subsequences of  $\mathbb{Z}_+$  (positive integers), and let  $\{a_k\}$  and  $\{b_k\}$  be sequences in  $\mathcal{B}_+$  such that the following statements hold:

- (1)  $m_k + 2 < m'_k < m'_k + 2 < m_{k+1}$  for all  $k$ .
- (2)  $n_k + 2 < n'_k < n'_k + 2 < n_{k+1}$  for all  $k$ .
- (3)  $\|a_k\| = \|b_k\| = 1$  for all  $k$ .
- (4)  $a_k \in \text{her}(e_{m'_k} - e_{m_k})$  for all  $k$ .
- (5)  $b_k \in \text{her}(e_{n'_k} - e_{n_k})$  for all  $k$ .
- (6) For all  $k$ , for all  $c \in \text{her}(a_k)_+$  with  $\|c\| = 1$ ,  $\|cA\| < \epsilon/10^{k+1}$ .
- (7) For all  $k$ , for all  $d \in \text{her}(b_k)_+$  with  $\|d\| = 1$ ,

$$\|dA\| > \|\pi(A)\| - \epsilon/10^{k+1}.$$

Since  $\mathcal{B}$  is simple, for all  $k$ , let  $x_k \in \mathcal{B}$  with  $\|x_k\| = 1$  be such that  $x_k^*x_k \in \text{her}(a_k)$  and  $x_kx_k^* \in \text{her}(b_k)$ . Then  $X =_{df} \sum_{k=1}^{\infty} x_k$  converges strictly to an element of  $\mathcal{M}(\mathcal{B})$ . Moreover,

$$\|\pi(A)\pi(X) - \pi(X)\pi(A)\| \geq \|\pi(AX)\| - \|\pi(XA)\| > \|\pi(A)\| - 2\epsilon.$$

This contradicts that  $\pi(A)$  is an element of the centre of  $\mathcal{M}(\mathcal{B})/\mathcal{B}$ . Since  $A$  was arbitrary and since the centre of  $\mathcal{M}(\mathcal{B})/\mathcal{B}$  is the linear span of its positive elements, we have that every element of the centre of  $\mathcal{M}(\mathcal{B})/\mathcal{B}$  is a scalar.  $\square$

**Lemma 4.** *Let  $\mathcal{B}$  be a  $C^*$ -algebra. Then there exists no sequence  $\{a_n\}$  of norm one elements in  $\mathcal{B} \otimes \mathcal{K} \otimes \mathcal{K}$  such that for all  $a \in \mathcal{M}(\mathcal{B} \otimes \mathcal{K}) \otimes 1_{\mathcal{M}(\mathcal{K})}$ ,*

$$\|aa_n - a_na\| \rightarrow 0$$

as  $n \rightarrow \infty$ .

**Proof.** The proof is exactly the same as that of [10] Lemma 2.1 (where we additionally assumed that  $\mathcal{B}$  was unital). The main change is to replace every occurrence of  $1_{\mathcal{B}}$  with  $1_{\mathcal{M}(\mathcal{B})}$ , and all the arguments will work verbatim.  $\square$

We next fix some notational conventions. Let  $\mathcal{B}$  be a  $C^*$ -algebra. Let  $\{e_{j,k}\}$  be a system of matrix units for  $\mathcal{K}$ . Since no confusion will occur, for all  $j, k$ , we often let  $e_{j,k}$  denote both the element in  $\mathcal{K}$  and  $1_{\mathcal{M}(\mathcal{B} \otimes \mathcal{K})} \otimes e_{j,k}$ . For all  $c \in \mathcal{M}(\mathcal{B} \otimes \mathcal{K} \otimes \mathcal{K})$ , for all  $j, k$ , let  $c_{j,k} =_{df} e_{j,j}ce_{k,k}$ .

**Lemma 5.** *Let  $\mathcal{B}$  be a simple  $\sigma$ -unital  $C^*$ -algebra. Let  $c \in \mathcal{M}(\mathcal{B} \otimes \mathcal{K} \otimes \mathcal{K})$  such that  $\pi(c)$  commutes with every element of  $\pi(\mathcal{M}(\mathcal{B} \otimes \mathcal{K}) \otimes 1_{\mathcal{M}(\mathcal{K})})$ . Then for all  $j, k$ ,*

$$c_{j,k} \in \mathbb{C}1_{\mathcal{M}(\mathcal{B} \otimes \mathcal{K})} \otimes e_{j,k} + \mathcal{B} \otimes \mathcal{K} \otimes e_{j,k}.$$

**Proof.** The proof is exactly the same as that of [10] Lemma 2.3, except that every occurrence of [10] Lemma 2.2 is replaced with (our present paper) Lemma 3.  $\square$

**Lemma 6.** *Let  $\mathcal{B}$  be a simple  $\sigma$ -unital  $C^*$ -algebra and  $c \in \mathcal{M}(\mathcal{B} \otimes \mathcal{K} \otimes \mathcal{K})$  such that  $\pi(c)$  commutes with every element of  $\pi(\mathcal{M}(\mathcal{B} \otimes \mathcal{K}) \otimes 1_{\mathcal{M}(\mathcal{K})})$ .*

*So by Lemma 5, for all  $j, k$ ,*

$$c_{j,k} = \alpha_{j,k} 1_{\mathcal{M}(\mathcal{B} \otimes \mathcal{K})} \otimes e_{j,k} + f_{j,k} \otimes e_{j,k}$$

where  $\alpha_{j,k} \in \mathbb{C}$  and  $f_{j,k} \in \mathcal{B} \otimes \mathcal{K}$ .

Then

$$g =_{df} \sum_{1 \leq j, k < \infty} \alpha_{j,k} 1_{\mathcal{M}(\mathcal{B} \otimes \mathcal{K})} \otimes e_{j,k} \in 1_{\mathcal{M}(\mathcal{B} \otimes \mathcal{K})} \otimes \mathbb{B}(l_2).$$

*In particular, the infinite sum, viewed as being the limit of the net of all finite sums, converges strictly.*

**Proof.** The proof is exactly the same as that of [10] Lemma 2.4, except that [10] Lemma 2.3 is replaced with (present paper) Lemma 5.  $\square$

**Lemma 7.** *Let  $\mathcal{B}$  be a simple  $\sigma$ -unital  $C^*$ -algebra and let  $c \in \mathcal{M}(\mathcal{B} \otimes \mathcal{K} \otimes \mathcal{K})$  be such that  $\pi(c)$  commutes with every element of  $\pi(\mathcal{M}(\mathcal{B} \otimes \mathcal{K}) \otimes 1_{\mathcal{M}(\mathcal{K})})$ .*

*Hence, by Lemma 5, for all  $j, k$ ,*

$$c_{j,k} = \alpha_{j,k} 1_{\mathcal{M}(\mathcal{B} \otimes \mathcal{K})} \otimes e_{j,k} + f_{j,k} \otimes e_{j,k}$$

where  $\alpha_{j,k} \in \mathbb{C}$  and  $f_{j,k} \in \mathcal{B} \otimes \mathcal{K}$ .

Then

$$\sum_{1 \leq j, k < \infty} f_{j,k} \otimes e_{j,k} \in \mathcal{B} \otimes \mathcal{K} \otimes \mathcal{K}.$$

*In particular, the above sum, as a limit of the net of finite sums, converges in norm.*

**Proof.** The proof is exactly the same as that of [10] Lemma 2.5, except that all occurrences of [10] Lemmas 2.1, 2.3 and 2.4 are replaced by (present paper) Lemmas 4, 5 and 6 respectively.  $\square$

**Lemma 8.** *Let  $\mathcal{B}$  be a simple  $\sigma$ -unital  $C^*$ -algebra.*

Then

$$\pi(\mathcal{M}(\mathcal{B} \otimes \mathcal{K}) \otimes 1_{\mathcal{M}(\mathcal{K})})' \subseteq \pi(1_{\mathcal{M}(\mathcal{B} \otimes \mathcal{K})} \otimes \mathbb{B}(l_2)).$$

**Proof.** This follows immediately from Lemmas 6 and 7.  $\square$

**Definition 1.** Let  $\mathcal{B}$  be a  $\sigma$ -unital simple  $C^*$ -algebra and let  $\mathcal{A}$  be a unital separable  $C^*$ -algebra.

Let  $\mathcal{T}$  denote the collection of all  $\alpha \in KK^1(\mathcal{A}, \mathcal{B})$  for which there exists a unital essential extension  $\tau : \mathcal{A} \rightarrow \pi(1_{\mathcal{M}(\mathcal{B})} \otimes \mathcal{M}(\mathcal{K})) \subseteq \mathcal{M}(\mathcal{B} \otimes \mathcal{K}) / (\mathcal{B} \otimes \mathcal{K})$  such that  $\alpha = [\tau]$ .

For the convenience of the reader, we briefly review some aspects of extension theory and KK-theory. We refer the reader to the references mentioned at the end of the Introduction for more information. Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $C^*$ -algebras with  $\mathcal{B}$  stable. Recall that an extension  $\phi : \mathcal{A} \rightarrow \mathcal{M}(\mathcal{B})/\mathcal{B}$  is *absorbing* if for every trivial extension  $\psi : \mathcal{A} \rightarrow \mathcal{M}(\mathcal{B})/\mathcal{B}$ ,  $\phi \oplus \psi$  is unitary equivalent to  $\phi$  where the sum is the BDF sum and the unitary comes from  $\mathcal{M}(\mathcal{B})$ . Recall that for separable nuclear  $\mathcal{A}$  and  $\sigma$ -unital stable  $\mathcal{B}$ ,  $KK^1(\mathcal{A}, \mathcal{B})$  is the group of unitary equivalence classes of extensions  $\phi : \mathcal{A} \rightarrow \mathcal{M}(\mathcal{B})/\mathcal{B}$  modulo the trivial extensions, where the sum is the BDF sum.  $KK^1(\mathcal{A}, \mathcal{B})$  can also be realized as the group of unitary equivalence classes of absorbing extensions. (E.g., see [2] 15.12.2, 15.12.4, and 17.6.5.) Assume that  $\mathcal{B}$  is separable and stable. Then  $\mathcal{B}$  has the *corona factorization property* means that for every unital separable nuclear  $C^*$ -algebra  $\mathcal{A}$ , if  $\phi : \mathcal{A} \rightarrow \mathcal{M}(\mathcal{B})/\mathcal{B}$  is a full extension such that  $1_{\mathcal{M}(\mathcal{B})/\mathcal{B}} - \phi(1)$  is a properly infinite full projection of  $\mathcal{M}(\mathcal{B})/\mathcal{B}$ , then  $\phi$  is absorbing. (See [8]. Recall that  $\phi$  is *full* means that for all  $x \in \mathcal{A} - \{0\}$ ,  $Ideal(\phi(x)) = \mathcal{M}(\mathcal{B})/\mathcal{B}$ .) Many  $C^*$ -algebras have the corona factorization property including all separable simple  $C^*$ -algebras that are purely infinite or have strict comparison of positive elements by traces.

**Theorem 1.** *Let  $\mathcal{B}$  be a separable simple  $C^*$ -algebra for which  $\mathcal{B} \otimes \mathcal{K}$  has the corona factorization property. Suppose that  $\mathcal{A} \subseteq \mathcal{M}(\mathcal{B} \otimes \mathcal{K})/(\mathcal{B} \otimes \mathcal{K})$  is a separable simple nuclear unital  $C^*$ -subalgebra. Suppose, in addition, that the inclusion map  $i$  (of the above inclusion) satisfies that  $[i] \in \mathcal{T}$ .*

*Then  $\mathcal{A}'' = \mathcal{A}$ .*

**Proof.** Since  $\mathcal{B} \otimes \mathcal{K} \cong \mathcal{B} \otimes \mathcal{K} \otimes \mathcal{K}$ , we may work with  $\mathcal{B} \otimes \mathcal{K} \otimes \mathcal{K}$  in place of  $\mathcal{B} \otimes \mathcal{K}$ . Since  $[i] \in \mathcal{T}$  and since  $\mathcal{B} \otimes \mathcal{K} \otimes \mathcal{K}$  has the corona factorization property, there exists a unital essential extension  $\phi : \mathcal{A} \rightarrow \pi(1_{\mathcal{M}(\mathcal{B} \otimes \mathcal{K})} \otimes \mathcal{M}(\mathcal{K}))$  and there exists a unitary  $u \in \mathcal{M}(\mathcal{B} \otimes \mathcal{K} \otimes \mathcal{K})/(\mathcal{B} \otimes \mathcal{K} \otimes \mathcal{K})$  such that for all  $a \in \mathcal{A}$ ,

$$i(a) = u\phi(a)u^*.$$

(Note that the unitary lives in the corona algebra and need not come from a unitary in  $\mathcal{M}(\mathcal{B} \otimes \mathcal{K} \otimes \mathcal{K})$ .)

Hence,

$$\mathcal{A}'' = (u\phi(\mathcal{A})u^*)'' = u\phi(\mathcal{A})''u^* = u\phi(\mathcal{A})u^* = \mathcal{A}.$$

(The third equality comes from Lemma 8 and the original Voiculescu Double Commutant Theorem.) □

Finally, we end this paper by providing (as an illustration) two applications of our theory.

Recall that the Razak algebras are approximately subhomogeneous  $C^*$ -algebras with trivial  $K_0$  and  $K_1$ . They are basic and important examples of simple stably projectionless  $C^*$ -algebras. (See [12].)

**Theorem 2.** *Let  $\mathcal{B}$  be a stable Razak algebra, and suppose that*

$$\mathcal{A} \subseteq \mathcal{M}(\mathcal{B} \otimes \mathcal{K})/(\mathcal{B} \otimes \mathcal{K})$$

*is a separable simple unital  $C^*$ -subalgebra.*

*Then  $\mathcal{A}'' = \mathcal{A}$ .*

**Proof.** This follows immediately from Theorem 1 and from the facts that the Razak algebra is a  $C^*$ -algebra with the corona factorization property and is KK-contractible.  $\square$

**Theorem 3.** *Let  $\mathcal{Z}$  be the Jiang–Su algebra, and suppose that  $\mathcal{A} \subseteq \mathcal{M}(\mathcal{Z} \otimes \mathcal{K})/(\mathcal{Z} \otimes \mathcal{K})$  is a separable simple nuclear unital  $C^*$ -subalgebra.*

*Then  $\mathcal{A}'' = \mathcal{A}$ .*

**Proof.** This follows immediately from Theorem 1 and from the facts that  $\mathcal{Z}$  is a  $C^*$ -algebra with the corona factorization property and is KK-equivalent to the complex numbers  $\mathbb{C}$ .  $\square$

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