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Automorphisms acting on the left-orderings of a bi-orderable group

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Abstract. We generalize a result of Koberda, 2011, by showing that the natural action of the automorphism group on the space of left-orderings is faithful for all nonabelian bi-orderable groups G, as well as for a certain class of left-orderable groups that includes the braid groups and mapping class groups of orientable surfaces with a single boundary component. As a corollary we show that the action of $\mathrm{Aut}(G)$ on ∂G is faithful whenever G is bi-orderable and hyperbolic, following the approach of Koberda. We also analyze the action of the commensurator of G on its space of virtual left-orderings.

Contents

1. Introduction

Let G be a group. We call a strict total ordering $<$ of the elements of G a *left-ordering* if $g < h$ implies $fg < fh$ for all $f, g, h \in G$. If G admits a left-ordering $<$ that is also right-invariant, in the sense that $g < h$ implies $gf < hf$ for all $f, g, h \in G$, then $<$ is a *bi-ordering* of G.

Each of these concepts can equivalently be defined in terms of positive cones. That is, given a left-ordering $<$ of G, we can identify $<$ with its positive cone

$$P = \{g \in G \mid g > 1\}$$

which is a subset of G satisfying:

(1) $P \cdot P \subset P$

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(2) $P \sqcup P^{-1} \sqcup \{1\} = G$.

Conversely, given a subset $P \subset G$ satisfying (1) and (2), it determines a positive cone according to the prescription $g < h$ if and only if $g^{-1}h \in P$ for all $g, h \in G$. Bi-orderings may be similarly defined in terms of positive cones, but the positive cone of any bi-ordering must also satisfy a third condition, namely $gPg^{-1} \subset P$ for all $g \in G$.

We write $\mathrm{LO}(G)$ for the set of all positive cones $P \subset G$ satisfying (1) and (2) above, and, thinking of it as a subset of 2^G (equipped with the product topology) we endow $\mathrm{LO}(G)$ with the subspace topology. Thus the open sets of $\mathrm{LO}(G)$ are finite intersections of sets of the form

$$U_g = \{P \in \mathrm{LO}(G) \mid g \in P\} \text{ and } U_g^c = \{P \in \mathrm{LO}(G) \mid g^{-1} \in P\}.$$

We call $\mathrm{LO}(G)$ the space of left-orderings of the group G. We similarly can define the space of bi-orderings of G, $\mathrm{BiO}(G)$, by taking all positive cones P that satisfy the additional third condition of $gPg^{-1} \subset P$ for all $g \in G$. Topologizing $\mathrm{BiO}(G)$ in the same way, we evidently have $\mathrm{BiO}(G) \subset \mathrm{LO}(G)$. Endowed with these topologies, both $\mathrm{LO}(G)$ and $\mathrm{BiO}(G)$ are compact spaces.

There is an action of G on $\mathrm{LO}(G)$ defined by $g(P) = gPg^{-1}$. More generally, there is an action of $\mathrm{Aut}(G)$ on $\mathrm{LO}(G)$ by observing that $\phi(P)$ is again a positive cone for all $P \in \mathrm{LO}(G)$ and $\phi \in \mathrm{Aut}(G)$. The action of $\mathrm{Aut}(G)$ on $\mathrm{LO}(G)$ is an action by homeomorphisms. Since the positive cones which are fixed under conjugation correspond to the bi-orderings of G, there is also an action of $\mathrm{Out}(G)$ on $\mathrm{BiO}(G)$.

With the topological structure and group actions as above, $\mathrm{LO}(G)$ has found many applications within the study of orderable groups (for example, it was used to show that every left-orderable group has finitely many or uncountably many left-orderings [Lin11], and was used to demonstrate a connection between orderability and amenability [Mor06]), though applications beyond the realm of orderability are few. In recent work Koberda provided an example of such an application, by showing that whenever G is a residually torsion-free nilpotent hyperbolic group, the natural action of $\mathrm{Aut}(G)$ on ∂G is faithful [Kob11]. This application relies on the following theorem, which was also extended in [Mor12] by replacing $\mathrm{Aut}(G)$ with the commensurator of G:

Theorem 1.1 ([Kob11, Theorem 1.1])**.** *If G is a finitely generated residually torsion-free nilpotent group, then the natural action of* $\mathrm{Aut}(G)$ *on* $\mathrm{LO}(G)$ *is faithful.*

In this paper, we characterize the action of $\mathrm{Aut}(G)$ on $\mathrm{LO}(G)$ when G is a bi-orderable group. Recall that finitely-generated residually torsion-free nilpotent groups are bi-orderable, though the converse is not true. For example, Thompson's group F is bi-orderable, but not residually nilpotent since $[F, F]$ is a simple group [DNR14, Section 1.2.4].

Note that for some bi-orderable groups, like $\mathbb{Q}^k$ for all $k > 0$, we should not expect the action of $\mathrm{Aut}(G)$ on $\mathrm{LO}(G)$ to be faithful. For if $G = \mathbb{Q}^k$ then multiplication by a positive rational p/q in each coordinate of $\mathbb{Q}^k$ can easily be seen to preserve all orderings of $\mathbb{Q}^k$. However, it turns out that these automorphisms of abelian groups are the only nontrivial automorphisms of bi-orderable groups which act trivially on the space of left-orderings. For an abelian group G and a fixed $p/q \in \mathbb{Q}$, we denote by $\tau_{p/q} : G \to G$ the automorphism satisfying $\tau_{p/q}(g^q) = g^p$ for all $g \in G$, when it exists. We prove:

Theorem 1.2. *Let G be a bi-orderable group.*

(1) *If G is nonabelian then* $\mathrm{Aut}(G)$ *acts faithfully on* $\mathrm{LO}(G)$*.*
(2) *If G is abelian then the kernel of the action of* $\mathrm{Aut}(G)$ *on* $\mathrm{LO}(G)$ *contains precisely the automorphisms $\tau_{p/q}$, if any such automorphisms exist.*

Note that part (2) of Theorem 1.2 already appears as [Mor12, Proposition 4.3(2)]. We are also able to analyze the behaviour of the action of $\mathrm{Aut}(G)$ on $\mathrm{LO}(G)$ with respect to certain kinds of extensions.

Theorem 1.3. *Suppose that G is left-orderable and that*

$$1 \to K \to G \to \mathbb{Z} \to 1$$

is a short exact sequence of groups. Suppose that $\mathrm{Aut}(K)$ *acts faithfully on* $\mathrm{LO}(K)$*. If conjugation by a generator of $\mathbb{Z}$ preserves a left-ordering of K, then* $\mathrm{Aut}(G)$ *acts faithfully on* $\mathrm{LO}(G)$*.*

Since bi-orderability is not preserved under extensions (even under extensions such as those in the statement of the theorem above), this allows us to create non-bi-orderable groups G for which $\mathrm{Aut}(G)$ acts faithfully on $\mathrm{LO}(G)$. See also Proposition 3.1.

As a corollary of Theorem 1.2 we can extend Koberda's result concerning the action of $\mathrm{Aut}(G)$ on ∂G to all bi-orderable hyperbolic groups.

Corollary 1.4. *If G is a bi-orderable hyperbolic group, then* $\mathrm{Aut}(G)$ *acts faithfully on ∂G.*

The proof of Corollary 1.4 is a combination of Theorem 1.2 and Proposition 4.1.

The paper is organized as follows. In Section 2 we provide additional background on left-orderings and bi-orderings of groups, and prove Theorem 1.2. In Section 3 we prove Theorem 1.3 and also study the braid groups B_n. In Section 4 show that the action of $\mathrm{Aut}(G)$ on ∂G is faithful when G is hyperbolic and bi-orderable, and describe the action of $\mathrm{Comm}(G)$ on $\mathrm{VLO}(G)$ for all bi-orderable groups.

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2. Automorphisms of bi-orderable groups acting on the space of orderings

By insisting that the group G be bi-orderable, we allow ourselves some flexibility in creating new left-orderings of G. The orderings that we will create arise from considering the action of G on itself by conjugation, which is an order-preserving action if G is bi-ordered (see Lemma 2.2). With this line of reasoning we will create sufficiently many left-orderings to show that whenever $\phi \in \mathrm{Aut}(G)$ and $\phi(g) \neq g$ for some $g \in G$, then there exists $P \in \mathrm{LO}(G)$ that contains g but not $\phi(g)$. It follows that the action of ϕ on $\mathrm{LO}(G)$ is nontrivial, because the positive cone P satisfies $\phi(P) \neq P$.

Recall that a subset $S \subset G$ is called *isolated* if $g^k \in S$ for some $k \in \mathbb{Z}$ implies that $g \in S$. The *isolator* of a subgroup H of G is the set

$$I(H) = \{g \in G \mid \text{ there exists } k \in \mathbb{Z} \text{ such that } g^k \in H\}.$$

In general, $I(H)$ is not a subgroup. However, when H is abelian and G is bi-orderable, then $I(H)$ is an abelian subgroup. Essential in proving this fact is the following property of bi-orderable groups: In a bi-orderable group, when g^k and h^ℓ commute for some $k, \ell \in \mathbb{Z}$, then so do g and h. This fact will also be used several times in the proofs of this section.

When H is a rank one abelian subgroup of G, so is $I(H)$. If g is a nonidentity element of a bi-orderable group G, then we will denote the isolator of the cyclic subgroup $\langle g \rangle$ by $I(g)$ for short. Thus $I(g)$ is always a rank one abelian group. We record the following fact for future use:

Lemma 2.1. *Let G be a group. If g, h are distinct elements of G, then either $I(g) = I(h)$ or $I(g) \cap I(h) = \{1\}$.*

Proof. Suppose there exists $f \in I(h) \cap I(g)$ where $f \neq 1$. Since $f \in I(g)$, there exist $n, m \in \mathbb{Z}$ such that $f^n = g^m$. But now $g^n \in I(h)$ and since $I(h)$ is isolated, g is also in $I(h)$ and $I(h) = I(g)$. □

Recall that a subset S in a left-ordered group G is called *convex* with respect to a given left-ordering $<$ if $g, h \in S$ and $g < f < h$ implies $f \in S$. Of particular importance is the case when a subgroup C of a left-ordered group G is convex, as the convex subgroups of a left-ordering determine its structure in a sense described below. The convex subgroups of a left-ordered group G are ordered by inclusion. A subgroup is *relatively convex* if there exists a left-ordering relative to which it is convex.

Given a subgroup C of a left-ordered group G, the natural quotient ordering of the left cosets G/C is well-defined if and only if C is convex, in this case the natural left-action of G/C preserves the quotient ordering. Therefore we can think of the ordering of G as lexicographic: it is constructed via inclusion of the left-ordered subgroup C and via pullback of the natural ordering on the cosets G/C.

Consequently, if C is a convex subgroup of a left-ordered group, then the left-ordering of G may be altered by replacing the left-ordering of C with any left-ordering that we please. It follows that relative convexity is transitive, in the sense that if K is relatively convex in H, and H is relatively convex in G, then K is relatively convex in G. This fact is needed in the proof of the following lemma.

Lemma 2.2 ([Cla12, Lemma 2.4]). *Suppose that G is a bi-orderable group, and that $g \in G$ is not the identity. Then $I(g)$ is relatively convex.*

Proof. Let G_i, $i = 1, 2$ denote two copies of the group G, and equip each copy with a given bi-ordering $<$. Create a total ordering of $G_1 \cup G_2$ using $<$ to order each G_i, and declare the elements of G_1 smaller than those of G_2.

Now consider the action of G on $G_1 \cup G_2$ defined by conjugation on the elements of G_1, and by left-multiplication on the elements of G_2. This defines an effective, order-preserving action of G on the totally ordered set $G_1 \cup G_2$. Fix a nonidentity element $g \in G_1$ and well-order $G_1 \cup G_2$ so that g is smallest. Then using the action of G on $G_1 \cup G_2$ one may create a left-ordering of G in the standard way, relative to which $Stab_G(g) = C_G(g)$ is convex. Here, $C_G(g)$ denotes the centralizer of g in G (See [Cla12, Proposition 2.3] or [CR16, Example 1.11 and Problem 2.16] for details of this construction). Now as $C_G(g)$ is bi-orderable, the centre $Z(C_G(g))$ is relatively convex in $C_G(g)$ by [BMR77, Theorem 2.4]. Moreover, $I(g) \subset Z(C_G(g))$ since every element of $I(g)$ has some power which lies in $\langle g \rangle$, and thus commutes with all elements of $C_G(g)$. Since $I(g)$ is an isolated subgroup and $Z(C_G(g))$ is abelian, $I(g)$ is relatively convex in $Z(C_G(g))$. Thus $I(g)$ is relatively convex in G. □

Proposition 2.3. *Suppose that G is a bi-orderable group, and that $\phi \in \mathrm{Aut}(G)$. If there exists $g \in G$ such that $\phi(I(g)) \neq I(g)$, or if there exists $g \in G$ such that $\phi(g)^n = g^{-m}$ for some $m, n > 0$, then the action of ϕ on $\mathrm{LO}(G)$ is nontrivial.*

Proof. Suppose there exists $g \in G$ such that $\phi(g)^n = g^{-m}$ for some $m, n > 0$. Consider an arbitrary positive cone $P \in LO(G)$. We can assume $g \in P$, if not we replace P by P^{-1}. Then $g \in P$ and $\phi(g) \notin P$, so we have $P \neq \phi(P)$.

Now suppose there exists g such that $I(g) \neq \phi(I(g))$, and note that $\phi(I(g)) = I(\phi(g))$. By Lemma 2.2 $I(g)$ is convex in some left-ordering of G with positive cone $P \in LO(G)$. Applying ϕ, one checks that $\phi(I(g)) = I(\phi(g))$ is convex relative to the ordering of G determined by $\phi(P)$.

To show that $\phi(P) \neq P$, we need only show that $I(g)$ is not convex relative to the ordering of G determined by $\phi(P)$. If it were, we would have either $I(g) \subset I(\phi(g))$ or $I(\phi(g)) \subset I(g)$, since convex subgroups are ordered by inclusion. By Lemma 2.1, either inclusion forces $I(g) = I(\phi(g)) = \phi(I(g))$, a contradiction. □

Therefore, by Proposition 2.3, when G is a bi-orderable group and $\phi \in \mathrm{Aut}(G)$ we know that ϕ acts nontrivially on $\mathrm{LO}(G)$ unless ϕ satisfies:

$$(*) \qquad \forall g \in \mathrm{domain}(\phi)\ \exists n, m > 0 \text{ such that } \phi(g)^n = g^m.$$

We therefore investigate the existence of such automorphisms of bi-orderable groups.

Our lemmas below are stated in a slightly more general setting than needed in this section, as we will also be using them in our investigation of the action of $\mathrm{Comm}(G)$ on $\mathrm{VLO}(G)$ in Section 4.

Recall that when G is abelian, we denote by $\tau_{p/q} : G \to G$ the automorphism satisfying $\tau_{p/q}(g^q) = g^p$ for all $g \in G$, when it exists. More generally, if H_1, H_2 are finite index abelian subgroups of a group G, we denote by $\tau_{p/q} : H_1 \to H_2$ the isomorphism satisfying $\tau_{p/q}(g^q) = g^p$ for all $g \in H_1$, when it exists.

Lemma 2.4. *Suppose G is a bi-orderable group with finite index torsion-free abelian subgroups H_1, H_2, and $\phi : H_1 \to H_2$ is an isomorphism satisfying $(*)$. Then there exist $p, q > 0$ such that $\phi(g)^q = g^p$ for all $g \in H_1$, so that $\phi = \tau_{p/q}$.*

Proof. This lemma is essentially Case 2 of the proof of [Mor12, Proposition 4.3]. Here is an alternative proof. Assume $\phi : H_1 \to H_2$ satisfies $(*)$ and that H_1 is torsion free abelian. Let $g, h \in H_1$ and suppose $\phi(g)^m = g^n$ and $\phi(h)^\ell = h^k$ for some $k, \ell, m, n > 0$. By uniqueness of roots, we may assume that $\gcd(m, n) = \gcd(k, \ell) = 1$, we wish to show that $m = \ell$ and $n = k$. If $I(g) = I(h)$ then the result follows by applying ϕ to a common power of g and h which lies in H_1, such a common power exists since $|G : H_1|$ is finite. So suppose $I(g) \neq I(h)$, and therefore $I(g) \cap I(h) = \{1\}$ by Lemma 2.1.

Considering $g^m h^\ell$, we see that $\phi(g^m h^\ell) = g^n h^k \in I(g^m h^\ell)$, so there exist relatively prime $s, t > 0$ such that $(g^m h^\ell)^s = (g^n h^k)^t$. Since H_1 is abelian $g^{ms-nt} = h^{tk-s\ell}$, and since both are in $I(g) \cap I(h)$, both are equal to 1. Since $\gcd(m, n) = \gcd(s, t) = 1$, from $ms - nt = 0$ we find $m = t$ and $s = n$. Similarly from $tk - s\ell$ we find $t = \ell$ and $k = s$, so we are done. □

Lemma 2.5. *Suppose G is a bi-orderable group with finite index subgroups H_1, H_2, that $\phi : H_1 \to H_2$ is an isomorphism satisfying $(*)$, and that ϕ is not the identity. Then for every $g \in H_1$ there exist $p, q > 0$ such that $\phi(g)^q = g^p$ where $p \neq q$.*

Proof. Since ϕ is not the identity there exists $g \in H_1$ with $\phi(g) \neq g$, say $\phi(g)^s = g^t$ with $s \neq t$ (necessarily $s \neq t$ since G is bi-orderable). Now let $h \in G$ be given. By $(*)$ there exists $n, m > 0$ such that $\phi(h)^n = h^m$. If $n = m$ then $\phi(h) = h$ since G is bi-orderable. But $\phi(g^s h) = g^t h$, so $g^t h \in I(g^s h)$. But then $(g^t h)(h^{-1} g^{-s}) = g^{t-s} \in I(g^s h)$. Therefore $g \in I(g^s h)$, and so $h \in I(g^s h)$, and $I(g) = I(h)$. Now since $I(g)$ is abelian we may apply Lemma 2.4 to the restriction isomorphism $\phi|_{I(g)} : I(g) \to I(g)$ arising from ϕ. We conclude that $n = s$ and $m = t$, contradicting the fact that $n = m$. Thus $n \neq m$. □

Note that we can improve the conclusion of the previous lemma, by using uniqueness of roots in a bi-orderable group to show that p, q exist with $\gcd(p, q) = 1$. However this is not needed for our purposes.

Lemma 2.6. *Suppose G is a bi-orderable group with finite index subgroups H_1, H_2 and that $\phi : H_1 \to H_2$ is an isomorphism satisfying $(*)$. Let $g, h \in H_1$ be given and suppose that $\phi(g)^m = g^n$ and $\phi(h)^\ell = h^k$. Then*

$$g^{n-m}hg^{m-n} \in I(h) \text{ and } h^{k-\ell}gh^{\ell-k} \in I(g).$$

Proof. By symmetry, it suffices to show only $g^{n-m}hg^{m-n} \in I(h)$. First, notice that $\phi(f) \in I(f)$ for all $f \in H_1$ by $(*)$. Therefore

$$\phi(g^m h^\ell g^{-m}) = g^n h^k g^{-n} \in I(g^m h g^{-m}),$$

and since $I(g^m h g^{-m})$ is isolated we conclude $g^n h g^{-n} \in I(g^m h g^{-m})$. Next, notice that if $x \in I(h)$ then $g^i x g^{-i} \in I(g^i h g^{-i})$ for all $i \in \mathbb{Z}$, and thus $g^{n-m}hg^{m-n} \in I(h)$. □

Lemma 2.7. *Suppose G is a bi-orderable group with finite index subgroups H_1, H_2, that $\phi : H_1 \to H_2$ is an isomorphism satisfying $(*)$, and that ϕ is not the identity. Then H_1 is abelian.*

Proof. Let $g, h \in H_1$ be given. If $I(g) = I(h)$ then g and h commute. Thus we assume $I(g) \neq I(h)$. By Lemma 2.5 there exist $m, n > 0$ and $k, \ell > 0$ with $m \neq n$ and $k \neq \ell$ such that $\phi(g)^m = g^n$ and $\phi(h)^\ell = h^k$. Consider $h^{k-\ell}g^{n-m}h^{\ell-k}g^{m-n}$. On one hand, we have

$$h^{k-\ell}g^{n-m}h^{\ell-k}g^{m-n} = (h^{k-\ell}g^{n-m}h^{\ell-k}) \cdot g^{m-n} \in I(g),$$

since it is a product of elements of $I(g)$ (here we use Lemma 2.6). On the other hand, $h^{k-\ell} \cdot (g^{n-m}h^{\ell-k}g^{m-n}) \in I(h)$ by similar reasoning. By Lemma 2.1 $I(g) \cap I(h) = \{1\}$ and so $h^{k-\ell}g^{n-m}h^{\ell-k}g^{m-n} = 1$. But this means the nontrivial powers $h^{k-\ell}$ and g^{n-m} commute, so h and g commute since G is bi-orderable. Thus H_1 is abelian. □

Proof of Theorem 1.2. Let G be a bi-orderable group and let $\phi \in \mathrm{Aut}(G)$ be nontrivial. If G is nonabelian, then by Lemma 2.7 ϕ cannot satisfy $(*)$. By Proposition 2.3 ϕ acts nontrivially on $\mathrm{LO}(G)$, so the action of $\mathrm{Aut}(G)$ on $\mathrm{LO}(G)$ is faithful.

If G is abelian, and if ϕ does not satisfy $(*)$, then Proposition 2.3 tells us that ϕ acts nontrivially on $\mathrm{LO}(G)$. If ϕ does satisfy $(*)$, then Lemma 2.4 tells us that $\phi = \tau_{p/q}$ for some $p/q \in \mathbb{Q}$. It is easy to see that in this case, ϕ acts trivially on $\mathrm{LO}(G)$. Thus the kernel of the action of $\mathrm{Aut}(G)$ on $\mathrm{LO}(G)$ consists exactly of the automorphisms $\tau_{p/q}$. □

3. Non-bi-orderable groups

For certain classes of left-orderable groups, it is sometimes sufficient to examine the action of $\mathrm{Aut}(G)$ on a small subset of $\mathrm{LO}(G)$ (perhaps even a finite subset) in order to determine that the action is faithful.

Recall that a left-ordering of G is *discrete* if there is a smallest positive element. If $\phi : G \to G$ is an automorphism, and if P is the positive cone of a discrete left-ordering with smallest positive element $g \in G$, then $\phi(P)$ is the positive cone of a discrete left-ordering whose smallest positive element is $\phi(g)$. Thus if $g \neq \phi(g)$, then $P \neq \phi(P)$. We apply this idea in the following proposition.

Proposition 3.1. *Suppose that G is a left-orderable group with generators $\{g_i\}_{i \in I}$, and that for each $i \in I$ there exists $P_i \in \mathrm{LO}(G)$ which is the positive cone of a discrete left-ordering with g_i as smallest positive element. Then $\mathrm{Aut}(G)$ acts faithfully on $\mathrm{LO}(G)$.*

Proof. If $\phi : G \to G$ is a nontrivial automorphism, then there exists a generator g_i such that $\phi(g_i) \neq g_i$. But then $\phi(P_i) \neq P_i$, so that ϕ acts nontrivially on $\mathrm{LO}(G)$. □

Example 3.2. Recall the Artin presentation of braid group B_n is given by

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ if } |i-j| = 1 \end{array} \right\rangle.$$

By Dehornoy, the braid groups B_n are left orderable for all n, as is the braid group B_∞ [Deh94]. The *Dehornoy ordering* of B_n is a left-ordering that is defined in terms of representative words of braids as follows: A word w in the generators $\sigma_1, \ldots, \sigma_{n-1}$ is called i-positive (respectively i-negative) if w contains at least one occurence σ_i, no occurence of $\sigma_1, \ldots, \sigma_{i-1}$, and every occurence of σ_i has positive (respectively negative) exponent. A braid $\beta \in B_n$ is called i-positive (respectively i-negative) if it admits a representative word w in the generators $\sigma_1, \ldots, \sigma_{n-1}$ that is i-positive (respectively i-negative). The Dehornoy ordering of the braid group B_n is the ordering whose positive cone P_D is the set of all braids $\beta \in B_n$ that are i-positive for some i. Using $sh^{n-j} : B_j \to B_n$ to denote the shift homomorphism sending σ_i to σ_{i+j}, the convex subgroups of B_n are $sh^{n-j}(B_j) = \langle \sigma_{n-j+1}, \ldots, \sigma_{n-1} \rangle \subset B_n$ [DDRW08], in particular the Dehornoy ordering is discrete with smallest positive element σ_{n-1}.

We can also define a related left-ordering as follows: a word w in generators $\sigma_1, \ldots, \sigma_{n-1}$ is called *i-reverse positive*, if it has no occurence of $\sigma_{i+1}, \ldots, \sigma_{n-1}$, and every occurence of σ_i has positive exponent. Now similar to Dehornoy ordering, define an ordering $<'_D$ on B_n, whose positive cone P'_D is consists of all braids $\beta \in B_n$ that are i-reverse positive for some i.

It is straightforward to check that $<'_D$ is a also a discrete ordering of B_n, with σ_1 as its least positive element. Moreover, the convex subgroups of B_n with respect to $<'_D$ are exactly the subgroups $B_j = \langle \sigma_1, \ldots, \sigma_{j-1} \rangle \subset B_n$ for $1 \leq j \leq n$.

Now given any i where $1 \leq i \leq n-1$, we can construct a left ordering $<_i$ on B_n with σ_i as its least positive element. First, we left-order B_n with $<'_D$. Since B_{i+1} is convex with respect to P'_D, we can replace the left ordering

$<'_D$ on B_{i+1} with the left ordering of $<_D$. Denote the resulting ordering of B_n by $<_i$. By construction, $<_i$ is a discrete ordering with σ_i as its least positive element. Based on this construction and Proposition 3.1, $\mathrm{Aut}(B_n)$ acts faithfully on $\mathrm{LO}(B_n)$.

This same construction can also be used to produce a left-ordering of B_∞ with σ_i as smallest positive element for all $i \geq 1$. Thus $\mathrm{Aut}(B_\infty)$ acts faithfully on $\mathrm{LO}(B_\infty)$ as well. □

More generally, it is possible to make an analysis of some mapping class groups using similar techniques.

Example 3.3. First, we sketch how to construct left-orderings of mapping class groups of punctured surfaces with boundary, following [RW00]. Fix a compact surface S with nonempty boundary and denote the set of punctures by $\mathcal{P}$. Define an *ideal arc* to be the image of a map

$$(I, \partial I, int(I)) \to (S, \partial S \cup \mathcal{P}, S \setminus (\partial S \cup \mathcal{P}))$$

which is injective on $int(I)$, here I is the unit interval. Two ideal arcs are isotopic if there is an isotopy deforming one into the other, fixing $\mathcal{P}$ and ∂S. A *curve diagram* is a collection Γ of nonisotopic ideal arcs satisfying: All ideal arcs are embedded and disjoint, the endpoints of the arcs lie in ∂S, and $S \setminus \Gamma$ is a disk (possibly with a single puncture).

Having fixed a curve diagram Γ, we choose an enumeration $\gamma_1, \dots, \gamma_n$ of the ideal arcs of the curve diagram, and orient the curves $\{\gamma_i\}_{i=1}^n$ and the boundary components of S however we please. Now given $\phi, \psi \in \mathrm{Mod}(S)$, define $\phi < \psi$ whenever the following situation occurs: Suppose that the curve diagrams $\phi(\Gamma)$ and $\psi(\Gamma)$ are reduced with respect to one another (see [RW00] for a description of reduced curve diagrams). Suppose that $\phi(\gamma_i)$ and $\psi(\gamma_i)$ coincide for $i = 1, \dots, k-1$, but that $\phi(\gamma_k)$ and $\psi(\gamma_k)$ do not coincide for some $k \leq n$. Declare $\phi < \psi$ if $\phi(\gamma_k)$ first branches off $\psi(\gamma_k)$ to the left, where "to the left" means in the direction of the chosen orientation of the component of ∂S containing the common initial endpoint of $\phi(\gamma_k)$ and $\psi(\gamma_k)$.

For $i = 1, \dots, n$ set

$$C_i = \{\phi \in \mathrm{Mod}(S) \mid \phi(\gamma_j) \text{ and } \gamma_j \text{ coincide for } j = 1, \dots, i\}.$$

It is not hard to check that $\{1\} = C_n \subset C_{n-1} \subset \ldots \subset C_1 \subset \mathrm{Mod}(S)$ are convex subgroups of the left-ordering defined in the previous paragraph (cf. [SW00, Lemma 4.5] and [NW11, Theorem 3.1]).

Now we restrict to the case where S is orientable of genus $g \geq 2$, has no punctures and a single boundary component. We fix a generating set of $\mathrm{Mod}(S)$, in our case it is easiest to use the Humphries generators [Hum79], which are Dehn twists about the red curves indicated in Figure 1. For ease of exposition we identify each curve with the corresponding generator, that is, a_i, b_i and c will simultaneously be used to denote the curves appearing in Figure 1 as well as the corresponding generators of $\mathrm{Mod}(S)$.

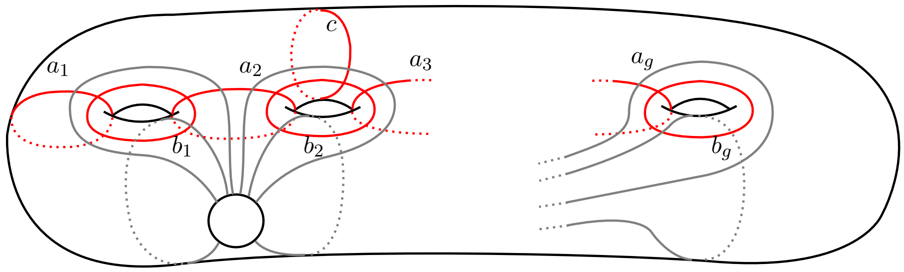

FIGURE 1. The Humphries generators in red, and our chosen curve diagram Γ in light grey.

Fix a curve diagram Γ consisting of the light grey curves appearing in Figure 1, orient the curves and the boundary component of S in any way. From this curve diagram it is possible to create, for each Humphries generator, a left-ordering of $\mathrm{Mod}(S)$ having the given generator as smallest positive element as follows: Given a generator b_i, enumerate the grey ideal arcs $\gamma_1, \ldots, \gamma_{2g}$ so that the arc intersecting b_i transversely is γ_{2g}. In this case the convex subgroup C_{2g-1} consists of $\phi \in \mathrm{Mod}(S)$ for which $\phi(\gamma_j) = \gamma_j$ (up to isotopy) for $j = 1, \ldots, 2g-1$, and thus the support of any $\phi \in C_{2g-1}$ is an annulus $A = S \setminus \{\gamma_1, \ldots, \gamma_{2g-1}\}$ whose central curve is b_i. Thus $C_{2g-1} = \mathrm{Mod}(A) \cong \mathbb{Z}$, generated by a Dehn twist about b_i. In any left-ordering where $C_{2g-1} = \langle b_i \rangle$ is convex, $b_i^{\pm 1}$ will be the smallest positive element; we may choose the ordering of $\langle b_i \rangle$ so that it is b_i. The same approach yields left-orderings of $\mathrm{Mod}(S)$ for which a_1 and c are smallest positive elements, as the curve diagram Γ intersects each of a_1 and c exactly once.

For $a_i \in \{a_2, \ldots, a_g\}$, enumerate the ideal arcs $\gamma_1, \ldots, \gamma_{2g}$ so that the two arcs intersecting a_i are γ_{2g-1} and γ_{2g}. In this case the convex subgroup C_{2g-2} consists of $\phi \in \mathrm{Mod}(S)$ with support a "pair of pants" $P = S \setminus \{\gamma_1, \ldots, \gamma_{2g-2}\}$ with a_i parallel to one of the boundary components of P. As such, one can create a left ordering of the convex subgroup $C_{2g-2} = \mathrm{Mod}(P) \cong \mathbb{Z}^3$ having a_i as a smallest positive element, and thus a left-ordering of $\mathrm{Mod}(S)$ with a_i as smallest positive element. We can now apply Proposition 3.1 to conclude that $\mathrm{Aut}(\mathrm{Mod}(S))$ acts faithfully on $\mathrm{LO}(\mathrm{Mod}(S))$.

It may also be possible to apply similar techniques to nonorientable surfaces, punctured surfaces, or surfaces with more than one boundary component. However, owing to the increased complexity of the generating sets in these cases, an analysis as above does not directly yield all of the required left-orderings and so is left to future work. $\square$

If K and H are bi-orderable groups and

$$1 \to K \to G \to H \to 1$$

is a short exact sequence, then G can be lexicographically bi-ordered if and only if there exists a bi-ordering of K whose positive cone is invariant under the conjugation action of H. By relaxing this condition, we are able to create groups which are *not* bi-orderable, but for which the automorphism group acts faithfully on the space of left-orderings.

Theorem 3.4. *Suppose that G is left-orderable and that*

$$1 \to K \to G \to \mathbb{Z} \to 1$$

is a short exact sequence of groups. Suppose that $\mathrm{Aut}(K)$ *acts faithfully on* $\mathrm{LO}(K)$*. If conjugation by the generator of $\mathbb{Z}$ preserves a left-ordering of K, then* $\mathrm{Aut}(G)$ *acts faithfully on* $\mathrm{LO}(G)$*.*

Proof. Suppose that $\phi : G \to G$ is a nontrivial automorphism. If $\phi(K) \neq K$, choose $g \in K$ with $\phi(g) \notin K$. Then by choosing signs appropriately, we may use the given short exact sequence to construct a positive cone $P \subset G$ for which $g \in P$ while $\phi(g) \notin P$. Thus $\phi(P) \neq P$.

On the other hand, suppose that $\phi(K) = K$. If there exists $k \in K$ for which $\phi(k) \neq k$, then we know there is a positive cone $P_K \in \mathrm{LO}(K)$ for which $\phi(P_K) \neq P_K$ since $\mathrm{Aut}(K)$ acts faithfully on $\mathrm{LO}(K)$. Using the given short exact sequence we may extend P_K to a positive cone $P \subset G$ satisfying $\phi(P) \neq P$.

Last, suppose that $\phi(k) = k$ for all $k \in K$, and choose $t \in G$ which maps to the generator of $\mathbb{Z}$. Equip K with a positive cone P_K that is preserved by conjugation by t, and proceed as in [LRR09, Lemma 3.4]. Note that every $g \in G$ can be written uniquely as kt^n for some $n \in \mathbb{Z}$ and $k \in K$, and since ϕ is nontrivial and satisfies $\phi(k) = k$ for all $k \in K$ it follows that $\phi(t) \neq t$. Construct a positive cone $P \subset G$ as follows: an element kt^n is in P if $k \in P_K$ or $k = 1$ and $n > 0$. Then P clearly satisfies $P \cup P^{-1} = G \setminus \{1\}$ and $P \cap P^{-1} = \emptyset$. Moreover if kt^n and $k't^m$ are both in P, then so is $kt^nk't^m = k(t^nk't^{-n})t^{m+n}$ since conjugation by t preserves P_K. One can easily verify that the subgroup $\langle t \rangle$ is convex relative to the ordering of G determined by P, so that P determines a discrete ordering of G with t as smallest positive element. The positive cone $\phi(P)$ will determine a left-ordering of G with $\phi(t)$ as smallest positive element. As $\phi(t) \neq t$, we conclude that $\phi(P) \neq P$. □

If K is a bi-orderable group, automorphisms $\phi : K \to K$ which preserve a left-ordering of K but not a bi-ordering are likely quite common. However, there is little in the literature dealing with automorphism-invariant left-orderings, as the focus has primarily been on automorphism-invariant bi-orderings [PR03, PR06, LRR08].

Here is an example of how an automorphism-invariant left-ordering (which is not a bi-ordering) may arise, which we use to illustrate an application of Theorem 3.4.

Example 3.5. Set $K = \mathbb{Q}^2 \rtimes \mathbb{Z}$ where the conjugation action of $\mathbb{Z}$ on $\mathbb{Q}^2$ is by the matrix $A = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)$. Then K is bi-orderable, since the action of A preserves the bi-ordering of $\mathbb{Z}^2$ defined by $(a, b) > (0, 0)$ if and only if $(a, b) \cdot (\sqrt{2}, 1) = b + \sqrt{2}a > 0$. In fact, since the eigenvectors of A are $\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$ with positive and negative eigenvalues respectively, the ordering described above (and its opposite) are the only orderings of $\mathbb{Q}^2$ preserved by A. Thus K is bi-orderable and nonabelian, so by Theorem 1.2 $\mathrm{Aut}(K)$ acts faithfully on $\mathrm{LO}(K)$.

Now we define $G = K \rtimes \mathbb{Z}$ where the action of the generator of $\mathbb{Z}$ on an element of K is $((a, b), c) \mapsto (-A(a, b)^T, c)$. The action of $-A$ on the subgroup $\mathbb{Q}^2 \subset K$, having the same eigenvectors as A but with eigenvalues of opposite sign, preserves only the ordering defined by $(a, b) > (0, 0)$ if and only if $(a, b) \cdot (-\sqrt{2}, 1) = b - \sqrt{2}a > 0$, and its opposite. Using this ordering on $\mathbb{Q}^2$, and lexicographically ordering K using the short exact sequence $1 \to \mathbb{Q}^2 \to K \to \mathbb{Z} \to 1$, we arrive at a left-ordering of K preserved by the action of the generator of $\mathbb{Z}$.

We conclude $\mathrm{Aut}(G)$ will act faithfully on $\mathrm{LO}(G)$ by Theorem 3.4.

Note that G is left-orderable by a straightforward short exact sequence argument, but is not bi-orderable since the actions of A and $-A$ on $\mathbb{Q}^2 \subset K$ do not preserve a common ordering, so Theorem 1.2 does not apply. Proposition 3.1 also cannot apply to G since any generator of $\mathbb{Q}^2 \subset G$ cannot be the smallest positive element of a left-ordering of G. □

Despite these extensions and examples, one cannot hope to replace “bi-orderable” in Theorem 1.2 with either the weaker condition of local indicability or the condition that G admit an ordering that is recurrent for every cyclic subgroup (See [Mor06] for more information on recurrent orderings). Koberda points out that for the Klein bottle group, $K = \langle x, y \mid xyx^{-1} = y^{-1} \rangle$, the action of $\mathrm{Aut}(K)$ on $\mathrm{LO}(K)$ is not faithful. Yet K is both locally indicable and admits recurrent orderings, as it only has four left-orderings.

4. Applications and generalizations

The action of $\mathrm{Aut}(G)$ on $\mathrm{LO}(G)$ is connected to the action of $\mathrm{Aut}(G)$ on ∂G by the following theorem. Though not stated in full generality in [Kob11], the proof below appears there as part of the proof of [Kob11, Theorem 1.2]. As it is relatively short, we repeat it here for the reader’s convenience. For background and futher information on hyperbolic groups, see [Kob11, Gro87, KB02].

Proposition 4.1. *If G is a left-orderable hyperbolic group and $\mathrm{Aut}(G)$ acts faithfully on $\mathrm{LO}(G)$, then it acts faithfully on ∂G.*

Proof. Recall that for each element $g \in G$, there are two distinct points in the boundary ∂G defined by $x_g = \lim_{n\to\infty} g^n$ and $y_g = \lim_{n\to\infty} g^{-n}$. Moreover, given $g, h \in G$ if $\langle g, h\rangle$ is not a virtually cyclic group, then g and h determine distinct points on the boundary.

Choose a nontrivial automorphism $\phi \in \mathrm{Aut}(G)$, $g \in G$ and $P \in \mathrm{LO}(G)$ such that $g \in P$ and $\phi(g) \notin P$ (and thus $\phi(P) \neq P$). Since we cannot have $g^k = \phi(g)^\ell$ for some $k, \ell > 0$, there are two cases. Recall we defined $I(g)$ in Section 2 to be the isolator of the cyclic subgroup $\langle g\rangle$.

Case 1. $\phi(g) \in I(g)$ and there exists $k, \ell > 0$ such that $g^{-k} = \phi(g)^\ell$. In this case, observe that $x_g = \lim_{n\to\infty}(g^\ell)^n$, so that

$$\phi(x_g) = \lim_{n\to\infty} \phi(g^\ell)^n = \lim_{n\to\infty}(g^{-k})^n = y_g,$$

so that ϕ acts nontrivially on ∂G.

Case 2. $\phi(g) \notin I(g)$. Then $\phi(g)$ and g do not generate a virtually cyclic subgroup, so x_g and $\phi(x_g) = x_{\phi(g)}$ are distinct. Thus ϕ acts nontrivially on ∂G. □

Consequently, by applying Theorem 1.2, we arrive at Corollary 1.4. If G is hyperbolic and satisfies the hypotheses of Theorem 3.4 or Proposition 3.1, then $\mathrm{Aut}(G)$ acts faithfully on ∂G, too. However it seems difficult to construct a hyperbolic group G satisfying the hypotheses of either result.

There are also two natural generalizations one may consider, both developed by Witte Morris in [Mor12]. First, one may replace the automorphism group with the commensurator group $\mathrm{Comm}(G)$ of G. Recall that a *commensuration* of a group G is an isomorphism $\phi : H_1 \to H_2$ of finite index subgroups $H_i \subset G$. Two commensurations $\phi : H_1 \to H_2$ and $\phi' : H_1' \to H_2'$ are equivalent if there exists a finite index subgroup $H \subset H_1 \cap H_1'$ such that $\phi|_H = \phi'|_H$. The set of equivalence classes of commensurations forms the *commensurator group* $\mathrm{Comm}(G)$ of G.

Witte Morris points out that for torsion free locally nilpotent groups, $\mathrm{Comm}(G)$ acts naturally on $\mathrm{LO}(G)$. This follows from an application of Koberda's theorem (Theorem 1.1), and the fact that for every subgroup H of a torsion-free locally nilpotent group G, the restriction map $r : \mathrm{LO}(G) \to \mathrm{LO}(H)$ is surjective. When G is a bi-orderable group, the restriction $r : \mathrm{LO}(G) \to \mathrm{LO}(H)$ is not a surjective map in general, so this generalization is not possible in our setting.

However, using the restriction map $r : \mathrm{LO}(G) \to \mathrm{LO}(H)$ for each finite index subgroup $H \subset G$, one can define the space of *virtual* left-orderings of G as the limit

$$\mathrm{VLO}(G) = \varinjlim \mathrm{LO}(H),$$

where the limit is over all finite-index subgroups H of G [Mor12]. When $P \in \mathrm{LO}(H)$ and H is a finite index subgroup of G, we will denote the corresponding element of $\mathrm{VLO}(G)$ by $[P]$. Then $\mathrm{Comm}(G)$ naturally acts on $\mathrm{VLO}(G)$: for each commensuration $\phi : H_1 \to H_2$ and each positive cone

$P \in \mathrm{LO}(H)$, set $\phi([P]) = [\phi(P \cap H_1)]$. It is straightforward to check that this definition respects the necessary equivalence relations.

Lemma 4.2. *Let G be a left-orderable group and $\phi : H_1 \to H_2$ a commensuration of G where H_1 is abelian. If $\phi = \tau_{p/q}$ for some $p/q \in \mathbb{Q}$ then the element of* $\mathrm{Comm}(G)$ *represented by ϕ acts trivially on* $\mathrm{VLO}(G)$.

Proof. Suppose that H is a finite index subgroup and $P \subset H$ is the positive cone of a left-ordering. Consider $P \cap H_1$ and $\phi(P \cap H_1)$. The first is the positive cone of a left-ordering of $H \cap H_1$, the second is the positive cone of a left-ordering of $H \cap H_2$. Using the fact that ϕ satisfies $(*)$, one can show that these orderings agree on the finite index subgroup $H \cap H_1 \cap H_2$ so that $[P] = [\phi(P \cap H_1)]$, and thus ϕ acts trivially on $\mathrm{VLO}(G)$. □

Theorem 4.3. *Let G be a bi-orderable group.*

1. *If G is not virtually abelian,* $\mathrm{Comm}(G)$ *acts faithfully on* $\mathrm{VLO}(G)$.
2. *If G is virtually abelian then the kernel of the action of* $\mathrm{Comm}(G)$ *on* $\mathrm{VLO}(G)$ *contains precisely the elements represented by commensurations $\tau_{p/q} : H_1 \to H_2$, if any such commensurations exist.*

Proof. First suppose that G is not virtually abelian, and let $\phi : H_1 \to H_2$ be a nontrivial commensuration of G. By Lemma 2.7, ϕ cannot satisfy $(*)$ since H_1 is not abelian. Thus there exists $g \in H_1$ such that $\phi(g) \neq g$ and either $\phi(g)^n = g^{-m}$ for some $m, n > 0$ or $I(g) \neq I(\phi(g))$. In either case we can construct a left-ordering of G with positive cone P satisfying $g \in P$ and $\phi(g) \notin P$ using arguments identical to those in the proof of Lemma 2.3. Then $[P] \neq [\phi(P \cap H_1)]$, so (the class of) $\phi : H_1 \to H_2$ acts nontrivially on $\mathrm{VLO}(G)$.

On the other hand, suppose G is virtually abelian, and let $\phi : H_1 \to H_2$ be a nontrivial commensuration of G. If ϕ does not satisfy $(*)$, then an argument identical to the previous paragraph shows that the class of ϕ acts nontrivially on $\mathrm{VLO}(G)$. On the other hand, if ϕ does satisfy $(*)$, then $\phi = \tau_{p/q}$ for some $p/q \in \mathbb{Q}$ by Lemma 2.4. In this case, $\phi = \tau_{p/q}$ acts trivially on $\mathrm{VLO}(G)$ by Lemma 4.2. □

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