

# The random graph embeds in the curve graph of any infinite genus surface

Edgar A. Bering IV and Jonah Gaster

ABSTRACT. The random graph is an infinite graph with the universal property that any embedding of  $G - v$  extends to an embedding of  $G$ , for any finite graph. In this paper we show that this graph embeds in the curve graph of a surface  $\Sigma$  if and only if  $\Sigma$  has infinite genus, showing that the curve system on an infinite genus surface is “as complicated as possible”.

## CONTENTS

1. Introduction	59
2. Proof of Theorem 1.1	61
Acknowledgments	65
References	65

## 1. Introduction

In this paper we will prove:

**Theorem 1.1.** *The random graph embeds into the curve graph  $\mathcal{C}(\Sigma)$  of a surface  $\Sigma$  if and only if  $\Sigma$  has infinite genus.*

We adopt the terminology that an *embedding* of a graph  $f : G \rightarrow H$  is a one-to-one map on the vertices so that  $(u, v)$  is an edge in  $G$  if and only if  $(f(u), f(v))$  is an edge in  $H$ . (This is also called an *induced subgraph* elsewhere in the literature.)

The one-ended, infinite genus, orientable surface with one boundary component is a subsurface of any orientable infinite genus surface [Ric63]. The choice of a disk on  $\Sigma$ , the one-ended orientable surface of infinite genus without boundary, thus induces an embedding of the curve graph  $\mathcal{C}(\Sigma)$  into the curve graph of an arbitrary orientable infinite genus surface. Therefore, for one direction of the theorem, it suffices to produce an embedding of the random graph into  $\mathcal{C}(\Sigma)$  when  $\Sigma$  is the one-ended orientable surface of infinite genus.

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The other direction is perhaps more surprising, since it is tempting to view an infinite-type surface of finite genus as already quite complicated. However, Ehrlich, Even, and Tarjan [EET76] showed that there are graphs that cannot be realized as the incidence graph of a collection of planar intervals (in their language, there are graphs not of *planar type*), and we employ their construction to demonstrate the necessity of infinite genus.

Rado [Rad64] showed that every countable graph embeds in the random graph, however we focus on the random graph for its combinatorial properties. The first order theory of the random graph, in the graph language, is not edge stable in the model-theoretic sense [TZ12]. It follows that

**Corollary 1.2.** *The first order theory of the curve graph of an infinite genus surface is not edge stable.*

The lack of edge stability implies that the theory of the curve graph is also unstable in the model-theoretic sense. With Gabriel Conant, we prove a complementary result for finite-type surfaces [BCG16]; the theory of  $\mathcal{C}(\Sigma)$  for a finite-type surface  $\Sigma$  is edge stable. The combination of these two results show that the model-theoretic dividing line of edge stability and the topological dividing line of finite-type coincide for curve graphs. It is unknown whether or not the curve graph of a finite-type surface is stable.

Erdős and Rényi introduced the random graph from a probabilistic point of view, constructing a graph on countably many vertices by adding edges with probability  $\frac{1}{2}$ . The result of this construction is almost surely isomorphic to a unique object, which we call *the random graph* [ER63]. Rado gave an explicit construction of the random graph: take as vertices the natural numbers  $\mathbb{N}$ . Given  $x < y$ , add an edge  $(x, y)$  if the  $x$ th bit of the binary expansion of  $y$  is 1 [Rad64]. The random graph enjoys a universal property, known as the extension property; for any finite graph  $G$ , if  $G - v$  embeds into the random graph then this embedding can be extended to  $G$ .

The other graph of interest in this article is the curve graph of an infinite genus surface, with or without boundary. A simple closed curve on a surface is essential if no component of the complement is a disk, and non-peripheral if no component of the complement is an annulus. For brevity, we will use curve to mean the isotopy class of an essential non-peripheral simple closed curve. The intersection number of two curves (denoted  $i(\alpha, \beta)$ ) is the minimum cardinality of the intersection taken over all transverse realizations of  $\alpha$  and  $\beta$ .

Fix a surface  $\Sigma$ . The curve graph  $\mathcal{C}(\Sigma)$  has as vertices the curves on  $\Sigma$ , and an edge between the vertices corresponding to curves  $\alpha$  and  $\beta$  when

$$i(\alpha, \beta) = 0.$$

(As an aside, this construction can be extended to a definition of a higher-dimensional simplicial complex of interest in its own right, but we will focus on the 1-skeleton [Har81, MM99, MM00].) When  $\Sigma$  is of finite-type,  $\mathcal{C}(\Sigma)$  is well-known to be  $\delta$ -hyperbolic and infinite-diameter, with automorphism

group isomorphic to the mapping class group of  $\Sigma$  [Iva97, MM99]. Recent work has focused on different analogues of  $\mathcal{C}(\Sigma)$  when  $\Sigma$  is of infinite-type, more satisfying from a geometric viewpoint [Bav16, AFP15, AV16, FP15, DFV16].

In fact, it is not hard to see that every finite graph  $G$  embeds in the curve graph of a surface for some closed surface of genus  $g$ . We outline a simple proof. Suppose  $G$  has  $n$  vertices. Let  $\Sigma_0$  indicate a closed surface of genus large enough so that  $\Sigma_0$  contains a collection of  $n$  curves in minimal position that pairwise intersect once<sup>1</sup>, and identify these curves with the vertices of  $G$  arbitrarily. For each edge between a pair of vertices of  $G$ , add a handle near the intersection of the corresponding curves on  $\Sigma_0$ , and thread one of the curves through the handle so that the new curves on the new surface do not intersect. The identification of the vertices of  $G$  with the resulting curve system extends to an embedding of  $G$  into the curve graph  $\mathcal{C}(\Sigma)$  of the resulting surface  $\Sigma$ , by construction.

**Remark 1.3.** This leaves open the problem of determining the minimal genus such that every finite graph on  $n$  vertices embeds in the curve graph of that genus (cf. [KK14, Question 1.1]). The above construction provides the bound  $O(n^2)$ .

This implies that every finite graph embeds into the curve graph of an infinite genus surface, and it suggests that this curve graph of an infinite genus surface should be quite complicated. Note, however, that this alone does not guarantee the presence of the random graph. Also note that it is apparent that the random graph does not embed in the curve graph of any finite-type surface: A complete subgraph of the curve graph of a surface of genus  $g$  with  $n$  punctures has size at most  $3g - 3 + n$ , whereas every finite complete graph embeds in the random graph.

Moreover, the curve graph of an infinite genus surface cannot itself be isomorphic to the random graph. Fix an infinite genus surface  $\Sigma$ . Pick a separating curve<sup>2</sup>  $\gamma$  and two curves  $\alpha, \beta$ , one in each component of  $\Sigma \setminus \gamma$ . Let  $G$  be the graph in Figure 1. We can embed  $G - v$  into  $\mathcal{C}(\Sigma)$  according to the labeling in the figure, but an extension to  $v$  would imply the existence of a curve disjoint from  $\gamma$  that intersects both  $\alpha$  and  $\beta$ , a contradiction since  $\gamma$  is separating.

## 2. Proof of Theorem 1.1

**Proof.** We deal with the forward implication first, showing that the presence of the random graph in the curve graph implies that  $\Sigma$  has infinite

<sup>1</sup>Genus  $\lceil \frac{n-1}{2} \rceil$  suffices. Such a system of curves has been referred to as a *complete 1-system* in the literature.

<sup>2</sup>Such a curve always exists: If  $\gamma$  is non-separating, choose a curve that intersects it once, and take a regular neighborhood of the union. The boundary of this neighborhood is a separating curve.

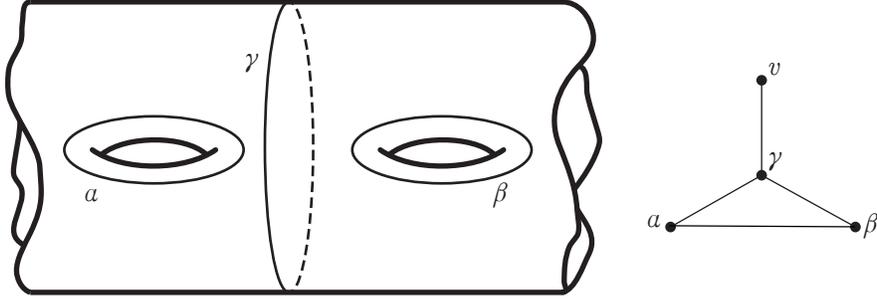


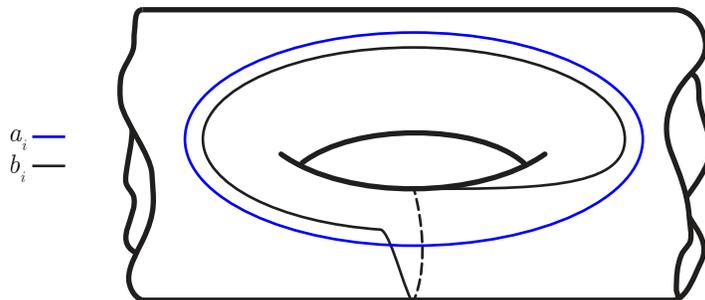
FIGURE 1. A collection of curves and a partial graph embedding that cannot extend.

genus. It is evidently enough to show that there is a finite graph which does not embed in  $\mathcal{C}(\Sigma)$  when  $\Sigma$  has finite genus, since any finite graph embeds in the random graph. This construction is due to Ehrlich, Even, and Tarjan when  $g = 0$  [EET76]; it is simple enough to include completely.

Suppose that  $\Sigma$  has genus  $g < \infty$ , and choose a finite graph  $G$  that admits no topological embedding  $G \rightarrow \Sigma_{g,0}$  into the closed surface of genus  $g$  (a large complete graph will do). Consider the graph  $G'$  obtained by adding a vertex that subdivides each edge of  $G$ , so that there are now  $|V(G)|$  old vertices and  $|E(G)|$  new vertices of  $G'$ . Let  $\hat{G}$  indicate the complementary graph of  $G'$ , and note that: (1) each new vertex  $v$  of  $\hat{G}$  is incident to all other vertices of  $\hat{G}$ , except for the two old vertices that are incident to the edge of  $G$  corresponding to  $v$ , and (2) each old vertex is incident to every other old vertex.

Suppose that  $\hat{G}$  is realized by a curve system  $\Gamma$  on  $\Sigma$  in minimal position. The subdivision of the vertices of  $\hat{G}$  into new and old vertices gives a subdivision of  $\Gamma$  into new and old curves. For each old curve  $\gamma$ , select a point  $p_\gamma \in \gamma$  in the complement of its intersections with the other curves of  $\Gamma$ , and contract  $\gamma \setminus p_\gamma$  to a point. Because the old vertices of  $\hat{G}$  are all incident to each other, when we do this contraction to each old curve one-by-one, we obtain  $|V(G)|$  vertices on  $\Sigma_{g,0}$ . Moreover, because the new vertices of  $\hat{G}$  are all incident to each other, the new curves become a system of disjoint arcs connecting these vertices. By construction the resulting arcs provide a topological embedding of  $G$  into  $\Sigma_{g,0}$ , a contradiction.

For the other direction, we will provide an explicit construction of a family of curves on the one-ended orientable infinite genus surface whose intersection relation is exactly described by the random graph. Our approach is in two parts; first we will give a countable collection of multicurves with this property, then describe how to add handles to convert these multicurves to curves without changing the intersection relation of the curve system or the homeomorphism type of the surface.

FIGURE 2. A possible choice for  $a_i$  and  $b_i$  at handle  $i$ .

Rado's construction fits more naturally into the setting of multicurves, so we first define a multicurve complex  $m\mathcal{C}(\Sigma)$  analogous to the curve complex. Let the vertices of  $m\mathcal{C}(\Sigma)$  be finite sets of disjoint curves. For multicurves  $U, V \in m\mathcal{C}(\Sigma)$ , let  $i(U, V)$  be the sum of intersection numbers  $i(\alpha, \beta)$  over all  $\alpha \in U, \beta \in V$ . In analogy with the curve graph, there is an edge in  $m\mathcal{C}(\Sigma)$  between  $U, V$  if  $i(U, V) = 0$ . (The multicurve graph has also been called the *clique graph* in the literature [KK14]). Below, we write multicurves additively, e.g.  $\alpha + \beta$  is the multicurve  $\{\alpha, \beta\}$ .

To fix notation, let  $\Sigma_1$  be the one-ended orientable surface of infinite genus. Note that the random graph is self-complementary (that is, the complement graph is isomorphic to the random graph), so we will work with the complement of Rado's model: let  $x, y \in \mathbb{N}$  with  $x < y$  be adjacent when the  $x$ th bit in the binary expansion of  $y$  is 0. We describe below a map  $[\cdot] : \mathbb{N} \rightarrow m\mathcal{C}(\Sigma_1)$  so that, for  $x < y$ , the intersection number  $i([x], [y])$  is equal to the  $x$ th bit in the binary expansion of  $y$ . Such a map induces an embedding of the random graph into  $m\mathcal{C}(\Sigma_1)$ .

Realize  $\Sigma_1$  in  $\mathbb{R}^3$  as the regular neighborhood of the lattice on points  $\mathbb{N} \times \{0, 1\} \times \{0\}$ . With this embedding the 'centers' of 'holes' of  $\Sigma_1$  occur at  $(n + \frac{1}{2}, \frac{1}{2}, 0)$  with  $n \in \mathbb{N}$ . The intersection of  $\Sigma_1$  with the coordinate plane  $\mathbb{R} \times \mathbb{R} \times \{0\}$  is the disjoint union of countably many circles and a real line. Let  $a_i$  be the circle component in the strip  $(i - 1, i) \times \mathbb{R} \times \{0\}$ , and let  $b_i$  be the Dehn twist of  $a_i$  around the intersection of the half-plane  $\{i - \frac{1}{2}\} \times (-\infty, \frac{1}{2}) \times \mathbb{R}$  with  $\Sigma$ . In other words,  $a_i$  winds around the  $i$ th hole of  $\Sigma$ , and  $i(a_i, b_j) = \delta_{i,j}$ , as pictured in Figure 2.

Given a natural number  $x$ , let  $x_i$  be the  $i$ th binary digit in the expansion of  $x$ . We define

$$[x] = b_x + \sum_{i=0}^{\lceil \log_2 x \rceil} x_i \cdot a_i.$$

Figure 3 shows  $[0]$  and  $[4]$ . By construction this is our desired map and the intersection relation among the multicurves  $\{[n]\}_{n \in \mathbb{N}}$  is given by the random graph.

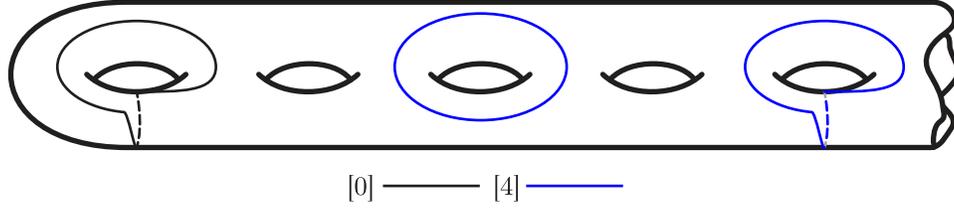


FIGURE 3. An illustration of geometric realizations of  $[0]$  and  $[4]$ .

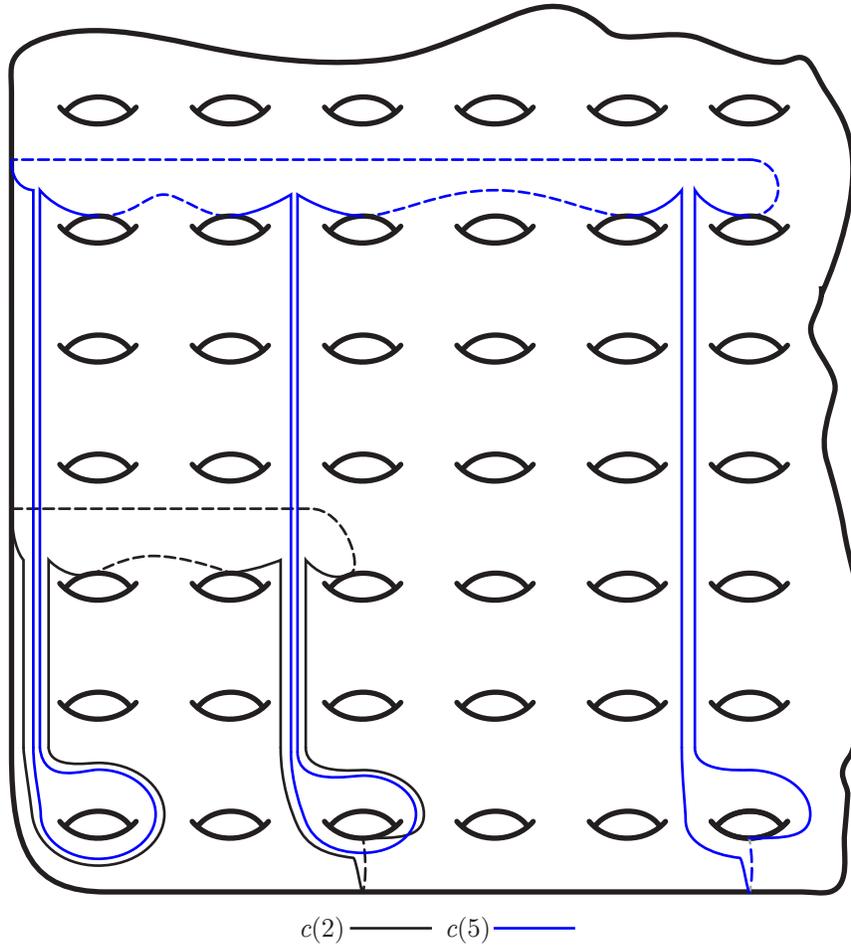


FIGURE 4. An illustration of  $c(2)$  and  $c(5)$  realizing  $i(c(2), c(5)) = 1$ .

At this point one would like to blindly add handles to realize these multicurves as curves. However, for each bit there are infinitely many curves that need to be connected to the handle encoding that bit, so care must be exercised. Consider a new realization of  $\Sigma_1$  in  $\mathbb{R}^3$ , as the regular neighborhood of the lattice on  $\mathbb{N} \times \mathbb{N} \times \{0\}$ . The centers of ‘holes’ are now at  $(x + \frac{1}{2}, y + \frac{1}{2}, 0)$  for  $x, y \in \mathbb{N}_{\geq 0}^2$ . This naturally indexes the rows and columns of the embedding (row  $n$  is the regular neighborhood of points of the form  $(x, n, 0)$ , and the columns are similarly indexed). We take  $a_i, b_j$  as before (along the  $x$ -axis). For a multicurve  $[x] = b_x + \sum_i x_i \cdot a_i$ , construct the curve  $c(x)$  by connecting each  $a_i$  or  $b_i$  in  $[x]$  to row  $x + 1$  by a pair of vertical lines along column  $i$ , and then connect these arcs to one another along the ‘back’ of  $\Sigma_1$ ; Figure 4 shows  $c(2)$  and  $c(5)$ . For  $x < y$ , we can realize  $c(x)$  and  $c(y)$  so that when  $x$  and  $y$  use a common column  $c(x)$  passes outside of  $c(y)$ ; hence  $i(c(x), c(y)) = i([x], [y])$ . (Note that, when curves intersect once, this intersection is necessarily essential [FM12].) We conclude that  $\{c(n)\}_{n \in \mathbb{N}}$  is the vertex set of an embedding of the random graph in  $\mathcal{C}(\Sigma_1)$ .  $\square$

**Remark 2.1.** The embedding of the random graph given by  $\{c(n)\}_{n \in \mathbb{N}}$  is far from unique. Each curve  $b_i$  is constructed by a single Dehn twist. Varying the powers of each twist defining a  $b_i$  individually yields systems of curves in distinct mapping class group orbits. The extended mapping class group of  $\Sigma$  is isomorphic to the graph automorphisms of  $\mathcal{C}(\Sigma)$  in the case of the infinite genus surface with one end, so these embeddings are also combinatorially inequivalent [HV14].

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(Edgar A. Bering IV) DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE, UNIVERSITY OF ILLINOIS AT CHICAGO, 851 S MORGAN ST. (M/C 249), CHICAGO, IL, 60607  
 eberin2@uic.edu

(Jonah Gaster) DEPARTMENT OF MATHEMATICS, BOSTON COLLEGE, CHESTNUT HILL, MA 02467  
 gaster@bc.edu

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