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On Fell bundles over inverse semigroups and their left regular representations

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Abstract. We prove a version of Wordingham's theorem for left regular representations in the setting of Fell bundles over inverse semigroups and use this result to discuss the various associated cross sectional C^*-algebras.

Contents

1. Introduction

Following an unpublished work of Sieben, the concept of a Fell bundle over a discrete group was generalized by Exel in [11], where the notion of a Fell bundle $\mathcal{A} = \{A_s\}_{s\in S}$ over an inverse semigroup S was introduced and the associated full cross sectional C^*-algebra $C^*(\mathcal{A})$ was defined as the universal C^*-algebra for C^*-algebraic representations of $\mathcal{A}$. This construction may be used to present other classes of C^*-algebras in a unified manner. For example, Buss and Exel show in [4] that to each partial action β of an inverse semigroup S on a C*-algebra A, one may associate a Fell bundle $\mathcal{A}_\beta$ over S such that $C^*(\mathcal{A}_\beta)$ is naturally isomorphic to the full crossed product of A by β. (In fact, Buss and Exel even consider twisted partial actions.) In another direction, the same authors establish in [5] that any Fell bundle $\mathcal{B}$ over an étale groupoid $\mathcal{G}$ gives rise to a Fell bundle $\mathcal{A}$ over S, where S

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is any inverse semigroup consisting of bisections (or slices) of $\mathcal{G}$ (as defined by Renault in [18]), and show that, under some mild assumptions, the full cross sectional C*-algebra of $\mathcal{B}$ is isomorphic to $C^*(\mathcal{A})$.

Exel also defines in [11] the reduced cross sectional C^*-algebra $C_r^*(\mathcal{A})$ associated to a Fell bundle $\mathcal{A}$ over an inverse semigroup S. One drawback of his construction is that it is somewhat involved (we summarize it in section 3.4). Another approach has recently been proposed in [7] when $\mathcal{A}$ is saturated and S is unital, relying on the result from [8] that $\mathcal{A}$ may then be identified with an action of S by Hilbert bimodules on the unit fibre A of $\mathcal{A}$, thus making it possible to define $C_r^*(\mathcal{A})$ as the reduced crossed product of A by this action of S. We believe that it would be helpful to find a more direct construction of $C_r^*(\mathcal{A})$, at least in some cases. For example, such a construction would make it easier to initiate a study of amenability for Fell bundles over inverse semigroups, having in mind that, as for Fell bundles over groups [12, 9], a natural definition of amenability for $\mathcal{A}$ is to require that the canonical $*$-homomorphism from $C^*(\mathcal{A})$ onto $C_r^*(\mathcal{A})$ is injective.

Given a Fell bundle $\mathcal{A}$ over an inverse semigroup S, our main goal in the present paper is to introduce a certain C^*-algebra $C^*_{\mathrm{r,alt}}(\mathcal{A})$ which may also be considered as a kind of reduced cross sectional C*-algebra for $\mathcal{A}$, and to compare it with Exel's $C_r^*(\mathcal{A})$. Inspired by the approach used by Khoshkam and Skandalis [14] in the case of an action of an inverse semigroup on a C^*-algebra, our first step (see Section 3) is to associate to $\mathcal{A}$ a full cross sectional C^*-algebra $C^*_{\mathrm{KS}}(\mathcal{A})$ which is universal for so-called pre-representations of $\mathcal{A}$ in C^*-algebras, or, equivalently, for C^*-algebraic representations of the convolution $*$-algebra $C_c(\mathcal{A})$ canonically attached to $\mathcal{A}$. Exel's $C^*(\mathcal{A})$ is then easily obtained as a quotient of $C^*_{\mathrm{KS}}(\mathcal{A})$. Our next step (see Section 4) is to show that $C_c(\mathcal{A})$ has a natural injective left regular C^*-algebraic representation Φ_Λ. The injectivity of Φ_Λ may be seen as an analog of Wordingham's theorem for $\ell^1(S)$ (cf. [17]), and our proof is related to his original proof, although some extra arguments are necessary. Letting $C_{\mathrm{r,KS}}(\mathcal{A})$ denote the C^*-algebra generated by the range of Φ_Λ, $C^*_{\mathrm{r,alt}}(\mathcal{A})$ is then defined as the quotient of $C_{\mathrm{r,KS}}(\mathcal{A})$ by a certain canonical ideal. From the naturality of our construction, it readily follows that there is a canonical $*$-homomorphism $\Psi_{\Lambda^{\mathrm{alt}}}$ from $C^*(\mathcal{A})$ onto $C^*_{\mathrm{r,alt}}(\mathcal{A})$.

In the case where S consists only of idempotents (hence is a semilattice) and $\mathcal{E}$ is Fell bundle over S, we check in Section 5 that $C^*_{\mathrm{KS}}(\mathcal{E}) = C^*_{\mathrm{r,KS}}(\mathcal{E})$ and $C^*(\mathcal{E}) = C^*_{\mathrm{r,alt}}(\mathcal{E}) \simeq C^*_{\mathrm{r}}(\mathcal{E})$. Next, given a Fell bundle $\mathcal{A}$ over an inverse semigroup S such that S is E^*-unitary (cf. Section 2) and such that $A_0 = \{0\}$ if S har a 0 element, we let $\mathcal{E}$ denote the Fell bundle obtained by restricting $\mathcal{A}$ to the semilattice E of idempotents in S and show (in Section 6) that there exists a faithful conditional expectation from $C^*_{\mathrm{r,KS}}(\mathcal{A})$ onto $C^*_{\mathrm{r,KS}}(\mathcal{E})$. In the final section, keeping the same assumptions, we describe $C_r^*(\mathcal{A})$ as a quotient of $C_{\mathrm{r,KS}}(\mathcal{A})$ and show that there exists a surjective canonical $*$-homomorphism Ψ' from $C^*_{\mathrm{r,alt}}(\mathcal{A})$ onto $C_r^*(\mathcal{A})$. We also characterize when Ψ'

is a $*$-isomorphism and end by showing that this happens frequently when S is strongly E^*-unitary.

2. Preliminaries

We recall that a semigroup is a set equipped with an associative binary operation, while a monoid is a semigroup with an identity. A commutative idempotent semigroup is called a semilattice. We also recall that an inverse semigroup is a semigroup S where for each $s \in S$ there is a unique $s^* \in S$ satisfying

$$ss^*s = s \text{ and } s^*ss^* = s^*.$$

The map $s \mapsto s^*$ is then an involution on S. Every inverse semigroup S contains a canonical semilattice, namely

$$E(S) = \{e \in S : e^2 = e\}$$

satisfying $E(S) = \{e \in S : e^2 = e = e^*\} = \{s^*s : s \in S\} = \{tt^* : t \in S\}$. Throughout this article, S will be a fixed inverse semigroup and $E = E(S)$ will denote its semilattice of idempotents. We refer to [16] and [17] for the basics of the theory of inverse semigroups. We recall below a few facts that we will need later.

There is a natural partial order relation $\leq$ on S given by $s \leq t$ if and only if $s = et$ for some $e \in E$, if and only if $s = tf$ for some $f \in E$, where e may be chosen to be ss^*, and f to be s^*s. For $e, f \in E$, we have $e \leq f$ if and only if $e = ef$.

Many inverse semigroups have a zero, that is, an element 0 satisfying $0s = s0 = 0$ for all $s \in S$. Such an element is necessarily unique and lies in E. If S has a zero, we set $S^\times = S \setminus \{0\}$ and $E^\times = E \setminus \{0\}$. Otherwise, we set $S^\times = S$ and $E^\times = E$.

We will say that S is *E^*-unitary* if the set $\{s \in S : e \leq s\}$ is contained in E for every $e \in E^\times$. If this holds for every $e \in E$, then S is called *E-unitary*. These two concepts clearly coincide if S does not have a zero. For inverse semigroups having a zero, E-unitarity is a too strong requirement, only satisfied by semilattices. We note that E^*-unitarity is usually only defined for inverse semigroups having a zero, in which case it is sometimes called 0-E-unitarity (and is defined as above). Our use of terminology will allow us to unify some statements. The class of E^*-unitary inverse semigroups has by far been the one who has received most attention from C^*-algebraists, most probably because they are easier to handle.

We will also need to refer to a stronger form of E^*-unitarity. We recall that a map σ from S into a group with identity 1 is called a *grading* if $\sigma(st) = \sigma(s)\sigma(t)$ whenever $s, t \in S$ and $st \in S^\times$, and that σ is said to be *idempotent pure* if $\sigma^{-1}(\{1\}) = E$. Then S is said to be *strongly E^*-unitary* [3] if there exists an idempotent pure grading from S into some group. It is known that S is E^*-unitary whenever it is strongly E^*-unitary.

A semigroup homomorphism from S into another inverse semigroup is necessarily $*$-preserving, so this provides the natural notion of homomorphism between inverse semigroups. If X is a set, then $\mathcal{I}(X)$ will denote the symmetric inverse semigroup on X, consisting of all partial bijections on X (with composition defined on the largest possible domain). An *action of S on X* is then a homomorphism of S into $\mathcal{I}(X)$. The essence of the Wagner–Preston theorem is that there always exists an injective action of S on some set.

When A is a C^*-algebra, we will let $\mathrm{PAut}(A)$ denote the inverse subsemigroup of $\mathcal{I}(A)$ consisting of all partial $*$-automorphisms of A; so $\phi \in \mathrm{PAut}(A)$ if and only if ϕ is a $*$-isomorphism between two ideals of A. Here, and in the sequel, ideals in C^*-algebras are always assumed to be two-sided and closed, unless otherwise specified. An *action of S on a C^*-algebra A* is an homomorphism α from S into $\mathrm{PAut}(A)$. Khoshkam and Skandalis show in [14] how to associate to such an action a full (resp. reduced) C^*-crossed product, which we will denote by $A \rtimes_{\alpha}^{\mathrm{KS}} S$ (resp. $A \rtimes_{\alpha,r}^{\mathrm{KS}} S$). In fact, their construction goes through for a more general kind of action of S on A, and the interested reader should consult [14] for more details, including a discussion of the relationship between their full crossed product and the crossed product construction previously introduced by Sieben in [19].

Following [6], one may define *partial* actions of S on sets and on C^*-algebras. As mentioned in the introduction, Buss and Exel actually consider *twisted* partial actions of S in [6], but we will restrict ourselves to the untwisted case to avoid many technicalities. Partial actions were first introduced in the case where S is a group, and the reader may consult [12] for a nice introduction to this subject, including many references to the literature. We recall a few relevant definitions and facts from [6]. A *partial homomorphism* of S in a semigroup H is a map $\pi : S \mapsto H$ such that:

(i) $\pi(s)\pi(t)\pi(t^*) = \pi(st)\pi(t^*)$,
(ii) $\pi(s^*)\pi(s)\pi(t) = \pi(s^*)\pi(st)$,
(iii) $\pi(s)\pi(s^*)\pi(s) = \pi(s)$,

hold for all $s, t \in S$. Note that if H is an inverse semigroup, then (iii) implies

(iv) $\pi(s^*) = \pi(s)^*$

for all $s \in S$; hence, in this case, π is a partial homomorphism if and only if (i), (ii) and (iv) hold. Moreover, still assuming that H is an inverse semigroup, this is equivalent to requiring that the three conditions:

(a) $\pi(s^*) = \pi(s)^*$,
(b) $\pi(s)\pi(t) \leq \pi(st)$,
(c) $\pi(s) \leq \pi(t)$ whenever $s \leq t$,

hold for $s, t \in S$, cf. [6, Proposition 3.1].

A *partial action* of S on a set X (resp. on a C^*-algebra A) is then defined as a partial homomorphism β from S into $\mathcal{I}(X)$ (resp. into $\mathrm{PAut}(A)$). As in [6], we will also require that a partial action β of S on a C^*-algebra A satisfies that the union $\cup_{s\in S} J_s = \cup_{s\in S}\operatorname{im}(\beta_s)$ spans a dense subspace of A.

3. Fell bundles over inverse semigroups

In [11], Exel defines a Fell bundle over S as a quadruple

$$\mathcal{A} = \big(\{A_s\}_{s\in S}, \{\mu_{s,t}\}_{s,t\in S}, \{*_s\}_{s\in S}, \{j_{t,s}\}_{s,t\in S, s\le t}\big)$$

where for $s,t\in S$ we have that:

(i) A_s is a complex Banach space.
(ii) $\mu_{s,t} : A_s \odot A_t \to A_{st}$ is a linear map.
(iii) $*_s : A_s \to A_{s^*}$ is a conjugate linear isometric map.
(iv) $j_{t,s} : A_s \hookrightarrow A_t$ is a linear isometric map whenever $s \le t$.

It is moreover required that for every $r,s,t\in S$, and every $a\in A_r, b\in A_s$, and $c\in A_t$, we have:

(v) $\mu_{rs,t}(\mu_{r,s}(a\otimes b)\otimes c) = \mu_{r,st}(a\otimes \mu_{s,t}(b\otimes c))$.
(vi) $*_{rs}(\mu_{r,s}(a\otimes b)) = \mu_{s^*,r^*}(*_s(b)\otimes *_r(a))$.
(vii) $*_{s^*}(*_s(a)) = a$.
(viii) $\|\mu_{r,s}(a\otimes b)\| \le \|a\|\,\|b\|$.
(ix) $\|\mu_{r^*,r}(*_r(a)\otimes a)\| = \|a\|^2$.
(x) $\mu_{r^*,r}(*_r(a)\otimes a) \ge 0$ in A_{r^*r}.
(xi) If $r\le s\le t$, then $j_{t,r} = j_{t,s}\circ j_{s,r}$.
(xii) If $r\le r'$ and $s\le s'$, then $j_{r's',rs}\circ \mu_{r,s} = \mu_{r',s'}\circ (j_{r',r}\otimes j_{s',s})$ and $j_{s',s}\circ *_s = *_{s'}\circ j_{s',s}$.

As shown by Exel, axioms (i)–(ix) imply that A_e is a C^*-algebra whenever $e\in E$, with $cd = \mu_{e,e}(c\otimes d)$ and $c^* = *_e(c)$ for $c,d\in A_e$. Hence, the requirement in axiom (x) is meaningful. Exel also shows that the following properties hold:

(xiii) $j_{s,s}$ is the identity map id_{A_s} for every $s\in S$.
(xiv) If $e,f\in E$ and $e\le f$, then $j_{f,e}(A_e)$ is an ideal in A_f.

When no confusion is possible, we will use the simplified notation

$$a\cdot b := \mu_{s,t}(a\otimes b) \text{ and } a^* := *_s(a)$$

whenever $a\in A_s$ and $b\in A_t$, and just write $\mathcal{A} = (\{A_s\}_{s\in S}, \{j_{t,s}\}_{s,t\in S,\, s\le t})$, or even only $\mathcal{A} = \{A_s\}_{s\in S}$ in some cases. If $s\in S$ and $e := s^*s\in E$, then one easily verifies that A_s becomes a right Hilbert A_e-module with respect to the right action given by $(a_s, a_e)\mapsto a_s\cdot a_e \in A_{ss^*s} = A_s$ for $a_s\in A_s$ and $a_e\in A_e$, and the A_e-valued inner product given by $\langle a_s, b_s\rangle = a_s^*\cdot b_s$ for $a_s, b_s\in A_s$. For later use, we also note the following fact:

$$\text{If } e,f\in E, e\le f, \text{and } c,d\in A_e, \text{then } j_{f,e}(c)\cdot d = cd. \tag{1}$$

Indeed, using properties (xii) and (xiii), we get

$$j_{f,e}(c)\cdot d = j_{f,e}(c)\cdot j_{e,e}(d) = j_{fe,ee}(c\cdot d) = j_{e,e}(c\cdot d) = c\cdot d = cd.$$

We also recall that a Fell bundle $\mathcal{A}$ is called *saturated* when the span of $\{a_s \cdot b_t : a_s \in A_s, b_t \in A_t\}$ is dense in A_{st} for all $s, t \in S$.

3.1. An important class of examples of Fell bundles over inverse semigroups arises from (twisted) partial actions of inverse semigroups on C^*-algebras (cf. [6, Section 8]). For the ease of the reader, we sketch this construction in the untwisted case.

Let $\beta : S \to \mathrm{PAut}(A)$ be a partial action of S on a C^*-algebra A. For each $s \in S$, set $J_{s^*} = \mathrm{dom}(\beta_s)$ and $J_s = \mathrm{im}(\beta_s)$, so β_s is a $*$-isomorphism from J_{s^*} onto J_s and $\beta_s^{-1} = \beta_{s^*}$. One may then show (cf. [6, Proposition 3.4, Proposition 3.8 and Proposition 6.3]) that the family $\{J_s\}$ satisfy certain compatibility properties, such as $\beta_s(J_{s^*} \cap J_t) = J_s \cap J_{st}$, $J_s \subset J_{ss^*}$ (and $J_s = J_{ss^*}$ if β is an action), $J_s \subset J_t$ whenever $s \leq t$, and $\beta_e = \mathrm{id}_{J_e}$ when $e \in E$.

Now, for each $s \in S$, we set $A_s = \{(a, s) : a \in J_s\}$ and organize A_s as a Banach space by identifying $a \in J_s$ with $(a, s) \in A_s$. Note that if $s, t \in S$, $a \in J_s$ and $b \in J_t$, then

$$\beta_s^{-1}(a)\, b \in J_{s^*} \cap J_t.$$

Thus,

$$\beta_s\Big(\beta_s^{-1}(a)b\Big) \in J_s \cap J_{st} \quad \text{and} \quad a^* \in J_s, \text{ so } \beta_s^{-1}(a^*) \in J_{s^*}\,;$$

hence one may define $\mu_{s,t}\big((a, s) \otimes (b, t)\big) =: (a, s) \cdot (b, t)$ and $*_s(a, s) =: (a, s)^*$ by

$$(a, s) \cdot (b, t) = \Big(\beta_s\big(\beta_s^{-1}(a)b\big), st\Big) \in A_{st},$$
$$(a, s)^* = \big(\beta_s^{-1}(a^*), s^*\big) \in A_{s^*}.$$

Moreover, if $s \leq t$, then $\beta_s \leq \beta_t$, so $J_s = \mathrm{im}(\beta_s) \subset \mathrm{im}(\beta_t) = J_t$, and one may then define $j_{t,s} : A_s \to A_t$ by

$$j_{t,s}(a, s) = (a, t), \quad \text{for all } a \in J_s.$$

It may then be checked that $\mathcal{A} = (\{A_s\}_{s\in S}, \{j_{t,s}\}_{s,t\in S,\, s\leq t})$ becomes a Fell bundle over S with respect to these operations (cf. [4] for the case of a global (twisted) action).

3.2. Still following [11], a *pre-representation* of a Fell bundle

$$\mathcal{A} = (\{A_s\}_{s\in S}, \{j_{t,s}\}_{s,t\in S,\, s\leq t})$$

in a complex $*$-algebra B is a family $\Pi = \{\pi_s\}_{s\in S}$, where for each $s \in S$,

$$\pi_s : A_s \to B$$

is a linear map such that for all $s, t \in S$, all $a \in A_s$, and all $b \in A_t$, one has:

(a) $\pi_{st}(a \cdot b) = \pi_s(a)\pi_t(b)$,
(b) $\pi_{s^*}(a^*) = \pi_s(a)^*$.

If, in addition, Π satisfies

(c) $\pi_t \circ j_{t,s} = \pi_s$ whenever $s \leq t$, then Π is called a *representation* of $\mathcal{A}$ in B.

We recall that if Π is a pre-representation of $\mathcal{A}$ in a C^*-algebra B, then for each $s \in S$ and $a \in A_s$, we have $\|\pi_s(a)\| \leq \|a\|$. Indeed, as $\pi_e : A_e \mapsto B$ is then a $*$-homomorphism between C^*-algebras for every $e \in E$, we get

$$\begin{aligned}\|\pi_s(a)\|^2 &= \|\pi_s(a)^*\pi_s(a)\| = \|\pi_{s^*}(a^*)\pi_s(a)\| \\ &= \|\pi_{s^*s}(a^* \cdot a)\| \leq \|a^* \cdot a\| = \|a\|^2.\end{aligned}$$

Consider now the direct sum of vector spaces

$$\mathcal{C}_c(\mathcal{A}) = \bigoplus_{s\in S} A_s.$$

We will often write an element $g \in \mathcal{C}_c(\mathcal{A})$ as a formal sum $g = \sum_{s\in S} a_s\delta_s$ where $a_s \in A_s$ for $s \in S$ and all but finitely many a_s are equal to 0. Then $\mathcal{C}_c(\mathcal{A})$ can be given the structure of a complex $*$-algebra by extending linearly the operations

$$\begin{aligned}(a_s\delta_s)(b_t\delta_t) &= (a_s \cdot b_t)\delta_{st}, \\ (a_s\delta_s)^* &= a_s^*\delta_{s^*}.\end{aligned}$$

Alternatively, if one prefers to write $\mathcal{C}_c(\mathcal{A})$ as

$$\mathcal{C}_c(\mathcal{A}) = \left\{ g \in \prod_{s\in S} A_s : g(s) = 0 \text{ for all but finitely many } s \right\},$$

one may define the product and the involution on $\mathcal{C}_c(\mathcal{A})$ by

$$(f * g)(r) = \sum_{s,\, t\in S,\, st\,=\,r} f(s) \cdot g(t) \quad \text{and} \quad f^*(r) = f(r^*)^*$$

for $f, g \in \mathcal{C}_c(\mathcal{A})$ and $r \in S$.

For each $s \in S$, let $\pi_s^0 : A_s \to \mathcal{C}_c(\mathcal{A})$ be defined by

$$\pi_s^0(a_s) = a_s\delta_s$$

for each $a_s \in A_s$. Then $\Pi^0 := \{\pi_s^0\}_{s\in S}$ is a pre-representation of $\mathcal{A}$ in $\mathcal{C}_c(\mathcal{A})$, which satisfies the following universal property (cf. [11, Proposition 3.7]):

To each pre-representation $\Pi = \{\pi_s\}_{s\in S}$ of $\mathcal{A}$ in a $*$-algebra B one may associate a $*$-homomorphism $\Phi_\Pi : \mathcal{C}_c(\mathcal{A}) \to B$ given by

$$\Phi_\Pi\left(\sum_{s\in S} a_s\delta_s\right) = \sum_{s\in S} \pi_s(a_s),$$

which satisfies $\Phi_\Pi \circ \pi_s^0 = \pi_s$ for all $s \in S$. Moreover, the map $\Pi \mapsto \Phi_\Pi$ gives a bijection between pre-representations of $\mathcal{A}$ in B and $*$-homomophisms from $\mathcal{C}_c(\mathcal{A})$ into B.

Consider $g = \sum_{s\in S} a_s\delta_s \in \mathcal{C}_c(\mathcal{A})$. If B is a C^*-algebra and Π is a pre-representation of $\mathcal{A}$ in B, then we have

$$\|\Phi_\Pi(g)\| = \left\|\sum_{s\in S} \pi_s(a_s)\right\| \le \sum_{s\in S} \|\pi_s(a_s)\| \le \sum_{s\in S} \|a_s\|.$$

Hence, if we define

$$\|g\|_{\mathrm{u}} := \sup_{\Phi} \big\{\|\Phi(g)\|\big\},$$

where the supremum is taken over all $*$-homomorphisms from $\mathcal{C}_c(\mathcal{A})$ into any C^*-algebra, then

$$\begin{aligned}\|g\|_{\mathrm{u}} &= \sup\big\{\|\Phi_\Pi(g)\| : \Pi \text{ is a pre-representation of } \mathcal{A} \text{ in some } C^*\text{-algebra}\big\}\\ &\le \sum_{s\in S} \|a_s\| \,<\, \infty.\end{aligned}$$

Hence $\|\cdot\|_{\mathrm{u}}$ gives a C^*-seminorm on $\mathcal{C}_c(\mathcal{A})$. As we will show in the next section, there always exists an injective $*$-representation of $\mathcal{C}_c(\mathcal{A})$ in some C^*-algebra (namely the one associated to the left regular representation of $\mathcal{C}_c(\mathcal{A})$). It follows that $\|\cdot\|_{\mathrm{u}}$ is in fact a C^*-norm, and we may therefore define the *full KS-cross sectional C^*-algebra of* $\mathcal{A}$, denoted by $C^*_{\mathrm{KS}}(\mathcal{A})$, as the completion of $\mathcal{C}_c(\mathcal{A})$ w.r.t. $\|\cdot\|_{\mathrm{u}}$.

We will use the same notation to denote the norm on $C^*_{\mathrm{KS}}(\mathcal{A})$ and will identify $\mathcal{C}_c(\mathcal{A})$ with its canonical copy in $C^*_{\mathrm{KS}}(\mathcal{A})$. We may therefore regard Π^0 as a pre-representation of $\mathcal{A}$ in $C^*_{\mathrm{KS}}(\mathcal{A})$, which is universal in the sense that given any pre-representation Π of $\mathcal{A}$ in a C^*-algebra B, then there exists a unique $*$-homomorphism from $C^*_{\mathrm{KS}}(\mathcal{A})$ into B, which we also denote by Φ_Π, satisfying $\Phi_\Pi \circ \pi^0_s = \pi_s$ for all $s \in S$.

If for example $\mathcal{A}_\alpha$ denotes the Fell bundle associated to an action α of S on a C^*-algebra A, then it is straightforward to verify that $C^*_{\mathrm{KS}}\big(\mathcal{A}_\alpha\big)$ coincides with the full KS-crossed product $A \rtimes^{\mathrm{KS}}_\alpha S$ constructed in [14]. Thus, if β is a partial action of S on a C^*-algebra A, it is natural to define the *full KS-crossed product* by $A \rtimes^{\mathrm{KS}}_\beta S := C^*_{\mathrm{KS}}\big(\mathcal{A}_\beta\big)$, where $\mathcal{A}_\beta$ denotes the Fell bundle over S associated to β in 3.1.

3.3. In [11], Exel defines the full cross sectional C^*-algebra $C^*(\mathcal{A})$ of a Fell bundle $\mathcal{A} = \big(\{A_s\}_{s\in S}, \{j_{t,s}\}_{s,t\in S,\, s\le t}\big)$. This algebra may be described as a quotient of $C^*_{\mathrm{KS}}(\mathcal{A})$. To explain this, we first have to review Exel's construction. Let $\mathcal{N}_\mathcal{A}$ denote the subspace of $\mathcal{C}_c(\mathcal{A})$ spanned by the set

$$\Big\{a_s\delta_s - j_{t,s}(a_s)\delta_t : s,t \in S, s \le t, a_s \in A_s\Big\}.$$

Exel shows in [11, Proposition 3.9] that $\mathcal{N}_\mathcal{A}$ is a two-sided selfadjoint ideal of $\mathcal{C}_c(\mathcal{A})$. It follows that $\mathcal{C}_c(\mathcal{A})/\mathcal{N}_\mathcal{A}$ becomes a complex $*$-algebra in the obvious way. Moreover, [11, Proposition 3.10] says that if Π is a pre-representation of $\mathcal{A}$ in a $*$-algebra B, then Π is a representation of $\mathcal{A}$ if and only if Φ_Π vanishes on $\mathcal{N}_\mathcal{A}$, in which case we will denote the associated $*$-homomorphism from

$\mathcal{C}_c(\mathcal{A})/\mathcal{N}_\mathcal{A}$ into B by $\widetilde{\Phi}_\Pi$. The map $\Pi \mapsto \widetilde{\Phi}_\Pi$ gives then a bijection between representations of $\mathcal{A}$ in B and $*$-homomorphisms from $\mathcal{C}_c(\mathcal{A})/\mathcal{N}_\mathcal{A}$ into B.

Now, for any $g = \sum_{s\in S} a_s\delta_s \in \mathcal{C}_c(\mathcal{A})$ and any representation Π of $\mathcal{A}$ in a C^*-algebra, we have

$$\|\widetilde{\Phi}_\Pi(g + \mathcal{N}_\mathcal{A})\| = \|\Phi_\Pi(g)\| \leq \|g\|_{\mathrm{u}}.$$

It follows that if we define

$$\|g + \mathcal{N}_\mathcal{A}\|_* := \sup_\Psi \big\{\|\Psi\big(g + \mathcal{N}_\mathcal{A}\big)\|\big\}$$

where the supremum is taken over all $*$-homomorphisms Ψ from $\mathcal{C}_c(\mathcal{A})/\mathcal{N}_\mathcal{A}$ into a C^*-algebra, we get

$$\begin{aligned}&\|g + \mathcal{N}_\mathcal{A}\|_* \\ &= \sup\big\{\|\Phi_\Pi(g)\| : \Pi \text{ is a representation of } \mathcal{A} \text{ in some } C^*\text{-algebra}\big\} \\ &\leq \|g\|_{\mathrm{u}},\end{aligned}$$

so $\|\cdot\|_*$ gives a C^*-seminorm on $\mathcal{C}_c(\mathcal{A})/\mathcal{N}_\mathcal{A}$. The *full (Exel) cross sectional C^*-algebra* $C^*(\mathcal{A})$ is then defined as the Hausdorff completion of $\mathcal{C}_c(\mathcal{A})/\mathcal{N}_\mathcal{A}$ w.r.t. to this seminorm.

Letting

$$Q_\mathcal{A} : \mathcal{C}_c(\mathcal{A}) \to \mathcal{C}_c(\mathcal{A})/\mathcal{N}_\mathcal{A}$$

denote the quotient map and

$$R_\mathcal{A} : \mathcal{C}_c(\mathcal{A})/\mathcal{N}_\mathcal{A} \to C^*(\mathcal{A})$$

denote the canonical map, we get that $\iota_\mathcal{A} := R_\mathcal{A} \circ Q_\mathcal{A}$ is a contractive $*$-homomorphism from $\mathcal{C}_c(\mathcal{A})$ into $C^*(\mathcal{A})$ having dense range. Thus, $\iota_\mathcal{A}$ extends to a $*$-homomorphism $q_\mathcal{A}$ from $C^*_{\mathrm{KS}}(\mathcal{A})$ onto $C^*(\mathcal{A})$ such that

$$C^*(\mathcal{A}) \simeq C^*_{\mathrm{KS}}(\mathcal{A}) \,/\, \mathrm{Ker}\, q_\mathcal{A}. \tag{2}$$

Now, for each $s \in S$, define $\pi_s^\mathcal{A} : A_s \to C^*(\mathcal{A})$ by

$$\pi_s^\mathcal{A} = q_\mathcal{A} \circ \pi_s^0 \ (= \iota_\mathcal{A} \circ \pi_s^0).$$

Then one checks (cf. [11, Proposition 3.12] and the proof of [11, Proposition 3.13]) that

$$\Pi^\mathcal{A} := \big\{\pi_s^\mathcal{A}\big\}_{s\in S}$$

is a representation of $\mathcal{A}$ in $C^*(\mathcal{A})$ satisfying the following universal property: given any representation $\Pi = \{\pi_s\}_{s\in S}$ of $\mathcal{A}$ in a C^*-algebra B, there exists a unique $*$-homomorphism $\Psi_\Pi : C^*(\mathcal{A}) \to B$ such that $\Psi_\Pi \circ \pi_s^\mathcal{A} = \pi_s$ for all $s \in S$. It follows immediately that $\Phi_\Pi = \Psi_\Pi \circ q_\mathcal{A}$ for every such representation Π.

The ideal $\mathrm{Ker}\, q_\mathcal{A}$ has a natural description in terms of $\mathcal{N}_\mathcal{A}$. Indeed, letting $\mathcal{M}_\mathcal{A}$ denote the ideal of $C^*_{\mathrm{KS}}(\mathcal{A})$ given by $\mathcal{M}_\mathcal{A} := \overline{\mathcal{N}_\mathcal{A}}^{\|\cdot\|_{\mathrm{u}}}$, we have

$$\mathrm{Ker}\, q_\mathcal{A} = \mathcal{M}_\mathcal{A}.$$

To prove this, we first note that since $q_{\mathcal{A}}(\mathcal{N}_{\mathcal{A}}) = \iota_{\mathcal{A}}(\mathcal{N}_{\mathcal{A}}) = \{0\}$, we have

$$\mathcal{M}_{\mathcal{A}} \subset \operatorname{Ker} q_{\mathcal{A}}.$$

Next, let $s \in S$ and define $\omega_s : A_s \to C^*_{\mathrm{KS}}(\mathcal{A})/\mathcal{M}_{\mathcal{A}}$ by

$$\omega_s(a_s) := \pi^0_s(a_s) + \mathcal{M}_{\mathcal{A}}$$

for every $a_s \in A_s$. As $\mathcal{M}_{\mathcal{A}}$ contains $\mathcal{N}_{\mathcal{A}}$, one easily verifies that $\Omega = \{\omega_s\}_{s\in S}$ is a representation of $\mathcal{A}$ in $C^*_{\mathrm{KS}}(\mathcal{A})/\mathcal{M}_{\mathcal{A}}$. The associated $*$-homomorphism Φ_Ω from $C^*_{\mathrm{KS}}(\mathcal{A})$ into $C^*_{\mathrm{KS}}(\mathcal{A})/\mathcal{M}_{\mathcal{A}}$ is then nothing but the quotient map. Since $\Phi_\Omega = \Psi_\Omega \circ q_{\mathcal{A}}$, it follows that

$$\operatorname{Ker} q_{\mathcal{A}} \subset \operatorname{Ker} \Phi_\Omega = \mathcal{M}_{\mathcal{A}}.$$

Thus we get $\operatorname{Ker} q_{\mathcal{A}} = \mathcal{M}_{\mathcal{A}}$, as desired. It follows that

$$C^*(\mathcal{A}) \simeq C^*_{\mathrm{KS}}(\mathcal{A}) \,/\mathcal{M}_{\mathcal{A}}. \tag{3}$$

3.4. In [11], Exel also constructs the reduced cross sectional C^*-algebra $C^*_r(\mathcal{A})$ of a Fell bundle $\mathcal{A} = (\{A_s\}_{s\in S}, \{j_{t,s}\}_{s,t\in S,\, s\le t})$. His construction, which is somewhat involved, may be summarized as follows.

Consider first $e \in E$ and $s \in S$ such that $e \le s$. Then $j_{s,e}$ gives an isometric embedding of A_e into A_s, so one may view A_e as a subspace of A_s. Let φ_e be a continuous linear functional on A_e. Exel shows in [11, Proposition 6.1] that φ_e extends to a continuous linear functional $\tilde{\varphi}^s_e$ on A_s satisfying $\|\tilde{\varphi}^s_e\| = \|\varphi_e\|$ and

$$\tilde{\varphi}^s_e(x) = \lim_i \varphi_e(xu_i) = \lim_i \varphi_e(u_i x) = \lim_i \varphi_e(u_i x u_i)$$

for every approximate unit $\{u_i\}_i$ for A_e and every $x \in A_s$.

Next, let $e \in E$ and let φ_e be a state on A_e. Define $\tilde{\varphi}_e$ on $\mathcal{C}_c(\mathcal{A})$ by

$$\tilde{\varphi}_e\left(\sum_{s\in S} a_s\delta_s\right) = \sum_{s\in S,\, s\ge e} \tilde{\varphi}^s_e(a_s).$$

Then, as shown in [11, Proposition 6.9], $\tilde{\varphi}_e$ is a state on $\mathcal{C}_c(\mathcal{A})$ when $\mathcal{C}_c(\mathcal{A})$ is considered as a normed $*$-algebra with respect to the norm $\|g\|_1 = \sum_{s\in S}\|a_s\|$ for $g = \sum_s a_s\delta_s \in \mathcal{C}_c(\mathcal{A})$.

Now, let $\mathcal{E} = \{A_e\}_{e\in E}$ denote the restriction of $\mathcal{A}$ to the semilattice E, let $\Pi^{\mathcal{E}} = \{\pi^{\mathcal{E}}_e\}_{e\in E}$ denote the universal representation of $\mathcal{E}$ in $C^*(\mathcal{E})$, and fix a pure state φ on $C^*(\mathcal{E})$. For each $e \in E$, one has that $\pi^{\mathcal{E}}_e(A_e)$ is an ideal of $C^*(\mathcal{E})$, and $\varphi_e := \varphi \circ \pi^{\mathcal{E}}_e$ is a state on A_e as long as $\varphi_e \ne 0$, that is, whenever φ does not vanish on $\pi^{\mathcal{E}}_e(A_e)$. Moreover, [11, Proposition 7.4] says that there exists a positive linear functional $\tilde{\varphi}$ on $\mathcal{C}_c(\mathcal{A})$ such that:

(i) For every $s \in S$ and $a_s \in A_s$, one has that

$$\tilde{\varphi}(a_s\delta_s) = \begin{cases} \tilde{\varphi}^s_e(a_s) & \text{if there exists } e \in E \text{ such that } \varphi_e \ne 0 \text{ and } e \le s, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) For every $e \in E$ and every $a_e \in A_e$ one has that

$$\tilde{\varphi}(a_e\delta_e) = \varphi_e(a_e) = \varphi(\pi_e^{\mathcal{E}}(a_e)).$$

(iii) $\|\tilde{\varphi}\| \leq \|\varphi\|$.

(iv) $\tilde{\varphi}$ vanishes on the ideal $\mathcal{N}_{\mathcal{A}}$.

For later use we note that if S is E^*-unitary, then (i) and (ii) together just say that

$$\tilde{\varphi}(a_s\delta_s) = \begin{cases} \varphi(\pi_s^{\mathcal{E}}(a_s)) & \text{if } s \in E, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Let $H_{\tilde{\varphi}}$ be the Hilbert space completion of $\mathcal{C}_c(\mathcal{A})$ with respect to the pre-inner-product given by

$$\langle g, h\rangle_{\tilde{\varphi}} = \tilde{\varphi}(h^*g) \quad \text{for } g, h \in \mathcal{C}_c(\mathcal{A}),$$

and let $h \mapsto \widehat{h}$ denote the canonical map $\mathcal{C}_c(\mathcal{A}) \to H_{\tilde{\varphi}}$. The GNS representation of $\tilde{\varphi}$, which is defined in the usual way by

$$\Upsilon_{\tilde{\varphi}}(g)\widehat{h} = \widehat{gh} \quad \text{for } g, h \in \mathcal{C}_c(\mathcal{A}),$$

gives a $*$-representation of $\mathcal{C}_c(\mathcal{A})$ on $H_{\tilde{\varphi}}$.

Exel's reduced cross sectional C^*-algebra $C_r^*(\mathcal{A})$ is then defined as the Hausdorff completion of $\mathcal{C}_c(\mathcal{A})$ with respect to the C^*-seminorm given by

$$\|g\|'_{\mathrm{r}} = \sup_{\varphi} \|\Upsilon_{\tilde{\varphi}}(g)\|$$

where the supremum is taken over the set $\mathcal{P}(C^*(\mathcal{E}))$ consisting of all pure states of $C^*(\mathcal{E})$. Note that the kernel of $\Upsilon_{\tilde{\varphi}}$ is given by

$$\mathrm{Ker}\Upsilon_{\tilde{\varphi}} = \{g \in \mathcal{C}_c(\mathcal{A}) : \tilde{\varphi}(h^*gh') = 0 \text{ for all } h, h' \in \mathcal{C}_c(\mathcal{A})\}.$$

So if $\mathcal{K}_{\mathcal{A}} := \{g \in \mathcal{C}_c(\mathcal{A}) : \|g\|'_{\mathrm{r}} = 0\}$, then

$$\mathcal{K}_{\mathcal{A}} = \bigcap_{\varphi \in \mathcal{P}(C^*(\mathcal{E}))} \mathrm{Ker}\Upsilon_{\tilde{\varphi}},$$

and $C_r^*(\mathcal{A})$ is the completion of $\mathcal{C}_c(\mathcal{A})/\mathcal{K}_{\mathcal{A}}$ with respect to the norm

$$\|g + \mathcal{K}_{\mathcal{A}}\|''_{\mathrm{r}} := \|g\|'_{\mathrm{r}}.$$

Letting $\iota_{\mathcal{A}}^{\mathrm{red}} : \mathcal{C}_c(\mathcal{A}) \to C_r^*(\mathcal{A})$ denote the canonical $*$-homomorphism, one gets the left regular representation $\Pi^{\mathrm{red}} = \{\pi_s^{\mathrm{red}}\}_{s\in S}$ of $\mathcal{A}$ in $C_r^*(\mathcal{A})$ by setting

$$\pi_s^{\mathrm{red}} = \iota_{\mathcal{A}}^{\mathrm{red}} \circ \pi_s^0$$

for each $s \in S$. The associated $*$-homomorphism $\Psi_{\Pi^{\mathrm{red}}} : C^*(\mathcal{A}) \to C_r^*(\mathcal{A})$ is then surjective (cf. [11, Proposition 8.6]).

4. The left regular representation of $C^*_{\mathbf{KS}}(\mathcal{A})$

Let $\mathcal{A}$ be a Fell bundle over S. In this section we will describe how one may define the left regular representation Φ_Λ of $\mathcal{C}_c(\mathcal{A})$ in a certain C^*-algebra B naturally associated with $\mathcal{A}$, and show that Φ_Λ is injective. We will first construct the left regular pre-representation Λ of $\mathcal{A}$ in B. The associated $*$-homomorphim Φ_Λ from $C^*_{\mathrm{KS}}(\mathcal{A})$ into B will then give the left regular representation of $C^*_{\mathrm{KS}}(\mathcal{A})$.

4.1. We begin by recalling some notation and a few facts that will be useful in our construction.

For each $u \in S$, we set

$$D(u) = \{s \in S : ss^* \le u^*u\},$$

so $D(u^*) = \{v \in S : vv^* \le uu^*\}$, and for each $e \in E$, we set

$$S_e = \{s \in S : s^*s = e\}.$$

The Wagner–Preston theorem (and its proof), see for example [17, Proposition 2.1.3], says that for each $u \in S$, the map $\gamma_u : D(u) \to D(u^*)$ given by $\gamma_u(s) = us$ is a bijection, with inverse given by $\gamma_{u^*} : D(u^*) \to D(u)$. Moreover, it says that the map $\gamma : u \mapsto \gamma_u$ is an injective homomorphism from S into $\mathcal{I}(S)$. A part of the last statement is that for $u_1, u_2, s \in S$, we have

(5) $s \in D(u_1u_2)$ if and only if $s \in D(u_2)$ and $u_2s \in D(u_1)$.

Consider $u \in S$ and assume $s \in S_e \cap D(u)$ for some $e \in E$. Then we have

$$(us)^*us = s^*u^*us = s^*u^*u\, ss^*s = s^*ss^*s = s^*s = e$$

so $us \in S_e \cap D(u^*)$. Hence, if $v \in S_e \cap D(u^*)$, then $u^*v \in S_e \cap D(u)$. It follows that the map $s \mapsto us$ gives a bijection from $S_e \cap D(u)$ onto $S_e \cap D(u^*)$, with inverse given by $v \mapsto u^*v$ for $v \in S_e \cap D(u^*)$.

4.2. Let now $\mathcal{A} = \{A_s\}_{s\in S}$ be a Fell bundle over S. Given $e \in E$, set

$$X_e = \left\{\xi \in \prod_{s\in S_e} A_s : \sum_{s\in S_e} \xi(s)^* \cdot \xi(s) \text{ is norm convergent in } A_e\right\}.$$

Note that the sum $\sum_{s\in S_e} \xi(s)^* \cdot \xi(s)$ makes sense since

$$\xi(s)^* \cdot \xi(s) \in A_{s^*s} = A_e$$

for each $s \in S_e$. Proceeding in the same way as for the direct sum of a family of right Hilbert C^*-modules over the same C^*-algebra [15], it is not difficult to check that X_e is a subspace of the product vector space $\prod_{s\in S_e} A_s$, which becomes a right Hilbert A_e-module with respect to the operations

$$(\xi \cdot a)(s) = \xi(s) \cdot a \in A_{se} = A_s,$$

$$\langle \xi, \eta\rangle_e = \sum_{s\in S_e} \xi(s)^* \cdot \eta(s) \in A_e,$$

for $\xi, \eta \in X_e$, $a \in A_e$ and $s \in S_e$.

Consider $e \in E$ and $u \in S$. For $a_u \in A_u$, we let

$$\lambda_{e,u}(a_u) : X_e \to X_e$$

be the linear operator defined by

$$\big(\lambda_{e,u}(a_u)\xi\big)(v) = \begin{cases} a_u \cdot \xi(u^*v) & \text{if } v \in D(u^*), \\ 0 & \text{otherwise.} \end{cases}$$

for $\xi \in X_e$ and $v \in S_e$. To see that $\lambda_{e,u}(a_u)$ is well defined, let $\xi \in X_e$. If $v \in S_e \cap D(u^*)$, then $u^*v \in S_e$, and $\xi(u^*v) \in A_{u^*v}$, so we get

$$a_u \cdot \xi(u^*v) \in A_u \cdot A_{u^*v} \subset A_{uu^*v} = A_{uu^*vv^*v} = A_{vv^*v} = A_v.$$

Thus we see that $\lambda_{e,u}(a_u)\xi$ lies in $\prod_{v\in S_e} A_v$. Moreover, if $v \in S_e \cap D(u^*)$, then one readily verifies that the map $b \mapsto a_u \cdot b$ is an adjointable linear map from A_{u^*v} into A_v (with adjoint map $c \mapsto a_u^* \cdot c$); thus, using [15, Proposition 1.2], we get

$$\big(a_u \cdot \xi(u^*v)\big)^* \cdot \big(a_u \cdot \xi(u^*v)\big) \;\le\; \|a_u\|^2 \; \xi(u^*v)^* \cdot \xi(u^*v).$$

Now, since $\xi \in X_e$, the sum

$$\sum_{v \in S_e \cap D(u^*)} \xi(u^*v)^* \cdot \xi(u^*v)$$

is norm-convergent in A_e, and it follows that

$$\sum_{v\in S_e} \big(\lambda_{e,u}(a_u)\xi\big)(v)^* \cdot \big(\lambda_{e,u}(a_u)\xi\big)(v) \;=\; \sum_{v\in S_e \cap D(u^*)} \big(a_u \cdot \xi(u^*v)\big)^* \cdot \big(a_u \cdot \xi(u^*v)\big)$$

is also norm-convergent in A_e. Thus, $\lambda_{e,u}(a_u)\xi \in X_e$, as desired.

Next, we show that $\lambda_{e,u}(a_u) \in \mathcal{L}(X_e)$. For $\xi, \eta \in X_e$, we have

$$\begin{aligned} \langle \lambda_{e,u}(a_u)\xi, \eta\rangle_e &= \sum_{v\in S_e} (\lambda_{e,u}(a_u)\xi)(v)^* \cdot \eta(v) \\ &= \sum_{v\in S_e\cap D(u^*)} (a_u \cdot \xi(u^*v))^* \cdot \eta(v) \\ &= \sum_{v\in S_e\cap D(u^*)} \xi(u^*v)^* \cdot a_u^* \cdot \eta(v) \\ &= \sum_{s\in S_e\cap D(u)} \xi(s)^* \cdot a_u^* \cdot \eta(us) \\ &= \sum_{s\in S_e} \xi(s)^* \cdot (\lambda_{e,u^*}(a_u^*)\eta)(s) \\ &= \langle \xi, \lambda_{e,u^*}(a_u^*)\eta\rangle_e, \end{aligned}$$

where we have used that the map $v \mapsto u^*v$ is a bijection from $S_e \cap D(u^*)$ onto $S_e \cap D(u)$. This shows that $\lambda_{e,u}(a_u)$ is an adjointable operator on X_e, with adjoint given by

$$\lambda_{e,u}(a_u)^* = \lambda_{e,u^*}(a_u^*). \tag{6}$$

Thus we get a map $\lambda_{e,u} : A_u \to \mathcal{L}(X_e)$ for each $e \in E$ and each $u \in S$. For each $e \in E$ we set

$$\Lambda^e := \{\lambda_{e,u}\}_{u\in S}.$$

To show that Λ^e is a pre-representation of $\mathcal{A}$ in $\mathcal{L}(X_e)$, in view of (6), we only have to show that for $u, u' \in S$, $a \in A_u$ and $a' \in A_{u'}$, we have

$$\lambda_{e,uu'}(a \cdot a') = \lambda_{e,u}(a)\, \lambda_{e,u'}(a'). \tag{7}$$

To prove this, consider $\xi \in X_e$ and $v \in S_e$. Then

$$\begin{aligned}\big(\lambda_{e,uu'}(a\cdot a')\xi\big)(v) &= \begin{cases} a\cdot a'\cdot \xi((uu')^*v) & \text{if } v \in D\big((uu')^*\big),\\ 0 & \text{otherwise,}\end{cases}\\ &= \begin{cases} a\cdot a'\cdot \xi(u'^*u^*v) & \text{if } v \in D\big(u'^*u^*\big),\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

while

$$\begin{aligned}&\big(\lambda_{e,u}(a)\lambda_{e,u'}(a')\xi\big)(v)\\ &= \begin{cases} a\cdot (\lambda_{e,u'}(a')\xi)(u^*v) & \text{if } v \in D(u^*),\\ 0 & \text{otherwise}\end{cases}\\ &= \begin{cases} a\cdot a'\cdot \xi(u'^*u^*v) & \text{if } v \in D(u^*) \text{ and } u^*v \in D(u'^*),\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Now, using (5) with $u_1 = u'^*$ and $u_2 = u^*$ gives that $v \in D\big(u'^*u^*\big)$ if and only if $v \in D(u^*)$ and $u^*v \in D(u'^*)$, so we see that

$$\big(\lambda_{e,uu'}(a\cdot a')\xi\big)(v) = \big(\lambda_{e,u}(a)\lambda_{e,u'}(a')\xi\big)(v).$$

It follows that (7) holds, as desired.

We can now form the product pre-representation $\Lambda = \prod_{e\in E}\Lambda^e$ of $\mathcal{A}$ in the product C^*-algebra $B := \prod_{e\in E}\mathcal{L}(X_e)$. It is natural to call Λ the *left regular pre-representation* of $\mathcal{A}$ in B. It is given by $\Lambda = \{\lambda_u\}_{u\in S}$, where $\lambda_u : A_u \to B$ is defined by

$$\lambda_u(a_u) = \Big(\lambda_{e,u}(a_u)\Big)_{e\in E}$$

for $u \in S$ and $a_u \in S$. The associated $*$-homomorphism $\Phi_\Lambda : \mathcal{C}_c(\mathcal{A}) \to B$ (resp. $C^*_{\mathrm{KS}}(\mathcal{A}) \to B$), which satisfies

$$\Phi_\Lambda\left(\sum_{u\in S} a_u\delta_u\right) = \left(\sum_{u\in S}\lambda_{e,u}(a_u)\right)_{e\in E},$$

will be called the *left regular representation of* $\mathcal{C}_c(\mathcal{A})$ (resp. $C^*_{\mathrm{KS}}(\mathcal{A})$) in B. Note that $\Phi_\Lambda = \prod_{e\in E}\Phi_{\Lambda^e}$, since

$$\left(\prod_{e\in E}\Phi_{\Lambda^e}\right)\left(\sum_{u\in S} a_u\delta_u\right) = \left(\Phi_{\Lambda^e}\left(\sum_{u\in S} a_u\delta_u\right)\right)_{e\in E} = \left(\sum_{u\in S}\lambda_{e,u}(a_u)\right)_{e\in E}.$$

4.3. Our aim is to show that Φ_Λ is injective on $\mathcal{C}_c(\mathcal{A})$ (cf. Theorem 4.3). The following lemma will be crucial.

Lemma 4.1. *Assume that $g = \sum_{u\in S} a_u\delta_u \in \mathcal{C}_c(\mathcal{A})$ satisfies $\Phi_\Lambda(g) = 0$ and let $e, f \in E$ with $e \leq f$.*

Then, for each $t \in S$ and each $b \in A_e$, we have

$$\sum_{u\in S,\, f\leq u^*u,\, ue=t} a_u \cdot b = 0.$$

Note that here (and elsewhere), we use the convention that a sum over an empty index set is equal to 0.

Proof. For each $s \in S$ and $a \in A_s$, we will let $a \odot \varepsilon_s$ denote the element of X_{s^*s} given for each $t \in S_{s^*s}$ by

$$(a \odot \varepsilon_s)(t) = \begin{cases} a & \text{if } t = s, \\ 0 & \text{if } t \neq s. \end{cases}$$

For every $v \in E$, we set $g_v = \sum_{u\in S_v} a_u\delta_u$. Then $g_v \in \mathcal{C}_c(\mathcal{A})$, $g_v = 0$ for all but finitely many v in E, and

$$g = \sum_{v\in E} g_v.$$

Moreover,

$$0 = \Phi_\Lambda(g) = \sum_{v\in E}\Phi_\Lambda(g_v) = \left(\sum_{v\in E}\sum_{u\in S_v}\lambda_{p,u}(a_u)\right)_{p\in E}. \tag{8}$$

Now, consider $v \in E$, $u \in S_v$, $a \in A_u$ and $a' \in A_f$.

Note first that $a' \odot \varepsilon_f \in X_{f^*f} = X_f$. Moreover, if $f \leq v$, then

$$(uf)^*uf = fu^*uf = fvf = f,$$

so $uf \in S_f$ and $(a\cdot a') \odot \varepsilon_{uf} \in X_{(uf)^*uf} = X_f$. We claim that

$$\lambda_{f,u}(a)(a' \odot \varepsilon_f) = \begin{cases} (a\cdot a') \odot \varepsilon_{uf} & \text{if } f \leq v, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

To prove this claim, let $t \in S_f$. Then we have $t \in D(u^*)$, that is, $tt^* \leq uu^*$, if and only if $f = t^*t \leq u^*u = v$. As

$$\big(\lambda_{f,u}(a)(a' \odot \varepsilon_f)\big)(t) = \begin{cases} a\cdot(a' \odot \varepsilon_f)(u^*t) & \text{if } t \in D(u^*) \\ 0 & \text{otherwise,} \end{cases}$$

we see that $\lambda_{f,u}(a)(a' \odot \varepsilon_f) = 0$ when $f \not\leq v$.

If $f \leq v$, thus $t \in S_f \cap D(u^*)$, then we have $u^*t \in S_f \cap D(u)$, with $u^*t = f$ if and only $uf = t$ (cf. 4.1), so we get

$$\begin{aligned}\big(\lambda_{f,u}(a)(a' \odot \varepsilon_f)\big)(t) &= a \cdot (a' \odot \varepsilon_f)(u^*t) \\ &= \begin{cases} a \cdot a' & \text{if } t = uf \\ 0 & \text{otherwise} \end{cases} = \big((a \cdot a') \odot \varepsilon_{uf}\big)(t).\end{aligned}$$

We have thus shown that $\lambda_{f,u}(a)(a' \odot \varepsilon_f) = (a \cdot a') \odot \varepsilon_{uf}$ whenever $f \leq v$, and this finishes the proof of (9).

Let now $b \in A_e$. By the Cohen–Hewitt factorization theorem [13, Theorem 32.22] we can write b as a product $b = cd$ where $c, d \in A_e$. As $e \leq f$, we get from (1) that

$$j_{f,e}(c) \cdot d = cd = b. \tag{10}$$

For each $v \in E$ we get from (9) that

$$\sum_{u \in S_v} \lambda_{f,u}(a_u)(j_{f,e}(c) \odot \varepsilon_f) = \begin{cases} \sum_{u \in S_v} (a_u \cdot j_{f,e}(c)) \odot \varepsilon_{uf} & \text{if } f \leq v, \\ 0 & \text{otherwise.} \end{cases}$$

Using (8) it then follows that

$$0 = \sum_{v \in E} \sum_{u \in S_v} \lambda_{f,u}(a_u)(j_{f,e}(c) \odot \varepsilon_f) = \sum_{\{v \in E : f \leq v\}} \sum_{u \in S_v} (a_u \cdot j_{f,e}(c)) \odot \varepsilon_{uf}.$$

By looking at individual coefficients we can then conclude that in $\mathcal{C}_c(\mathcal{A})$,

$$0 = \sum_{\{v \in E : f \leq v\}} \sum_{u \in S_v} (a_u \cdot j_{f,e}(c))\, \delta_{uf} \tag{11}$$

Since $e \leq f$ we get from (11) and (10) that

$$0 = \left(\sum_{\{v \in E : f \leq v\}} \sum_{u \in S_v} (a_u \cdot j_{f,e}(c))\, \delta_{uf} \right) \big(d\delta_e\big) = \sum_{\{v \in E : f \leq v\}} \sum_{u \in S_v} (a_u \cdot b)\, \delta_{ue}.$$

We see that given $t \in S$, the t-coefficient of the sum on the right hand side of the above equation is

$$\sum_{u \in S,\, f \leq u^*u, ue = t} a_u \cdot b,$$

which must then be equal to 0. $\square$

We will need another lemma. Let F be a semilattice and A be a Banach space. As usual, we will denote the dual space of A, consisting of all continuous linear functionals on A, by A^*. We let $\mathcal{C}_c(F, A)$ denote the vector space of all finitely supported functions from F to A. We will describe an element of $\mathcal{C}_c(F, A)$ as a formal sum $\sum_{f \in F} a_f \delta_f$ where each $a_f \in A$ and

$a_f = 0$ for all but finitely many f in F. Given $\psi \in A^*$ and $e \in F$, we define $\theta_{\psi,e} : \mathcal{C}_c(F, A) \to \mathbb{C}$ to be the linear functional given by

$$\theta_{\psi,e}\left(\sum_{f\in F} a_f \delta_f\right) = \sum_{f\in F,\, f\geq e} \psi(a_f).$$

Lemma 4.2. *Let A be a Banach space and F be a semilattice. Then the set $\{\theta_{\psi,e} : e \in F, \psi \in A^*\}$ separates the elements of $\mathcal{C}_c(F, A)$.*

Proof. Suppose $\sum_{f\in F} a_f \delta_f \neq 0$. Since $a_f = 0$ for all but finitely many f in F, we can choose $e \in F$ such that $a_e \neq 0$ and $a_f = 0$ for all $f \in F \setminus \{e\}$ satisfying $f \geq e$. We may then pick $\psi \in A^*$ such that $\psi(a_e) \neq 0$, and this gives

$$\theta_{\psi,e}\left(\sum_{f\in F} a_f \delta_f\right) = \sum_{f\in F,\, f\geq e} \psi(a_f) = \psi(a_e) \neq 0. \qquad \square$$

The following theorem is a generalization of Wordingham's theorem [17, Theorem 2.1.1], and our proof follows the pattern of Wordingham's original proof.

Theorem 4.3. *Let $\mathcal{A} = \{A_s\}_{s\in S}$ be a Fell bundle over an inverse semigroup S. Then the left regular representation of $\mathcal{C}_c(\mathcal{A})$ is injective.*

Proof. Let $g \in \mathcal{C}_c(\mathcal{A})$ and express g as a sum $g = \sum_{u\in S} a_u \delta_u$, where $\operatorname{supp}(g) = \{u \in S : a_u \neq 0\}$ is finite. Assume $\Phi_\Lambda(g) = 0$. We want to show that $a_t = 0$ for each $t \in S$. Since $g = \sum_{e\in E} \sum_{u\in S_e} a_u \delta_u$ it is sufficient to show that for any $e \in E$, $a_t = 0$ when $t \in S_e$.

Fix $e \in E$ and consider $t \in S$. Let F be the subsemilattice of E given by

$$F = \{v \in E : e \leq v\}.$$

Also, let $f \in F$ and $b \in A_e$. For each $v \in F$ set

$$\beta^t_v = \sum_{u\in S,\, v=u^*u,\, ue=t} a_u \cdot b \ \in A_t,$$

$$\beta^t = \sum_{v\in F} \beta^t_v \,\delta_v \ \in \mathcal{C}_c(F, A_t).$$

Note that β^t has finite support since $\beta^t_v = 0$ if $v \notin \{u^*u : u \in \operatorname{supp}(g)\}$, and $\operatorname{supp}(g)$ is finite. Now, for each $\psi \in (A_t)^*$, we get from Lemma 4.1 that

$$\begin{aligned}\theta_{\psi,f}\big(\beta^t\big) &= \sum_{v\in F,\, f\leq v} \psi\big(\beta^t_v\big)\\ &= \psi\left(\sum_{u\in S,\, f\leq u^*u,\, ue=t} a_u \cdot b\right) = 0.\end{aligned}$$

Then $\beta^t = 0$ by Lemma 4.2, so $\beta^t_v = 0$ for each $v \in F$. In particular, since $e \in F$, we get

$$\sum_{u \in S_e,\, ue = t} a_u \cdot b = \beta^t_e = 0. \tag{12}$$

Assume now that $t \in S_e$. If $u \in S_e$ satisfies that $ue = t$, then we have $ut^*t = t$ and $u^*u = t^*t$, which together imply that $u = t$. So (12) gives that $a_t \cdot b = 0$. Choosing $b = a_t^* \cdot a_t \in A_{t^*t} = A_e$, we get $a_t \cdot a_t^* \cdot a_t = 0$, so $(a_t^* \cdot a_t)^2 = a_t^* \cdot a_t \cdot a_t^* \cdot a_t = 0$, hence $a_t^* \cdot a_t = 0$, and axiom (x) in the definition of a Fell bundle gives that $a_t = 0$, as desired. $\square$

4.4. We define the *reduced KS-cross sectional C^*-algebra $C^*_{r,\mathrm{KS}}(\mathcal{A})$ of $\mathcal{A}$* as the completion of $\mathcal{C}_c(\mathcal{A})$ with respect to the norm $\|\cdot\|_r$ given by

$$\|g\|_{\mathrm{r}} := \|\Phi_\Lambda(g)\|$$

for $g \in \mathcal{C}_c(\mathcal{A})$. Alternatively, we may consider $C^*_{r,\mathrm{KS}}(\mathcal{A})$ to be given as the norm-closure of $\Phi_\Lambda(\mathcal{C}_c(\mathcal{A}))$ in B, or, equivalently, as $\Phi_\Lambda(C^*_{\mathrm{KS}}(\mathcal{A}))$.

Recall that $\mathcal{N}_{\mathcal{A}}$ denotes the two-sided selfadjoint ideal of $\mathcal{C}_c(\mathcal{A})$ spanned by the set

$$\{a_s\delta_s - j_{t,s}(a_s)\delta_t : s, t \in S, s \le t, a_s \in A_s\}.$$

We define $\mathcal{I}_{\mathcal{A}}$ to be the closure of $\mathcal{N}_{\mathcal{A}}$ inside $C^*_{r,\mathrm{KS}}(\mathcal{A})$. In other words, we set

$$\mathcal{I}_{\mathcal{A}} = \overline{\Phi_\Lambda(\mathcal{N}_{\mathcal{A}})}.$$

It is easy to check that $\mathcal{I}_{\mathcal{A}}$ is an ideal of $C^*_{r,\mathrm{KS}}(\mathcal{A})$. Hence we may form the quotient C^*-algebra

$$C^*_{r,\mathrm{alt}}(\mathcal{A}) := C^*_{r,\mathrm{KS}}(\mathcal{A})/\mathcal{I}_{\mathcal{A}},$$

which provides an alternative version of the reduced cross sectional C^*-algebra of $\mathcal{A}$. We will let $q^r_{\mathcal{A}} : C^*_{r,\mathrm{KS}}(\mathcal{A}) \to C^*_{r,\mathrm{alt}}(\mathcal{A})$ denote the quotient map. It is not clear whether $C^*_{r,\mathrm{alt}}(\mathcal{A})$ is isomorphic to Exel's reduced C^*-algebra $C^*_r(\mathcal{A})$ (cf. 3.4). We will show in Section 7 that this is true whenever S is strongly E^*-unitary and $A_0 = \{0\}$ if S has a zero. For $u \in S$ let $\lambda^{\mathrm{alt}}_u : A_u \to C^*_{r,\mathrm{alt}}(\mathcal{A})$ be defined by

$$\lambda^{\mathrm{alt}}_u(a_u) = \lambda_u(a_u) + \mathcal{I}_{\mathcal{A}}$$

for all $a_u \in A_u$. It is then almost immediate that $\Lambda^{\mathrm{alt}} := \{\lambda^{\mathrm{alt}}_u\}_{u \in S}$ is a representation of $\mathcal{A}$ in $C^*_{r,\mathrm{alt}}(\mathcal{A})$. Using the universal property of $C^*(\mathcal{A})$ we get a surjective $*$-homomorphism $\Psi_{\Lambda^{\mathrm{alt}}}$ from $C^*(\mathcal{A})$ onto $C^*_{r,\mathrm{alt}}(\mathcal{A})$ satisfying

$$\Psi_{\Lambda^{\mathrm{alt}}}\big(\pi^{\mathcal{A}}_u(a_u)\big) = \lambda^{\mathrm{alt}}_u(a_u) = \lambda_u(a_u) + \mathcal{I}_{\mathcal{A}}$$

for all $u \in S$ and $a_u \in A_u$. Similarly, we get a surjective $*$-homomorphism $\Phi_{\Lambda^{\mathrm{alt}}}$ from $C^*_{\mathrm{KS}}(\mathcal{A})$ onto $C^*_{r,\mathrm{alt}}(\mathcal{A})$, which satisfies

$$\Phi_{\Lambda^{\mathrm{alt}}} = q^r_{\mathcal{A}} \circ \Phi_\Lambda = \Psi_{\Lambda^{\mathrm{alt}}} \circ q_{\mathcal{A}}.$$

The following commutative diagram sums up the relationship between the various algebras and some of the $*$-homomorphisms defined so far.

$$\begin{tikzcd}
\mathcal{C}_c(\mathcal{A}) \arrow[r, hook] \arrow[d, two heads, "Q_{\mathcal{A}}"] \arrow[dr, "\iota_{\mathcal{A}}"] \arrow[ddr, bend right=60, "\iota^{\mathrm{red}}_{\mathcal{A}}"'] & C^*_{\mathrm{KS}}(\mathcal{A}) \arrow[r, two heads, "\Phi_\Lambda"] \arrow[d, two heads, "q_{\mathcal{A}}"] \arrow[dr, two heads, "\Phi_{\Lambda^{\mathrm{alt}}}"] & C^*_{r,\mathrm{KS}}(\mathcal{A}) \arrow[d, two heads, "q^{\mathrm{r}}_{\mathcal{A}}"] \\
\mathcal{C}_c(\mathcal{A})/\mathcal{N}_{\mathcal{A}} \arrow[r, "R_{\mathcal{A}}"] & C^*(\mathcal{A}) \arrow[r, two heads, "\Psi_{\Lambda^{\mathrm{alt}}}"] \arrow[d, two heads, "\Psi_{\Pi^{\mathrm{red}}}"] & C^*_{r,\mathrm{alt}}(\mathcal{A}) \\
 & C^*_r(\mathcal{A}) &
\end{tikzcd}$$

5. Fell bundles over semilattices

In this section we look at the case where $S = E$ is a semilattice, and consider a Fell bundle $\mathcal{E} = \{A_e\}_{e\in E}$. Since $E_e = \{f \in E : f^*f = e\} = \{e\}$ for each $e \in E$, the Hilbert A_e-module X_e that occurs in the definition of the pre-representation $\Lambda^e = \big(\lambda_{e,f}\big)_{f\in E}$ of $\mathcal{E}$ in $\mathcal{L}(X_e)$ is nothing but A_e itself (with its standard structure). Thus $C^*_{r,\mathrm{KS}}(\mathcal{E})$ can be viewed as a C^*-subalgebra of $\prod_{e\in E}\mathcal{L}(A_e)$, and for $e, f \in E$ and $a_f \in A_f$, $\lambda_{e,f}(a_f) : A_e \to A_e$ is given by

$$\lambda_{e,f}(a_f)b = \begin{cases} a_f \cdot b & \text{if } e \le f, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for all } b \in A_e. \tag{13}$$

As before, let $\Phi_{\Lambda^e} : \mathcal{C}_c(\mathcal{E}) \to \mathcal{L}(A_e)$ be the corresponding $*$-homomorphism given by

$$\Phi_{\Lambda^e}\left(\sum_{f\in E} a_f\delta_f\right) = \sum_{f\in E} \lambda_{e,f}(a_f).$$

Let (a^i) be an approximate unit for A_e, let $f \in E$ be such that $f \ge e$, and let $a_f \in A_f$. Then for all $b \in A_e$ we have

$$\lim_i\,(a^i \cdot a_f)\cdot b = \lim_i\, a^i \cdot (a_f \cdot b) = a_f \cdot b,$$

and $\lim_i\, a_f \cdot (a^i \cdot b) = a_f \cdot b$. Thus,

$$\lambda_{e,f}(a_f) = \lim_i a^i \cdot a_f = \lim_i a_f \cdot a^i \tag{14}$$

where the limits in equation (14) are taken in the strict topology of $\mathcal{L}(A_e)$.

We recall that a *character* (sometimes called a semicharacter) on E is a nonzero homomorphism from E into the semilattice $\{0, 1\}$, see, e.g., [17].

Lemma 5.1. *Let $\pi : C^*_{\mathrm{KS}}(\mathcal{E}) \to \mathcal{B}(H)$ be a nonzero irreducible representation of $C^*_{\mathrm{KS}}(\mathcal{E})$ on a Hilbert space H. For $e \in E$, let p_e denote the orthogonal projection of H onto the norm-closure of $\pi(A_e\delta_e)H$ in H. Then $p_e \in \{0, I_H\}$. Moreover, the map $\widehat{\pi} : E \to \{0,1\}$ defined by*

$$\widehat{\pi}(e) = \begin{cases} 1 & \text{if } p_e = I_H, \\ 0 & \text{if } p_e = 0, \end{cases}$$

is a character on E.

Proof. Let (a^i) be an approximate unit for A_e. It is straightforward to check that $\pi(a^i\delta_e)$ converges strongly to p_e.

Let $f \in E$ be such that $f \le e$ and let $a \in A_f$. Since $j_{e,f}(a^i \cdot a) = a^i \cdot j_{e,f}(a)$, and $a^i \cdot j_{e,f}(a)$ converges to $j_{e,f}(a)$ in norm, it follows, using that $j_{e,f}$ is isometric, that $a^i \cdot a$ converges to a in norm. Hence, for any $\xi \in H$, we have

$$p_e\pi(a\delta_f)\xi = \lim_i \pi(a^i\delta_e)\pi(a\delta_f)\xi = \lim_i \pi(a^i \cdot a\delta_f)\xi = \pi(a\delta_f)\xi.$$

It follows that $p_f H \subset p_e H$, that is, $p_f \le p_e$.

Consider now $e' \in E$. Then $e'e \le e$, so $p_{e'e}H \subset p_e H$.

Hence for $a \in A_{e'}$ we get

$$\begin{aligned} \pi(a\delta_{e'})p_e H &= \pi(a\delta_{e'})\overline{\pi(A_e\delta_e)H} \\ &\subset \overline{\pi(A_{e'} \cdot A_e\delta_{e'e})H} \\ &\subset \overline{\pi(A_{e'e}\delta_{e'e})H} \\ &= p_{e'e}H \subset p_e H. \end{aligned}$$

This implies that $p_e H$ is a closed invariant subspace for $\pi(C^*_{\mathrm{KS}}(\mathcal{E}))$, hence that $p_e \in \{0, I_H\}$ since π is irreducible. Moreover, for any $e, e' \in E$, $p_{e'}p_e = p_e p_{e'}$ is then a projection, and, as seen above, we have $p_{e'e} \le p_e$, and similarly $p_{e'e} = p_{ee'} \le p_{e'}$, so $p_{e'e} \le p_{e'}p_e$. On the other hand, for $a \in A_{e'}$, we know that $\pi(a\delta_{e'})p_e H \subset p_{e'e}H$. Hence, using an approximate unit for $A_{e'}$, one easily deduces that $p_{e'}p_e \le p_{e'e}$. Thus we get $p_{e'e} = p_{e'}p_e$. Since π is nonzero, it clearly follows that $\widehat{\pi}$ is a character on E. $\square$

Given a character ψ on E, we set $F_\psi = \{e \in E : \psi(e) = 1\}$. Then F_ψ is an example of a filter in E (and every filter on E can be obtained this way), cf. [17]. We recall that a filter in E is a nonempty subsemilattice F of E such that if $f \in E$ and $f \ge e$ for some $e \in F$, then $f \in F$.

Proposition 5.2. *We have $C^*_{r,\mathrm{KS}}(\mathcal{E}) = C^*_{\mathrm{KS}}(\mathcal{E})$.*

Proof. It suffices to check that every irreducible representation of $C^*_{\mathrm{KS}}(\mathcal{E})$ is dominated in norm by the left regular representation Φ_Λ. More precisely, it suffices to show that given a Hilbert space H and a nonzero irreducible representation $\pi : C^*_{\mathrm{KS}}(\mathcal{E}) \to \mathcal{B}(H)$, we have

$$\|\pi(g)\| \le \|\Phi_\Lambda(g)\|$$

for all $g \in \mathcal{C}_c(\mathcal{E})$. Let $\widehat{\pi}$ be the character on E described in Lemma 5.1, and let $F = F_{\widehat{\pi}}$ be the corresponding filter in E. Note that if p_e is defined as in Lemma 5.1, then $p_e = I_H$ when $e \in F$, while $p_e = 0$ when $e \in E \setminus F$. Hence for all $e \in E \setminus F$ and all $a_e \in A_e$ we have $\pi(a_e\delta_e) = 0$. Indeed, letting (a^i) be an approximate unit for A_e, we then have

$$\pi(a_e\delta_e)\xi = \lim_i \pi(a_e a^i \delta_e)\xi = \lim_i \pi(a_e\delta_e)\pi(a^i\delta_e)\xi = \pi(a_e\delta_e)p_e\xi = 0$$

for all $\xi \in H$. Consider now $g = \sum_{u \in E} a_u\delta_u \in \mathcal{C}_c(\mathcal{E})$. Using the observation we just made, we get

$$\pi(g) = \sum_{f \in K \cap F} \pi(a_f\delta_f),$$

where $K := \{u \in E : a_u \neq 0\}$ is finite. Since F is a semilattice and $K \cap F$ is a finite subset of F, there exists some $e \in F$ such that $e \leq f$ for all $f \in K \cap F$. Let (a^i) be an approximate unit for A_e. Since the restriction π_e of π to $A_e\delta_e \simeq A_e$ is a nondegenerate representation of $A_e\delta_e$ on H, it may be extended to a representation $\overline{\pi_e} : \mathcal{L}(A_e) \to \mathcal{B}(H)$ (see for instance [1, Theorem II.7.3.9]). Moreover, if $(b^i\delta_e)$ is a net in $A_e\delta_e$ converging strictly to some $x \in \mathcal{L}(A_e)$, then $\pi_e(b^i\delta_e)$ converges strongly to $\overline{\pi_e}(x)$ in $\mathcal{B}(H)$. Thus for any $\xi \in H$, using equations (13) and (14), we get

$$\begin{aligned}
\pi(g)\xi = p_e\pi(g)\xi &= \lim_i \pi(a^i\delta_e) \sum_{f \in K \cap F} \pi(a_f\delta_f)\xi \\
&= \lim_i \sum_{f \in K \cap F} \pi\Big((a^i \cdot a_f)\delta_e\Big)\xi \\
&= \sum_{f \in K \cap F} \overline{\pi_e}(\lambda_{e,f}(a_f))\xi \\
&= \overline{\pi_e}(\Phi_{\Lambda^e}(g))\xi.
\end{aligned}$$

It follows that $\pi(g) = \overline{\pi_e}(\Phi_{\Lambda^e}(g))$, so $\|\pi(g)\| \leq \|\Phi_{\Lambda^e}(g)\| \leq \|\Phi_\Lambda(g)\|$. $\square$

Proposition 5.2 implies that $\mathcal{M}_{\mathcal{E}} = \mathcal{I}_{\mathcal{E}}$, hence that $C^*(\mathcal{E}) = C^*_{r,\mathrm{alt}}(\mathcal{E})$. Since it follows from [11, Corollary 8.10] that $C^*_r(\mathcal{E}) \simeq C^*(\mathcal{E})$, we get:

Corollary 5.3. $C^*_{r,\mathrm{alt}}(\mathcal{E}) = C^*(\mathcal{E}) \simeq C^*_r(\mathcal{E})$.

6. Conditional expectations onto the diagonal

Let $\mathcal{A} = \{A_s\}_{s \in S}$ be a Fell bundle over S and let $\mathcal{E} = \{A_e\}_{e \in E}$ denote the Fell bundle obtained by restricting $\mathcal{A}$ to the semilattice E of idempotents in S. Recall that $C^*_{r,\mathrm{KS}}(\mathcal{A})$ can be viewed as a C^*-subalgebra of $\prod_{e \in E} \mathcal{L}(X_e)$ and that $C^*_{r,\mathrm{KS}}(\mathcal{E})$ can be viewed as a C^*-subalgebra of $\prod_{e \in E} \mathcal{L}(A_e)$. When it is necessary to distinguish them, we will denote by $\Phi_\Lambda^{\mathcal{A}}$ the left regular representation of $\mathcal{C}_c(\mathcal{A})$ and by $\Phi_\Lambda^{\mathcal{E}}$ the left regular representation of $\mathcal{C}_c(\mathcal{E})$. Similarly, we will write $\{\lambda_{e,s}^{\mathcal{A}}\}$ and $\{\lambda_{e,f}^{\mathcal{E}}\}$ for the respective pre-representations.

Let $e \in E$ and $s \in S_e$ (so that $s^*s = e$). Recalling that A_s is a (right) A_e-module, we define an A_e-module map $\gamma_s : A_s \to X_e$ by $\gamma_s(a) = a \odot \varepsilon_s$, i.e.,

$$\gamma_s(a)(t) = \begin{cases} a & \text{if } s = t \\ 0 & \text{otherwise.} \end{cases}$$

One readily checks that γ_s is adjointable with adjoint given by $\gamma_s^*(\xi) = \xi(s)$ for every $\xi \in X_e$. Then $\gamma_s^*\gamma_s$ is clearly the identity map on A_s, so γ_s is isometric. Moreover, we have $\gamma_s\gamma_s^*\xi = \xi(s) \odot \varepsilon_s$ for every $\xi \in X_e$, and it follows that $\sum_{s\in S_e} \gamma_s\gamma_s^*\xi = \xi$ for every $\xi \in X_e$, where the sum converges in the norm topology on X_e.

Lemma 6.1. *Let $e \in E$, $s, t \in S_e$, $u \in S$ and $a \in A_u$. Then the map $\gamma_s^*\lambda_{e,u}(a)\gamma_t : A_t \to A_s$ is given by*

$$\big(\gamma_s^*\lambda_{e,u}(a)\gamma_t\big)(b) = \begin{cases} a \cdot b & \text{if } u \geq st^*, \\ 0 & \text{otherwise} \end{cases}$$

for all $b \in A_t$.

Proof. For $b \in A_t$ we have

$$\begin{aligned} \gamma_s^*\lambda_{e,u}(a)\gamma_t(b) &= \big(\lambda_{e,u}(a)\gamma_t(b)\big)(s) \\ &= \begin{cases} a \cdot \gamma_t(b)(u^*s) & \text{if } s \in D(u^*), \\ \quad 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} a \cdot b & \text{if } u^*s = t \text{ and } ss^* \leq uu^*, \\ \ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Suppose first that $u \geq st^*$. Then $uu^* \geq (st^*)(st^*)^* = ss^*$ since $t^*t = s^*s$. Moreover, $u^* \geq ts^*$, so $ts^* = u^*(ts^*)^*(ts^*) = u^*ss^*$, hence $u^*s = ts^*s = tt^*t = t$. Conversely, if $ss^* \leq uu^*$ and $u^*s = t$, then

$$u^*(ts^*)^*(ts^*) = u^*st^*ts^* = u^*uu^*ss^*ss^* = u^*ss^* = ts^*,$$

so $u^* \geq ts^*$, hence $u \geq st^*$. □

Lemma 6.2. *Let $t \in S$ and $b \in A_t$. Then there exist $c \in A_{tt^*}$ and $d \in A_t$ such that $b = c \cdot d$.*

Proof. Let (u^i) be an approximate unit for A_{tt^*}. Since $||u^i|| \leq 1$ for all i, we get

$$\begin{aligned} ||u^i \cdot b - b||^2 &= ||(u_i \cdot b - b)(u_i \cdot b - b)^*|| \\ &= ||u_i \cdot b \cdot b^* - b \cdot b^* + (u_i \cdot b \cdot b^* - b \cdot b^*) \cdot u_i|| \\ &\leq 2||u_i \cdot b \cdot b^* - b \cdot b^*||. \end{aligned}$$

So $u^i \cdot b$ converges to b. Regarding A_t as a left A_{tt^*}-module in the obvious way, we may then apply the Cohen–Hewitt factorization theorem [13, Theorem 32.22] to b and deduce that $b = c \cdot d$ for some $c \in A_{tt^*}$ and $d \in A_t$. □

Lemma 6.3. *Let $t \in S$ and $b \in A_t$. Let $b = c \cdot d$ be any factorization of b with $c \in A_{tt^*}$ and $d \in A_t$. Let $(T_e)_{e\in E} \in C^*_{r,\mathrm{KS}}(\mathcal{A})$, and let $s \in S$ be such that $s^*s = t^*t$. Then*

$$(\gamma_s^* T_{t^*t}\gamma_t)(b) = (\gamma_{st^*}^* T_{tt^*}\gamma_{tt^*})(c) \cdot d.$$

Proof. Note first that the expression on the right-hand side is well-defined since $(st^*)^*(st^*) = ts^*st^* = tt^*$ as $s^*s = t^*t$. By linearity and continuity, it suffices to prove that for any $u \in S$ and $a \in A_u$, we have

$$(\gamma_s^* \lambda_{t^*t,u}(a)\gamma_t)(b) = (\gamma_{st^*}^* \lambda_{tt^*,u}(a)\gamma_{tt^*})(c) \cdot d.$$

This follows immediately by applying Lemma 6.1 to both sides, and using that $st^*(tt^*)^* = st^*$. □

Lemma 6.4. *For any $e, f \in E$, and $a_f \in A_f$ we have*

$$\gamma_e^* \lambda^{\mathcal{A}}_{e,f}(a_f)\gamma_e = \lambda^{\mathcal{E}}_{e,f}(a_f).$$

Moreover, if S is E^*-unitary and $A_0 = \{0\}$ (if S has a 0-element), then for any $e \in E$, $u \in S$ and $a_u \in A_u$, we have

$$\gamma_e^* \lambda^{\mathcal{A}}_{e,u}(a_u)\gamma_e = \begin{cases} \lambda^{\mathcal{E}}_{e,u}(a_u) & \textit{if } u \in E, \\ 0 & \textit{otherwise.} \end{cases} \tag{15}$$

Proof. We prove the second statement. The proof of the first statement follows from a small adjustment to the argument and is left to the reader. Assume that S is E^*-unitary and $A_0 = \{0\}$ (if S has a 0-element). Let $b \in A_e$. Lemma 6.1 gives that

$$\big(\gamma_e^* \lambda^{\mathcal{A}}_{e,u}(a_u)\gamma_e\big)(b) = \begin{cases} a_u \cdot b & \text{if } e \le u, \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Since S is E^*-unitary, $e \le u$ implies that u is idempotent or $e = 0$. If $e \neq 0$, the right hand side of (16) is equal to

$$\begin{cases} a_u \cdot b & \text{if } e \le u \text{ and } u \in E, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} \lambda^{\mathcal{E}}_{e,u}(a_u)b & \text{if } u \in E, \\ 0 & \text{otherwise,} \end{cases}$$

so we see that (15) holds in this case. If $e = 0$ (so S has a 0-element), then both sides of (15) are equal to 0 since $A_0 = \{0\}$ by assumption. □

Lemma 6.5. *There is an embedding of $C^*_{r,\mathrm{KS}}(\mathcal{E})$ into $C^*_{r,\mathrm{KS}}(\mathcal{A})$ extending the inclusion $\mathcal{C}_c(\mathcal{E}) \subset \mathcal{C}_c(\mathcal{A})$.*

Proof. Let $\sum_{f\in E} a_f\delta_f \in \mathcal{C}_c(\mathcal{E})$. We need to prove that

$$\left\| \Phi^{\mathcal{E}}_{\Lambda}\left(\sum_{f\in E} a_f\delta_f\right)\right\| = \left\| \Phi^{\mathcal{A}}_{\Lambda}\left(\sum_{f\in E} a_f\delta_f\right)\right\|. \tag{17}$$

Since $C^*_{r,\mathrm{KS}}(\mathcal{E}) = C^*_{\mathrm{KS}}(\mathcal{E})$, cf. Proposition 5.2, the expression on the left-hand side of (17) is the same as the universal norm of $\sum_{f\in E} a_f\delta_f$ in $C^*_{\mathrm{KS}}(\mathcal{E})$. As

$\Phi_\Lambda^{\mathcal{A}}$ restricts to a $*$-homomorphism of $C_c(\mathcal{E})$, we see that the $\geq$ inequality in (17) must hold. On the other hand, since γ_e is an isometry for each $e \in E$, Lemma 6.4 implies that

$$\sup_{e\in E}\left\|\sum_{f\in E}\lambda^{\mathcal{E}}_{e,f}(a_f)\right\| \leq \sup_{e\in E}\left\|\sum_{f\in E}\lambda^{\mathcal{A}}_{e,f}(a_f)\right\|.$$

This shows that $\leq$ inequality in (17) holds. □

We will identify $C^*_{r,\mathrm{KS}}(\mathcal{E})$ with its canonical image in $C^*_{r,\mathrm{KS}}(\mathcal{A})$, and call it the *diagonal* (C^*-subalgebra) of $C^*_{r,\mathrm{KS}}(\mathcal{A})$. Define $\mathfrak{E} : \mathcal{C}_c(\mathcal{A}) \to \mathcal{C}_c(\mathcal{E})$ by

$$\mathfrak{E}\left(\sum_{u\in S} a_u\delta_u\right) = \sum_{e\in E} a_e\delta_e$$

for all $\sum_{u\in S} a_u\delta_u \in \mathcal{C}_c(\mathcal{A})$. Moreover, define a positive linear map $\mathfrak{E}_{\mathrm{KS}}$ from $C^*_{r,\mathrm{KS}}(\mathcal{A})$ into $\prod_{e\in E}\mathcal{L}(A_e)$ by

$$\mathfrak{E}_{\mathrm{KS}}\big((T_e)_{e\in E}\big) = (\gamma_e^* T_e\gamma_e)_{e\in E}$$

for all $(T_e)_{e\in E} \in C^*_{r,\mathrm{KS}}(\mathcal{A})$.

Proposition 6.6. *The map $\mathfrak{E}_{\mathrm{KS}} : C^*_{r,\mathrm{KS}}(\mathcal{A}) \to \prod_{e\in E}\mathcal{L}(A_e)$ is faithful.*

If S is E^*-unitary and $A_0 = \{0\}$ (if S has a 0-element), then $\mathfrak{E}_{\mathrm{KS}}$ satisfies

$$\mathfrak{E}_{\mathrm{KS}}\Big(\Phi^{\mathcal{A}}_\Lambda(g)\Big) = \Phi^{\mathcal{E}}_\Lambda\Big(\mathfrak{E}(g)\Big) \tag{18}$$

for all $g \in \mathcal{C}_c(\mathcal{A})$. Moreover, in this case, $\mathfrak{E}_{\mathrm{KS}}$ is a faithful conditional expectation from $C^*_{r,\mathrm{KS}}(\mathcal{A})$ onto $C^*_{r,\mathrm{KS}}(\mathcal{E})$.

Proof. Let $e \in E$, $T_e \in \mathcal{L}(X_e)$ and $a \in A_e$. For each $s \in S_e$, we have

$$(\gamma_s^* T_e\gamma_e)(a) = \big(T_e\gamma_e(a)\big)(s),$$

so we get

$$\begin{aligned}\Big\langle(\gamma_e^*T_e^*T_e\gamma_e)(a), a\Big\rangle_{A_e} &= \Big\langle T_e\gamma_e(a), T_e\gamma_e(a)\Big\rangle_{X_e}\\ &= \sum_{s\in S_e}\big(T_e\gamma_e(a)\big)(s)^*\big(T_e\gamma_e(a)\big)(s)\\ &= \sum_{s\in S_e}\big(\gamma_s^*T_e\gamma_e\big)(a)^*\big(\gamma_s^*T_e\gamma_e\big)(a).\end{aligned}$$

So we see that if $\gamma_e^*T_e^*T_e\gamma_e = 0$, then $\gamma_s^*T_e\gamma_e = 0$ for each $s \in S_e$.

Consider $T = (T_e)_{e\in E} \in C^*_{r,\mathrm{KS}}(\mathcal{A})$. If $\gamma_s^*T_e\gamma_e = 0$ for all $e \in E$ and $s \in S_e$, then for any $e \in E$ and $s, t \in S_e$, we have in particular that $\gamma^*_{st^*}T_{tt^*}\gamma_{tt^*} = 0$, so Lemma 6.2 and Lemma 6.3 imply that $\gamma_s^*T_e\gamma_t = 0$. Combining this with our first observation, we get that if $\gamma_e^*T_e^*T_e\gamma_e = 0$ for each $e \in E$, then $\gamma_s^*T_e\gamma_t = 0$ for each $e \in E$ and $s, t \in S_e$.

Assume now that $\mathfrak{E}_{\mathrm{KS}}(T^*T) = 0$. This means that $\gamma_e^* T_e^* T_e \gamma_e = 0$ for all $e \in E$. Hence, for $e \in E$ and $\xi \in X_e$, we get

$$T_e \xi = \sum_{s \in S_e} \gamma_s \gamma_s^* T_e \xi = \sum_{s \in S_e} \sum_{t \in S_e} \gamma_s \gamma_s^* T_e \gamma_t \gamma_t^* \xi = 0.$$

Thus $T_e = 0$ for every $e \in E$, so $T = 0$. This proves that $\mathfrak{E}_{\mathrm{KS}}$ is faithful.

Next, assume that S is E^*-unitary and $A_0 = \{0\}$ (if S has a 0-element). To show that (18) holds amounts to show that for any $e \in E$, we have

$$\gamma_e^* \left(\sum_{u \in S} \lambda_{e,u}(a_u) \right) \gamma_e = \sum_{f \in E} \lambda_{e,f}(a_f)$$

for all $\sum_{u \in S} a_u \delta_u \in \mathcal{C}_c(\mathcal{A})$. This follows readily from Lemma 6.4. It is then clear that the image of $\mathfrak{E}_{\mathrm{KS}}$ is $C^*_{r,\mathrm{KS}}(\mathcal{E})$. Note also that $\mathfrak{E}_{\mathrm{KS}}$ is contractive since γ_e is an isometry for each $e \in E$. Moreover, it is immediate from (18) that $\mathfrak{E}_{\mathrm{KS}}$ is a projection map. Hence, Tomiyama's theorem (see for instance [1, Theorem II.6.10.2]) gives that $\mathfrak{E}_{\mathrm{KS}}$ is a conditional expectation. □

Remark 6.7. Suppose that S is strongly E^*-unitary and and $A_0 = \{0\}$ (if S has a zero). Let σ be an idempotent pure grading from $S^\times$ into a group G. Then for each $g \in G$ one can form the Banach space

$$B_g := \overline{\bigoplus_{s \in S,\, \sigma(s) = g} \Phi_\Lambda(A_s \delta_s)} \subset C^*_{r,\mathrm{KS}}(\mathcal{A}).$$

It is straightforward to check that $\mathcal{B} := \{B_g\}_{g \in G}$ is a Fell bundle over G, giving a G-grading for $C^*_{r,\mathrm{KS}}(\mathcal{A})$ in the sense of [9, Definition 3.1]. Moreover, since σ is idempotent pure, we have $\{s \in S : \sigma(s) = 1_G\} = E$, so $B_{1_G} = C^*_{r,\mathrm{KS}}(\mathcal{E})$. Since $\mathfrak{E}_{\mathrm{KS}}$ is faithful by the previous proposition, it then follows from [9, Proposition 3.7] that $C^*_{r,\mathrm{KS}}(\mathcal{A})$ is naturally isomorphic to the reduced cross sectional C^*-algebra $C^*_r(\mathcal{B})$ associated with $\mathcal{B}$.

The following covariance property of $\mathfrak{E}_{\mathrm{KS}}$ will be useful later.

Lemma 6.8. *Suppose S is E^*-unitary and $A_0 = \{0\}$ (if S has a 0 element). Then for all $s \in S$, $b \in A_s$ and $T \in C^*_{r,\mathrm{KS}}(\mathcal{A})$ we have*

$$\mathfrak{E}_{\mathrm{KS}}\big(\lambda_s(b)^* T \lambda_s(b)\big) = \lambda_s(b)^* \mathfrak{E}_{\mathrm{KS}}(T) \lambda_s(b). \tag{19}$$

Proof. Let $s \in S$ and $b \in A_s$. Consider $\sum_{t \in S} a_t \delta_t \in \mathcal{C}_c(\mathcal{A})$. Then for any $t \in S$ we have that $s^*ts = 0$ if and only if $ss^*tss^* = 0$. Moreover, $s^*ts \in E$ if and only if $ss^*tss^* \in E$; thus, since S is E^*-unitary and $t \geq ss^*tss^*$, we get that $s^*ts \in E$ if and only if $t \in E$ or $s^*ts = 0$. Hence, using that $A_0 = \{0\}$

(if S has a 0 element), we get

$$\begin{aligned}\mathfrak{E}\left((b\delta_s)^*\left(\sum_{t\in S} a_t\delta_t\right)(b\delta_s)\right) &= \mathfrak{E}\left(\sum_{t\in S,\, s^*ts\neq 0}(b^*\cdot a_t\cdot b)\delta_{s^*ts}\right)\\ &= \sum_{t\in S,\, s^*ts\in E^\times}(b^*\cdot a_t\cdot b)\delta_{s^*ts}\\ &= (b\delta_s)^*\mathfrak{E}\left(\sum_{t\in S} a_t\delta_t\right)(b\delta_s).\end{aligned}$$

Using Equation (18), we then see that (19) holds whenever $T = \Phi_\Lambda^{\mathcal{A}}(g)$ for some $g \in \mathcal{C}_c(\mathcal{A})$. By linearity and continuity of $\mathfrak{E}_{\mathrm{KS}}$ and density of $\Phi_\Lambda^{\mathcal{A}}(\mathcal{C}_c(\mathcal{A}))$ in $C^*_{r,\mathrm{KS}}(\mathcal{A})$, it then holds for all $T \in C^*_{r,\mathrm{KS}}(\mathcal{A})$. □

7. Comparison with Exel's reduced cross sectional C^*-algebras

Throughout this section we consider a Fell bundle $\mathcal{A} = \{A_s\}_{s\in S}$ over an E^*-unitary inverse semigroup S and assume that $A_0 = \{0\}$ (if S has a 0 element). Our aim is to show that Exel's $C^*_r(\mathcal{A})$ is a quotient of $C^*_{r,\mathrm{alt}}(\mathcal{A})$ and that these C^*-algebras are canonically isomorphic under certain assumptions.

As in the previous section, we let $\mathcal{E} = \{A_e\}_{e\in E}$ denote the Fell bundle obtained by restricting $\mathcal{A}$ to the semilattice $E = E(S)$. From Proposition 6.6, we see that $\mathfrak{E}_{\mathrm{KS}}\big(\Phi_\Lambda^{\mathcal{A}}(\mathcal{N}_{\mathcal{A}})\big) = \Phi_\Lambda^{\mathcal{E}}(\mathcal{N}_{\mathcal{E}})$, and it easily follows that $\mathfrak{E}_{\mathrm{KS}}(\mathcal{I}_{\mathcal{A}}) = \mathcal{I}_{\mathcal{E}}$.

For any ideal $\mathcal{K}$ of $C^*_{r,\mathrm{KS}}(\mathcal{A})$ satisfying $\mathfrak{E}_{\mathrm{KS}}(\mathcal{K}) = \mathcal{I}_{\mathcal{E}}$ we can define a surjective linear map $\mathfrak{E}_{\mathcal{K}} : C^*_{r,\mathrm{KS}}(\mathcal{A})/\mathcal{K} \to C^*_{r,\mathrm{alt}}(\mathcal{E})$ by

$$\mathfrak{E}_{\mathcal{K}}(T + \mathcal{K}) = \mathfrak{E}_{\mathrm{KS}}(T) + \mathcal{I}_{\mathcal{E}}.$$

It is straightforward to check that $\mathfrak{E}_{\mathcal{K}}$ is contractive. Note also that for each $T \in \Phi_\Lambda^{\mathcal{A}}(\mathcal{C}_c(\mathcal{E}))$ we have

$$\|T + \mathcal{I}_{\mathcal{E}}\| = \|\mathfrak{E}_{\mathcal{K}}(T + \mathcal{K})\| \leq \|T + \mathcal{K}\| \leq \|T + \mathcal{I}_{\mathcal{E}}\|$$

where the last inequality uses that the map

$$g \mapsto \Phi_\Lambda^{\mathcal{A}}(g) + \mathcal{K}$$

is a representation of $\mathcal{C}_c(\mathcal{E})$ in $C^*_{r,\mathrm{KS}}(\mathcal{A})/\mathcal{K}$ and that $C^*_{r,\mathrm{alt}}(\mathcal{E}) = C^*(\mathcal{E})$. It follows that $\|T + \mathcal{I}_{\mathcal{E}}\| = \|T + \mathcal{K}\|$ for each $T \in C^*_{r,\mathrm{KS}}(\mathcal{E})$, so we can identify $C^*_{r,\mathrm{alt}}(\mathcal{E})$ with the image of $C^*_{r,\mathrm{KS}}(\mathcal{E})$ in the quotient $C^*_{r,\mathrm{KS}}(\mathcal{A})/\mathcal{K}$. Using Tomiyama's theorem (see for instance [1, Theorem II.6.10.2]), we get that $\mathfrak{E}_{\mathcal{K}}$ is a conditional expectation; in particular it is completely positive.

Proposition 7.1. *Define*

$$\mathcal{J}_{\mathcal{A}} = \{T \in C^*_{r,\mathrm{KS}}(\mathcal{A}) : \mathfrak{E}_{\mathrm{KS}}(T^*T) \in \mathcal{I}_{\mathcal{E}}\}.$$

Then we have

$$\mathcal{J}_{\mathcal{A}} = \{T \in C^*_{r,\mathrm{KS}}(\mathcal{A}) \,:\, \mathfrak{E}_{\mathrm{KS}}(QTR) \in \mathcal{I}_{\mathcal{E}} \ \text{for all } Q, R \in C^*_{r,\mathrm{KS}}(\mathcal{A})\}. \tag{20}$$

Thus $\mathcal{J}_{\mathcal{A}}$ is an ideal of $C^*_{r,\mathrm{KS}}(\mathcal{A})$, satisfying $\mathcal{I}_{\mathcal{A}} \subset \mathcal{J}_{\mathcal{A}}$ and $\mathfrak{E}_{\mathrm{KS}}(\mathcal{J}_{\mathcal{A}}) = \mathcal{I}_{\mathcal{E}}$. Moreover, the conditional expectation $\mathfrak{E}_{\mathcal{J}_{\mathcal{A}}}$ from $C^*_{r,\mathrm{KS}}(\mathcal{A})/\mathcal{J}_{\mathcal{A}}$ onto $C^*_{r,\mathrm{alt}}(\mathcal{E})$ is faithful.

Proof. Let $\mathcal{K}$ be the ideal of $C^*_{r,\mathrm{KS}}(\mathcal{A})$ defined by the right hand side of equation (20). Then by using an approximate unit for $C^*_{r,\mathrm{KS}}(\mathcal{A})$ one easily deduce that $\mathfrak{E}_{\mathrm{KS}}(\mathcal{K}) = \mathcal{I}_{\mathcal{E}}$ and $\mathcal{K} \subset \mathcal{J}_{\mathcal{A}}$.

Let $T \in \mathcal{J}_{\mathcal{A}}$. Then we have

$$\|\mathfrak{E}_{\mathcal{K}}(T^*T + \mathcal{K})\| = \|\mathfrak{E}_{\mathrm{KS}}(T^*T) + \mathcal{I}_{\mathcal{E}}\| = 0 \tag{21}$$

since $\mathfrak{E}_{\mathrm{KS}}(T^*T) \in \mathcal{I}_{\mathcal{E}}$. Consider now $Q \in C^*_{r,\mathrm{KS}}(\mathcal{A})$. Then, by using the Cauchy–Schwarz inequality (cf. [15]) and equation (21), we get

$$\begin{aligned}\|\mathfrak{E}_{\mathrm{KS}}\big((QT)^*(QT)\big) + \mathcal{I}_{\mathcal{E}}\|^2 &= \|\mathfrak{E}_{\mathcal{K}}\big((QT)^*(QT) + \mathcal{K}\big)\|^2 \\ &= \|\mathfrak{E}_{\mathcal{K}}\big((T+\mathcal{K})^*(Q^*QT + \mathcal{K})\big)\|^2 \\ &\le \|\mathfrak{E}_{\mathcal{K}}(T^*T + \mathcal{K})\|\|\mathfrak{E}_{\mathcal{K}}((Q^*QT)^*(Q^*QT) + \mathcal{K})\| \\ &= 0.\end{aligned}$$

So $\mathfrak{E}_{\mathrm{KS}}((QT)^*(QT)) \in \mathcal{I}_{\mathcal{E}}$, hence $QT \in \mathcal{J}_{\mathcal{A}}$.

Next, consider $R = \lambda_s(b)$ for $s \in S$ and $b \in A_s$. Lemma 6.8 gives that

$$\mathfrak{E}_{\mathrm{KS}}(R^*T^*TR) = R^*\mathfrak{E}_{\mathrm{KS}}(T^*T)R.$$

So $\mathfrak{E}_{\mathrm{KS}}(R^*T^*TR) \in \mathcal{I}_{\mathcal{A}}$ since $\mathfrak{E}_{\mathrm{KS}}(T^*T) \in \mathcal{I}_{\mathcal{E}} \subset \mathcal{I}_{\mathcal{A}}$ and $\mathcal{I}_{\mathcal{A}}$ is an ideal. As the range of $\mathfrak{E}_{\mathrm{KS}}$ is $C^*_{r,\mathrm{KS}}(\mathcal{E})$ we also get that $\mathfrak{E}_{\mathrm{KS}}(R^*T^*TR) \in C^*_{r,\mathrm{KS}}(\mathcal{E})$, so $\mathfrak{E}_{\mathrm{KS}}(R^*T^*TR) \in \mathcal{I}_{\mathcal{E}}$. By the Schwarz inequality (sometimes called the Kadison inequality), see for instance [1, Proposition II.6.9.14], we have

$$\mathfrak{E}_{\mathrm{KS}}(TR)^*\mathfrak{E}_{\mathrm{KS}}(TR) \le \mathfrak{E}_{\mathrm{KS}}((TR)^*TR) = \mathfrak{E}_{\mathrm{KS}}(R^*T^*TR) \in \mathcal{I}_{\mathcal{E}}.$$

Hence $\mathfrak{E}_{\mathrm{KS}}(TR)^*\mathfrak{E}_{\mathrm{KS}}(TR) \in \mathcal{I}_{\mathcal{E}}$ since $\mathcal{I}_{\mathcal{E}}$ (being an ideal) is a hereditary subalgebra of $C^*_{r,\mathrm{KS}}(\mathcal{E})$, and it therefore follows that $\mathfrak{E}_{\mathrm{KS}}(TR) \in \mathcal{I}_{\mathcal{E}}$ (cf. [1, Proposition II.5.1.1]). By linearity and continuity of $\mathfrak{E}_{\mathrm{KS}}$ and density of $\Phi^{\mathcal{A}}_{\Lambda}(\mathcal{C}_c(\mathcal{A}))$ in $C^*_{r,\mathrm{KS}}(\mathcal{A})$, we get that $\mathfrak{E}_{\mathrm{KS}}(TR) \in \mathcal{I}_{\mathcal{E}}$ for all $R \in C^*_{r,\mathrm{KS}}(\mathcal{A})$.

If now $T \in \mathcal{J}_{\mathcal{A}}$ and $Q, R \in C^*_{r,\mathrm{KS}}(\mathcal{A})$, then we get that $T' := QT \in \mathcal{J}_{\mathcal{A}}$, and this implies that $\mathfrak{E}_{\mathrm{KS}}(QTR) = \mathfrak{E}_{\mathrm{KS}}(T'R) \in \mathcal{I}_{\mathcal{E}}$. This shows that $\mathcal{J}_{\mathcal{A}} \subset \mathcal{K}$, hence that $\mathcal{J}_{\mathcal{A}} = \mathcal{K}$.

Since we have shown that $\mathcal{J}_{\mathcal{A}}$ is an ideal of $C^*_{r,\mathrm{KS}}(\mathcal{A})$ satisfying

$$\mathfrak{E}_{\mathrm{KS}}(\mathcal{J}_{\mathcal{A}}) = \mathcal{I}_{\mathcal{E}},$$

the canonical conditional expectation $\mathfrak{E}_{\mathcal{J}_{\mathcal{A}}}$ from $C^*_{r,\mathrm{KS}}(\mathcal{A})/\mathcal{J}_{\mathcal{A}}$ onto $C^*_{r,\mathrm{alt}}(\mathcal{E})$ is well defined. Showing that $\mathfrak{E}_{\mathcal{J}_{\mathcal{A}}}$ is faithful amounts to verifying that $T^*T \in \mathcal{J}_{\mathcal{A}}$ whenever $\mathfrak{E}_{\mathrm{KS}}(T^*T) \in \mathcal{I}_{\mathcal{E}}$. This readily follows from the definition of $\mathcal{J}_{\mathcal{A}}$ and the fact that $\mathcal{J}_{\mathcal{A}}$ is an ideal of $C^*_{r,\mathrm{KS}}(\mathcal{A})$. $\square$

Proposition 7.2. *Let $q_{\mathcal{J}_{\mathcal{A}}}$ denote the quotient map from $C^*_{r,\mathrm{KS}}(\mathcal{A})$ onto $C^*_{r,\mathrm{KS}}(\mathcal{A})/\mathcal{J}_{\mathcal{A}}$. Then there exists a canonical isomorphism*

$$\Psi : C^*_{r,\mathrm{KS}}(\mathcal{A})/\mathcal{J}_{\mathcal{A}} \to C^*_r(\mathcal{A})$$

satisfying $\Psi \circ q_{\mathcal{J}_{\mathcal{A}}} \circ \Phi^{\mathcal{A}}_{\Lambda} = \iota^{\mathrm{red}}_{\mathcal{A}}$.

Proof. The strategy for proving the proposition is to show that there exists a $*$-homomorphism $\Psi : C^*_{r,\mathrm{KS}}(\mathcal{A})/\mathcal{J}_{\mathcal{A}} \to C^*_r(\mathcal{A})$ and a linear map $\mathfrak{E}_{\mathrm{r}} : C^*_r(\mathcal{A}) \to C^*(\mathcal{E})$ making the following diagram commute:

$$\begin{tikzcd}
C^*_r(\mathcal{A}) \arrow[ddd, dashed, "\mathfrak{E}_{\mathrm{r}}"] & & & C^*_{r,\mathrm{KS}}(\mathcal{A})/\mathcal{J}_{\mathcal{A}} \arrow[lll, dashed, "\Psi"'] \arrow[ddd, "\mathfrak{E}_{\mathcal{J}_{\mathcal{A}}}"] \\
& \mathcal{C}_c(\mathcal{A}) \arrow[ul, "\iota^{\mathrm{red}}_{\mathcal{A}}"'] \arrow[r, hook, "\Phi^{\mathcal{A}}_{\Lambda}"] \arrow[d, "\mathfrak{E}"] & C^*_{r,\mathrm{KS}}(\mathcal{A}) \arrow[ur, two heads, "q_{\mathcal{J}_{\mathcal{A}}}"] \arrow[d, "\mathfrak{E}_{\mathrm{KS}}"] & \\
& \mathcal{C}_c(\mathcal{E}) \arrow[r, hook, "\Phi^{\mathcal{E}}_{\Lambda}"] \arrow[dl, "\iota_{\mathcal{E}}"'] & C^*_{r,\mathrm{KS}}(\mathcal{E}) \arrow[dr, two heads] & \\
C^*(\mathcal{E}) & & & C^*_{r,\mathrm{alt}}(\mathcal{E}) \arrow[lll, "="']
\end{tikzcd}$$

It will then follow that Ψ is an isomorphism by considering the outer square in this diagram and using that $\mathfrak{E}_{\mathcal{J}_{\mathcal{A}}}$ is faithful, as shown in the previous proposition.

Let φ be a pure state on $C^*(\mathcal{E})$. It is easy to deduce from equation (4) that the functional $\tilde{\varphi}$ on $\mathcal{C}_c(\mathcal{A})$ defined in section 3.4 is given by

$$\tilde{\varphi} = \varphi \circ \iota_{\mathcal{E}} \circ \mathfrak{E}$$

where $\iota_{\mathcal{E}}$ denotes the canonical map from $\mathcal{C}_c(\mathcal{E})$ to $C^*(\mathcal{E})$. Moreover, it is straightforward to see that we have $\iota_{\mathcal{E}} \circ \mathfrak{E} = \mathfrak{E}_{\mathcal{J}_{\mathcal{A}}} \circ q_{\mathcal{J}_{\mathcal{A}}} \circ \Phi^{\mathcal{A}}_{\Lambda}$, so we get

$$\tilde{\varphi} = \varphi \circ \mathfrak{E}_{\mathcal{J}_{\mathcal{A}}} \circ q_{\mathcal{J}_{\mathcal{A}}} \circ \Phi^{\mathcal{A}}_{\Lambda}.$$

Let $\varphi' = \varphi \circ \mathfrak{E}_{\mathcal{J}_{\mathcal{A}}} \circ q_{\mathcal{J}_{\mathcal{A}}}$. Then φ' is a state on $C^*_{r,\mathrm{KS}}(\mathcal{A})$. As before, let $(\Upsilon_{\tilde{\varphi}}, H_{\tilde{\varphi}})$ be the GNS representation associated to $\tilde{\varphi}$, with $x \mapsto \hat{x}$ denoting the canonical map $\mathcal{C}_c(\mathcal{A}) \to H_{\tilde{\varphi}}$. Form also the GNS-representation $(\pi_{\varphi'}, H_{\varphi'})$ associated to φ', with $T \mapsto \widehat{T}$ denoting the canonical map from $C^*_{r,\mathrm{KS}}(\mathcal{A})$ into $H_{\varphi'}$. For any $x \in \mathcal{C}_c(\mathcal{A})$, we obtain

$$\|\hat{x}\|^2 = \tilde{\varphi}(x^*x) = \varphi'\big(\Phi^{\mathcal{A}}_{\Lambda}(x^*x)\big) = \|\widehat{\Phi^{\mathcal{A}}_{\Lambda}(x)}\|^2.$$

Since $\{\hat{x} : x \in \mathcal{C}_c(\mathcal{A})\}$ is dense in $H_{\tilde{\varphi}}$, the assignment $\hat{x} \mapsto \widehat{\Phi_\Lambda^{\mathcal{A}}(x)}$ extends to an isometry $V : H_{\tilde{\varphi}} \to H_{\varphi'}$.

Consider now $g \in \mathcal{C}_c(\mathcal{A})$. For any $x, y \in \mathcal{C}_c(\mathcal{A})$ we get

$$\begin{aligned}\Big\langle V^*\pi_{\varphi'}\big(\Phi_\Lambda^{\mathcal{A}}(g)\big)V\hat{x}, \hat{y}\Big\rangle &= \Big\langle \pi_{\varphi'}\big(\Phi_\Lambda^{\mathcal{A}}(g)\big)\widehat{\Phi_\Lambda^{\mathcal{A}}(x)}, \widehat{\Phi_\Lambda^{\mathcal{A}}(y)}\Big\rangle \\ &= \varphi'\big(\Phi_\Lambda^{\mathcal{A}}(y^*gx)\big) = \tilde{\varphi}(y^*gx) \\ &= \Big\langle \Upsilon_{\tilde{\varphi}}(g)\hat{x}, \hat{y}\Big\rangle.\end{aligned}$$

So $\Upsilon_{\tilde{\varphi}}(g) = V^*\pi_{\varphi'}\big(\Phi_\Lambda^{\mathcal{A}}(g)\big)V$, and it follows that $\|\Upsilon_{\tilde{\varphi}}(g)\| \le \|\pi_{\varphi'}\big(\Phi_\Lambda^{\mathcal{A}}(g)\big)\|$ since V is an isometry. Moreover, as φ' annihilates $\mathcal{J}_{\mathcal{A}}$, the kernel of $\pi_{\varphi'}$ contains $\mathcal{J}_{\mathcal{A}}$, so we get $\|\pi_{\varphi'}\big(\Phi_\Lambda^{\mathcal{A}}(g)\big)\| \le \|q_{\mathcal{J}_{\mathcal{A}}}(\Phi_\Lambda^{\mathcal{A}}(g))\|$. Hence we conclude that

$$\|\Upsilon_{\tilde{\varphi}}(g)\| \le \|q_{\mathcal{J}_{\mathcal{A}}}(\Phi_\Lambda^{\mathcal{A}}(g))\|.$$

Since this holds for all $\varphi \in \mathcal{P}(C^*(\mathcal{E}))$ we get $\|\iota_{\mathcal{A}}^{\mathrm{red}}(g)\| \le \|q_{\mathcal{J}_{\mathcal{A}}}(\Phi_\Lambda^{\mathcal{A}}(g))\|$. It follows that there exists a $*$-homomorphism $\Psi : C^*_{r,\mathrm{KS}}(\mathcal{A})/\mathcal{J}_{\mathcal{A}} \to C^*_r(\mathcal{A})$ satisfying $\Psi\big(q_{\mathcal{J}_{\mathcal{A}}}(\Phi_\Lambda^{\mathcal{A}}(g))\big) = \iota_{\mathcal{A}}^{\mathrm{red}}(g)$ for all $g \in \mathcal{C}_c(\mathcal{A})$, as desired.

Next, we will show that the map $\mathfrak{E}'_{\mathrm{r}} : \iota_{\mathcal{A}}^{\mathrm{red}}(\mathcal{C}_c(\mathcal{A})) \to C^*(\mathcal{E})$ given by $\mathfrak{E}'_{\mathrm{r}}(\iota_{\mathcal{A}}^{\mathrm{red}}(g)) = \iota_{\mathcal{E}}(\mathfrak{E}(g))$ is well defined, linear and contractive. By density, it will then extend to a (contractive) linear map $\mathfrak{E}_{\mathrm{r}} : C^*_r(\mathcal{A}) \to C^*(\mathcal{E})$, as desired.

To see that $\mathfrak{E}'_{\mathrm{r}}$ is well defined, note that if $g \in \mathcal{C}_c(\mathcal{A})$ and $\iota_{\mathcal{A}}^{\mathrm{red}}(g) = 0$, then $0 = \tilde{\varphi}(x^*gy) = \varphi(\iota_{\mathcal{E}}(\mathfrak{E}(x^*gy)))$ for all $x, y \in \mathcal{C}_c(\mathcal{A})$ and all $\varphi \in \mathcal{P}(C^*(\mathcal{E}))$. Letting x and y range over $\mathcal{C}_c(\mathcal{E})$ so that $\mathfrak{E}(x^*gy) = x^*\mathfrak{E}(g)y$ (which follows from Proposition 6.6 since $\mathfrak{E}_{\mathrm{KS}}$ is a conditional expectation), and using the density of $\iota_{\mathcal{E}}(\mathcal{C}_c(\mathcal{E}))$ in $C^*(\mathcal{E})$ we get $\iota_{\mathcal{E}}(\mathfrak{E}(g)) = 0$. It readily follows that $\mathfrak{E}'_{\mathrm{r}}$ is well defined, and its linearity is then obvious.

Further, consider $g \in \mathcal{C}_c(\mathcal{A})$. For $x, y \in \mathcal{C}_c(\mathcal{E})$ with $\|\iota_{\mathcal{E}}(x)\|, \|\iota_{\mathcal{E}}(y)\| \le 1$, we have

$$\begin{aligned}\|\iota_{\mathcal{E}}(x^*\mathfrak{E}(g)y)\| = \|\iota_{\mathcal{E}}(\mathfrak{E}(x^*gy))\| &= \sup_{\varphi \in \mathcal{P}(C^*(\mathcal{E}))} \varphi(\iota_{\mathcal{E}}(\mathfrak{E}(x^*gy))) \\ &\le \sup_{\varphi \in \mathcal{P}(C^*(\mathcal{E}))} \|\Upsilon_{\tilde{\varphi}}(g)\| = \|\iota_{\mathcal{A}}^{\mathrm{red}}(g)\|.\end{aligned}$$

For every $\varepsilon > 0$ it is not difficult to see that we can find x and y as above such that

$$\|\iota_{\mathcal{E}}(\mathfrak{E}(g))\| \le \|\iota_{\mathcal{E}}(x^*\mathfrak{E}(g)y)\| + \varepsilon,$$

so we get

$$\|\mathfrak{E}'_{\mathrm{r}}(\iota_{\mathcal{A}}^{\mathrm{red}}(g))\| = \|\iota_{\mathcal{E}}(\mathfrak{E}(g))\| \le \|\iota_{\mathcal{E}}(x^*\mathfrak{E}(g)y)\| + \varepsilon \le \|\iota_{\mathcal{A}}^{\mathrm{red}}(g)\| + \varepsilon.$$

Thus we conclude that $\mathfrak{E}'_{\mathrm{r}}$ is contractive.

The reader will have no problem to check that the maps Ψ and $\mathfrak{E}_{\mathrm{r}}$ we have constructed make the above diagram commutative, thus finishing the proof. $\square$

From Proposition 7.2 and its proof we get the following result which may be worthy of being stated separately. It is proved for saturated Fell bundles over unital inverse semigroups in [7] using a different approach.

Proposition 7.3. *There exists a faithful conditional expectation*

$$\mathfrak{E}_{\mathrm{r}} : C^*_r(\mathcal{A}) \to C^*(\mathcal{E})$$

satisfying $\mathfrak{E}_{\mathrm{r}} \circ \iota^{\mathrm{red}}_{\mathcal{A}} = \iota_{\mathcal{A}} \circ \mathfrak{E}$.

Since $\mathcal{I}_{\mathcal{A}} \subset \mathcal{J}_{\mathcal{A}}$, there is a natural surjective $*$-homomorphism $p_{\mathcal{A}}$ from $C^*_{r,\mathrm{alt}}(\mathcal{A}) = C^*_{r,\mathrm{KS}}(\mathcal{A})/\mathcal{I}_{\mathcal{A}}$ onto $C^*_{r,\mathrm{KS}}(\mathcal{A})/\mathcal{J}_{\mathcal{A}}$. Using Propositions 7.2 and 7.3, the relationship between the reduced C^*-algebra $C^*_{r,\mathrm{alt}}(\mathcal{A})$ introduced in the present article and Exel's $C^*_r(\mathcal{A})$ can be described as follows:

Theorem 7.4. *There exists a surjective canonical $*$-homomorphism*

$$\Psi' : C^*_{r,\mathrm{alt}}(\mathcal{A}) \to C^*_r(\mathcal{A})$$

satisfying $\Psi' \circ \Psi_{\Lambda^{\mathrm{alt}}} = \Psi_{\Pi^{\mathrm{red}}}$.

Moreover, the conditional expectation $\mathfrak{E}^{\mathrm{alt}}_{\mathrm{r}} : C^*_{r,\mathrm{alt}}(\mathcal{A}) \to C^*_{r,\mathrm{alt}}(\mathcal{E}) = C^*(\mathcal{E})$ *given by* $\mathfrak{E}^{\mathrm{alt}}_{\mathrm{r}} = \mathfrak{E}_{\mathrm{r}} \circ \Psi'$ *is canonical in the sense that*

$$\mathfrak{E}^{\mathrm{alt}}_{\mathrm{r}}\big(\Phi^{\mathcal{A}}_{\Lambda^{\mathrm{alt}}}(g)\big) = \Phi^{\mathcal{E}}_{\Lambda^{\mathrm{alt}}}(\mathfrak{E}(g))$$

for all $g \in C_c(\mathcal{A})$, and the following conditions are equivalent:

- Ψ' *is an isomorphism.*
- $\mathcal{I}_{\mathcal{A}} = \mathcal{J}_{\mathcal{A}}$.
- $\mathfrak{E}^{\mathrm{alt}}_{\mathrm{r}}$ *is faithful.*

Proof. It suffices to set $\Psi' = \Psi \circ p_{\mathcal{A}}$ and observe that $\mathfrak{E}^{\mathrm{alt}}_{\mathrm{r}} = \mathfrak{E}_{\mathcal{J}_{\mathcal{A}}} \circ p_{\mathcal{A}}$. □

We don't know whether $C^*_{r,\mathrm{alt}}(\mathcal{A})$ is isomorphic to $C^*_r(\mathcal{A})$ in general. When S is strongly E^*-unitary this happens quite often.

Corollary 7.5. *Assume S is strongly E^*-unitary and let $\sigma : S^\times \to G$ be an idempotent pure grading into a group G. Let $\mathcal{B}$ be the associated Fell bundle over G defined in Remark 6.7. If G is exact* [2]*, or if $\mathcal{B}$ satisfies Exel's approximation property* [9]*, then $C^*_{r,\mathrm{alt}}(\mathcal{A})$ is canonically isomorphic to $C^*_r(\mathcal{A})$.*

Proof. By using [10, Theorem 5.1] if G is exact, or [9, Proposition 4.10] if $\mathcal{B}$ satisfies Exel's approximation property, one deduces easily that $\mathcal{I}_{\mathcal{A}} = \mathcal{J}_{\mathcal{A}}$ after making appropriate identifications of these ideals in $C^*_r(\mathcal{B})$. Hence, the result follows from Theorem 7.4. □

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