

Noncommutative T-duality. The dynamical duality theory and 2-dimensional examples

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ABSTRACT. A duality theory for bundles of C^* -algebras whose fibres are twisted transformation group algebras is established. Classical T-duality is obtained as a special case, where all fibres are commutative tori, i.e., untwisted group algebras for \mathbb{Z}^n . Our theory also includes the bundles considered by Mathai and Rosenberg in their work on noncommutative T-duals, in which they allow twisted group algebras on one side of the duality.

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1. Introduction

In classical (or topological) T-duality one starts with a circle bundle $p : E \rightarrow B$ over a locally compact base space B together with a class $\delta \in H^3(E, \mathbb{Z})$ (often called H -flux). Then T-duality provides an involution $(E, \delta) \mapsto (E^\#, \delta^\#)$ of circle bundles over B with H -flux, satisfying a number of interesting properties. We refer to Rosenberg's CBMS memoir [Ro09] for a detailed axiomatic definition of this classical notion of T-duality (see [Ro09, Axiomatics 6.1.2]) and an explanation of how this relates to string theory. There is a completely topological construction of T-duality due to Bunke and Schick [BuS05]. Another approach due to Raeburn and Rosenberg in [RaR88] (see also [Ro09, Chapter 7]) is completely C*-algebraic in nature: given the data (E, δ) as above, there is a unique stable continuous-trace algebra $CT(E, \delta)$ (i.e., an algebra of continuous C_0 -sections of a locally trivial field of compact operators over E) with Dixmier–Douady invariant $\delta \in H^3(E, \mathbb{Z})$, and the results in [RaR88] show that there is an essentially unique \mathbb{R} -action on $CT(E, \delta)$ which covers the action of \mathbb{R} on E given by inflating the given circle action. Moreover, they show that there is a unique circle bundle $E^\#$ over B together with a class $\delta^\# \in H^3(E^\#, \mathbb{Z})$ such that

$$CT(E, \delta) \rtimes \mathbb{R} \cong CT(E^\#, \delta^\#).$$

Note that this C*-algebraic description of T-duality has the advantage that it gives a direct connection of the (twisted) K -theories of (E, δ) and $(E^\#, \delta^\#)$: by the Connes–Thom-isomorphism there is a natural isomorphism

$$K^*(E, \delta) = K_*(CT(E, \delta)) \cong K_{*+1}(CT(E, \delta) \rtimes \mathbb{R}) = K^{*+1}(E^\#, \delta^\#).$$

Note that Takai-duality provides an isomorphism

$$CT(E^\#, \delta^\#) \rtimes \widehat{\mathbb{R}} \cong CT(E, \delta) \rtimes \mathbb{R} \rtimes \widehat{\mathbb{R}} \cong CT(E, \delta) \otimes \mathcal{K}(L^2(\mathbb{R})) \cong CT(E, \delta).$$

which shows that this construction really provides an involution on the category of circle bundles over B with H -flux.

Unfortunately, for higher dimensional torus bundles the theory becomes much more involved since in general there does not exist a (classical) dual pair $(E^\#, \delta^\#)$ for a given pair (E, δ) for a principal \mathbb{T}^n -bundle $p : E \rightarrow B$ if $n \geq 2$. However, in [MaR05] Mathai and Rosenberg show that in case $n = 2$

there always exist (nonunique) noncommutative T-duals which are section algebras of bundles of stable noncommutative tori. Moreover, the possible duality pairs can be classified with the help of a certain Mackey-obstruction map. The construction extends the ideas explained above: recall that the noncommutative 2-tori are just the twisted group algebras $C^*(\mathbb{Z}^2, \omega)$ for $[\omega] \in H^2(\mathbb{Z}^2, \mathbb{U}(1))$ (and similarly for higher dimensional tori). Now given a pair (E, δ) as above in which $p : E \rightarrow B$ is a principal \mathbb{T}^2 -bundle, there exists (in general a nonunique) action of \mathbb{R}^2 on $CT(E, \delta)$ which covers the given \mathbb{T}^2 -action on E and then the T-dual is constructed as the crossed product $CT(E, \delta) \rtimes \mathbb{R}^2$. Indeed, writing E locally as $U_i \times \mathbb{T}^2$, the action of \mathbb{R}^2 on the invariant ideal $CT(U_i \times \mathbb{T}^2, \delta) \subseteq CT(E, \delta)$ is induced from an action of the lattice $\mathbb{Z}^2 \subseteq \mathbb{R}^2$ on $CT(U_i, \delta|_{U_i})$. Hence the ideal $CT(U_i \times \mathbb{T}^2, \delta) \rtimes \mathbb{R}^2$ is stably isomorphic to $CT(U_i, \delta|_{U_i}) \rtimes \mathbb{Z}^2$, which is a section algebra of a continuous C*-bundle with fibres $\mathcal{K} \rtimes_{\alpha_x} \mathbb{Z}^2 \cong \mathcal{K} \otimes C^*(\mathbb{Z}^2, \omega_x)$ at $x \in U_i$, where $[\omega_x] \in H^2(\mathbb{Z}^2, \mathbb{U}(1))$ is the Mackey obstruction for implementing the action α_x on the fibre \mathcal{K} at x as the adjoint action of a homomorphism $U : \mathbb{Z}^2 \rightarrow \mathbb{U}(\mathcal{H})$ (if $\mathcal{K} = \mathcal{K}(\mathcal{H})$). Later, in [MaR06] the authors extended these results to higher dimensional tori, where the obstruction for the existence of a possibly noncommutative T-dual is given by the requirement that the class $\delta \in H^3(E, \mathbb{Z})$ vanishes on the fibres of $p : E \rightarrow B$ (which is automatic if $n \leq 2$). Again, the Connes–Thom-isomorphism always provides an isomorphism (up to dimension shift by n) of the relevant K -theories.

It is obvious that the resulting duality theory lacks symmetry: while on one side we have a classical pair (E, δ) , there is often a noncommutative torus bundle on the dual side. In this paper we start to investigate a symmetric version of noncommutative T-duality in which we allow *noncommutative torus bundles on both sides* of the duality.

The framework we develop in this article is general enough to work for general locally compact abelian groups G with discrete and co-compact subgroups $N \subset G$ and is not necessarily coupled to the motivating example $G = \mathbb{R}^n, N = \mathbb{Z}^n$. Let us summarise some of its main content: we introduce some notation and review some basic knowledge about C*-dynamical systems in Section 2. Then Section 3 starts with a recapitulation of Mathai’s and Rosenberg’s T-duality over the one-point space. It serves as a motivation for our framework for general noncommutative T-duality over the one point space as it is given in 3.2. This means to identify a subcategory $NCT(N; \widehat{G})^\natural$ of the category of all C*-dynamical systems whose objects are stable, twisted transformation group algebras of N , equipped with transverse \widehat{G} -actions (Definition 3.9). This subcategory is dual to $NCT(N^\perp; G)^\natural$ by the duality functor ${}_-\rtimes G$ (Theorem 3.12). In particular, for $G = \mathbb{R}^n, N = \mathbb{Z}^n$, we obtain a self-duality of the category of stable, noncommutative tori $NCT(\mathbb{Z}^n; \mathbb{R}^n)^\natural$ with transverse \mathbb{R}^n -actions. In Section 3.3 we construct a cohomological invariant

$$[NCT(N; \widehat{G})^\natural] \rightarrow H^2(N, \mathbb{U}(1))$$

on the set of isomorphism classes of these objects. This allows us to re-obtain the classical (i.e., commutative) subtheory inside our theory: it is the kernel of this map (Theorem 3.20). Section 3.4 describes the complete picture of noncommutative T-duality in two and three dimensions, i.e., for $G = \mathbb{R}^n, N = \mathbb{Z}^n, n = 2, 3$, over the one-point space.

In Section 4 we turn to the global situation of noncommutative T-duality. The objects which we consider therein are bundles of C^* -algebras whose fibres are stable, twisted group algebras which satisfy a certain local triviality property which we call ω -triviality (Definition 4.2). For $G = \mathbb{R}^n, N = \mathbb{Z}^n$ the theory over the one-point space directly generalises to the bundle case. In particular, we establish a duality for these bundles, and we re-obtain the classical T-duality as a subcategory characterised point-wise by a trivial cohomological invariant. For more general groups some technical assumptions have to be made.

Section 5 presents in detail some examples of noncommutative T-duality over the circle. In particular, we give an example of a locally ω -trivial bundle which does not arise as a dual of a commutative bundle, thereby showing that in our framework the class of objects is bigger than in the approaches made so far.

2. Notation and preliminaries

2.1. Abelian groups. Throughout this paper G will always denote an abelian, Hausdorff, second-countable, locally compact group. Its dual group, the group of characters $\widehat{G} := \text{Hom}(G, U(1))$, is equipped with the compact-open topology. It is again an abelian, Hausdorff, second-countable and locally compact group [Ru90]. The bidual $\widehat{\widehat{G}}$ is canonically isomorphic to G , and we use both of the notations $\langle g, \chi \rangle$ and $\langle \chi, g \rangle$ to denote $\chi(g) \in U(1)$, for $g \in G, \chi \in \widehat{G}$. We assume that we have given a discrete and co-compact subgroup $N \subset G$. Then its annihilator $N^\perp := \{\chi \in \widehat{G} : \chi|_N = 1\}$ is discrete and co-compact in \widehat{G} , and there are canonical identifications $\widehat{N} = \widehat{G}/N^\perp$ and $\widehat{N^\perp} = G/N$.

The most prominent and guiding example of groups that fit into this situation is for $n \in \mathbb{N}$ the self-dual example

$$\mathbb{Z}^n \hookrightarrow \mathbb{R}^n \twoheadrightarrow \mathbb{T}^n,$$

where \mathbb{T}^n is the n -fold torus $\mathbb{R}^n/\mathbb{Z}^n$. Another self dual example for $n \in \mathbb{N}$ is

$$\mathbb{Q}^n \hookrightarrow \mathbb{A}^n \twoheadrightarrow \mathbb{S}^n,$$

where the (discrete) rational numbers \mathbb{Q} sit inside the adeles \mathbb{A} , and the quotient is the solenoid \mathbb{S} , the dual group of the rationals (see [HeR]).

2.2. C^* -dynamical systems. By the term C^* -algebra we will typically mean a separable C^* -algebra. The C^* -automorphism group $\text{Aut}(A)$ of a C^* -algebra A is equipped with the topology of point-wise convergence. An

action $\alpha : G \rightarrow \text{Aut}(A)$ is just a continuous group homomorphism (usually called strongly continuous), and such a triple (A, G, α) is called a C^* -dynamical system.

If (A, G, α) is a C^* -dynamical system (with G abelian), then its dual is the system $(A \rtimes_\alpha G, \widehat{G}, \widehat{\alpha})$, where the crossed product $A \rtimes_\alpha G$ is the enveloping C^* -algebra of the Banach $*$ -algebra $L^1(G, \alpha, *)$ which is $L^1(G, A)$ equipped with the product

$$(f * f')(g) := \int_G f(h)\alpha_h(f'(g - h)) dh$$

and with involution $f^*(g) := \alpha_g(f(-g))^*$, for $f, f' \in L^1(G, A)$. The dual action $\widehat{\alpha}$ is given on the dense subspace $L^1(G, A)$ just by point-wise multiplication: $\widehat{\alpha}_\chi(f) := \langle \chi, - \rangle f(-)$, for $\chi \in \widehat{G}, f \in L^1(G, A)$.

Recall the famous Takai duality theorem (e.g., see [W07, Theorem 7.1]):

Theorem 2.1. *There is a G -equivariant isomorphism*

$$\left((A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}, G, \widehat{\widehat{\alpha}} \right) \cong \left(A \otimes \mathcal{K}(L^2(G)), G, \alpha \otimes (\text{Ad} \circ \rho) \right),$$

where $\mathcal{K}(L^2(G))$ is the algebra of compact operators on $L^2(G)$, ρ is the right regular representation of G on $L^2(G)$, and Ad denotes the conjugation action of the unitary operators on the compacts.

2.3. Morita equivalent actions. Assume that (A, G, α) and (B, G, β) are two C^* -dynamical systems. Recall that a Morita equivalence (E, γ) between (A, G, α) and (B, G, β) consists of an A - B -equivalence bimodule E together with an action $\gamma : G \rightarrow \text{Aut}(E)$ which is compatible with the given actions α and β in the sense that

$$\alpha_g(A\langle \xi, \eta \rangle)\gamma_g(\zeta) = \gamma_g(A\langle \xi, \eta \rangle\zeta) = \gamma_g(\xi\langle \eta, \zeta \rangle_B) = \gamma_g(\xi)\beta_g(\langle \eta, \zeta \rangle_B)$$

holds for all $\xi, \eta, \zeta \in E$ and $g \in G$. If (E, γ) is such a Morita equivalence, then $C_c(G, E)$ becomes a $C_c(G, A)$ - $C_c(G, B)$ -bimodule by defining the left and right actions and the inner products by the convolution formulas

$$\begin{aligned} f \cdot \xi(g) &= \int_G f(h)\gamma_g(\xi(g - h)) dh \\ \xi \cdot f'(g) &= \int_G \xi(h)\beta_h(f'(g - h)) dh \\ C_{c(G,A)}\langle \xi, \eta \rangle(g) &= \int_G A\langle \xi(h), \gamma_g(\eta(g - h)) \rangle dh \\ \langle \xi, \eta \rangle_{C_c(G,B)}(g) &= \int_G \beta_{-g}(\langle \xi(h), \eta(g + h) \rangle_B) dh \end{aligned}$$

for all $\xi, \eta \in C_c(G, E), f \in C_c(G, A)$ and $f' \in C_c(G, B)$. With these operations, $C_c(G, E)$ completes to a $(A \rtimes_\alpha G)$ - $(B \rtimes_\beta G)$ -equivalence bimodule

$E \rtimes_{\gamma} G$ which equipped with the dual action $\widehat{\gamma} : G \rightarrow \text{Aut}(E \rtimes_{\gamma} G)$ given by point-wise multiplication,

$$(\widehat{\gamma}_{\chi}\xi)(g) = \langle g, \chi \rangle \xi(g), \quad \text{for all } \chi \in \widehat{G}, \xi \in C_c(G, E),$$

gives an equivariant Morita equivalence $(E \rtimes_{\gamma} G, \widehat{\gamma})$ for the dual systems $(A \rtimes_{\alpha} G, \widehat{\alpha})$ and $(B \rtimes_{\beta} G, \widehat{\beta})$ (see [Co84, E94] for further details).

Example 2.2. If there is an α - β -equivariant isomorphism $\Phi : A \rightarrow B$, we consider A as an A - B -bimodule with inner products given by

$${}_A \langle a, b \rangle = ab^* \quad \text{and} \quad \langle a, b \rangle_B = \Phi(a^*b),$$

then (A, α) gives a Morita equivalence between (A, G, α) and (B, G, β) .

Moreover, for any system (A, G, α) , the system

$$\left(A \otimes \mathcal{K}(L^2(G)), G, \alpha \otimes (\text{Ad} \circ \rho) \right)$$

is Morita equivalent to (A, G, α) with bimodule $A \otimes L^2(G)$ and action

$$\alpha \otimes \rho : G \rightarrow \text{Aut}(A \otimes L^2(G)).$$

Thus it follows from Takai’s duality theorem that (A, G, α) is Morita equivalent to the double dual system $\left((A \rtimes_{\alpha} G) \rtimes_{\widehat{\alpha}} \widehat{G}, G, \widehat{\widehat{\alpha}} \right)$.

The Morita equivalence between α and $\widehat{\widehat{\alpha}}$ is a primal case of an important Morita equivalence which will play a fundamental rôle in this paper. Let us review some other special cases of Morita equivalences which will appear in this work. (For reference, see [Pe79, Section 8.11].)

Example 2.3.

- (1) (Exterior Equivalence). If α and β are actions on the same C^* -algebra A , then they are called *exterior equivalent* if there exists a strictly continuous map to the unitary group of the multiplier algebra of A , $v : G \rightarrow \text{UM}(A); g \mapsto v_g$ such that

$$\alpha_g(a) = v_g \beta_g(a) v_g^*, \quad v_{g+h} = v_g \beta_g(v_h), \quad \text{for all } g, h \in G, a \in A.$$

Let $E = A$ be the canonical A - A -equivalence and define

$$\gamma : G \rightarrow \text{Aut}(E)$$

by $\gamma_g(a) = v_g \beta_g(a)$ for $g \in G$ and $a \in E$. Then it is easily checked that this implements a Morita equivalence between (A, G, α) and (A, G, β) . We say that v implements the exterior equivalence between α and β . In this case there is a canonical isomorphism

$$\Phi_v : A \rtimes_{\beta} G \rightarrow A \rtimes_{\alpha} G, \quad \Phi_v(f)(g) = f(g) v_g^*, \quad f \in C_c(G, A),$$

which is $\widehat{\alpha}$ - $\widehat{\beta}$ -equivariant, hence induces an isomorphism for the dual actions.

- (2) (Outer Conjugacy). Two systems (A, G, α) and (B, G, β) are *outer conjugate*, if there exists an isomorphism $\Phi : A \rightarrow B$ such that β is exterior equivalent to the action $\alpha' = \Phi \circ \alpha \circ \Phi^{-1}$.
- (3) (Stable Outer Conjugacy). This is rather the most general form of Morita equivalence, in fact, it is an equivalent notion for C^* -algebras which have a countable approximate identity (or, equivalently, which contain strictly positive elements) [Co84]. Two systems (A, G, α) and (B, G, β) are *stably outer conjugate* if the two systems

$$(A \otimes \mathcal{K}, G, \alpha \otimes \text{id}) \quad \text{and} \quad (B \otimes \mathcal{K}, G, \beta \otimes \text{id})$$

are outer conjugate, where \mathcal{K} is the algebra of compact operators on some separable Hilbert space.

If $\alpha : G \rightarrow \text{Aut}(A)$ is an action, we can restrict it to the subgroup $N \subset G$, and then $C_c(G, A)$ completes to give an $(A \rtimes_\alpha N)$ -Hilbert module, $E_N^G(A)$, if the right action of $A \rtimes_\alpha N$ and the $A \rtimes_\alpha N$ -valued inner products are defined on the level of $C_c(N, A)$ as

$$\begin{aligned} \langle \xi, \eta \rangle_{A \rtimes N}(n) &= \int_G \alpha_h(\xi(h))^* \eta(m-h) dh, \\ \xi \cdot f(g) &= \int_N \xi(g+n) \alpha_{g+n}(f(-n)) dn \end{aligned}$$

for $\xi, \eta \in C_c(G, A), f \in C_c(N, A)$.

There is a canonical left action of the dual system $(A \rtimes_\alpha G, N^\perp, \hat{\alpha})$ on $E_N^G(A)$ which is given by the covariant representation (Φ, U) in which $\Phi(f)\xi = f * \xi$ for $f, \xi \in C_c(G, A)$ and $U_\chi \xi = \chi \cdot \xi$ (point-wise multiplication). The integrated form $\Phi \times U : A \rtimes_\alpha G \rtimes_{\hat{\alpha}} N^\perp \rightarrow \mathcal{L}(E_N^G(A))$ defines a left action of $A \rtimes_\alpha G \rtimes_{\hat{\alpha}} N^\perp$ on $E_N^G(A)$, which by [E94, Proposition 2.1] implements an isomorphism onto $\mathcal{K}(E_N^G(A))$ ($\mathcal{L}(\cdot)$ and $\mathcal{K}(\cdot)$ denote the adjointable and compact operators of a module, respectively). Of course, this is just a reformulation of Green's famous imprimitivity theorem [G78, Theorem 17]. In [E94, Proposition 3.4 and Lemma 3.6] it is shown that the resulting Morita equivalence between $A \rtimes_\alpha G \rtimes_{\hat{\alpha}} N^\perp$ and $A \rtimes_\alpha N$ is equivariant with respect to certain actions by G and \hat{G} . For notation, given an action $\alpha : G \rightarrow \text{Aut}(A)$ for the abelian group G and $N \subseteq G$ is a closed subgroup of G we let $\alpha^{\text{dec}} : G \rightarrow \text{Aut}(A \rtimes_\alpha N)$ denote the action given by

$$\alpha_g^{\text{dec}}(f)(n) = \alpha_g(f(n)) \quad g \in G, n \in N, f \in C_c(N, A)$$

and if $\beta : G/N \rightarrow \text{Aut}(B)$ is an action of the quotient group, we denote by $\text{inf } \beta : G \rightarrow \text{Aut}(B)$ the inflation of β to G . Combining the results of [E94, Proposition 3.4 and Lemma 3.6] we then get:

Proposition 2.4. *In the above situation $E_N^G(A)$ becomes an $(A \rtimes_\alpha G \rtimes_{\hat{\alpha}} N^\perp)$ - $(A \rtimes_\alpha N)$ -imprimitivity bimodule. Moreover, if we define actions γ and $\hat{\gamma}$ of*

G and \widehat{G} on $E_N^G(A)$ by

$$(\gamma_g \xi)(h) := \xi(g+h) \quad \text{and} \quad (\widehat{\gamma}_\chi \xi)(g) := \langle g, \chi \rangle \xi(g)$$

for $\xi \in C_c(G, A)$, then $(E_N^G(A), \gamma)$ is a Morita equivalence between

$$(A \rtimes_\alpha G \rtimes_{\widehat{\alpha}} N^\perp, G, \inf \widehat{\alpha}|_{N^\perp}) \quad \text{and} \quad (A \rtimes_\alpha N, G, \alpha^{\text{dec}})$$

and $(E_N^G(A), \widehat{\gamma})$ is a Morita equivalence between

$$(A \rtimes_\alpha G \rtimes_{\widehat{\alpha}} N^\perp, \widehat{G}, \widehat{\alpha}^{\text{dec}}) \quad \text{and} \quad (A \rtimes_\alpha N, \widehat{G}, \inf \widehat{\alpha}|_N).$$

2.4. Actions on \mathcal{K} , twisted group algebras, and 2-cocycles. Consider the short exact sequence

$$1 \rightarrow \text{U}(1) \rightarrow \text{U}(\mathcal{H}) \xrightarrow{\text{Ad}} \text{PU}(\mathcal{H}) \rightarrow 1,$$

where $\text{U}(\mathcal{H})$ is the unitary group of some separable Hilbert space \mathcal{H} , and $\text{PU}(\mathcal{H})$ is the projective unitary group, the quotient of the unitaries by its center. It induces a (not very long) exact sequence in Borel cohomology

$$(1) \quad \dots \rightarrow H^1(G, \text{U}(\mathcal{H})) \rightarrow H^1(G, \text{PU}(\mathcal{H})) \xrightarrow{\text{Ma}} H^2(G, \text{U}(1))$$

which terminates at $H^2(G, \text{U}(1))$ due to the noncommutativity of the involved coefficient groups. The (negative of the usual) connecting homomorphism¹ Ma is called *Mackey obstruction*. Now, because $\text{U}(\mathcal{H}) = \text{UM}(\mathcal{K})$, the unitary group of the multiplier algebra of the compacts $\mathcal{K} = \mathcal{K}(\mathcal{H})$, and because all automorphisms of \mathcal{K} are inner, conjugation defines a canonical isomorphism $\text{PU}(\mathcal{H}) = \text{Aut}(\mathcal{K})$. So if $\alpha : G \rightarrow \text{Aut}(\mathcal{K})$ is an action on the compacts, i.e., $\alpha \in H^1(G, \text{PU}(\mathcal{H}))$, it defines a class $\text{Ma}(\alpha) \in H^2(G, \text{U}(1))$, the Mackey obstruction of α . The action α is said to be *unitary* if its Mackey obstruction vanishes. Note that any class $[\omega] \in H^2(G, \text{U}(1))$ arises as a Mackey obstruction of some action $\alpha : G \rightarrow \text{Aut}(\mathcal{K})$: Just put $\alpha_\omega(g) := \text{Ad}(L_\omega(g))$, where $L_\omega : G \rightarrow \text{U}(L^2(G))$ is the left regular ω -representation of G , i.e.,

$$(2) \quad (L_\omega(g)\xi)(h) = \omega(g, h-g)\xi(h-g), \quad \xi \in L^2(G), g, h \in G.$$

Then $\text{Ma}(\alpha_\omega) = [\omega^{-1}] = -[\omega]$.

The following shows that actions on \mathcal{K} are classified up to Morita (or exterior) equivalence by the Borel cohomology group $H^2(G, \text{U}(1))$. We refer to [CrKRW93, Section 6.3] for a more general result.

Proposition 2.5. *Suppose $\alpha, \beta : G \rightarrow \text{Aut}(\mathcal{K})$ are two actions of G on \mathcal{K} . Then the following are equivalent:*

- (1) α and β are exterior equivalent.
- (2) α and β are Morita equivalent.
- (3) $\text{Ma}(\alpha) = \text{Ma}(\beta) \in H^2(G, \text{U}(1))$.

¹ We choose the convention $\text{Ma}(\alpha) := -[\partial V]$, for a Borel lift $V : G \rightarrow \text{U}(\mathcal{H})$ of $\alpha : G \rightarrow \text{PU}(\mathcal{H})$.

If $\text{Ma}(\alpha) = [\omega]$, then the crossed product $\mathcal{K} \rtimes_{\alpha} G$ is isomorphic to²

$$\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega} G),$$

where $\mathbb{C} \rtimes_{\omega} G$ denotes the twisted group C*-algebra of G with respect to the cocycle ω . This is the enveloping C*-algebra of the Banach *-algebra $L^1(G, \omega, *)$ given by the Banach space $L^1(G)$ with convolution and involution given by

$$(3) \quad \begin{aligned} (f * f')(g) &:= \int_G f(h)f'(g-h)\omega(h, g-h) \, dh \quad \text{and} \\ f^*(g) &:= \overline{\omega(g, -g)f(-g)}. \end{aligned}$$

The isomorphism $\Psi : \mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega} G) \rightarrow \mathcal{K} \rtimes_{\alpha} G$ is given on the level of L^1 -functions by

$$(4) \quad \Psi(k \otimes f)(g) = f(g)kV(g), \quad k \in \mathcal{K}, f \in L^1(G),$$

where $V : G \rightarrow U(\mathcal{H})$ is a Borel lift of α (i.e., 1-cochain) such that its boundary is ω^{-1} (see [E96, Theorem 1.4.15]). Note that this isomorphism is equivariant with respect to the canonical (dual) actions of \widehat{G} on $\mathcal{K} \rtimes_{\alpha} G$ and on $\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega} G)$ given by point-wise multiplication with characters. It is an immediate consequence of Proposition 2.5 and of Takai duality that, up to equivariant Morita equivalence, the twisted group algebra equipped with the dual \widehat{G} -action is also classified by the cohomology class $[\omega] \in H^2(G, U(1))$.

One should regard the twisted group algebra $\mathbb{C} \rtimes_{\omega} G$ as a deformation of $C_0(\widehat{G}) \cong \mathbb{C} \rtimes_1 G$, where 1 denotes the trivial cocycle on G . In particular, the twisted group algebras $\mathbb{C} \rtimes_{\omega} \mathbb{Z}^n$ are deformations of $C(\mathbb{T}^n)$, and they are called noncommutative n -tori. In this picture, the dual action of \mathbb{T}^n on $\mathbb{C} \rtimes_{\omega} \mathbb{Z}^n$ is the analogue of the translation action of \mathbb{T}^n on $C(\mathbb{T}^n)$ in the commutative case. We have a natural isomorphism between the additive group $M^u(n, \mathbb{R})$ of strictly upper triangular real matrices and $H^2(\mathbb{R}^n, U(1))$ which is given by sending a matrix A to the class of the cocycle ω_A given by

$$\omega_A(x, y) := \exp(2\pi i(Ax)^t y).$$

Under this identification, the restriction map

$$H^2(\mathbb{R}^n, U(1)) \rightarrow H^2(\mathbb{Z}^n, U(1)); \quad [\omega] \mapsto [\omega|_{\mathbb{Z}^n \times \mathbb{Z}^n}],$$

which is surjective, has kernel given by the collection of all classes corresponding to the set $M^u(n, \mathbb{Z})$ of strictly upper triangular matrices with integer coefficients. So the noncommutative n -tori are classified (up to \mathbb{T}^n -equivariant Morita equivalence) by

$$H^2(\mathbb{Z}^n, U(1)) \cong M^u(n, \mathbb{R})/M^u(n, \mathbb{Z}) \cong \mathbb{T}^{n(n-1)/2}.$$

²Here we see that our sign convention of Ma is the right one, for otherwise we would obtain

$$\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega^{-1}} G) \cong \mathcal{K} \rtimes_{\alpha} G.$$

We refer to [BaK73] for further details. Moreover, for every action α of \mathbb{Z}^n on \mathcal{K} we can find an action β of \mathbb{R}^n on \mathcal{K} such that the restriction $\beta|_{\mathbb{Z}^n}$ is Morita equivalent to α (choose an action β corresponding to any class $[\eta] \in H^2(\mathbb{R}^n, \mathbb{U}(1))$ which restricts to $\text{Ma}(\alpha)$).

If $\omega \in H^2(G, \mathbb{U}(1))$ for the abelian group G , then ω determines a continuous homomorphism $h_\omega : G \rightarrow \widehat{G}$ given by

$$(5) \quad \langle h_\omega(g), h \rangle := \omega(g, h)\omega(h, g)^{-1}, \quad \text{for all } g, h \in G.$$

h_ω only depends on the class $[\omega] \in H^2(G, \mathbb{U}(1))$ and $h_\omega = 0$ if and only if $[\omega] = 0$. The kernel $S \subset G$ of h_ω is called the *symmetry group* of ω . A cocycle ω is said to be *totally skew* if $S = \{0\}$, and ω is said to be *type I* if the image $h_\omega(G)$ is closed in \widehat{G} . Recall that for any C*-algebra A , $\text{Prim}(A)$ denotes the space of primitive ideals of A equipped with the Jacobson topology. With these notations, the following results have been shown by Baggett and Kleppner in [BaK73, Section 3].

Theorem 2.6. *For $\omega \in Z^2(G, \mathbb{U}(1))$ the following are true:*

- (1) *There is a canonical bijection between \widehat{S} and $\text{Prim}(\mathbb{C} \rtimes_\omega G)$ given by induction of representations. In particular, $\mathbb{C} \rtimes_\omega G$ is simple if and only if ω is totally skew.*
- (2) *The image $h_\omega(G)$ is always a dense subgroup of $S^\perp \subset \widehat{G}$, thus ω is type I if and only if $h_\omega(G) = S^\perp$.*
- (3) *The C*-algebra $\mathbb{C} \rtimes_\omega G$ is type I if and only if ω is type I.*

Moreover, the map

$$(6) \quad h_- : H^2(G, \mathbb{U}(1)) \hookrightarrow \text{Hom}(G, \widehat{G}); \quad [\omega] \mapsto h_\omega$$

is injective.

Combining (1) and (3) of the above theorem we see that $\mathbb{C} \rtimes_\omega G$ is simple and type I if and only if ω is type I and totally skew, i.e., $h_\omega : G \xrightarrow{\cong} \widehat{G}$ is an isomorphism. Since every separable, simple, type I C*-algebra is (isomorphic to) an algebra of compact operators on some separable Hilbert space, we see that $\mathbb{C} \rtimes_\omega G \cong \mathcal{K}(\mathcal{H})$ for some separable Hilbert space \mathcal{H} if (and only if) ω is totally skew and type I. Then the dual action

$$\widehat{G} \rightarrow \text{Aut}(\mathbb{C} \rtimes_\omega G) = \text{Aut}(\mathcal{K}(\mathcal{H}))$$

is again classified by a class $[\widehat{\omega}] \in H^2(\widehat{G}, \mathbb{U}(1))$. This class has been computed by one of the authors in [E96, Lemma 3.3.5]:

Proposition 2.7. *Suppose that $\omega \in Z^2(G, \mathbb{U}(1))$ is a type I and totally skew 2-cocycle. Then the Mackey-obstruction for the dual action*

$$\widehat{G} \rightarrow \text{Aut}(\mathbb{C} \rtimes_\omega G)$$

is given by the class of the cocycle $(h_\omega)_*\omega^{-1} \in Z^2(\widehat{G}, \mathbb{U}(1))$, where the push-forward is pullback along the inverse h_ω^{-1} , i.e.,

$$(h_\omega)_*\omega^{-1}(\chi, \psi) = \omega(h_\omega^{-1}(\chi), h_\omega^{-1}(\psi))^{-1}, \quad \chi, \psi \in \widehat{G}.$$

We call $(h_\omega)_*\omega^{-1}$ the *dual cocycle* of ω , and we leave the following lemma as an exercise for the reader.

Lemma 2.8. *Suppose that $\omega \in Z^2(G, \mathbb{U}(1))$ is type I and totally skew. Then $h_{(h_\omega)_*\omega^{-1}} = h_\omega^{-1}$ which implies that the dual cocycle $(h_\omega)_*\omega^{-1}$ is also totally skew and type I. This also implies that the double dual cocycle agrees with the original one:*

$$(h_{(h_\omega)_*\omega^{-1}})_*((h_\omega)_*\omega^{-1})^{-1} = \omega.$$

Example 2.9. Let $G = \mathbb{R}^n$ and let us identify $\widehat{\mathbb{R}^n}$ with \mathbb{R}^n via the canonical isomorphism $x \mapsto \chi_x$ with $\langle \chi_x, y \rangle = \exp(2\pi i x^t y)$. Let

$$\omega_A(x, y) := \exp(2\pi i (Ax)^t y)$$

for some strictly upper diagonal matrix $A \in M(n, \mathbb{R})$. Then

$$\langle h_{\omega_A}(x), y \rangle = \exp(2\pi i ((Ax)^t y - (Ay)^t x)) = \exp(2\pi i (\Sigma_A x)^t y) = \langle \chi_{\Sigma_A x}, y \rangle$$

with $\Sigma_A := A - A^t$ the skew symmetric matrix corresponding to A . Thus we see that up to the identification $\widehat{\mathbb{R}^n} \cong \mathbb{R}^n$ the homomorphism h_{ω_A} is given by the linear map $x \mapsto \Sigma_A x$. It follows that ω_A is always type I and ω_A is totally skew if and only if Σ_A is invertible. The dual cocycle is then given by ω_B with $B = \Sigma_A^{-1} A \Sigma_A^{-1}$.

3. Stable NC tori — NC T-duality over the one-point space

3.1. Introduction. We start our discussion of T-duality with bundles over a point. The trivial principal \mathbb{T}^n -bundle over the point is the n -torus \mathbb{T}^n equipped with the translation action of \mathbb{T}^n on itself. Suppose $\delta \in H^3(\mathbb{T}^n, \mathbb{Z})$ allows an action α of \mathbb{R}^n on the corresponding stable continuous-trace C^* -algebra $CT(\mathbb{T}^n, \delta)$ which covers the inflated action of \mathbb{R}^n on

$$\mathbb{T}^n = \text{Prim}(CT(\mathbb{T}^n, \delta)).$$

Since $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, it follows from [E90, Theorem] that $CT(\mathbb{T}^n, \delta)$ is equivariantly isomorphic to the induced algebra $\text{Ind}_{\mathbb{Z}^n}^{\mathbb{R}^n}(\mathcal{K}, \tilde{\alpha})$, where $\tilde{\alpha}$ denotes the action of the stabiliser \mathbb{Z}^n on the fibre $\mathcal{K} = CT(\mathbb{T}^n, \delta)|_z$ of $CT(\mathbb{T}^n, \delta)$ over some chosen point $z \in \mathbb{T}^n$. Recall that for any action $\beta : N \rightarrow \text{Aut}(B)$ on a C^* -algebra B , the induced system $(\text{Ind}_N^G(B, \beta), G, \text{Ind}(\beta))$ is given by

$$\begin{aligned} & \text{Ind}_N^G(B, \beta) \\ & := \{F \in C_b(G, B) : F(g+n) = \beta_{-n}(F(g)), \text{ for all } g \in G, n \in N, \\ & \quad \text{and } (gN \mapsto \|f(g)\|) \in C_0(G/N)\}, \end{aligned}$$

equipped with the point-wise operations, and with G -action $\text{Ind}(\beta)$ which is just given by left (sign!) translation. By the discussion in the previous section, we may assume, up to equivariant Morita equivalence, that $\tilde{\alpha} = \mu|_{\mathbb{Z}^n}$ for some action $\mu : \mathbb{R}^n \rightarrow \text{Aut}(\mathcal{K})$. But then we obtain an isomorphism

$$\Phi : \text{Ind}_{\mathbb{Z}^n}^{\mathbb{R}^n}(\mathcal{K}, \tilde{\alpha}) \rightarrow C(\mathbb{T}^n, \mathcal{K}) \cong \mathcal{K} \otimes C(\mathbb{T}^n), \quad \Phi(f)(j) = \mu_g(f(g)),$$

which transforms the induced action $\text{Ind}(\tilde{\alpha})$ to the diagonal action³ $\mu \otimes \text{inf}$. Thus we learn the following facts (which have been observed before by Mathai and Rosenberg in [MaR05, MaR06]): Firstly, the class δ is trivial, i.e., $CT(\mathbb{T}^n, \delta) \cong \mathcal{K} \otimes C(\mathbb{T}^n)$. Secondly, up to equivariant Morita equivalence the action α is given by a diagonal action $\mu \otimes \text{inf}$.

Recall from [MaR05, MaR06] that in the above setting the (possibly non-commutative) dual torus is given (again up to equivariant Morita equivalence) by the crossed product $\mathcal{K} \otimes C(\mathbb{T}^n) \rtimes_{\mu \otimes \text{inf}} \mathbb{R}^n$, equipped with the dual action of $\widehat{\mathbb{R}^n} \cong \mathbb{R}^n$. By the equivariant version of Green’s imprimitivity theorem (this is a special case of [EKQR06, Theorem 4.11]), this system is Morita equivalent to $\mathcal{K} \rtimes_{\mu} \mathbb{Z}^n$ equipped with the action of \mathbb{R}^n which is inflated from the dual action of \mathbb{T}^n on $\mathcal{K} \rtimes_{\tilde{\alpha}} \mathbb{Z}^n$. By the discussion in the previous section we know that $\mathcal{K} \rtimes_{\mu} \mathbb{Z}^n$ is equivariantly isomorphic to $\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega} \mathbb{Z}^n)$ equipped with the inflated action $\text{id} \otimes \text{inf}$ if $\text{Ma}(\mu) = [\omega]$.

To summarise the point-wise duality picture of Mathai and Rosenberg, there are stabilised commutative tori $\mathcal{K} \otimes C(\mathbb{T}^n)$ with a diagonal action $\mu \otimes \text{inf}$ of \mathbb{R}^n on one side, and there are stabilised noncommutative tori $\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega} \mathbb{Z}^n)$ with action $\text{id} \otimes \text{inf}$ on the other side. This motivates the content of this section which is the investigation of C^* -dynamical systems

$$(7) \quad \left(\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega} N), \widehat{G}, \widehat{\mu} \otimes \text{inf} \right),$$

where we will typically assume that the 2-cocycle ω on N has an extension to G , and $\widehat{\mu} : \widehat{G} \rightarrow \text{Aut}(\mathcal{K})$ is an action which is not necessarily trivial. By Takai duality the study of (7) is equivalent to the study of its dual system

$$(8) \quad \left((\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega} N)) \rtimes_{\widehat{\mu} \otimes \text{inf}} \widehat{G}, G, \widehat{\mu} \otimes \text{inf} \right),$$

and we need to understand under which circumstances there is a Morita equivalence

$$(8) \sim \left(\mathcal{K} \otimes (\mathbb{C} \rtimes_{\tilde{\omega}} N^{\perp}), G, \mu \otimes \text{inf} \right).$$

3.2. Iterated crossed products and transversality. For an action

$$\widehat{\mu} : \widehat{G} \rightarrow \text{Aut}(\mathcal{K}),$$

let us analyse the iterated crossed product

$$(9) \quad \left(\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega} N) \right) \rtimes_{\widehat{\mu} \otimes \text{inf}} \widehat{G},$$

where we assume that the 2-cocycle ω has an extension to G which we again denote by ω . We need

Definition 3.1. Suppose that G is a locally compact abelian group.

³To keep notation down, whenever there is a canonical action γ of a quotient G/N (or \widehat{G}/N^{\perp}) we will simply denote by inf (rather than by $\text{inf } \gamma$) the inflated action of G (or \widehat{G}).

- (1) By the *Heisenberg cocycle* \vee on $G \times \widehat{G}$ we understand the 2-cocycle given by

$$(h, \psi) \vee (g, \chi) := \langle \psi, g \rangle.$$

- (2) The Heisenberg cocycle on $\widehat{G} \times G$ is denoted by \wedge , i.e.,

$$(\psi, h) \wedge (\chi, g) := \langle h, \chi \rangle.$$

- (3) For $\omega \in Z^2(G, U(1))$ and $\widehat{\omega} \in Z^2(\widehat{G}, U(1))$ we denote by $\omega \vee \widehat{\omega}$ the product $\omega \cdot \vee \cdot \widehat{\omega}$ in which we regard ω and $\widehat{\omega}$ as cocycles on $G \times \widehat{G}$ by pullback along the projections to G and \widehat{G} , respectively. Similarly, we define $\widehat{\omega} \wedge \omega \in Z^2(\widehat{G} \times G, U(1))$.

Lemma 3.2. *If $[\widehat{\omega}]$ is the Mackey obstruction of $\widehat{\mu}$, then (9) is G -equivariantly isomorphic to*

$$(10) \quad \mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega \vee \widehat{\omega}} (N \times \widehat{G})),$$

where G acts dually on the second factor of $N \times \widehat{G}$.

Proof. The product and the involution of the crossed product (9) on the basis of the inflation action are given in terms of the pairing

$$\langle \cdot, \cdot \rangle : N \times \widehat{G} \rightarrow U(1)$$

which on the level of L^1 -functions can be expressed in terms of the cocycle \vee . In fact, similar to (4), we can define an isomorphism

$$\begin{array}{ccc} \mathcal{K} \otimes L^1(N \times G, \omega \vee \widehat{\omega}) & \xrightarrow[\Psi]{\cong} & L^1(N \times G, \mathcal{K}) \cong L^1(G, \mathcal{K} \otimes L^1(N, \omega)) \\ \bigcap & & \bigcap \\ \mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega \vee \widehat{\omega}} (N \times \widehat{G})) & & L^1(G, \mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega} N)) \\ & & \bigcap \\ & & (\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega} N)) \rtimes_{\widehat{\mu} \otimes \text{inf}} \widehat{G} \end{array}$$

by $\Psi(k \otimes f)(n, \chi) = f(n, \chi)k\widehat{V}(\chi)$ where $\widehat{V} : \widehat{G} \rightarrow U(\mathcal{H})$ is a Borel map with $\widehat{\mu} = \text{Ad} \circ \widehat{V}$ and such that $\widehat{\omega}^{-1} = \partial \widehat{V}$. It is straightforward to check that this isomorphism is G -equivariant with respect to the dual actions. \square

Applying Proposition 2.4 to (10) for the subgroup $N \times \widehat{G} \subset G \times \widehat{G}$ we obtain the Morita equivalent system

$$(11) \quad \mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega \vee \widehat{\omega}} (G \times \widehat{G})) \rtimes_{\text{dual}} (N^\perp \times 0),$$

whereon G acts by the dual action on the second group factor of the inner crossed product. We want to apply Proposition 2.7 to this inner crossed product, so we are interested in the properties of

$$h_{\omega \vee \widehat{\omega}} : G \times \widehat{G} \rightarrow \widehat{G} \times G$$

defined in (5).

Lemma 3.3. *The homomorphism $h_{\omega \vee \hat{\omega}} : G \times \hat{G} \rightarrow \hat{G} \times G$ is given by*

$$h_{\omega \vee \hat{\omega}} = \begin{pmatrix} h_{\omega} & \text{id}_{\hat{G}} \\ -\text{id}_G & h_{\hat{\omega}} \end{pmatrix} : (g, \chi) \mapsto (h_{\omega}(g) + \chi, h_{\hat{\omega}}(\chi) - g).$$

Proof. We have

$$\begin{aligned} \langle h_{\omega \vee \hat{\omega}}(g, \chi), (h, \psi) \rangle &= \omega \vee \hat{\omega}((g, \chi), (h, \psi)) \omega \vee \hat{\omega}((h, \psi), (g, \chi))^{-1} \\ &= \langle h_{\omega}(g), h \rangle \langle h_{\hat{\omega}}(\chi), \psi \rangle \langle \chi, h \rangle \langle \psi, g \rangle^{-1} \\ &= \langle h_{\omega}(g) + \chi, h \rangle \langle h_{\hat{\omega}}(\chi) - g, \psi \rangle. \end{aligned}$$

The canonical identification $G \cong \hat{\hat{G}}$ yields the result. \square

Lemma 3.4. *The following three conditions are equivalent:*

- (1) $\phi := \text{id}_G + h_{\hat{\omega}} \circ h_{\omega} : G \rightarrow G$ is an isomorphism.
- (2) $\hat{\phi} := \text{id}_{\hat{G}} + h_{\omega} \circ h_{\hat{\omega}} : \hat{G} \rightarrow \hat{G}$ is an isomorphism.
- (3) $h_{\omega \vee \hat{\omega}}$ is an isomorphism.

Proof. The equivalence of the first two statements follows from the observation that $\hat{\phi}$ is the dual of ϕ . Alternatively, if ϕ^{-1} exists, then a one-line calculation shows that $\text{id}_{\hat{G}} - h_{\omega} \circ \phi^{-1} \circ h_{\hat{\omega}}$ is an inverse for $\hat{\phi}$.

Now, if $h_{\omega \vee \hat{\omega}}$ is an isomorphism, then

$$h_{\omega \vee \hat{\omega}}^{\dagger} := \text{flip} \circ h_{\omega \vee \hat{\omega}} \circ \text{flip} = \begin{pmatrix} h_{\hat{\omega}} & -\text{id}_G \\ \text{id}_{\hat{G}} & h_{\omega} \end{pmatrix}$$

is an isomorphism, where $\text{flip} : \hat{G} \times G \rightarrow G \times \hat{G}$ is transposition. Then the composed isomorphism is

$$h_{\omega \vee \hat{\omega}} \circ h_{\omega \vee \hat{\omega}}^{\dagger} = \begin{pmatrix} \text{id}_{\hat{G}} + h_{\omega} \circ h_{\hat{\omega}} & 0 \\ 0 & \text{id}_G + h_{\hat{\omega}} \circ h_{\omega} \end{pmatrix} = \begin{pmatrix} \hat{\phi} & 0 \\ 0 & \phi \end{pmatrix}.$$

Conversely, if $\phi, \hat{\phi}$ are isomorphisms then

$$h_{\omega \vee \hat{\omega}}^{-1} := h_{\omega \vee \hat{\omega}}^{\dagger} \circ \begin{pmatrix} \hat{\phi}^{-1} & 0 \\ 0 & \phi^{-1} \end{pmatrix}$$

exists and is obviously a right inverse. The property of being a left inverse requires a small calculation. First we have

$$h_{\omega \vee \hat{\omega}}^{-1} \circ h_{\omega \vee \hat{\omega}} = \begin{pmatrix} h_{\hat{\omega}} \circ \hat{\phi}^{-1} \circ h_{\omega} + \phi^{-1} & h_{\omega} \circ \phi^{-1} - \hat{\phi}^{-1} \circ h_{\hat{\omega}} \\ \hat{\phi}^{-1} \circ h_{\omega} - h_{\omega} \circ \phi^{-1} & \hat{\phi}^{-1} + h_{\omega} \circ \phi^{-1} \circ h_{\hat{\omega}} \end{pmatrix}.$$

Inserting now $\hat{\phi}^{-1} = \text{id}_{\hat{G}} - h_{\omega} \circ \phi^{-1} \circ h_{\hat{\omega}}$ into this, we derive at

$$h_{\omega \vee \hat{\omega}}^{-1} \circ h_{\omega \vee \hat{\omega}} = \begin{pmatrix} \text{id}_G & 0 \\ 0 & \text{id}_{\hat{G}} \end{pmatrix}. \quad \square$$

Let's assume one of the equivalent conditions of Lemma 3.4. Then the inner crossed product in (11) is isomorphic to the compacts. Having determined $h_{\omega \vee \hat{\omega}}^{-1}$ we can compute the dual cocycle $(h_{\omega \vee \hat{\omega}})_*(\omega \vee \hat{\omega})^{-1}$ according to Proposition 2.7. The result helps us to understand the remaining outer

crossed product with N^\perp in (11) and also the remaining G -action on it. To state the result, some more notation is useful.

Definition 3.5. For a 2-cocycle ω on G and a 2-cocycle $\hat{\omega}$ on \hat{G} , we define new cocycles on G and \hat{G} by

$$\omega \rtimes \hat{\omega} := \omega \cdot h_\omega^* \hat{\omega}^{-1}, \quad \hat{\omega} \rtimes \omega := \hat{\omega} \cdot h_{\hat{\omega}}^* \omega^{-1}.$$

Moreover, if $\phi = \text{id}_G + h_\omega \circ h_\omega$ is an isomorphism, we define

$$\omega \bar{\rtimes} \hat{\omega} := \phi_*(\omega \rtimes \hat{\omega}), \quad \hat{\omega} \bar{\rtimes} \omega := \hat{\phi}_*(\hat{\omega} \rtimes \omega).$$

Lemma 3.6. *In $H^2(\hat{G} \times G, U(1))$ the following equality holds:⁴*

$$(12) \quad [(h_{\omega \vee \hat{\omega}})_*(\omega \vee \hat{\omega})^{-1}] = [\hat{\omega} \bar{\rtimes} \omega] + [\omega \bar{\rtimes} \hat{\omega}] - (\hat{\phi} \times \text{id}_G)_*[\wedge],$$

where the classes on \hat{G} and G are understood as classes on $\hat{G} \times G$ by pullback along the projections.

Proof. Before we start with the actual computation we need to be aware of some general cocycle properties. Let ν be a 2-cocycle on any abelian group. Twofold application of the cocycle identity gives

$$\begin{aligned} \nu(x + y, -(x + y)) &= \nu(y, -x - y) \nu(x, -x)^{-1} \nu(x, y)^{-1} \\ &= \nu(0, x) \nu(-x, -y)^{-1} \nu(y, -y) \nu(x, -x)^{-1} \nu(x, y)^{-1} \end{aligned}$$

which means that $c(x) := \nu(x, -x)\nu(0, 0)$ is a cochain that implements $\nu(x, y) \sim \nu(-y, -x)^{-1}$.

Furthermore, a fourfold application of the cocycle identity gives

$$\begin{aligned} \nu(a + x, b + y) &= \nu(a, b) \nu(x, y) \\ &\quad \cdot \nu(x, b) \nu(b, x)^{-1} \\ &\quad \cdot \nu(a, x)^{-1} \nu(b, y)^{-1} \nu(a + b, x + y). \end{aligned}$$

Or, if we define a 2-cocycle $\tilde{\nu}$ on the product of the group with itself by $\tilde{\nu}((a, x), (b, y)) := \nu(a + x, b + y)^{-1}$, then

$$\begin{aligned} \tilde{\nu}((a, x), (b, y)) &= \nu(a, b)^{-1} \nu(x, y)^{-1} \langle h_\nu(x), b \rangle^{-1} (d\nu)((a, x), (b, y)) \\ &\sim \nu(a, b)^{-1} \nu(x, y)^{-1} \langle x, h_\nu(b) \rangle, \end{aligned}$$

where d is the boundary operator on the product of the group with itself.

Let us now turn to the actual computation. By definition we have to compute $(h_{\omega \vee \hat{\omega}}^{-1})^*(\omega \vee \hat{\omega})^{-1}$, for

$$h_{\omega \vee \hat{\omega}}^{-1} = \begin{pmatrix} h_{\hat{\omega}} & -\text{id}_G \\ \text{id}_{\hat{G}} & h_\omega \end{pmatrix} \circ \begin{pmatrix} \hat{\phi}^{-1} & 0 \\ 0 & \phi^{-1} \end{pmatrix}.$$

⁴We denote the group operation on $U(1)$ -valued 2-cocycles multiplicatively, whereas we denote the group operation on any cohomology group additively.

Let us use the shorthands $\chi' := \hat{\phi}^{-1}\chi$ and $g' := \phi^{-1}g$. Then we have

$$(13) \quad (h_{\omega \vee \hat{\omega}})_*(\omega \vee \hat{\omega})^{-1}((\chi, g), (\psi, h)) \\ = \omega(h_{\hat{\omega}}(\chi') - g', h_{\hat{\omega}}(\psi') - h')^{-1}$$

$$(14) \quad \cdot \langle \chi' + h_{\omega}(g'), h_{\hat{\omega}}(\psi') - h' \rangle^{-1}$$

$$(15) \quad \cdot \hat{\omega}(\chi' + h_{\omega}(g'), \psi' + h_{\omega}(h'))^{-1}$$

The middle term (14) decomposes to

$$(14) = \langle \chi', h' \rangle \langle h_{\omega}(g'), h_{\hat{\omega}}(\psi') \rangle^{-1} \hat{\omega}(\chi', \psi') \hat{\omega}(\psi', \chi')^{-1} \omega(g', h') \omega(h', g')^{-1} \\ \sim \langle \chi', h' \rangle \langle h_{\omega}(g'), h_{\hat{\omega}}(\psi') \rangle^{-1} \hat{\omega}(\chi', \psi') \hat{\omega}(-\chi', -\psi') \omega(g', h') \omega(h', g')^{-1}.$$

But as $((\chi, g), (\psi, h)) \mapsto \langle \chi, h \rangle$ is cohomologous to $((\chi, g), (\psi, h)) \mapsto \langle \psi, g \rangle^{-1}$ (just by the cochain $(\chi, g) \mapsto \langle g, \chi \rangle$) this can be transformed to

$$(14) \sim \langle g', \psi' \rangle^{-1} \langle h_{\omega}(g'), h_{\hat{\omega}}(\psi') \rangle^{-1} \hat{\omega}(\chi', \psi') \hat{\omega}(\psi', \chi')^{-1} \omega(g', h') \omega(h', g')^{-1}.$$

To transform (13) and (15), we apply the above identity for $\tilde{\nu}$. We find

$$(15) \sim \hat{\omega}(\chi', \psi')^{-1} \hat{\omega}(h_{\omega}(g'), h_{\omega}(h'))^{-1} \langle (h_{\omega}(g'), h_{\hat{\omega}}(\psi')) \rangle$$

and

$$(13) \sim \omega(h_{\hat{\omega}}(\chi'), h_{\hat{\omega}}(\psi'))^{-1} \omega(-g', -h')^{-1} \langle -g', h_{\omega}(h_{\hat{\omega}}(\psi')) \rangle \\ \sim \omega(h_{\hat{\omega}}(\chi'), h_{\hat{\omega}}(\psi'))^{-1} \omega(h', g') \langle -g', h_{\omega}(h_{\hat{\omega}}(\psi')) \rangle.$$

Multiplying these partial results we get

$$(16) \quad (13) \cdot (14) \cdot (15) \sim \langle g', \psi' + h_{\omega}(h_{\hat{\omega}}(\psi')) \rangle^{-1} \\ \cdot \omega(g', h') h_{\hat{\omega}}^* \hat{\omega}(g', h')^{-1} \\ \cdot \hat{\omega}(-\chi', -\psi') h_{\hat{\omega}}^* \omega(\chi', \psi')^{-1}$$

which is the claimed formula up to the minus sign inside the argument of $\hat{\omega}$. However, the injectivity of the map $[\hat{\omega}] \mapsto h_{\hat{\omega}} \in \text{Hom}(\widehat{G}, G)$ implies that $\hat{\omega}(\chi, \psi) \sim \hat{\omega}(-\chi, -\psi)$. So the lemma is proven. \square

Remark 3.7. It is very important for later purposes (Theorem 4.8) to observe at this point that the computation done in the proof of Lemma 3.6 is based on explicit cochains (given in terms of $\hat{\omega}, \omega$ or $\langle -, - \rangle$) up to and including (16). Only the very last step $\hat{\omega}(\chi, \psi) \sim \hat{\omega}(-\chi, -\psi)$ required an abstract argument. If $\hat{\omega}$ is cohomologous to a bicharacter then this last relation can also be made explicit: Let $\hat{\omega} = dc \hat{\eta}$ for some bicharacter $\hat{\eta}$, then

$$\hat{\omega}(\chi, \psi) = dc(\chi, \psi) \hat{\eta}(\chi, \psi) \\ = dc(\chi, \psi) \hat{\eta}(-\chi, -\psi) \\ = dc(\chi, \psi) dc(-\chi, -\psi)^{-1} dc(-\chi, -\psi) \hat{\eta}(-\chi, -\psi) \\ = d\tilde{c}(\chi, \psi) \hat{\omega}(-\chi, -\psi),$$

for $\tilde{c}(\chi) := c(\chi)c(-\chi)^{-1}$. These explicit cohomology relations are abstract cocycle identities and do not depend on $U(1)$ as a module. In fact, Lemma 3.6 remains valid if (1) the involved cocycles are not just $U(1)$ -valued but $C(B, U(1))$ -valued (trivial module structure) for some space B , if (2) they are cohomologous to bihomomorphisms (rather than bicharacters), and if (3) we have control over the continuity properties of the quantities $h_\omega, h_{\hat{\omega}}$ which then should be regarded as bundle maps

$$\begin{array}{ccccc} B \times G & \xrightarrow{h_\omega} & B \times \hat{G} & \xrightarrow{h_{\hat{\omega}}} & B \times G \\ \downarrow & & \downarrow & & \downarrow \\ B & \xlongequal{\quad} & B & \xlongequal{\quad} & B. \end{array}$$

This will ensure that if a cocycle with values in $C(B, U(1))$ is point-wise pulled back by $h_\omega, h_{\hat{\omega}}$ or pushed forward by ϕ (which involves inversion in $\text{Aut}(G)!$), then the resulting point-wise defined object is again mapping to $C(B, U(1))$. Equality (12) then holds in $H^2(\hat{G} \times G, C(B, U(1)))$.

The structure of $[(h_{\omega \vee \hat{\omega}})_*(\omega \vee \hat{\omega})^{-1}]$ given by its three summands now immediately yields the following corollary.

Corollary 3.8. *If ϕ is an isomorphism, the crossed product (11) is G -equivariantly isomorphic to*

$$(17) \quad \mathcal{K} \otimes \mathcal{K} \otimes (\mathbb{C} \rtimes_{\hat{\omega} \bar{\omega}} N^\perp),$$

where G acts by $\text{id} \otimes \mu \otimes \text{in}^\phi$ for an action μ with Mackey obstruction

$$\text{Ma}(\mu) = [\omega \bar{\omega}] \in H^2(G, U(1)),$$

and

$$(\text{in}^\phi_g(f))(n) := \langle g, \hat{\phi}^{-1}(n) \rangle^{-1} f(n).$$

So except from the part of the action given by in^ϕ we have found a structure rather similar to the one with which we have started. To manipulate it a little further we need an extra assumption.

Definition 3.9.

- (1) A homomorphism $G \rightarrow G$ is said to be an *automorphism of (N, G)* if it is an automorphism of G and if it maps N bijectively to itself. We denote by $\text{Aut}(N, G)$ the set of all of those.
- (2) A pair of cocycles $\omega : G \times G \rightarrow U(1), \hat{\omega} : \hat{G} \times \hat{G} \rightarrow U(1)$ (or their cohomology classes) is called *transverse* if $\phi = \text{id}_G + h_{\hat{\omega}} \circ h_\omega \in \text{Aut}(N, G)$. (Equivalently, one might require that $\hat{\phi} = \text{id}_{\hat{G}} + h_\omega \circ h_{\hat{\omega}}$ is an automorphism of (N^\perp, \hat{G}) .)
- (3) The binary relation defined by transversality is denoted by

$$\pitchfork \subset H^2(G, U(1)) \times H^2(\hat{G}, U(1)),$$

i.e., $\omega \pitchfork \hat{\omega}$ if and only if ω and $\hat{\omega}$ are transverse.

- (4) Let $\hat{\mu} \otimes \text{inf}$ be a \widehat{G} -action on $\mathcal{K} \otimes (\mathbb{C} \rtimes_{\eta} N)$. The actions $\hat{\mu}$ or $\hat{\mu} \otimes \text{inf}$ or the dynamical system $(\mathcal{K} \otimes (\mathbb{C} \rtimes_{\eta} N), \widehat{G}, \hat{\mu} \otimes \text{inf})$ are called *transverse* if the cocycle $\eta : N \times N \rightarrow \text{U}(1)$ has an extension ω to G such that $\omega \upharpoonright \text{Ma}(\hat{\mu})$.

Corollary 3.10. *If the dynamical system $(\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega} N), \widehat{G}, \hat{\mu} \otimes \text{inf})$ is transverse, then its dual system*

$$\left((\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega} N)) \rtimes_{\hat{\mu} \otimes \text{inf}} \widehat{G}, G, \widehat{\hat{\mu} \otimes \text{inf}} \right) \cong \left(\mathcal{K} \otimes \mathcal{K} \otimes (\mathbb{C} \rtimes_{\hat{\omega} \rtimes \omega} N^{\perp}), G, \text{id} \otimes \mu \otimes \text{in}^{\phi} \right)$$

is G -equivariantly isomorphic to

$$\mathcal{K} \otimes (\mathbb{C} \rtimes_{\hat{\omega} \rtimes \omega} N^{\perp}),$$

where G acts by $\mu \otimes \text{inf}$ with Mackey obstruction

$$\text{Ma}(\mu) = [\omega \rtimes \hat{\omega}] \quad (= [\phi_*(\omega \rtimes \hat{\omega})]).$$

Proof. Using $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$ and Corollary 3.8 it suffices to show that

$$\left(\mathcal{K} \otimes (\mathbb{C} \rtimes_{\hat{\omega} \rtimes \omega} N^{\perp}), G, \mu \otimes \text{in}^{\phi} \right) \cong \left(\mathcal{K} \otimes (\mathbb{C} \rtimes_{\hat{\omega} \rtimes \omega} N^{\perp}), G, \mu \otimes \text{inf} \right).$$

By transversality, $\hat{\phi}$ induces an isomorphism of N^{\perp} , so it induces an isomorphism

$$(18) \quad \hat{\phi}^* : \mathbb{C} \rtimes_{\phi_*(\hat{\omega} \rtimes \omega)} N^{\perp} \cong \mathbb{C} \rtimes_{\hat{\omega} \rtimes \omega} N^{\perp}$$

given by pullback: $f \mapsto f \circ \hat{\phi}$. Similarly, the inversion on the group $\Theta : N^{\perp} \rightarrow N^{\perp}$ induces an automorphism by pullback

$$(19) \quad \Theta^* : \mathbb{C} \rtimes_{\hat{\omega} \rtimes \omega} N^{\perp} \cong \mathbb{C} \rtimes_{\Theta^*(\hat{\omega} \rtimes \omega)} N^{\perp}.$$

Note that

$$(\text{in}^{\phi}_g(f))(\phi(-n)) = \langle g, -n \rangle^{-1} f(\phi(-n)) = \langle g, n \rangle f(\phi(-n)),$$

so the composition $\Theta^* \circ \hat{\phi}^*$ turns in^{ϕ} into the ordinary inflation action. However, pullback of a 2-cocycle along the inversion gives a cocycle that is similar to the original one, i.e., they have the same cohomology class (see Remark 3.7). Then their twisted group algebras are equivariantly isomorphic. \square

By Takai duality, we know that the dual of the constructed system

$$(\mathcal{K} \otimes (\mathbb{C} \rtimes_{\hat{\omega} \rtimes \omega} N^{\perp}), \mu \otimes \text{inf})$$

(i.e., the bidual of the original system) is Morita equivalent to the original (transverse) system. The following lemma shows that the dual system of a transverse system is transverse again.

Lemma 3.11. *The assignment $(\omega, \hat{\omega}) \mapsto (\omega \bar{\times} \hat{\omega}, \hat{\omega} \times \omega)$ defines a bijection $\mathfrak{h} \rightarrow \mathfrak{h}$ such that*

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\cong} & \mathfrak{h} \\ \downarrow & & \downarrow \\ \text{Aut}(N, G) & \xrightarrow{\text{id}} & \text{Aut}(N, G) \end{array}$$

commutes, where the vertical arrows are given by the tautological map.

Proof. Firstly, $\phi' := \text{id} + h_{\hat{\omega} \times \omega} \circ h_{\omega \bar{\times} \hat{\omega}}$ is an isomorphism of (N, G) : A one-line computation gives $h_{h_{\hat{\omega}}^* \omega^{-1}} = h_{\hat{\omega}} \circ h_{\omega} \circ h_{\hat{\omega}} : \hat{G} \rightarrow \hat{G}$, and so

$$h_{\hat{\omega} \times \omega} = h_{\hat{\omega} \cdot h_{\hat{\omega}}^* \omega^{-1}} = h_{\hat{\omega}} + h_{\hat{\omega}} \circ h_{\omega} \circ h_{\hat{\omega}} = h_{\hat{\omega}} \circ \hat{\phi},$$

wherein as before $\hat{\phi} = \text{id} + h_{\omega} \circ h_{\hat{\omega}}$. The same algebra gives

$$h_{\omega \bar{\times} \hat{\omega}} = h_{\phi_* (\omega \times \hat{\omega})} = \hat{\phi}^{-1} \circ h_{\omega \times \hat{\omega}} \circ \phi^{-1} = \hat{\phi}^{-1} \circ h_{\omega}.$$

Therefore

$$\phi' = \text{id} + h_{\hat{\omega} \times \omega} \circ h_{\omega \bar{\times} \hat{\omega}} = \text{id} + (h_{\hat{\omega}} \circ \hat{\phi}) \circ (\hat{\phi}^{-1} \circ h_{\omega}) = \phi$$

which is an automorphism of (N, G) by assumption. This also shows that the diagram of the lemma commutes.

Secondly, another straight forward calculation shows that the inverse of $(\omega, \hat{\omega}) \mapsto (\omega \bar{\times} \hat{\omega}, \hat{\omega} \times \omega)$ is given by

$$(\omega, \hat{\omega}) \mapsto (\omega \times \hat{\omega}, \hat{\omega} \bar{\times} \omega). \quad \square$$

Corollary 3.10 tells that transversality gives a sufficient condition to answer the question raised in (8). Combining it together with Lemma 3.11 we have found a class of C^* -dynamical systems which is closed under taking crossed products: Let us denote by $NCT(N; \hat{G})$ the 2-category of systems $(\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega} N), \hat{G}, \hat{\mu} \otimes \text{inf})$, which has \hat{G} -equivariant Morita equivalences as 1-morphisms and equivariant isomorphisms between them as 2-morphisms. There is a proper subcategory $NCT(N; \hat{G})^{\mathfrak{h}} \subset NCT(N; \hat{G})$ which consists of systems which are 2-isomorphic (i.e., Morita equivalent) to a transverse representative. This whole section is summarised in the following theorem:

Theorem 3.12. *The duality functor $_ \times G$ defined on all C^* -dynamical systems with group G restricts to a duality of transverse dynamical systems:*

$$\begin{array}{ccc} C^*\text{-Dynamical Systems} & \xrightarrow{\sim} & C^*\text{-Dynamical Systems} \\ \text{with Group } G & & \text{with Group } \hat{G} \\ \cup & & \cup \\ NCT(N^{\perp}; G) & & NCT(N; \hat{G}) \\ \cup & & \cup \\ NCT(N^{\perp}; G)^{\mathfrak{h}} & \xrightarrow{\sim} & NCT(N; \hat{G})^{\mathfrak{h}} \end{array} .$$

3.3. Classification remarks. Recall from Section 2.4 that the C^* -dynamical systems $\mathcal{K} \rtimes_{\alpha} N$ or $\mathbb{C} \rtimes_{\omega} N$ equipped with their canonical \widehat{N} -actions are classified up to equivariant Morita equivalence by second Borel cohomology

$$\text{Ma}(\alpha), [\omega] \in H^2(N, U(1)).$$

We start with an example that illustrates that the objects with which we are dealing are more involved.

Example 3.13. Let $G = \mathbb{R}^2$ and $N = \mathbb{Z}^2$, and choose

$$\frac{1}{3}, \frac{2}{3} \in \mathbb{R}/\mathbb{Z} = \mathbb{T} \cong H^2(N, U(1)).$$

Denote by $\hat{\mu}_3$ an action of $\widehat{G} = \mathbb{R}^2$ on \mathcal{K} with Mackey obstruction

$$3 \in \mathbb{R} \cong H^2(\mathbb{R}^2, U(1)).$$

Then there is a \widehat{G} -equivariant Morita equivalence

$$\left(\mathcal{K} \otimes (\mathbb{C} \rtimes_{\frac{2}{3}} N), \hat{\mu}_3 \otimes \text{inf}\right) \sim \left(\mathcal{K} \otimes (\mathbb{C} \rtimes_{\frac{1}{3}} N), \text{id} \otimes \text{inf}\right).$$

Proof. A lengthy but direct proof is given in Appendix A. Using our theory of transversality one can significantly shorten the proof. This is done in Section 3.4 below. \square

To understand why this example could possibly be true let us try to understand what can be said in general about an equality of classes

$$[\omega_1], [\omega_2] \in H^2(N, U(1))$$

if there is a \widehat{G} -equivariant Morita equivalence

$$(20) \quad \mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega_1} N) \sim \mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega_2} N),$$

where \widehat{G} acts on both sides diagonally by actions on the compacts

$$\hat{\mu}_i : \widehat{G} \rightarrow \text{Aut}(\mathcal{K}), \quad i = 1, 2,$$

tensor the inflated actions

$$\text{inf}_i : \widehat{G} \rightarrow \text{Aut}(\mathbb{C} \rtimes_{\omega_i} N), \quad i = 1, 2,$$

on the respective twisted group C^* -algebras. Denote by $S \subset N$ the symmetry group of ω_2 that is the kernel of the map $h_{\omega_2} : N \rightarrow \widehat{N}$. The dual group \widehat{S} of S is homeomorphic to the primitive spectrum of $\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega_2} N)$, and, by the Dauns–Hofmann Theorem (see [W07]), the $U(1)$ -valued functions thereon are isomorphic to the center of the unitary group of its multiplier algebra, i.e., there is a short exact sequence

$$1 \rightarrow C(\widehat{S}, U(1)) \rightarrow \text{UM}(\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega_2} N)) \xrightarrow{\text{Ad}} \text{Inn}(\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega_2} N)) \rightarrow 1,$$

and this sequence is \widehat{N} -equivariant for the actions that are induced by the dual action on $\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega_2} N)$. On $C(\widehat{S}, U(1))$ this action is just translation in the argument after restriction $\widehat{N} \rightarrow \widehat{S}$. Using the polish topology on $\text{Inn}(A)$

and let $\langle \eta, - \rangle_i := (\text{id} \otimes \text{id} \otimes \text{inf}_i)(\eta)$. We just compute:

$$\begin{aligned} \gamma(\chi + \psi) &= \Phi \circ \langle \chi + \psi, - \rangle_1 \circ \Phi^{-1} \circ \langle \chi + \psi, - \rangle_2^{-1} \\ &= \Phi \circ \langle \chi, - \rangle_1 \circ \Phi^{-1} \circ \Phi \circ \langle \psi, - \rangle_1 \circ \Phi^{-1} \circ \langle \chi + \psi, - \rangle_2^{-1} \\ &= \left(\Phi \circ \langle \chi, - \rangle_1 \circ \Phi^{-1} \circ \langle \chi, - \rangle_2^{-1} \right) \circ \langle \chi, - \rangle_2 \\ &\quad \circ \left(\Phi \circ \langle \psi, - \rangle_1 \circ \Phi^{-1} \circ \langle \psi, - \rangle_2^{-1} \right) \circ \langle \chi, - \rangle_2^{-1} \\ &= \gamma(\chi) \circ \langle \chi, - \rangle_2 \circ \gamma(\psi) \circ \langle \chi, - \rangle_2^{-1}. \end{aligned}$$

The multiplication action on the unitary group of the multiplier algebra turns into conjugation when passing to the inner automorphisms by Ad, so we obtain the cocycle identity $\gamma(\chi + \psi) = \gamma(\chi) \circ (\chi|_N \cdot \gamma(\psi))$ for γ . In particular, it follows that $\gamma(n^\perp + \psi) = \gamma(n^\perp) \circ \gamma(\psi) = \gamma(\psi)$, for $n^\perp \in N^\perp$, i.e., γ is constant on the cosets. \square

The cocycle $\dot{\gamma}$ determines a cohomology class which – by abuse of notation – we again denote by

$$\gamma \in H^1\left(\widehat{N}, \text{Inn}(\mathcal{K} \otimes \mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega_2} N))\right).$$

By the (not very long) exact sequence it defines an obstruction class

$$\delta(\gamma) \in H^2(\widehat{N}, C(\widehat{S}, U(1))).$$

If this obstruction class vanishes, γ has a unitary lift, and this lift then implements a Morita equivalence between the two actions $\text{inf}_i, i = 1, 2$. The next lemma shows that in this case the classes $[\omega_1], [\omega_2] \in H^2(N, U(1))$ agree if we assume that these classes extend to G :

Lemma 3.15. *Let $[\omega_1], [\omega_2] \in H^2(N, U(1))$ be two classes in the image of the restriction map $H^2(G, U(1)) \rightarrow H^2(N, U(1))$. If*

$$\mathbb{C} \rtimes_{\omega_1} N \sim \mathbb{C} \rtimes_{\omega_2} N$$

is a \widehat{G} -equivariant Morita equivalence, where \widehat{G} acts on both sides by the inflated actions, then

$$[\omega_1] = [\omega_2] \in H^2(N, U(1)).$$

Proof. If μ_i is a G -action on \mathcal{K} with Mackey obstruction $[\omega_i]$, then

$$\mathcal{K} \otimes C(G/N) \quad \text{with} \quad \mu_i \otimes (\text{left translation})$$

is the (pre-)dual of $\mathbb{C} \rtimes_{\omega_i} N, i = 1, 2$. But these systems are induced systems $\text{Ind}_N^G(\mathcal{K}, \mu_i|_N)$ by [E90, Theorem] which are classified up to Morita equivalence by their actions $\mu_i|_N : N \rightarrow \text{Aut}(\mathcal{K})$. \square

Under certain circumstances one might indeed conclude that the obstruction class $\delta(\gamma)$ vanishes:

Corollary 3.16. *Assume that N is torsion-free, and assume ω_2 (or ω_1) is totally skew. If there is a \widehat{G} -equivariant Morita equivalence*

$$\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega_1} N) \sim \mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega_2} N),$$

with diagonal \widehat{G} -actions on both algebras as assumed in (20), then

$$[\omega_1] = [\omega_2] \in H^2(N, U(1)).$$

Proof. A totally skew cocycle has trivial symmetry group $S = \{0\}$, and if N is torsion-free, then \widehat{N} is connected, and so the whole obstruction group vanishes:

$$H^2(\widehat{N}, C(\widehat{S}, U(1))) = H^2(\widehat{N}, U(1)) \hookrightarrow \text{Hom}(\widehat{N}, N) = 0. \quad \square$$

In Example 3.13 both of the involved noncommutative tori were rational. However, if one of the involved classes is irrational, then its symmetry group is trivial, so one can apply the above corollary.

Example 3.17. Let $G = \mathbb{R}^2$, $N = \mathbb{Z}^2$, and let $r \in \mathbb{R}/\mathbb{Z} \cong H^2(\mathbb{Z}^2, U(1))$ be an irrational number, and let ω_r be a corresponding cocycle. If there is an \mathbb{R}^2 -equivariant Morita equivalence

$$\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega} \mathbb{Z}^2) \sim \mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega_r} \mathbb{Z}^2),$$

where \mathbb{R}^2 acts on both algebras diagonally as in (20), then

$$[\omega] = [\omega_r] \in H^2(\mathbb{Z}^2, U(1)).$$

Let us continue with some further analysis of the class $\delta(\gamma)$ constructed above. Consider the following diagram of induced maps

$$\begin{array}{ccccc} H^2(\widehat{N}, C(\widehat{S}, U(1))) & \ni & \delta(\gamma) & & \\ \downarrow q^* & & & & \\ H^2(\widehat{G}, C(\widehat{S}, U(1))) & \xleftarrow{c_*} & H^2(\widehat{G}, U(1)) & \ni & \text{Ma}(\hat{\mu}_1^{op} \otimes \hat{\mu}_2) \\ \downarrow i^* & & \downarrow i^\# & & \downarrow ? \\ H^2(N^\perp, C(\widehat{S}, U(1))) & \xrightleftharpoons[\text{ev}_\#]{c_\#} & H^2(N^\perp, U(1)) & \ni & 0, \end{array}$$

where $c : U(1) \rightarrow C(\widehat{S}, U(1))$ is the obvious inclusion, and

$$\text{ev} : C(\widehat{S}, U(1)) \rightarrow U(1)$$

is the evaluation at $0 \in \widehat{S}$ which is N^\perp -equivariant, because N^\perp acts trivially on both sides. As $\text{ev}_\# \circ c_\# = \text{id}$, $c_\#$ is injective. It is easily seen that the Mackey obstruction class $\text{Ma}(\hat{\mu}_1^{op} \otimes \hat{\mu}_2)$ of the action

$$\hat{\mu}_1^{op} \otimes \hat{\mu}_2 : \widehat{G} \rightarrow \text{Aut}(\mathcal{K} \otimes \mathcal{K}),$$

is connected to $\delta(\gamma)$ by⁵ $q^*(\delta(\gamma)) = -c_*(\text{Ma}(\hat{\mu}_1^{op} \otimes \hat{\mu}_2))$. But the composition $i^* \circ q^*$ is zero, so by commutativity of the square and by injectivity of $c_\#$, we see that $i^\#(\text{Ma}(\hat{\mu}_1^{op} \otimes \hat{\mu}_2)) = 0$, and we have just proven the following statement:

Lemma 3.18. *Let $\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega_1} N) \sim \mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega_2} N)$ be a \widehat{G} -equivariant Morita equivalence, where \widehat{G} acts on both sides diagonally by actions on the compacts $\hat{\mu}_i : \widehat{G} \rightarrow \text{Aut}(\mathcal{K})$, $i = 1, 2$, tensor the inflated actions*

$$\text{inf}_i : \widehat{G} \rightarrow \text{Aut}(\mathbb{C} \rtimes_{\omega_i} N), \quad i = 1, 2,$$

on the respective crossed products. Then the Mackey obstructions of the restricted actions $\hat{\mu}_i|_{N^\perp}$ agree:

$$\text{Ma}(\hat{\mu}_1|_{N^\perp}) = \text{Ma}(\hat{\mu}_2|_{N^\perp}) \in H^2(N^\perp, \text{U}(1)).$$

An instance of Lemma 3.18 already appeared in Example 3.13, wherein the \mathbb{R}^2 -action $\hat{\mu}_3$ has trivial Mackey obstruction when restricted to \mathbb{Z}^2 . In general, Lemma 3.18 gives an obstruction map

$$(21) \quad \text{Ma}_{(N; \widehat{G})} : [NCT(N; \widehat{G})] \rightarrow H^2(N^\perp, \text{U}(1))$$

(here the brackets $[\cdot]$ denote 2-isomorphism classes, i.e., Morita equivalence classes). This map will enable us to identify the ‘‘commutative theory’’ inside our theory as we explain next.

If $\hat{\mu} \otimes \text{inf}$ is a transverse \widehat{G} -action on $\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega} N)$, then the previous section shows that its crossed product $\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega} N) \rtimes_{\hat{\mu} \otimes \text{inf}} \widehat{G}$ together with the dual G -action is Morita equivalent to an algebra $\mathcal{K} \otimes (\mathbb{C} \rtimes_{\hat{\eta}} N^\perp)$ equipped with a transverse action $\mu \otimes \text{inf}$. Corollary 3.10 determines the class $[\hat{\eta}] \in H^2(N^\perp, \text{U}(1))$ to a certain extend. Namely

$$(22) \quad [\hat{\eta}] \in (\text{Ma}(\hat{\mu})|_{N^\perp})^\natural \subset H^2(N^\perp, \text{U}(1)),$$

where, for $\hat{\theta} \in H^2(N^\perp, \text{U}(1))$, we have used the notation

$$(23) \quad \hat{\theta}^\natural := \left\{ [\hat{\omega} \rtimes \omega]|_{N^\perp} : [\hat{\omega}]|_{N^\perp} = \hat{\theta}, \omega \natural \hat{\omega} \right\}.$$

For $\hat{\theta}$ in the image of the restriction map $\widehat{\text{res}} : H^2(\widehat{G}, \text{U}(1)) \rightarrow H^2(N^\perp, \text{U}(1))$ this set is never empty. In fact, the trivial cocycle 1 is transverse to any cocycle $\hat{\omega}$, so for $\hat{\theta} = [\hat{\omega}]|_{N^\perp}$ we have

$$\hat{\theta} = [\hat{\omega}]|_{N^\perp} = [\hat{\omega} \rtimes 1]|_{N^\perp} \in \hat{\theta}^\natural.$$

The element relation (22) restricts the possible classes for building the dual algebra to the set $(\text{Ma}(\hat{\mu})|_{N^\perp})^\natural$. This set is a Morita invariant of both the original system $(\mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega} N), \hat{\mu} \otimes \text{inf})$ and of its dual system. (Just because of Lemma 3.18 which states that the class $\text{Ma}(\hat{\mu})|_{N^\perp}$ is a Morita invariant of the original system.) However, the element $[\hat{\eta}]$ itself is not a specified

⁵The sign reflects the fact that the Mackey obstruction is defined to be the negative of the actual connecting homomorphism in (1).

element in $(\text{Ma}(\hat{\mu})|_{N^\perp})^{\hat{\eta}}$. In fact, it varies with the choices of the extension of ω from N to G , and these choices can make a difference (see Section 3.4 for a detailed example). Nevertheless, the following lemma shows that in a very important case the set $(\text{Ma}(\hat{\mu})|_{N^\perp})^{\hat{\eta}}$ reduces to a singleton.

Lemma 3.19. *For $\hat{\theta} \in H^2(N^\perp, \text{U}(1))$ the following three statements are equivalent:*

- (1) $\hat{\theta} = 0$.
- (2) $\hat{\theta}^{\hat{\eta}} = \{0\}$.
- (3) $0 \in \hat{\theta}^{\hat{\eta}}$.

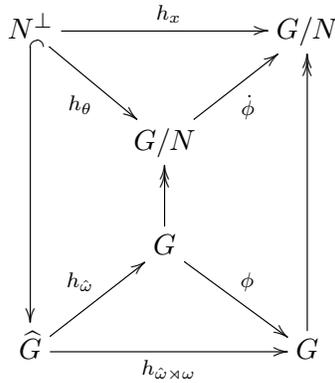
Proof. Let us compute the image of $\hat{\theta}^{\hat{\eta}}$ under

$$h_- : H^2(N^\perp, \text{U}(1)) \hookrightarrow \text{Hom}(N^\perp, G/N).$$

Let $x := [\hat{\omega} \rtimes \omega]|_{N^\perp} \in \hat{\theta}^{\hat{\eta}}$. In the proof of Lemma 3.11 we have already seen that

$$h_{\hat{\omega} \rtimes \omega} = \phi \circ h_{\hat{\omega}},$$

for $\phi = \text{id} + h_{\hat{\omega}} \circ h_\omega$. By transversality, $\phi : G \xrightarrow{\cong} G$ induces an isomorphism $\dot{\phi} : G/N \xrightarrow{\cong} G/N$. Now, the commutativities of the outer square, of the bottom triangle and of the two trapezoids in



imply that the upper triangle commutes, i.e., $h_x = \dot{\phi} \circ h_\theta$. But as $\dot{\phi}$ is an isomorphism, h_x vanishes if and only if h_θ vanishes. The lemma is then obvious. □

The important thing about this last lemma is that we have found an invariant that can distinguish the commutative systems, i.e., those which are equivariantly Morita equivalent to a system $(\mathcal{K} \otimes C(G/N), \mu \otimes \text{inf})$, from those who are genuinely noncommutative. Let us denote by $CT(N; \hat{G}) \subset NCT(N; \hat{G})$ the corresponding subcategory. Together with the results of the previous section we have found:

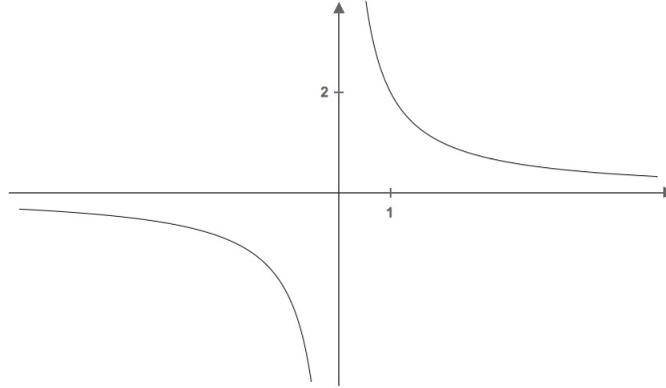


FIGURE 1. $\natural \subset \mathbb{R} \times \mathbb{R}$

Theorem 3.20. *The set of 2-isomorphism classes $[CT(N; \widehat{G})]$ is the kernel of the composition $[NCT(N; \widehat{G})^\natural] \rightarrow [NCT(N^\perp; G)^\natural] \rightarrow H^2(N, U(1))$ in*

$$\begin{array}{ccc}
 H^2(N^\perp, U(1)) & \xleftarrow{\text{Ma}_{(N; \widehat{G})}} & [NCT(N; \widehat{G})] & & [NCT(N^\perp; G)] & \xrightarrow{\text{Ma}_{(N^\perp; G)}} & H^2(N, U(1)) \\
 & & \cup & & \cup & & \\
 & & [NCT(N; \widehat{G})^\natural] & \rightleftharpoons & [NCT(N^\perp; G)^\natural] & & \\
 & & \cup & & \cup & & \\
 & & [CT(N; \widehat{G})] & & [CT(N^\perp; G)] & &
 \end{array}$$

3.4. Example: duality for the NC-torus in dimension 2. Let

$$G := \mathbb{R}^2 = \widehat{G}, \quad \text{and} \quad N := \mathbb{Z}^2 = N^\perp.$$

Recall the isomorphisms $H^2(\mathbb{Z}^2, U(1)) \cong \mathbb{T}$ and $H^2(\mathbb{R}^2, U(1)) \cong \mathbb{R}$. Let us identify the transversality relation $\natural \subset \mathbb{R} \times \mathbb{R}$: A cocycle corresponding to $\theta \in \mathbb{R}$ is given by $\omega_\theta(x, y) := \exp(2\pi i \theta x_2 y_1)$, so one obtains

$$h_\theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} : \mathbb{R}^2 \rightarrow \mathbb{R}^2.$$

Then for some $\hat{\theta} \in \mathbb{R}$ we have

$$\phi = \text{id}_{\mathbb{R}^2} + h_{\hat{\theta}} \circ h_\theta = (1 - \theta \hat{\theta}) \cdot \text{id}_{\mathbb{R}^2}$$

which is invertible as long as $\theta \hat{\theta} \neq 1$. Moreover, it restricts to an isomorphism of \mathbb{Z}^2 if and only if $\theta \hat{\theta} = 0$ or $\theta \hat{\theta} = 2$. Therefore $\natural \subset \mathbb{R} \times \mathbb{R}$ consists of the union of the coordinate axis and of the graph of $x \mapsto \frac{2}{x}$ (see Figure 1).

We can now answer the following question about transverse actions completely:

Q: If $\hat{\theta} \in \mathbb{T}$, what are the transverse actions $\hat{\mu} \otimes \text{inf}$ on $\mathcal{K} \otimes (\mathbb{C} \rtimes_{\hat{\theta}} N)$?

A: If $\dot{\theta} = \dot{0} \in \mathbb{T}$, then every action $\hat{\mu} \otimes \text{inf}$ is transverse. If $\dot{\theta} \neq \dot{0} \in \mathbb{T}$, and $\theta \in \mathbb{R}$ is some lift of $\dot{\theta}$, then every action $\hat{\mu} \otimes \text{inf}$ is transverse which has a Mackey obstruction

$$\text{Ma}(\hat{\mu}) \in \left\{ 0, \frac{2}{\theta + n} \mid n \in \mathbb{Z} \right\}.$$

Let us compute the dual system of $(\mathcal{K} \otimes (\mathbb{C} \rtimes_{\dot{\theta}} \mathbb{Z}^2), \hat{\mu} \otimes \text{inf})$, for a transverse action $\hat{\mu}$ with $\text{Ma}(\hat{\mu}) = \dot{\theta}$:

(i) In the simple case of $\dot{\theta} = 0$ we can choose $\theta = 0$ as a transverse lift to $\dot{\theta}$. Then we have

$$\mathcal{K} \otimes C(\mathbb{T}) \rtimes_{\hat{\mu} \otimes \text{inf}} \mathbb{R}^2 \sim \mathcal{K} \otimes (\mathbb{C} \rtimes_{\dot{\theta}} \mathbb{Z}^2),$$

on which \mathbb{R}^2 acts by $\text{id} \otimes \text{inf}$, and $\dot{\theta}$ is the restriction of $\hat{\theta}$ to \mathbb{Z}^2 .

(ii) In case of $\dot{\theta} \neq \dot{0}$, there are two subcases. First, if $\hat{\theta} = 0$, then let θ be any lift of $\dot{\theta}$, and we have

$$\mathcal{K} \otimes (\mathbb{C} \rtimes_{\dot{\theta}} \mathbb{Z}^2) \rtimes_{\hat{\mu} \otimes \text{inf}} \mathbb{R}^2 \sim \mathcal{K} \otimes C(\mathbb{T}^2),$$

on which \mathbb{R}^2 acts by $\mu \otimes \text{inf}$ with $\text{Ma}(\mu) = \theta$. Second, if $\hat{\theta} \neq 0$, then transversality means that there is a lift θ of $\dot{\theta}$ such that $\hat{\theta} = 2/\theta$. We now only need to compute the cocycles occurring in Corollary 3.8:

$$\begin{aligned} h_{\dot{\theta}}^* \omega_{\theta}^{-1}(x, y) &= \exp(2\pi i \theta (-\hat{\theta} x_1)(\hat{\theta} y_2))^{-1} \\ &= \exp(2\pi i 2\hat{\theta} x_1 y_2) \\ &\sim \exp(2\pi i 2\hat{\theta} x_2 y_1)^{-1}, \end{aligned}$$

so $[\hat{\omega}_{\hat{\theta}} \cdot h_{\dot{\theta}}^* \omega_{\theta}^{-1}] = -\hat{\theta}$, and similarly $[\omega_{\theta} \cdot h_{\dot{\theta}}^* \hat{\omega}_{\hat{\theta}}^{-1}] = -\theta$. Pushing these classes forward along $\phi = \hat{\phi} = -\text{id}_{\mathbb{R}^2}$ doesn't change the class. So if we denote by $\hat{\theta} \in \mathbb{T}$ the restricted class of $\hat{\theta}$ and if $\mu_{-\theta}$ is an action with Mackey obstruction $-\theta$, then the dual is given by

$$(24) \quad (\mathcal{K} \otimes (\mathbb{C} \rtimes_{-\dot{\theta}} \mathbb{Z}^2), \mu_{-\theta} \otimes \text{inf}).$$

Note that we re-obtain Example 3.13 at this stage: In fact, for $\theta = \frac{2}{3}$ and $\hat{\theta} = 3$ the dual system of $(\mathcal{K} \otimes (\mathbb{C} \rtimes_{\dot{\theta}} \mathbb{Z}^2), \hat{\mu} \otimes \text{inf})$ is according to (24)

$$(\mathcal{K} \otimes C(\mathbb{T}^2), \mu_{-\frac{2}{3}} \otimes \text{inf}).$$

Then we might take the bidual according to (i) which is

$$(\mathcal{K} \otimes (\mathbb{C} \rtimes_{-\frac{2}{3}} \mathbb{Z}^2), \text{id} \otimes \text{inf}).$$

However, $\frac{1}{3} = -\frac{2}{3} \pmod{\mathbb{Z}}$, and this is exactly what we observed in Example 3.13.

3.5. Example: duality for 3-dimensional NC-tori. We consider now the standard lattice $\mathbb{Z}^n \subseteq \mathbb{R}^n$. Recall from [BaK73] that for every 2-cocycle ω on \mathbb{R}^n there is a unique strictly upper triangular real $n \times n$ matrix A such that ω is cohomologous to $\omega_A : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{U}(1)$ given by $\omega_A(x, y) = e^{2\pi i \langle Ax, y \rangle}$. Moreover, every 2-cocycle on \mathbb{Z}^n is cohomologous to the restriction of some ω_A to

$$\mathbb{Z}^n \times \mathbb{Z}^n \subseteq \mathbb{R}^n \times \mathbb{R}^n$$

and two such restrictions lie in the same class in $H^2(\mathbb{Z}^n, \mathbb{U}(1))$ if and only if the difference $A - A'$ has integer entries. If we identify \mathbb{R}^n with $\widehat{\mathbb{R}^n}$ via $x \mapsto \chi_x$ with $\chi_x(y) = e^{2\pi i \langle x, y \rangle}$, then a straight-forward computation shows that

$$h_{\omega_A} : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

is given by matrix multiplication with the skew symmetric matrix $A - A^t$. Now, given another strictly upper triangular matrix B defining a 2-cocycle $\widehat{\omega} = \omega_B$ on $\widehat{\mathbb{R}^n} \cong \mathbb{R}^n$, then the pair (ω_A, ω_B) is transverse if and only if the matrix

$$\Phi := I_n + (B - B^t)(A - A^t)$$

lies in $\text{GL}(n, \mathbb{Z})$. As a consequence, given an action $\widehat{\mu} \otimes \text{inf}$ of $\widehat{\mathbb{R}^n} \cong \mathbb{R}^n$ on $\mathcal{K} \otimes (\mathbb{C} \rtimes_{\eta} \mathbb{Z}^n)$ such that $\text{Ma}(\widehat{\mu}) = [\omega_B]$, then the system

$$(\mathcal{K} \otimes (\mathbb{C} \rtimes_{\eta} \mathbb{Z}^n), \widehat{\mathbb{R}^n}, \widehat{\mu} \otimes \text{inf})$$

is transversal if and only if there exists a strictly upper triangular matrix A such that $\eta \sim \omega_A|_{\mathbb{Z}^n \times \mathbb{Z}^n}$ and such that $I_n + (B - B^t)(A - A^t) \in \text{GL}(n, \mathbb{Z})$.

Of course, this will always be the case if $B = 0$, which results to give Mathai–Rosenberg duality between commutative and noncommutative tori. For the case $n = 2$ we completely solved this question in the previous example. In particular we saw that every noncommutative 2-torus has noncommutative duals. We shall now show that this is not always the case in higher dimensions.

For this we have a look at the case $n = 3$: Suppose that

$$A = \begin{pmatrix} 0 & a_1 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & b_1 & b_2 \\ 0 & 0 & b_3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then a short computation shows that

$$\begin{aligned} \Phi &:= I_3 + (B - B^t) \cdot (A - A^t) \\ &= \begin{pmatrix} 1 - (b_1 a_1 + b_2 a_2) & -b_2 a_3 & b_1 a_3 \\ -b_3 a_2 & 1 - (b_1 a_1 + b_3 a_3) & -b_1 a_2 \\ b_3 a_1 & -b_2 a_1 & 1 - (b_2 a_2 + b_3 a_3) \end{pmatrix}. \end{aligned}$$

A lengthy but straightforward computation gives

$$(25) \quad \det \Phi = 1 - 2(c_1 + c_2 + c_3) + 2(c_1 c_2 + c_1 c_3 + c_2 c_3) + c_1^2 + c_2^2 + c_3^2,$$

with $c_i := a_i b_i, i \in \{1, 2, 3\}$. In order that $\Phi \in \text{GL}(3, \mathbb{Z})$ we need $\det \Phi = \pm 1$. Since all entries of the matrix Φ need to be integers, it follows that all mixed terms $c_i c_j$ with $i \neq j$ are integers as well. Also, all sums $c_i + c_j$ with $i \neq j$ need to be integers. Using this, it easily follows that $2c_i \in \mathbb{Z}$ for all $i \in \{1, 2, 3\}$ (e.g., we get $2c_1 = (c_1 + c_2) + (c_1 + c_3) - (c_2 + c_3) \in \mathbb{Z}$). But then it follows that $c_1^2 + c_2^2 + c_3^2 \in \mathbb{Z}$. It is an easy exercise to check that the sum of three squares of half-integers is an integer if and only if all three numbers are integers themselves. Thus we get $c_1, c_2, c_3 \in \mathbb{Z}$ and we need to search for integer solutions c_1, c_2, c_3 for the equation $\det \Phi = \pm 1$. Using standard forms for quadratic equations one can check that there are precisely the following two sets of solutions

$$c_3 = -c_1 - c_2 \quad \text{or} \quad c_3 = 2 - c_1 - c_2 \quad \forall c_1, c_2 \in \mathbb{Z},$$

in which cases the determinant will be 1. Moreover, the matrix Φ will have integer entries if and only if $n_{ij} := a_i b_j \in \mathbb{Z}$ for all $1 \leq i, j \leq 3$. If $B = 0$, the conditions will always be fulfilled. On the other hand, if just one entry $b_i \neq 0$, this will force any of the pairs a_j, a_k to be rationally dependent, since if both $a_j, a_k \neq 0$, then the equations $0 \neq a_j b_i = n_{ji}$ and $0 \neq a_k b_i = n_{ki}$ implies that $a_j = \frac{n_{ji}}{n_{ki}} a_k$.

Thus, a general noncommutative 3-torus which admits a noncommutative dual will be attached to a matrix of the form $A = \theta \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ 0 & 0 & \alpha_3 \\ 0 & 0 & 0 \end{pmatrix}$ with $0 \neq \theta \in \mathbb{R}$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Q}$. Then $B = \frac{1}{\theta} \begin{pmatrix} 0 & \beta_1 & \beta_2 \\ 0 & 0 & \beta_3 \\ 0 & 0 & 0 \end{pmatrix}$ with $\beta_1, \beta_2, \beta_3 \in \mathbb{Q}$ such that $\alpha_i \cdot \beta_j \in \mathbb{Z}$ for all $1 \leq i, j \leq 3$ and $\alpha_1 \beta_1 + \alpha_2 \beta_2 + \alpha_3 \beta_3 \in \{0, 2\}$ will match up with A to give a dual pair of noncommutative three dimensional tori. As an example: if all $\alpha_i = 1$, then $(\beta_1, \beta_2, \beta_3) = (n, k, -n - k)$ as well as $(\beta_1, \beta_2, \beta_3) = (n, k, 2 - n - k)$, with $n, k \in \mathbb{Z}$, will give compatible parameters for B .

3.6. Example: tensor products of duality pairs. Duality pairs are closed under taking tensor products: Suppose that (ω_A, ω_B) is a duality pair for $\mathbb{Z}^n \subseteq \mathbb{R}^n$ and (ω_C, ω_D) is a duality pair for $\mathbb{Z}^m \subseteq \mathbb{R}^m$. Then $(\omega_A \cdot \omega_C, \omega_B \cdot \omega_D)$ is a duality pair for $\mathbb{Z}^{n+m} \subseteq \mathbb{R}^{n+m}$, where $\omega_A \cdot \omega_C$ is the cocycle on \mathbb{R}^{n+m} given by the product

$$\omega_A \cdot \omega_C((x_1, y_1), (x_2, y_2)) = \omega_A(x_1, x_2) \omega_C(y_1, y_2) \quad \forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^{n+m}$$

(and similarly for $\omega_B \cdot \omega_D$). Note that $\omega_A \cdot \omega_C = \omega_{\text{diag}(A,C)}$ if we denote by $\text{diag}(A, C)$ the matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M_{n+m}(\mathbb{R})$. Since

$$\begin{aligned} &I_{n+m} + (\text{diag}(B, D) - \text{diag}(B, D)^t)(\text{diag}(A, C) - \text{diag}(A, C)^t) \\ &= \text{diag}\left(I_n + (B - B^t)(A - A^t), I_m + (D - D^t)(C - C^t)\right) \end{aligned}$$

it follows that $(\omega_A \cdot \omega_C, \omega_B \cdot \omega_D)$ is a dual pair if and only if (ω_A, ω_B) and (ω_C, ω_D) are both dual pairs.

4. Noncommutative C*-dynamical T-duality

In classical (commutative) C*-dynamical T-duality the objects are principal torus bundles $E \rightarrow B$ equipped with a locally trivial bundle of compact operators $F \rightarrow E$ that is trivialisable over the fibres $E_b \rightarrow b$. The C*-algebra of sections vanishing at infinity $A := \Gamma_0(E, F)$ is a bundle of C*-algebras whose fibres are stable commutative tori $\mathcal{K} \otimes C(\mathbb{T}^n)$. We consider more general bundles whoses fibres are twisted group algebras.

In this section the word *space* will always mean a second countable, locally compact Hausdorff space.

4.1. $C_0(B)$ -algebras and continuous bundles of C*-algebras. Recall that a C*-algebra A is called a $C_0(B)$ -algebra for a space B , if A is equipped with a fixed nondegenerate *-homomorphism $\Phi : C_0(B) \rightarrow \text{ZM}(A)$, the center of the multiplier algebra $\text{M}(A)$ of A . If A is a $C_0(B)$ -algebra, then for any closed subset $X \subset B$ we let I_X denote the closed ideal $\Phi(C_0(B \setminus X))A$ of A and the quotient $A|_X := A/I_X$ is called the *restriction of A to X* . For a single point $\{b\} = X$, $A|_b$ is called the *fibre of A over b* . The elements of A can be viewed as sections of a fibre-bundle over B with fibres $A|_b$ by writing $a(b) := a + I_b \in A|_b$, for $a \in A$ and $b \in B$. A $C_0(B)$ -algebra A is called a *continuous bundle of C*-algebras over B* if these sections are continuous in the sense that $b \mapsto \|a(b)\|$ is continuous. We refer to [W07, Appendix C] for a detailed treatment of $C_0(B)$ -algebras.

An action $\alpha : G \rightarrow \text{Aut}(A)$ on a $C_0(B)$ -algebra A , is called *fibre-preserving* if it is $C_0(B)$ -linear, i.e., if $\alpha_g(f \cdot a) = f \cdot \alpha_g(a)$ for all $a \in A$, $g \in G$ and $f \in C_0(B)$, where we write $f \cdot a$ for $\Phi(f)a$. If α is $C_0(B)$ -linear, it induces actions $\alpha|_b$ on each fibre $A|_b$ by setting $(\alpha|_b(g))(a(b)) := (\alpha_g(a))(b)$. Then α is completely determined by the actions on the fibres.

If A_1 and A_2 are two $C_0(B)$ -algebras, then we say that an A_1 - A_2 -equivalence bimodule E is $C_0(B)$ -linear, if $f \cdot \xi = \xi \cdot f$ for all $f \in C_0(B)$, $\xi \in E$, where the left and right actions of $C_0(B)$ on E are given via extending the left and right actions of A_1 and A_2 to their multiplier algebras. We then say that A_1 and A_2 are $C_0(B)$ -linearly Morita equivalent.

We will typically deal with a situation in which the Morita equivalences are assumed to be both, G -equivariant and $C_0(B)$ -linear.

4.2. Families of twisted group algebras and ω -triviality. Let B be a locally compact space, and consider a $C_0(B)$ -linear action on $\mathcal{K} \otimes C_0(B)$, i.e., a continuous homomorphism $\mu : G \rightarrow C(B, \text{PU}(\mathcal{H}))$, where $C(B, \text{PU}(\mathcal{H}))$ is equipped with the compact open topology (this is the $C_0(B)$ -linear automorphism group of $\mathcal{K} \otimes C_0(B)$ with the topology of point-wise convergence). We can construct such a homomorphism μ literally by formula (2) from a cocycle $\omega \in Z^2(G, C(B, \text{U}(1)))$ by the left regular ω -representation. I.e., one just has to apply (2) point-wise for $\omega|_b \in Z^2(G, \text{U}(1))$ (the evaluation of ω at $b \in B$). The resulting function indeed is continuous as a map $G \rightarrow C(B, \text{PU}(\mathcal{H}))$ [HuORR86, Proof of Prop. 3.1]. This gives rise to a map

$$(26) \quad \xi_{B,G} : H^2(G, C(B, \text{U}(1))) \rightarrow \mathcal{E}_G(B),$$

where $\mathcal{E}_G(B)$ denotes the Morita equivalence classes of systems

$$(\mathcal{K} \otimes C_0(B), \mu, G).$$

It is shown in [CrKRW93] that $\mathcal{E}_G(B)$ is a group by forming the balanced tensor product over B , and the above map is a homomorphism. For the one-point space $B = \text{pt}$, Proposition 2.5 tells that the above map is an isomorphism, namely the inverse of the Mackey obstruction. Yet, for general B this fails:

Proposition 4.1 ([CrKRW93, Sec. 6.3]). *If the second Čech cohomology $\check{H}^2(B, \mathbb{Z})$ is countable, then there is an exact sequence*

$$0 \rightarrow H^2(G, C(B, \text{U}(1))) \xrightarrow{\xi_{B,G}} \mathcal{E}_G(B) \rightarrow \text{Hom}(G, \check{H}^2(B, \mathbb{Z})).$$

In the remainder of this article we will exclusively deal with actions μ on $\mathcal{K} \otimes C_0(B)$ which are in the image of $\xi_{B,G}$. Instead of analysing the crossed product $(\mathcal{K} \otimes C_0(B)) \rtimes_{\mu} G$ we will therefore mostly consider twisted transformation group algebras defined as follows: If $\omega \in Z^2(G, C(B, \text{U}(1)))$, then the Banach space $L^1(G, C_0(B))$ is turned into a Banach $*$ -algebra by the same formulas as in (3). Its enveloping C^* -algebra is denoted by

$$C_0(B) \rtimes_{\omega} G.$$

It has a canonical action of the dual group \widehat{G} by point-wise multiplication of characters. Furthermore, it is a C^* -algebra over B and its fibres are $(C_0(B) \rtimes_{\omega} G)|_b \cong \mathbb{C} \rtimes_{\omega|_b} G$, where again $\omega|_b$ denotes the restriction of ω to $b \in B$. (We refer to [EW02] for a more detailed discussion of twisted transformation group algebras.) If now $\mu : G \rightarrow C(B, \text{PU})$ is a given action in the image of $\xi_{B,G}$, then there is a \widehat{G} -equivariant and $C_0(B)$ -linear isomorphism

$$\Psi : \mathcal{K} \otimes (C_0(B) \rtimes_{\omega} G) \cong (\mathcal{K} \otimes C_0(B)) \rtimes_{\mu} G$$

given literally by formula (4). In other words, we can go back and forth between families of twisted group algebras and their corresponding crossed product algebras without losing information.

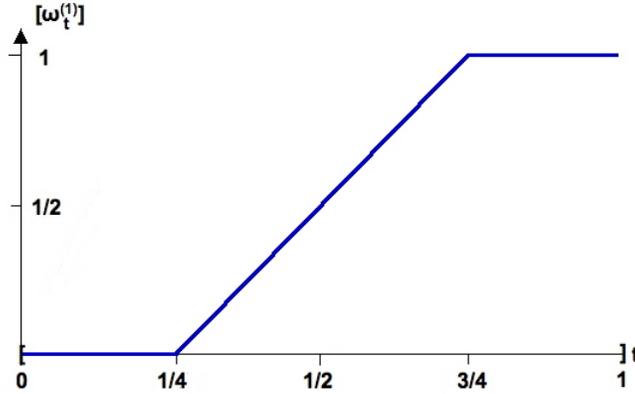


FIGURE 2. A function for defining a bundle of algebras.

We are now coming back to our general situation in which $N \subseteq G$ is a cocompact discrete subgroup of the abelian group G .

Definition 4.2. Let B be a space and let A be a (separable and stable) $C_0(B)$ -algebra.

- (1) A is called ω -trivial if there exists $\omega \in Z^2(G, C(B, U(1)))$ together with a $C_0(B)$ -linear Morita equivalence $A \sim C_0(B) \rtimes_{\omega} N$. The pair of such a cocycle together with such a Morita equivalence is called an ω -trivialisation.
- (2) A is called locally ω -trivial if there exists an open covering $(U_i)_{i \in I}$ of B such that for all $i \in I$ the restricted algebras $A|_{\bar{U}_i}$ over the closure $\bar{U}_i \supset U_i$ are ω -trivial.
- (3) The open sets together with their ω -trivialisations are called charts, and a collection of charts covering all of B is called an atlas.

(We use the term (local) $\hat{\omega}$ -triviality if we consider the groups N^{\perp}, \hat{G} instead of N, G .)

Note that although the Morita equivalence in this definition only refers to the cocycle on N , we require the cocycle to extend to G .

Example 4.3. Let $G := \mathbb{R}^2, N := \mathbb{Z}^2$. Let $B := [0, 1]$ be the interval and $S^1 := B/(0 \sim 1)$ be the circle.

- (1) Let $\omega^{(1)} \in Z^2(\mathbb{R}^2, C(B, U(1)))$ be given by the class function⁶

$$t \mapsto [\omega_t^{(1)}] \in \mathbb{R} \cong H^2(\mathbb{R}^2, U(1))$$

which is defined by Figure 2.

⁶I.e., $\omega_t^{(1)}(x, y) := \exp(2\pi i[\omega_t^{(1)}]x_2y_1)$.

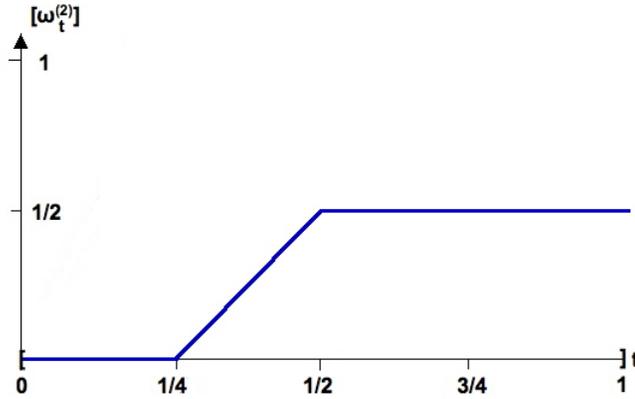


FIGURE 3. Another function for defining a bundle of algebras.

Let $X := C(B) \rtimes_{\omega^{(1)}} \mathbb{Z}^2$, which is a globally ω -trivial (nonstable) algebra over B . Note that $\omega_0^{(1)}|_{\mathbb{Z}^2 \times \mathbb{Z}^2} = \omega_1^{(1)}|_{\mathbb{Z}^2 \times \mathbb{Z}^2}$, so the fibres $X|_0$ and $X|_1$ over the two endpoints of B are canonically isomorphic (they are equal after identifying $X|_t \cong \mathbb{C} \rtimes_{\omega_t} \mathbb{Z}^2$ by the obvious map). Let us denote by A^1 the algebra which is obtained by gluing along this isomorphism, i.e., the pullback in

$$\begin{array}{ccc} A^1 & \xrightarrow{\dots\dots\dots} & X \\ \downarrow & & \downarrow i_0^* \times i_1^* \\ X|_0 & \xrightarrow{\text{id} \times \text{can}} & X|_0 \oplus X|_1 \end{array} \quad ,$$

and denote by $A^{(1)}$ its stabilisation $\mathcal{K} \otimes A^1$. If we let

$$\eta \in Z^2(N, C(S^1, U(1)))$$

be the cocycle given point-wise by $\omega^{(1)}|_{N \times N}$, then $A^{(1)}$ is canonically isomorphic to $\mathcal{K} \otimes C(S^1) \rtimes_{\eta} \mathbb{Z}^2$. This is not an ω -trivial algebra any more, yet it is still a locally ω -trivial algebra over the circle S^1 .

- (2) Let us do a slightly more involved construction with the cocycle $\omega^{(2)} \in Z^2(\mathbb{R}^2, C(B, U(1)))$ whose class function is given by Figure 3.

Let $Y := C(B) \rtimes_{\omega^{(2)}} \mathbb{Z}^2$. In this case, the fibres $Y|_0, Y|_1$ over the two endpoints of B are two nonisomorphic but Morita equivalent noncommutative tori with classes 0 and $\frac{1}{2}$. So there is a stable isomorphism $\varphi : \mathcal{K} \otimes Y|_0 \cong \mathcal{K} \otimes Y|_1$. We use this isomorphism to glue the stabilisation $\mathcal{K} \otimes Y$ over the endpoints to itself: Let $A^{(2)}$ be

the pullback in

$$\begin{array}{ccc}
 A^{(2)} & \xrightarrow{\dots\dots\dots} & \mathcal{K} \otimes Y \\
 \downarrow & & \downarrow i_0^* \times i_1^* \\
 \mathcal{K} \otimes Y|_0 & \xrightarrow{\text{id} \times \varphi} & (\mathcal{K} \otimes Y|_0) \oplus (\mathcal{K} \otimes Y|_1)
 \end{array}$$

We claim that $A^{(2)}$ is a locally ω -trivial algebra over the circle S^1 . It is clear that Y itself gives a chart for, say, $V := (\frac{1}{8}, \frac{7}{8}) \subset S^1$. The critical issue is to find a local ω -trivialisation around the gluing point: For, say, $U := [0, \frac{1}{4}] \sqcup (\frac{3}{4}, 1]/(0 \sim 1) \subset S^1$ we have

$$A^{(2)}|_{\bar{U}} \cong \left\{ (f, g) \in (\mathcal{K} \otimes Y|_{[0, -\frac{1}{4}]}) \oplus (\mathcal{K} \otimes Y|_{[\frac{3}{4}, 1]}) \mid \varphi(f|_0) = g|_1 \right\}.$$

But the isomorphism φ extends to a fibre-wise isomorphism

$$\tilde{\varphi} : \mathcal{K} \otimes Y|_{[0, \frac{1}{4}]} \xrightarrow{\cong} \mathcal{K} \otimes Y|_{[\frac{3}{4}, 1]}$$

just because all fibres are the same. So we obtain

$$\begin{aligned}
 A^{(2)}|_{\bar{U}} &\cong \left\{ (\tilde{\varphi}f, g) \in (\mathcal{K} \otimes Y|_{[\frac{3}{4}, 1]}) \oplus (\mathcal{K} \otimes Y|_{[\frac{3}{4}, 1]}) \mid (\tilde{\varphi}f)|_1 = g|_1 \right\} \\
 &\sim C(\bar{U}) \rtimes_{\omega|_{\bar{U}}} \mathbb{Z}^2,
 \end{aligned}$$

and this means that V, U give an atlas for $A^{(2)}$.

4.3. Duality for polarisable pairs. Recall that for a fibre-preserving action $\alpha : G \rightarrow \text{Aut}(A)$, i.e., a $C_0(B)$ -linear action, the crossed product C^* -algebra $A \rtimes_{\alpha} G$ is again a $C_0(B)$ -algebra.

Definition 4.4. Let B be a space.

- (1) An action $\alpha : G \rightarrow \text{Aut}(A)$ on a locally $\hat{\omega}$ -trivial $C_0(B)$ -algebra A is called *locally transverse* if there exists an atlas $(U_i, \hat{\omega}_i)_{i \in I}$ for A and cocycles $\omega_i \in Z^2(G, C(\bar{U}_i, U(1)))$ which are point-wise transverse to $\hat{\omega}_i$ together with \hat{G} -equivariant and $C_0(\bar{U}_i)$ -linear Morita equivalences

$$(A \rtimes_{\alpha} G)|_{\bar{U}_i} \sim C_0(\bar{U}_i) \rtimes_{\hat{\omega}_i \wedge \omega_i} (N^{\perp} \times G).$$

(The \hat{G} -equivariance is required for the dual actions on both sides.) These local data $U_i, \hat{\omega}_i, \omega_i$ and the Morita equivalences are called *transverse charts* which altogether constitute a *transverse atlas*.

- (2) A *NC pair* (A, α) over B is a locally $\hat{\omega}$ -trivial $C_0(B)$ -algebra A together with a locally transverse action.
- (3) We use the term *NC dual pair* for a locally ω -trivial algebra with a locally transverse \hat{G} -action.

Remark 4.5. Local transversality of an action α can be rephrased in terms of the local actions $\alpha|_{\bar{U}_i}$ on $A|_{\bar{U}_i} \cong \mathcal{K} \otimes C_0(\bar{U}_i) \rtimes_{\hat{\omega}_i} N^{\perp}$. It is

equivalent to require, firstly, that $\alpha|_{\bar{U}_i}$ factorises as $\mu_i \otimes \text{inf}$, and, secondly, that the element in $\mathcal{E}_G(\bar{U}_i)$ given by $\mu_i : \bar{U}_i \rightarrow C(G, \text{PU}(\mathcal{H}))$ is in the image of $H^2(G, C(\bar{U}_i, \text{U}(1))) \hookrightarrow \mathcal{E}_G(\bar{U}_i)$ such that its pre-image in $H^2(G, C(\bar{U}_i, \text{U}(1)))$ is point-wise transverse to $\hat{\omega}_i$.

If (A, α) is a NC pair, then the crossed product $A \rtimes_{\alpha} G$ is a $C_0(B)$ -algebra and its fibres can be computed point-wise according to Section 3.2 which shows that the dual action of \hat{G} on $A \rtimes_{\alpha} G$ is transverse in each fibre. However, the crossed product algebra $A \rtimes_{\alpha} G$ need not to be locally ω -trivial, and even if it is, then the point-wise computation of Section 3.2 determines the dual action only up to Morita equivalence in each fibre which is in general not enough to determine a unique action with these properties. So we cannot simply conclude that the dual action is locally transverse.

Definition 4.6. Let B be a space.

- (1) A local polarisation over $U \subset B$ is just a continuous family of group automorphisms $\varphi : \bar{U} \rightarrow \text{Aut}(\hat{G} \times G)$, where $\text{Aut}(\hat{G} \times G)$ is equipped with the Braconier topology. This is the topology generated by the compact open topology and the pre-images of the open sets by inversion in $\text{Aut}(\hat{G} \times G)$. By the exponential law ([Br64]) this means that φ is continuous if and only if

$$\begin{aligned} \bar{U} \times \hat{G} \times G &\rightarrow \bar{U} \times \hat{G} \times G \\ (u, \chi, g) &\mapsto (u, \varphi(u)(\chi, g)) \end{aligned}$$

is a homeomorphism.

- (2) Let (A, α) be a NC pair, and let $(U_i, \hat{\omega}_i, \omega_i)_{i \in I}$ be a transverse atlas. The atlas is called *polarisable* if there exist local polarisations φ_i over U_i such that

$$[\hat{\omega}_i \wedge \omega_i] = \varphi_i^*[\wedge] \in H^2(\hat{G} \times G, C(\bar{U}_i, \text{U}(1))).$$

- (3) A NC pair (A, α) is called polarisable if it permits a polarisable atlas.

Polarisability is a slight restriction on the class of objects we consider. In fact, if an atlas is polarisable, then it follows that the involved cocycles $\hat{\omega}_i, \omega_i$ are cohomologous to bihomomorphisms, just because

$$(27) \quad \omega_i = (\hat{\omega}_i \wedge \omega_i)|_{(0 \times G) \times (0 \times G)} \sim (\varphi_i^* \wedge)|_{(0 \times G) \times (0 \times G)},$$

and the latter cocycle is a bihomomorphism $G \times G \rightarrow C(\bar{U}_i, \text{U}(1))$. This means that the bundle theory does not cover all the cases of the point-wise theory in the previous sections. However, for $G = \mathbb{R}^n$ and $N = \mathbb{Z}^n$ all cocycles are cohomologous to bicharacters, and, as we see next, polarisability is no restriction at all:

Proposition 4.7. For $G = \mathbb{R}^n, N = \mathbb{Z}^n$ every NC pair is polarisable.

Proof. Let $U_i, \hat{\omega}_i, \omega_i$ be a chart. The first thing to note is that for \mathbb{R}^{2n} we always have an isomorphism⁷ [EW01, Sec. 5]

$$H^2(\mathbb{R}^{2n}, C(\bar{U}_i, U(1))) \cong C(\bar{U}_i, H^2(\mathbb{R}^{2n}, U(1))).$$

which is given by point-wise evaluation. Secondly, the Braconnier topology coincides with the usual topology on $\text{Aut}(\mathbb{R}^n \times \mathbb{R}^n) = \text{Gl}(2n, \mathbb{R})$.

Now, if $\hat{\eta} \wedge \eta$ is a type I and totally skew cocycle on \mathbb{R}^{2n} , then $h_{\hat{\eta} \wedge \eta} : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is an invertible, anti-symmetric matrix. But if $\text{As}(2n, \mathbb{R})$ denotes the set of all anti-symmetric $2n \times 2n$ -matrices, then

$$\begin{aligned} c : \text{Gl}(2n, \mathbb{R}) &\rightarrow \text{Gl}(2n, \mathbb{R}) \cap \text{As}(2n, \mathbb{R}), \\ \varphi &\mapsto \hat{\varphi} \circ h_\wedge \circ \varphi, \end{aligned}$$

where

$$h_\wedge = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix},$$

is surjective and admits local sections. (Here the dual map $\hat{\varphi}$ coincides with the transpose map φ^t .) In fact, surjectivity is a standard fact about anti-symmetric matrices. To conclude the existence of local sections, it is sufficient to observe that c is a submersion, i.e., we claim that its derivative

$$\begin{aligned} dc(\varphi) : \text{Mat}(2n, \mathbb{R}) &\rightarrow \text{As}(2n, \mathbb{R}) \\ M &\mapsto \hat{M} \circ \hat{\varphi} \circ h_\wedge + h_\wedge \circ \varphi \circ M \end{aligned}$$

is surjective for all $\varphi \in \text{Gl}(2n, \mathbb{R})$. But this is immediate: for any X in $\text{As}(2n, \mathbb{R})$ we can choose a $Y \in \text{Mat}(2n, \mathbb{R})$ such that $X = Y - \hat{Y}$, then $M_Y := \varphi^{-1} \circ h_\wedge^{-1} \circ Y$ satisfies

$$dc(\varphi)(M_Y) = \hat{Y} \circ \hat{h}_\wedge^{-1} \circ h_\wedge + Y = \hat{Y} \circ (-\mathbb{1}) + Y = X.$$

So (after passing to a possibly finer covering, again called U_i) we find lifts in

$$\begin{array}{ccc} & & \text{Gl}(2n, \mathbb{R}) \\ & \nearrow \exists \varphi_i & \downarrow \\ \bar{U}_i & \xrightarrow{h_{\hat{\omega}_i \wedge \omega_i}} & \text{Gl}(2n, \mathbb{R}) \cap \text{As}(2n, \mathbb{R}) \end{array}$$

and so $h_{(\varphi_i)^* \wedge} = \hat{\varphi}_i \circ h_\wedge \circ \varphi_i = h_{\hat{\omega}_i \wedge \omega_i}$ which means that

$$[(\varphi_i)^* \wedge] = [\hat{\omega}_i \wedge \omega_i]. \quad \square$$

⁷ $Z^2(G, U(1))$ has the topology of almost everywhere point-wise convergence, and $H^2(G, U(1))$ has the corresponding quotient topology. Then $H^2(\mathbb{R}^n, U(1)) \cong \mathbb{R}^{n(n-1)/2}$ as topological groups.

If (A, α) is a C^* -dynamical system over B (i.e., A is a $C_0(B)$ -algebra and α is fibre-preserving) and (A', α') is a C^* -dynamical system over B' , then a morphism

$$(f, M) : (A, \alpha) \rightarrow (A', \alpha')$$

consists of a continuous function $f : B \rightarrow B'$ together with a Morita equivalence ${}_A M_{f^* A'}$ over B . If

$$(f, M) : (A, \alpha) \rightarrow (A', \alpha') \quad \text{and} \quad (f', M') : (A', \alpha') \rightarrow (A'', \alpha'')$$

are two morphisms, then the composition is $(f \circ f', M \otimes_{f^* A'} f^* M')$ which is only associative up to isomorphism. So define a 2-morphism between two morphisms (g, L) and (g, L') to be an isomorphism $L \cong L'$ over B , and we have just defined the 2-category of C^* -dynamical systems over spaces. Apparently, NC pairs and polarisable NC pairs form 2-subcategories.

Theorem 4.8. *The duality functor $-\rtimes G$ defined on all C^* -dynamical systems over spaces with group G restricts to a duality of polarisable NC pairs:*

$$\begin{array}{ccc}
 \begin{array}{c} C^*\text{-Dynamical Systems} \\ \text{over Spaces with Group } G \end{array} & \xrightarrow{\sim} & \begin{array}{c} C^*\text{-Dynamical Systems} \\ \text{over Spaces with Group } \widehat{G} \end{array} \\
 \cup & & \cup \\
 \text{NC Pairs} & & \text{NC Dual Pairs} \\
 \cup & & \cup \\
 \begin{array}{c} \text{Polarisable} \\ \text{NC Pairs} \end{array} & \xrightarrow{\sim} & \begin{array}{c} \text{Polarisable} \\ \text{NC Dual Pairs} \end{array}
 \end{array}$$

Moreover, if (A, α) is a polarisable NC pair with polarisable atlas $U_i, \hat{\omega}_i, \omega_i$, then $U_i, \omega_i \rtimes \hat{\omega}_i, \hat{\omega}_i \overline{\rtimes} \omega_i$ (defined point-wise as in Definition 3.5) is a polarisable atlas for the dual.

Proof. Let (A, α) be a polarisable NC pair. We show first that $A \rtimes_\alpha G$ is locally ω -trivial and that the dual \widehat{G} -action on it is locally transverse in the sense of Definition 4.4(1). The proof is organised in 8 steps.

Step 0. Fix a polarisable atlas $U_i, \hat{\omega}_i, \omega_i$ with polarisations φ_i , i.e.,

$$\hat{\omega}_i \wedge \omega_i \sim (\varphi_i)^* \wedge .$$

Step 1. We have an isomorphism

$$\begin{aligned}
 C_0(\overline{U}_i) \rtimes_{\hat{\omega}_i \wedge \omega_i} (\widehat{G} \times G) &\xrightarrow{\cong} C_0(\overline{U}_i) \rtimes_{\wedge} (\widehat{G} \times G) \\
 f &\mapsto c_i \cdot f \circ \varphi_i^{-1},
 \end{aligned}$$

where $c_i : \widehat{G} \times G \rightarrow C(\overline{U}_i, U(1))$ relates $\hat{\omega}_i \wedge \omega_i \sim (\varphi_i)^* \wedge$. (Note: it is not difficult to show that φ_i is measure-preserving. If it weren't, there would occur a positive factor in this isomorphism to rescale the Haar measure.)

The dual action of $G \times \widehat{G}$ is transformed into the dual action pre-composed with $\widehat{\varphi}_i^{-1}$.

Step 2. We have to make a version of the Takai duality isomorphism explicit:

$$T : C_0(\overline{U}_i) \rtimes_{\wedge} (\widehat{G} \times G) \xrightarrow{\cong} \mathcal{K}(L^2(\widehat{G})) \otimes C_0(\overline{U}_i)$$

which is given for $f \in L^1(\widehat{G} \times G, C_0(\overline{U}_i))$ by

$$\tilde{f}(\chi, \chi') := \int_G f(\chi - \chi', g) \langle g, \chi' \rangle dg \in C_0(\overline{U}_i),$$

where \tilde{f} is the integral kernel for Tf , i.e., for $\xi \in L^2(\widehat{G})$ and $u \in \overline{U}_i$ we have

$$(Tf)|_u(\xi)(\chi) = \int_{\widehat{G}} \tilde{f}(\chi, \chi')|_u \xi(\chi') d\chi'.$$

In fact, it is a lengthy but straight forward calculation that T defines a *-homomorphism which is injective, because T is composed by injective transformations such as the Fourier transformation. The image of $L^1(\widehat{G} \times G, C_0(\overline{U}_i))$ is dense in $C_0(\overline{U}_i) \otimes \mathcal{K}(L^2(\widehat{G}))$ and thus T is an isomorphism.

Step 3. The dual action of $G \times \widehat{G}$ on $C_0(\overline{U}_i) \rtimes_{\wedge} (\widehat{G} \times G)$ transforms under T into⁸

$$(28) \quad \hat{\alpha}_i(g, \chi) := T \circ \langle (g, \chi), (-, -) \rangle \circ T^{-1} = \text{Ad}(\langle g, - \rangle R(\chi)) \otimes \text{id}_{C_0(\overline{U}_i)},$$

where $R : \widehat{G} \rightarrow \text{U}(L^2(\widehat{G}))$ is the right regular representation. $\hat{\alpha}_i$ is Morita equivalent to the action $\hat{\alpha}_i \otimes \text{id}_{\mathcal{K}(L^2(G))}$ on $\mathcal{K}(L^2(\widehat{G})) \otimes C_0(\overline{U}_i) \otimes \mathcal{K}(L^2(G))$, and this action is exterior equivalent to $\text{id}_{\mathcal{K}(L^2(\widehat{G}))} \otimes \hat{\alpha}'_i$ with

$$\hat{\alpha}'_i(g, \chi) := \text{id}_{C_0(\overline{U}_i)} \otimes \text{Ad}(L(g) \langle -, \chi \rangle),$$

where $L : \widehat{G} \rightarrow \text{U}(L^2(\widehat{G}))$ is the left regular representation. The cocycle which implements the exterior equivalence is given by

$$v_{(g, \chi)} := \langle g, - \rangle R(\chi) \otimes \text{id}_{C_0(\overline{U}_i)} \otimes \langle -, \chi \rangle^{-1} L(g)^{-1}.$$

The important thing to notice here is that $\hat{\alpha}'_i$ is in the image of $\xi_{\overline{U}_i, G \times \widehat{G}}$ from (26): It is the image of $\vee^{-1} \in Z^2(G \times \widehat{G}, C(\overline{U}_i, \text{U}(1)))$ which reads explicitly

$$(g, \chi) \vee^{-1} (h, \psi) = \langle h, \chi \rangle^{-1}.$$

As a consequence a crossed product with the action $\hat{\alpha}_i$ (or $\hat{\alpha}'_i$) pre-composed with $\widehat{\varphi}_i^{-1}$ is Morita equivalent to a twisted group algebra with cocycle $(\widehat{\varphi}_i)_* \vee^{-1}$.

⁸We denote multiplication operators on L^2 by the same symbol as their defining functions, e.g., $\langle g_0, - \rangle : \xi \mapsto \langle g_0, - \rangle \xi$, for $\xi \in L^2(\widehat{G})$.

Step 4. Note that for the Heisenberg cocycle \wedge on $\widehat{G} \times G$ we have

$$\langle h_\wedge(\chi, g), (\psi, h) \rangle = \langle (g, -\chi), (\psi, h) \rangle$$

which means that h_\wedge is the map $(\chi, g) \mapsto (g, -\chi)$ and so

$$((h_\wedge)_* \wedge)((g, \chi), (h, \psi)) = \langle g, -\psi \rangle \sim (g, \chi) \vee (h, \psi).$$

This relation holds in $Z^2(G \times \widehat{G}, U(1))$ but also in $Z^2(G \times \widehat{G}, C(\overline{U}_i, U(1)))$, where we consider the above cocycles as constant in the fibres. We can use this intermediate step to compute the cocycle of $\hat{\alpha}'_i$ from above:

$$\begin{aligned} ((\hat{\varphi}_i)_* \vee)^{-1} &\sim ((\hat{\varphi}_i)_*(h_\wedge)_* \wedge)^{-1} \\ &= ((\hat{\varphi}_i)_*(h_\wedge)_*(\varphi_i)_*(\varphi_i)^* \wedge)^{-1} \\ &= ((\hat{\varphi}_i \circ h_\wedge \circ \varphi_i)_* ((\varphi_i)^* \wedge))^{-1} \\ &= ((h_{(\varphi_i)^* \wedge})_* ((\varphi_i)^* \wedge))^{-1} \\ &\sim ((h_{\hat{\omega}_i \wedge \omega_i})_* (\hat{\omega}_i \wedge \omega_i))^{-1}. \end{aligned}$$

So apart from an interchange of G and \widehat{G} , this is exactly the formula we analysed in Lemma 3.6.

Step 5. By the continuity and openness of φ_i , the equality

$$\hat{\varphi}_i(u) \circ h_\wedge \circ \varphi_i(u) = h_{\hat{\omega}_i \wedge \omega_i|_u} = \begin{pmatrix} h_{\hat{\omega}_i|_u} & \text{id}_G \\ -\text{id}_{\widehat{G}} & h_{\omega_i|_u} \end{pmatrix}$$

shows that $(u, g) \mapsto (u, h_{\omega_i|_u}(g))$ is continuous and open, so

$$\begin{aligned} \phi_i : \overline{U}_i &\rightarrow \text{Aut}(G) \\ u &\mapsto (g \mapsto g + h_{\hat{\omega}_i|_u}(h_{\omega_i|_u}(g))) \end{aligned}$$

is continuous for the Braconier topology on $\text{Aut}(G)$. We already remarked in (27) that the involved cocycles are all bihomomorphisms, so as explained in Remark 3.7 we have

$$((\hat{\varphi}_i)_* \vee)^{-1} \sim \omega_i \overline{\omega}_i \cdot \hat{\omega}_i \overline{\omega}_i \cdot (\phi_i \times \text{id}_{\widehat{G}})_* \vee^{-1}.$$

This relation holds in $\in Z^2(G \times \widehat{G}, C(\overline{U}_i, U(1)))$, i.e., this is a local statement rather than just a point-wise statement.

Step 6. We can now locally compute the dual

$$\begin{aligned}
 (A \rtimes_{\alpha} G)|_{\overline{U}_i} &\sim C_0(\overline{U}_i) \rtimes_{\hat{\omega}_i \wedge \omega_i} (N^{\perp} \times G) \\
 &\sim C_0(\overline{U}_i) \rtimes_{\hat{\omega}_i \wedge \omega_i} (\widehat{G} \times G) \rtimes_{\text{dual}} (N \times 0) \\
 &\cong C_0(\overline{U}_i) \rtimes_{\varphi_i^* \wedge} (\widehat{G} \times G) \rtimes_{\text{dual}} (N \times 0) \\
 &\cong C_0(\overline{U}_i) \rtimes_{\wedge} (\widehat{G} \times G) \rtimes_{\text{dual} \circ \hat{\phi}_i^{-1}} (N \times 0) \\
 &\cong \mathcal{K}(L^2(\widehat{G})) \otimes C_0(\overline{U}_i) \rtimes_{\alpha_i \circ \hat{\phi}_i^{-1}} (N \times 0) \\
 &\sim C_0(\overline{U}_i) \rtimes_{(\hat{\phi}_i)_* \vee^{-1}} (N \times 0) \\
 &\cong C_0(\overline{U}_i) \rtimes_{\omega_i \overline{\rtimes} \hat{\omega}_i} (N \times 0) \\
 &\cong C_0(\overline{U}_i) \rtimes_{\omega_i \rtimes \hat{\omega}_i} (N \times 0),
 \end{aligned}$$

where the last isomorphism is pullback by ϕ_i . This implies the local ω -triviality of $A \rtimes_{\alpha} G$ and shows that $\omega \rtimes \hat{\omega}_i$ gives an atlas.

Step 7. Using these Morita equivalences, we can similarly compute the dual action of \widehat{G} on $(A \rtimes_{\alpha} G)|_{\overline{U}_i}$ (note that all equivalences are G -equivariant):

$$\begin{aligned}
 (A \rtimes_{\alpha} G)|_{\overline{U}_i} \rtimes_{\text{dual}} \widehat{G} &\sim C_0(\overline{U}_i) \rtimes_{\hat{\omega}_i \wedge \omega_i} (N^{\perp} \times G) \rtimes_{\text{dual}} \widehat{G} \\
 &\sim C_0(\overline{U}_i) \rtimes_{\hat{\omega}_i \wedge \omega_i} (\widehat{G} \times G) \rtimes_{\text{dual}} (N \times \widehat{G}) \\
 &\sim C_0(\overline{U}_i) \rtimes_{(\hat{\phi}_i)_* \vee^{-1}} (N \times \widehat{G}) \\
 &\cong C_0(\overline{U}_i) \rtimes_{\omega_i \overline{\rtimes} \hat{\omega}_i \cdot (\phi_i \times \text{id}_{\widehat{G}})_* \vee^{-1} \cdot \hat{\omega}_i \overline{\rtimes} \omega_i} (N \times \widehat{G}) \\
 &\cong C_0(\overline{U}_i) \rtimes_{\omega_i \rtimes \hat{\omega}_i \cdot \vee^{-1} \cdot \hat{\omega}_i \overline{\rtimes} \omega_i} (N \times \widehat{G}) \\
 &\cong C_0(\overline{U}_i) \rtimes_{\omega_i \rtimes \hat{\omega}_i \cdot \vee \cdot \hat{\omega}_i \overline{\rtimes} \omega_i} (N \times \widehat{G}),
 \end{aligned}$$

where the last step is pullback along the inversion \ominus in N together with the isomorphism induced by the cochain that relates $\ominus^*(\omega_i \rtimes \hat{\omega}_i) \sim \omega_i \rtimes \hat{\omega}_i$. This again requires the computation of Remark 3.7. As $\omega_i \rtimes \hat{\omega}_i$ and $\hat{\omega}_i \overline{\rtimes} \omega_i$ are point-wise transverse, this completes the proof. \square

By Lemma 3.18, we can associate to a given NC pair (A, α) an obstruction function

$$(29) \quad \theta : B \rightarrow H^2(N, \text{U}(1))$$

which at each point of B has the Mackey obstruction of the local and restricted N -action as its value. This function is continuous. In fact, it is locally continuous as it is given by point-wise evaluation⁹

$$Z^2(N, C(\overline{U}_i, \text{U}(1))) \cong C(\overline{U}_i, Z^2(N, \text{U}(1))) \rightarrow C(\overline{U}_i, H^2(N, \text{U}(1))).$$

⁹Here evaluation is an isomorphism because N is discrete, so the topology of almost everywhere point-wise convergence on $Z^2(N, \text{U}(1))$ coincides with the compact open topology for which we can apply the exponential law for locally compact Hausdorff spaces X, Y, Z : $C(X \times Y, Z) \cong C(X, C(Y, Z))$

Similarly, to a NC dual pair we associate an obstruction function

$$(30) \quad \hat{\theta} : B \rightarrow H^2(N^\perp, U(1)).$$

Using these functions we can talk about the commutative subtheory inside polarisable NC pairs:

Definition 4.9. A polarisable NC pair (A, α) is called *(point-wise) commutative* if the obstruction function $\hat{\theta}$ defined by its dual $(A \rtimes_\alpha G, \hat{\alpha})$ vanishes.

If we restrict to the groups $G = \mathbb{R}^n, N = \mathbb{Z}^n$ the next proposition shows that this notion of commutativity reproduces the familiar objects from classical T-duality.

Proposition 4.10. *Let $G = \mathbb{R}^n, N = \mathbb{Z}^n$, and let (A, α) be a (polarisable) NC pair. Then (A, α) is point-wise commutative if and only if there is a locally trivial principal G/N -bundle $E \rightarrow B$ and a locally trivial bundle of compact operators $F \rightarrow E$ such that*

$$(31) \quad A \cong \Gamma_0(E, F).$$

Moreover, $F \rightarrow E$ is trivialisable over (a neighbourhood of) all fibres $E|_b, b \in B$, and there is a G -action on F that covers the G/N action on E such that the isomorphism in (31) can be chosen to be G -equivariant.

Proof. It is clear that if an isomorphism (31) exists, then $\hat{\theta} = 0$.

Conversely, if $\hat{\theta}(b) = 0$, then by Lemma 3.19 $\hat{\theta}(b)^\# = \{0\}$ which determines the class of each fibre to be 0. So we have locally

$$A|_{\bar{U}_i} \sim C_0(\bar{U}_i) \rtimes_{\hat{\omega}_i} N^\perp,$$

where the cocycle $\hat{\omega}_i : Z^2(\hat{G}, C(\bar{U}_i, U(1)))$ is such that its restrictions

$$\hat{\eta}_i := \hat{\omega}_i|_{N^\perp} \in Z^2(N^\perp, C(\bar{U}_i, U(1))) \cong C(\bar{U}_i, Z^2(N^\perp, U(1)))$$

are point-wise in the image of the boundary operator

$$d : C^1(N^\perp, U(1)) \rightarrow C^2(N^\perp, U(1)) : \hat{\eta}_i|_b \in B^2(N^\perp, U(1)).$$

The kernel of d is $\text{Hom}(N^\perp, U(1)) \cong G/N = \mathbb{T}^n$ which is a Lie group. So just as in the proof of [Ro86, Theorem 2.1] we apply the Palais cross-section theorem [Pa61, 4.1] which implies that $d : C^1(N^\perp, U(1)) \rightarrow B^2(N^\perp, U(1))$ is a locally trivial G/N -bundle. Then, after passing to a possibly finer covering of B which we again call U_i , we have lifts $\hat{\nu}_i$ in

$$\begin{array}{ccc} & C^1(N^\perp, U(1)) & \\ & \nearrow \hat{\nu}_i & \downarrow d \\ \bar{U}_i & \xrightarrow{\hat{\eta}_i} & B^2(N^\perp, U(1)) \end{array}$$

which implies the existence of local G -equivariant isomorphisms

$$A|_{\bar{U}_i} \cong \mathcal{K} \otimes C_0(\bar{U}_i) \rtimes_{d \circ \hat{\nu}_i} N^\perp \cong \mathcal{K} \otimes C_0(\bar{U}_i) \rtimes_1 N^\perp \cong C_0(U_i \times G/N, \mathcal{K}).$$

Now, consider the transition on the overlap $U_{ij} := U_i \cap U_j$ of two charts

$$\begin{array}{ccc} C_0(\overline{U}_j \times G/N, \mathcal{K}) & & C_0(\overline{U}_i \times G/N, \mathcal{K}) \\ \downarrow & & \downarrow \\ C_0(\overline{U}_{ji} \times G/N, \mathcal{K}) & \xrightarrow[\varphi_{ij}]{\cong} & C_0(\overline{U}_{ij} \times G/N, \mathcal{K}). \end{array}$$

It is G -equivariant, and it induces a G -equivariant automorphism on the spectrum $\gamma_{ji} : \overline{U}_{ij} \times G/N \cong \overline{U}_{ji} \times G/N$. By equivariance γ_{ji} is of the form $\gamma_{ji}(u, z) = (u, g_{ji}(u) + z)$ for some $g_{ji} : \overline{U}_{ij} \rightarrow G/N$. Clearly, the g_{ji} define a Čech cocycle and thus a principal G/N -bundle $E \rightarrow B$. Precomposition of φ_{ij} with $(\gamma_{ji}^{-1})_*$ yields then a spectrum fixing automorphism of

$$C_0(\overline{U}_{ij} \times G/N, \mathcal{K}),$$

i.e., a function $\zeta_{ji} : \overline{U}_{ij} \times G/N \rightarrow \text{PU}$ such that

$$\varphi_{ij}(f)(u, z) = \zeta_{ji}^{-1}(u, z)[f(u, g_{ji}(u) + z)]$$

for $f \in C_0(\overline{U}_{ji} \times G/N, \mathcal{K}) \leftarrow C_0(\overline{U}_j \times G/N, \mathcal{K})$. The ζ_{ij} satisfy the twisted Čech identity

$$\zeta_{kj}(u, g_{ji}(u) + z) \zeta_{ji}(u, z) = \zeta_{ki}(u, z)$$

and hence define a bundle of compact operators $F \rightarrow E$. It is then clear from the construction that F is trivialisable over a neighbourhood of each fibre $E|_b \subset E$, and F carries a G -action which covers the principal G/N -action on E .

For any $a \in A$ the quotient maps $A \rightarrow C_0(\overline{U}_i \times G/N, \mathcal{K})$ define a compatible family of functions, i.e., they define a section of $F \rightarrow E$. Hence we get a G -equivariant map

$$A \rightarrow \Gamma(E, F).$$

Because this map is $C_0(B)$ -linear, one can use a partition of unity on B as an approximate identity on $C_0(B)$ to show that this map takes values in the sections vanishing at infinity only and that this assignment is injective and surjective

$$A \xrightarrow{\cong} \Gamma_0(E, F) \subset \Gamma(E, F). \quad \square$$

4.4. NC bundles. If (A, α) is a NC pair we have seen that it is not reasonable to ask for the cohomology classes of the fibres of A rather than to handle this issue with the ambiguity which is measured by the set $\hat{\theta}^{\text{th}}$ as defined in (23). So if we want to deal with bundles without actions the following definition is appropriate.

Definition 4.11. A NC bundle $(A, \hat{\theta})$ is a locally $\hat{\omega}$ -trivial algebra together with a continuous function $\hat{\theta} : B \rightarrow H^2(N^\perp, \text{U}(1))$ such that there exists an atlas $U_i, \hat{\omega}_i$ of A which has the property that for all i we have $[\hat{\omega}_i|_b]_N \in \hat{\theta}(b)^{\text{th}}$, for all $b \in U_i$. A NC bundle $(A, \hat{\theta})$ is called *commutative* if $\hat{\theta} = 0$.

(The notion of *(commutative) NC dual bundles* is defined analogously.)

These are the objects of the category of NC bundles. The morphisms $(A, \hat{\theta}) \rightarrow (A', \hat{\theta}')$ are pairs (f, M) of a continuous function $f : B \rightarrow B'$ such that $\hat{\theta} = \hat{\theta}' \circ f$ and a $C_0(B)$ -linear Morita equivalence ${}_A M_{f^* A'}$. Here we assume that A' is a $C_0(B')$ -algebra.

If (A, α) is any polarisable NC pair, we can use the obstructions function $\hat{\theta} : B \rightarrow H^2(N^\perp, U(1))$ of its dual $(A \rtimes_\alpha G, \hat{\alpha})$ to define an assignment on objects $(A, \alpha) \mapsto (A, \hat{\theta})$ which extends to a functor

$$\Theta : \begin{array}{c} \text{Polarisable} \\ \text{NC Pairs} \end{array} \longrightarrow \text{NC Bundles}$$

that gives the *underlying NC bundle* of a polarisable NC pair. Similarly, there is a functor

$$\hat{\Theta} : \begin{array}{c} \text{Polarisable} \\ \text{NC Dual Pairs} \end{array} \longrightarrow \begin{array}{c} \text{NC Dual} \\ \text{Bundles} \end{array}$$

that gives the underlying NC dual bundle of a polarisable NC dual pair. An *extension* of a NC (dual) bundle $(A, \hat{\theta})$ is a polarisable NC (dual) pair (A, α) such that $\Theta(A, \alpha) = (A, \hat{\theta})$, and a NC (dual) bundle is called *dualisable* if it has an extension. Now, if (\hat{A}, θ) is a dualisable NC dual bundle, one might ask the question whether it has an extension that is the dual of a commutative polarisable NC pair. In other words, we ask whether the functor

$$\Xi : \begin{array}{c} \text{Polarisable} \\ \text{NC Pairs} \end{array} \xrightarrow{\sim} \begin{array}{c} \text{Polarisable} \\ \text{NC Dual Pairs} \end{array} \xrightarrow{\hat{\Theta}} \begin{array}{c} \text{NC Dual} \\ \text{Bundles} \end{array}$$

and its restriction

$$\Xi|_{\text{com}} : \begin{array}{c} \text{Commutative} \\ \text{Polarisable} \\ \text{NC Pairs} \end{array} \subset \begin{array}{c} \text{Polarisable} \\ \text{NC Pairs} \end{array} \begin{array}{c} \downarrow \sim \\ \begin{array}{c} \text{Polarisable} \\ \text{NC Dual Pairs} \end{array} \xrightarrow{\hat{\Theta}} \begin{array}{c} \text{NC Dual} \\ \text{Bundles} \end{array} \end{array}$$

have the same (essential) image. If the answer to this question were yes, then the theory we developed wouldn't be significantly richer than the classical commutative and semi-commutative theory of Mathai and Rosenberg, because all NC bundles could be understood as crossed products of classical bundles (with all their possible G -actions). However, this is not the case:

Proposition 4.12. *Let $G := \mathbb{R}^2$ and $N := \mathbb{Z}^2$. Then the essential images of the functors Ξ and $\Xi|_{\text{com}}$ do not coincide.*

Proof. In Section 5.2 below we give an example of a NC bundle that cannot be obtained as the dual of a commutative polarisable NC pair. \square

5. Example: The Heisenberg bundle and its relatives

In this section we always let $G := \mathbb{R}^2$ and $N := \mathbb{Z}^2$ and we use the canonical identifications $\widehat{\mathbb{R}^2} \cong \mathbb{R}^2$ and $(\mathbb{Z}^2)^\perp \cong \mathbb{Z}^2$. We will typically denote elements in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ or $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ by x, y, z or by \dot{g} if g is in \mathbb{R} or \mathbb{R}^2 .

By S^1 we denote the unit interval $[0, 1]$ with glued end-points $0 \sim 1$. Sometimes it is convenient to have one of the endpoints, say 1, thickened to a whole (nonempty) interval $\mathbf{1} := [1_-, 1_+]$, i.e., we then consider the space $S^\mathbf{1} := ([0, 1] \sqcup \mathbf{1})/\sim$, where $1 \sim 1_-$ and $0 \sim 1_+$ which, of course, is homeomorphic to S^1 . For $t \in \mathbb{R}$ we denote by ω_t the 2-cocycle on \mathbb{R}^2 given by $\omega_t(g, h) := \exp(2\pi itg_2h_1)$. We will use the notation λ^t , $t \in \mathbb{R}$, for the actions $\mathbb{R}^2 \rightarrow \text{Aut}(\mathcal{K})$ which are given by $\lambda^t_g := \text{Ad}(L_{-t}(g))$ for the left regular ω_{-t} -representation

$$(L_{-t}(g)(\xi))(h) := \omega_{-t}(g, h - g)\xi(h - g), \quad \xi \in L^2(\mathbb{R}^2).$$

The actions λ^t have Mackey obstruction $+t \in \mathbb{R} \cong H^2(\mathbb{R}^2, \mathbb{U}(1))$.

5.1. The Heisenberg bundle. Consider the function

$$\omega_{S^1} : S^1 \rightarrow Z^2(N, \mathbb{U}(1))$$

given by $\omega_{S^1}(\dot{s}) := \omega_s|_{N \times N}$ which is well-defined as $\omega_n|_{N \times N} = 1$, for $n \in \mathbb{Z}$. The Heisenberg bundle is the C*-algebra over S^1 given by

$$\hat{A}_0 := \mathcal{K} \otimes (C(S^1) \rtimes_{\omega_{S^1}} N).$$

Its fibres are the stable noncommutative tori $\hat{A}|_{\dot{s}} \cong \mathcal{K} \otimes (\mathbb{C} \rtimes_{\omega_s|_{N \times N}} N)$. We equip it with the canonical \widehat{G} -action $\hat{\alpha}_0 := \text{id} \otimes \text{inf}$. The following proposition is immediate.

Proposition 5.1. *($\hat{A}_0, \hat{\alpha}_0$) is a (polarisable) NC dual pair.*

The obstruction function $\theta_0 : S^1 \rightarrow H^2(\mathbb{Z}^2, \mathbb{U}(1))$ defined by $(\hat{A}_0, \hat{\alpha}_0)$ is just the composition of the canonical identifications $S^1 \cong \mathbb{T} \cong H^2(\mathbb{Z}^2, \mathbb{U}(1))$.

Let us construct a (commutative) NC pair which will turn out to be the dual of the Heisenberg bundle. Consider the trivial principal \mathbb{T}^2 -bundle $E_0 := S^1 \times \mathbb{T}^2 \rightarrow S^1$. Its total space E_0 is a compact orientable three manifold so its 3rd (Čech) cohomology is $\check{H}^3(E_0, \mathbb{Z}) \cong \mathbb{Z}$. Let $F_1 \rightarrow E$ be a locally trivial \mathcal{K} -bundle representing the canonical generator of 3rd cohomology:

$$\begin{aligned} \check{H}^3(E_0, \mathbb{Z}) &\xrightarrow{\cong} \mathbb{Z}. \\ [F_1] &\mapsto 1 \end{aligned}$$

Let $\Gamma(E_0, F_1)$ be the section C*-algebra of $F_1 \rightarrow E$. We construct an action on this algebra which covers the principal \mathbb{T}^2 -action on

$$E_0 = \text{Prim}(\Gamma(E_0, F_1)).$$

To do so, observe first that one can describe the bundle F_1 slightly different. Up to isomorphism any \mathcal{K} -bundle over E_0 can be obtained from a function

$\mathbb{T}^2 \rightarrow \text{Aut}(\mathcal{K})$ which is used to glue the two boundary parts of the trivial \mathcal{K} -bundle

$$(32) \quad [0, 1] \times \mathbb{T}^2 \times \mathcal{K} \rightarrow [0, 1] \times \mathbb{T}^2$$

to each other. In particular, the bundle F_1 is obtained by using a classifying map $\mathbb{T}^2 \rightarrow \mathcal{B}U(1) = \mathcal{P}U(\mathcal{H}) = \text{Aut}(\mathcal{K})$ of the canonical $U(1)$ -bundle over \mathbb{T}^2 , i.e., a function whose class in 2nd cohomology $\check{H}^2(\mathbb{T}^2, \mathbb{Z}) \cong \mathbb{Z}$ corresponds to 1.

Let us identify such a function $\mathbb{T}^2 \rightarrow \text{Aut}(\mathcal{K})$ by the following construction. Choose $\mathcal{K} = \mathcal{K}(L^2(\mathbb{R}^2))$ and define a $C([0, 1])$ -linear action β on the sections $C([0, 1] \times \mathbb{T}^2, \mathcal{K})$ of (32) by $\beta_g(f)(t, x) = \lambda^t_g(f(t, x - \dot{g}))$. For $t = 1$ the cocycle involved in $\lambda^1, \omega_{-1} : (g, h) \mapsto \exp(2\pi i(-1)g_2h_1)$, becomes trivial when restricted to $N = \mathbb{Z}^2$. To continue we need a lemma:

Lemma 5.2. *The canonical isomorphism*

$$H^2(\mathbb{R}^2, L^\infty(\mathbb{R}^2/\mathbb{Z}^2, U(1))) \rightarrow H^2(\mathbb{Z}^2, U(1))$$

of [Mo76, Thm. 6] makes the diagram

$$\begin{array}{ccc} H^2(\mathbb{R}^2, U(1)) & & \\ \downarrow & \searrow & \\ H^2(\mathbb{Z}^2, U(1)) & \xleftarrow{\cong} & H^2(\mathbb{R}^2, L^\infty(\mathbb{R}^2/\mathbb{Z}^2, U(1))) \end{array}$$

commute, where the vertical map is restriction and the diagonal map is induced by the inclusion of coefficients.

Proof. See Appendix B. □

Because of this lemma there is a Borel function¹⁰

$$c : \mathbb{R}^2 \rightarrow L^\infty(\mathbb{T}^2, U(1)) \subset U(L^2(\mathbb{T}^2))$$

such that $\omega_{-1} = dc$, i.e.,

$$(33) \quad \omega_{-1}(g, h) = c(h)(z - \dot{g}) c(g + h)(z)^{-1} c(g)(z),$$

for almost all $z \in \mathbb{T}^2$. Note at this point that (33) implies that the restriction $c|_{\mathbb{Z}^2} : \mathbb{Z}^2 \rightarrow L^\infty(\mathbb{T}^2, U(1))$ is a homomorphism, i.e., there is a measure-one set $S \subset \mathbb{T}^2$ such that $n \mapsto c(n)(z)$ is in $\widehat{\mathbb{Z}^2}$, for all $z \in S$. The function c determines a function $\tilde{c} \in L^\infty(\mathbb{R}^2 \times \mathbb{T}^2, U(1)) \subset U(L^2(\mathbb{R}^2 \times \mathbb{T}^2))$ which for each g satisfies

$$(34) \quad \omega_{-1}(g, h) = \tilde{c}(h, z - \dot{g}) \tilde{c}(g + h, z)^{-1} c(g)(z),$$

for almost all $(h, z) \in \mathbb{R}^2 \times \mathbb{T}^2$. (By choosing a Borel representative of \tilde{c}) we consider for $z \in \mathbb{T}^2$ the unitary multiplication operator $\tilde{c}(\cdot, z) \in U(L^2(\mathbb{R}^2))$

¹⁰We always consider $L^\infty(X, U(1))$ as a subspace of $U(L^2(X))$ by associating to an L^∞ -function the multiplication operator that multiplies L^2 -functions point-wise with the given L^∞ -function.

which sends a function $\xi \in L^2(\mathbb{R}^2)$ to $h \mapsto \tilde{c}(h, z)\xi(h)$. These operators define a Borel function $z \mapsto \eta_0(z) := \text{Ad}(\tilde{c}(-, z)) \in \text{PU}(L^2(\mathbb{R}^2))$.

Lemma 5.3.

(1) *There exists a continuous function*

$$\eta : \mathbb{T}^2 \rightarrow \text{PU}(L^2(\mathbb{R}^2))$$

which agrees almost everywhere with η_0 .

(2) *The class of η in $\check{H}^2(\mathbb{T}^2, \mathbb{Z}) \cong \mathbb{Z}$ is the generator $+1$, i.e., η is a classifying map for the canonical line bundle over \mathbb{T}^2 .*

(3) *The equality*

$$\eta(z) \lambda_g^{k+1} = \lambda_g^k \eta(z - \dot{g}) \in \text{PU}(L^2(\mathbb{R}^2))$$

holds, for all $k \in \mathbb{Z}$, $g \in \mathbb{R}^2$, $z \in \mathbb{T}^2$.

Proof. (1) Consider the countable dense subgroup $\mathbb{Q}^2 \subset \mathbb{R}^2$. By (34) there is for each $g \in \mathbb{Q}^2$ a set $S_g \subset \mathbb{T}^2$ of measure one such that

$$\omega_{-1}(g, -) = \tilde{c}(-, z - \dot{g}_k) \tilde{c}(g_k + -, z)^{-1} c(g_k)(z) \in U(L^2(\mathbb{R}^2))$$

holds for all $z \in S_g$. Choose some $z_0 \in S \cap \bigcap_{g \in \mathbb{Q}^2} S_g$. Then

$$\omega_{-1}(g, -) = \tilde{c}(-, z_0 - \dot{g}) \tilde{c}(g + -, z_0)^{-1} c(g)(z_0) \in U(L^2(\mathbb{R}^2))$$

holds for all $g \in \mathbb{Q}^2$. The map

$$\begin{aligned} g &\mapsto \text{Ad}(\tilde{c}(-, z_0 - \dot{g})) \\ &= \text{Ad}(\omega_{-1}(g, -)\tilde{c}(g + -, z_0)) \\ &= \text{Ad}(\omega_{-1}(g, -)L_0(-g)\tilde{c}(-, z_0)L_0(g)) \end{aligned}$$

is clearly continuous on \mathbb{Q}^2 (as one can see in line three) and clearly factors over the quotient $\mathbb{Q}^2/\mathbb{Z}^2$ (as one can see in line one). Then define

$$\eta_{z_0} : \mathbb{T}^2 \rightarrow \text{PU}(L^2(\mathbb{R}^2))$$

to be its continuous extension (which is just given by the formula in line three, for all $g \in \mathbb{R}^2$), and let $\eta(z) := \eta_{z_0}(z_0 - \dot{g})$. By construction it's clear that η satisfies (1) of the lemma.

(2) The Čech classes of η and of $z \mapsto \eta(z - z_0)$ agree. We compute the Čech class of the latter. Choose continuous, local sections $\sigma_k : V_k \rightarrow \mathbb{R}^2$ of the quotient map $\mathbb{R}^2 \rightarrow \mathbb{T}^2$ such that the open domains $V_k \subset \mathbb{T}^2$ cover \mathbb{T}^2 . Then by line two from above

$$\eta_k(z) := \omega_{-1}(-\sigma_k(z), -) \tilde{c}(-\sigma_k(z) + -, z_0) \in U(L^2(G))$$

are continuous unitary local lifts of $z \mapsto \eta(z - z_0)$, so its class in

$$\check{H}^1(\mathbb{T}^2, U(1)) \cong \check{H}^2(\mathbb{T}^2, \mathbb{Z})$$

is given by the cocycle

$$\begin{aligned}
 \eta_{kl}(z) &:= \eta_k(z)\eta_l(z)^{-1} \\
 &= \omega_{-1}(-\sigma_k(z), -) \tilde{c}(-\sigma_k(z) + -, z_0) \tilde{c}(-\sigma_l(z) + -, z_0)^{-1} \\
 &\quad \cdot \omega_{-1}(-\sigma_l(z), -)^{-1} \\
 &= \omega_{-1}(-\sigma_k(z) + \sigma_l(z), -) \tilde{c}(-\sigma_k(z) + -, z_0) \\
 &\quad \cdot \tilde{c}((-\sigma_k(z) + -) + (\sigma_k(z) - \sigma_l(z)), z_0)^{-1} \\
 &= \omega_{-1}(-\sigma_k(z) + \sigma_l(z), -) \omega_{-1}(\sigma_k(z) - \sigma_l(z), \sigma_k(z) + -) \\
 &\quad \cdot c(\sigma_k(z) - \sigma_l(z))(z_0)^{-1} \\
 &= \omega_{-1}(\sigma_k(z) - \sigma_l(z), \sigma_k(z)) c((\sigma_k(z) - \sigma_l(z)))(z_0)^{-1} \in U(1).
 \end{aligned}$$

We claim that $c((\sigma_k(z) - \sigma_l(z)))(z_0)^{-1}$ is a coboundary term. In fact,

$$\sigma_k(z) - \sigma_l(z) \in \mathbb{Z}^2,$$

and so choose by surjectivity of $\widehat{\mathbb{R}^2} \rightarrow \widehat{\mathbb{Z}^2}$ an extension $\chi \in \widehat{\mathbb{R}^2}$ of the character $n \mapsto c(n)(z_0)$ which gives $\chi(\sigma_k(z))\chi(\sigma_l(z))^{-1} = c((\sigma_k(z) - \sigma_l(z)))(z_0)$. This indeed is a coboundary term and does not effect the class of η . The expressions

$$\begin{aligned}
 z &= (z_1, z_2) \mapsto \omega_{-1}(\sigma_l(z) - \sigma_k(z), \\
 \sigma_k(z)) &= \langle \sigma_k(z)_2 - \sigma_l(z)_2, z_1 \rangle,
 \end{aligned}$$

are transition functions for the canonical line bundle on $\mathbb{T} \times \mathbb{T}$ [S07, Sec. 2.6], i.e., they give the class of the canonical line bundle.

(3) Just multiply both sides of (33) by the left regular representation $L_{-k}(g)$ and apply $\text{Ad} : U(L^2(G)) \rightarrow \text{PU}(L^2(G))$ to both sides. \square

Part (3) of the lemma just says for $k = 0$ that

$$\begin{array}{ccc}
 C(1 \times \mathbb{T}^2, \mathcal{K}) & \xrightarrow{\beta|_1(g)} & C(1 \times \mathbb{T}^2, \mathcal{K}) \\
 \downarrow \eta_* & & \downarrow \eta_* \\
 C(0 \times \mathbb{T}^2, \mathcal{K}) & \xrightarrow{\beta|_0(g)} & C(0 \times \mathbb{T}^2, \mathcal{K})
 \end{array}$$

commutes, where $\beta|_0, \beta|_1$ are the actions on the fibres over $t = 0, 1$ given by the fibre-wise action β , and $\eta_*(f)(1, z) := \eta(z)(f(0, z))$. So by construction we have shown the following proposition.

Proposition 5.4. *Let (A_0, α_0) be the pullback in the category of C^* -dynamical systems of the diagram*

$$\begin{array}{ccc}
 (A_0, \alpha_0) & \xrightarrow{\hspace{10em}} & (C([0, 1] \times \mathbb{T}^2, \mathcal{K}), \beta) \\
 \downarrow \check{y} & & \downarrow i_0^* \times i_1^* \\
 (C(1 \times \mathbb{T}^2, \mathcal{K}), \beta|_1) & \xrightarrow{\eta_* \times \text{id}} & (C(0 \times \mathbb{T}^2, \mathcal{K}) \oplus C(1 \times \mathbb{T}^2, \mathcal{K}), \beta|_0 \times \beta|_1)
 \end{array}$$

then there is a canonical isomorphism $A_0 \cong \Gamma(E_0, F_1)$.

It is clear that (A_0, α_0) is a polarisable NC pair with a trivial obstruction function $\hat{\theta}_0 = 0 : S^1 \rightarrow H^2(\mathbb{Z}, U(1))$, so its dual is commutative:

Proposition 5.5. *(A_0, α_0) and $(\hat{A}_0, \hat{\alpha}_0)$ are dual to each other, i.e.,*

$$A_0 \rtimes_{\alpha_0} \mathbb{R}^2$$

with its dual $\widehat{\mathbb{R}^2}$ -action is Morita equivalent to $(\hat{A}_0, \hat{\alpha}_0)$.

Proof. Consider the function $\omega_{[0,1]} : [0, 1] \rightarrow Z^2(\mathbb{Z}^2, U(1))$ given by

$$\omega_{[0,1]}(t) := \omega_t|_{\mathbb{Z}^2 \times \mathbb{Z}^2},$$

then the Heisenberg bundle \hat{A}_0 with its induced action $\hat{\alpha}_0$ is the pullback in (35)

$$\begin{array}{ccc}
 (\hat{A}_0, \hat{\alpha}_0) & \xrightarrow{\hspace{10em}} & (\mathcal{K} \otimes (C([0, 1]) \rtimes_{\omega_{[0,1]}} \mathbb{Z}^2), \text{id} \otimes \text{inf}) \\
 \downarrow \check{y} & & \downarrow i_0^* \times i_1^* \\
 (C(1 \times \mathbb{T}^2, \mathcal{K}), \text{inf}) & \xrightarrow{\text{diag}} & (C(0 \times \mathbb{T}^2, \mathcal{K}) \oplus C(1 \times \mathbb{T}^2, \mathcal{K}), \text{inf} \otimes \text{inf}).
 \end{array}$$

In the diagram of Proposition 5.4 we can take crossed product with \mathbb{R}^2 . Because taking crossed products is a continuous functor, this leads to another pullback diagram. But this new diagram is stably isomorphic to diagram (35). □

Remark 5.6. The duality stated in Proposition 5.5 has already been observed in [MaR06, Section 4]. However, they follow a different approach which is less explicit than the one presented here, and we can use the intermediate steps of our approach for the construction of the twisted Heisenberg bundle which we do next.

5.2. The twisted Heisenberg bundle. Let us use the pullback description of the two NC (dual) pairs from above to construct a chimaera out of the two. Consider the algebra $\mathcal{K} \otimes (C([0, 1]) \rtimes_{\omega_{[0,1]}} \mathbb{Z}^2)$, where $\omega_{[0,1]}$ is as in (35). This algebra carries the fibre-wise action γ that is given in each fibre by $\gamma|_t := \lambda^{\frac{2}{1+t}} \otimes \text{inf}$. As t moves from 0 to 1, the Mackey obstruction of $\lambda^{\frac{2}{1+t}}$ moves from 2 to 1. For $k = 1$ part 3. of Lemma 5.3 implies that $\eta(z)\lambda^2_g = \lambda^1_g\eta(z - \dot{g})$, i.e.,

$$\begin{CD} C(1 \times \mathbb{T}^2, \mathcal{K}) @>\gamma|_1(g)>> C(1 \times \mathbb{T}^2, \mathcal{K}) \\ @VV\eta^*V @VV\eta^*V \\ C(0 \times \mathbb{T}^2, \mathcal{K}) @>\gamma|_0(g)>> C(0 \times \mathbb{T}^2, \mathcal{K}) \end{CD}$$

commutes, for $\eta^*(f)(1, z) := \eta(z)^{-1}(f(0, z))$. So naively, what we now could consider is the pullback in

$$\begin{CD} (\hat{A}_1, \hat{\alpha}_1) @>>> (\mathcal{K} \otimes (C([0, 1]) \rtimes_{\omega_{[0,1]}} \mathbb{Z}^2), \gamma) \\ @VVV @VV i_0^* \times i_1^* V \\ (C(1 \times \mathbb{T}^2, \mathcal{K}), \gamma|_1) @>\eta^* \times \text{id}>> (C(0 \times \mathbb{T}^2, \mathcal{K}) \oplus C(1 \times \mathbb{T}^2, \mathcal{K}), \gamma|_0 \times \gamma|_1). \end{CD}$$

Indeed \hat{A}_1 is a bundle over S^1 with fibers $\hat{A}_1|_s \cong \mathcal{K} \otimes (C \rtimes_{\omega_s} N)$. However, it fails to be ω -trivial around $\dot{0} = \dot{1} \in S^1$. We can get around this by thickening the gluing-point to the interval $\mathbf{1}$: We define the *twisted Heisenberg bundle* $\hat{A}_{\mathbf{1}}$ together with its action $\hat{\alpha}_{\mathbf{1}}$ to be the pullback in

$$\begin{CD} (\hat{A}_{\mathbf{1}}, \hat{\alpha}_{\mathbf{1}}) @>>> (\mathcal{K} \otimes (C([0, 1]) \rtimes_{\omega_{[0,1]}} \mathbb{Z}^2), \gamma) \\ @VVV @VV i_0^* \times i_1^* V \\ (C(\mathbf{1} \times \mathbb{T}^2, \mathcal{K}), \gamma|_{\mathbf{1}}) @>(\eta^* \circ i_{1+}^*) \times i_{1-}^*>> (C(0 \times \mathbb{T}^2, \mathcal{K}) \oplus C(1 \times \mathbb{T}^2, \mathcal{K}), \gamma|_0 \times \gamma|_1), \end{CD}$$

where the action $\gamma|_{\mathbf{1}}$ is just $\gamma|_1$ in each fibre.

Proposition 5.7. *The twisted Heisenberg bundle $(\hat{A}_{\mathbf{1}}, \hat{\alpha}_{\mathbf{1}})$ is a (polarisable) NC dual pair over $S^{\mathbf{1}}$.*

Proof. Let $1_{\pm} \in \mathbf{1}$ be the middle. We consider the open cover of $S^{\mathbf{1}}$ given by the two open sets $U := ((0, 1] \sqcup [1_-, 1_{\pm}))/\sim$ and $V := ([0, \frac{1}{2}] \sqcup (1_-, 1_+))/\sim$.

Then

$$\begin{aligned} \hat{A}_{\mathbf{1}}|_{\bar{U}} &\sim \left\{ (f, g) \in C([0, 1]) \rtimes_{\omega_{[0,1]}} \mathbb{Z}^2 \times C([1_-, 1_{\pm}] \times \mathbb{T}) \mid f|_1 = g|_{1_-} \right\} \\ &\sim C(\bar{U}) \rtimes_{\omega_{\bar{U}}} \mathbb{Z}^2, \end{aligned}$$

where we let $\omega_{\bar{U}}(s) := \omega_{1+s}$, for $s \in [0, 1]$ and $\omega_{\bar{U}}(s) := \omega_2$, for $s \in [1_-, 1_{\pm}]$. If we let $\hat{\omega}_{\bar{U}}(s) := \omega_{\frac{2}{1+s}}$, for $s \in [0, 1]$ and $\hat{\omega}_{\bar{U}}(s) := \omega_1$, for $s \in [1_-, 1_{\pm}]$, then $\omega_{\bar{U}}$ and $\hat{\omega}_{\bar{U}}$ are point-wise transverse (cp. Section 3.4). Moreover, by construction of the action $\hat{a}_{\mathbf{1}}$ we have

$$(\hat{A}_{\mathbf{1}} \rtimes_{\hat{a}_{\mathbf{1}}} \hat{\mathbb{R}}^2)|_{\bar{U}} \cong \mathcal{K} \otimes (C(\bar{U}) \rtimes_{\omega_{\bar{U}} \vee \hat{\omega}_{\bar{U}}} (\mathbb{Z}^2 \times \hat{\mathbb{R}}^2)).$$

We have shown that U is a chart for $(\hat{A}_{\mathbf{1}}, \hat{a}_{\mathbf{1}})$. To show that V is a chart we need one more step in between. In fact, we use that we can extend the isomorphism η^* (fibre-wise) to an isomorphism

$$(36) \quad \eta^* : C(\mathbf{1} \times \mathbb{T}^2, \mathcal{K}) \rightarrow C(\mathbf{0} \times \mathbb{T}^2, \mathcal{K}),$$

where $\mathbf{0} := [0_-, 0_+] := \mathbf{1} - 1$ is just the intervall $\mathbf{1}$ shifted by 1. This isomorphism turns the action $\gamma|_{\mathbf{1}}$ into $\gamma|_{\mathbf{0}}$, which in each fibre $s \in \mathbf{0}$ just is $\gamma|_{\mathbf{0}}|_s = \gamma|_0$. Then we find

$$\begin{aligned} &\hat{A}_{\mathbf{1}}|_{\bar{V}} \\ &\cong \downarrow \\ &\left\{ (f, g) \in \mathcal{K} \otimes (C([0, \frac{1}{2}]) \rtimes_{\omega_{[0,1]}} \mathbb{Z}^2) \times C(\mathbf{1} \times \mathbb{T}, \mathcal{K}) \mid f|_0 = \eta^*g|_{1_+} \right\} \\ &\cong \downarrow (f,g) \mapsto (f, \eta^*g) \\ &\left\{ (f, h) \in \mathcal{K} \otimes (C([0, \frac{1}{2}]) \rtimes_{\omega_{[0,1]}} \mathbb{Z}^2) \times C(\mathbf{0} \times \mathbb{T}, \mathcal{K}) \mid f|_0 = h|_{0_+} \right\} \\ &\cong \downarrow \\ &\mathcal{K} \otimes (C(\bar{V}) \rtimes_{\omega_{\bar{V}}} \mathbb{Z}^2) \end{aligned}$$

wherein $\omega_{\bar{V}}(s) := \omega_{1+s}$, for $s \in [0, \frac{1}{2}]$ and $\omega_{\bar{V}}(s) := \omega_1$, for $s \in \mathbf{1}$. We have $\theta_{\mathbf{1}}|_{\bar{V}} = [\omega_{\bar{V}}|_{\mathbb{Z}^2 \times \mathbb{Z}^2}]$, and we let $\hat{\omega}_{\bar{V}}(s) := \omega_{\frac{2}{1+s}}$, for $s \in [0, \frac{1}{2}]$ and $\hat{\omega}_{\bar{V}}(s) := \omega_2$, for $s \in \mathbf{1}$. By construction $\omega_{\bar{V}}$ and $\hat{\omega}_{\bar{V}}$ are point-wise transverse, and by construction of the action $\hat{a}_{\mathbf{1}}$ we have

$$(\hat{A}_{\mathbf{1}} \rtimes_{\hat{a}_{\mathbf{1}}} \hat{\mathbb{R}}^2)|_{\bar{V}} \cong \mathcal{K} \otimes (C(\bar{V}) \rtimes_{\omega_{\bar{V}} \vee \hat{\omega}_{\bar{V}}} (\mathbb{Z}^2 \times \hat{\mathbb{R}}^2)). \quad \square$$

Remark 5.8.

- (1) The possibility of finding the extension (36) is the crucial step in proving the local triviality of $\hat{A}_{\mathbf{1}}$. This extension exists on a bundle of commutative tori, but could not have been found if we had not thickened 1 to $\mathbf{1}$.

- (2) The essential point for point-wise transversality is that the point-wise cocycles $\omega_{[0,1]}(s) : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow U(1)$ have extensions

$$\omega_{1+s} : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow U(1)$$

whose cohomology classes vary in the interval $[1, 2]$ which (together with $[-2, -1]$) is the only interval of integer length which is preserved by the map $s \mapsto \frac{2}{s}$. The relevance of the second interval $[-2, -1]$ becomes clear below: It is the interval from which the (pre-)dual of $(\hat{A}_{\mathbf{1}}, \hat{\alpha}_{\mathbf{1}})$ takes its Mackey obstructions.

Let us identify the pre-dual of the twisted Heisenberg bundle. Knowing its local structure from the proof of Proposition 5.7 and knowing the calculation in Section 3.4 it is rather clear how the pre-dual looks like. Let $(A_{\mathbf{1}}, \alpha_{\mathbf{1}})$ to be the pullback in

$$\begin{array}{ccc} (A_{\mathbf{1}}, \alpha_{\mathbf{1}}) & \xrightarrow{\hspace{10em}} & (\mathcal{K} \otimes (C([0, 1]) \rtimes_{\hat{\omega}_{[0,1]}} \mathbb{Z}^2), \delta) \\ \downarrow \text{dotted} & & \downarrow i_0^* \times i_1^* \\ (C(\mathbf{1} \times \mathbb{T}^2, \mathcal{K}), \delta|_{\mathbf{1}}) & \xrightarrow{(\eta^* \circ i_{1+}^*) \times i_{1-}^*} & (C(0 \times \mathbb{T}^2, \mathcal{K}) \oplus C(1 \times \mathbb{T}^2, \mathcal{K}), \delta|_0 \times \delta|_1), \end{array}$$

where the action δ is point-wise given by $\delta|_t := \lambda^{-(1+t)} \otimes \text{inf}$, $t \in [0, 1]$, the action $\delta|_{\mathbf{1}}$ is just $\delta|_1$ in each fibre, and the cocycle $\hat{\omega}_{[0,1]}$ is defined by

$$\hat{\omega}_{[0,1]}(t) := \omega_{\frac{-2}{1+t}} \Big|_{N^\perp \times N^\perp}, \quad t \in [0, 1] \quad \text{and} \quad N = \mathbb{Z}^2.$$

Then one can prove that $(A_{\mathbf{1}}, \alpha_{\mathbf{1}})$ is a (polarisable) NC pair over $S^{\mathbf{1}}$ just along the lines of Proposition 5.7. Moreover, the computation of its dual is a point-wise repetition of Section 3.4:

Proposition 5.9. *The NC pair $(A_{\mathbf{1}}, \alpha_{\mathbf{1}})$ is dual to $(\hat{A}_{\mathbf{1}}, \hat{\alpha}_{\mathbf{1}})$.*

Let $(\hat{A}_{\mathbf{1}}, \theta_{\mathbf{1}}) := \hat{\Theta}(\hat{A}_{\mathbf{1}}, \alpha_{\mathbf{1}})$ be the underlying NC bundle of the twisted Heisenberg bundle as explained in Section 4.4. Note that

$$\theta_{\mathbf{1}} : S^{\mathbf{1}} \rightarrow H^2(\mathbb{Z}^2, U(1)) \cong \mathbb{T}$$

has winding number 1, and this is the key to proof what we have claimed in Proposition 4.12. Let us denote by $F_n \rightarrow E$ a locally trivial \mathcal{K} -bundle on $E := S^{\mathbf{1}} \times \mathbb{T}^2$ such that $[F_n] \in \check{H}^3(E, \mathbb{Z}) \cong \mathbb{Z}$ corresponds to $n \in \mathbb{Z}$. Then (up to isomorphism) all commutative NC pairs over $S^{\mathbf{1}}$ are given by $\Gamma(E, F_n)$ with suitable \mathbb{R}^2 -actions.

Proposition 5.10. *$(\hat{A}_{\mathbf{1}}, \theta_{\mathbf{1}})$ is not in the essential image of $\Xi|_{\text{com}}$.*

Proof. The proof makes use of the K-theory bundle introduced in [ENO09]. This is the group bundle over $S^{\mathbf{1}}$ whose fibres are given by the K-theory of the fibres. According to [ENO09, Sec. 5], the K_0 -bundle of the Heisenberg

bundle \hat{A}_0 is the \mathbb{Z}^2 -bundle over $S^{\mathbf{1}}$ which is given by gluing the trivial \mathbb{Z}^2 -bundle over the interval with the matrix

$$M := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The Heisenberg bundle is a crossed product of $\Gamma(E, F_1)$ by a fibre-preserving \mathbb{R}^2 -action, so by Connes' Thom isomorphism their fibres have the same K-theory, and thus the K-theory bundles are the same (see [ENO09, Theorem 3.5 and Remark 4.4]). This implies that η_* induces the above map M in K-theory. The twisted Heisenberg bundle then has a trivial K_0 -bundle, because it has the additional gluing with η^* (rather than with η_*). (The bundle $\hat{A}_{-\mathbf{1}}$ which is obtained from the Heisenberg bundle with twist η_* has K_0 -bundle glued by M^2 .) Therefore, if $\Gamma(E, F_n)$ is a crossed product of the twisted Heisenberg bundle it also must have a trivial K-theory bundle. But as F_n is obtained by gluing with η^n , the K_0 -bundle of $\Gamma(E, F_n)$ is given by gluing the trivial \mathbb{Z}^2 -bundle with M^n which is trivial if and only if $n = 0$. Now, consider $\Gamma(E, F_0) \cong \mathcal{K} \otimes C(E)$ with any (transverse) \mathbb{R}^2 -action α . Since $E = S^{\mathbf{1}} \times \mathbb{T}^2$ is a trivial \mathbb{T}^2 -bundle, it follows from [E90, Theorem] that the crossed product $\mathcal{K} \otimes C(E) \rtimes_{\alpha} \mathbb{R}^2$ is equivariantly Morita equivalent to $\mathcal{K} \otimes C(S^{\mathbf{1}} \times \{0\}) \rtimes_{\alpha|_{\mathbb{Z}^2}} \mathbb{Z}^2$ with the \mathbb{R}^2 -action that is inflated from the dual action of \mathbb{T}^2 (see the discussion in Section 3.1). However, $\mathcal{K} \otimes C(S^{\mathbf{1}}) \rtimes_{\alpha} \mathbb{Z}^2$ with the dual \mathbb{T}^2 -action is a NCP-torus bundle in the sense of [ENO09] which has a trivial K-theory bundle. Then [ENO09, Theorem 7.2] implies that the Mackey obstruction function

$$S^{\mathbf{1}} \rightarrow H^2(\mathbb{Z}^2, U(1)) \cong \mathbb{T}$$

of $\alpha|_{\mathbb{Z}^2} \in \text{Aut}(\mathcal{K} \otimes C(S^{\mathbf{1}} \times \{0\}))$ is null-homotopic. Since every null-homotopic map $S^{\mathbf{1}} \rightarrow \mathbb{T} \cong H^2(\mathbb{Z}^2, U(1))$ can be lifted to a continuous map $S^{\mathbf{1}} \rightarrow \mathbb{R} \cong H^2(\mathbb{R}^2, U(1))$, this implies that there exists an action $\mu : \mathbb{R}^2 \rightarrow \text{Aut}(\mathcal{K} \otimes C(S^{\mathbf{1}} \times \{0\}))$ such that $\mu|_{\mathbb{Z}^2}$ is Morita equivalent to $\alpha|_{\mathbb{Z}^2}$. But this implies (see Section 3.1) that α is Morita equivalent to the \mathbb{R}^2 -action $\mu \otimes \text{inf}$ on $(\mathcal{K} \otimes C(S^{\mathbf{1}})) \otimes C(\mathbb{T}^2)$. This means we have a global chart, and so the Mackey obstruction function of $\alpha|_{\mathbb{Z}^2}$ coincides with the obstruction function θ_1 defined by the polarisable NC pair $(\mathcal{K} \otimes C(E), \alpha)$ which has winding number 1. This is a contradiction. \square

Appendix A. Example 3.13

Let $G = \mathbb{R}^2$ and $N = \mathbb{Z}^2$, and choose $\frac{1}{3}, \frac{2}{3} \in \mathbb{R}/\mathbb{Z} = \mathbb{T} \cong H^2(N, U(1))$. Denote by $\hat{\mu}_3$ an action of $\hat{G} = \mathbb{R}^2$ on \mathcal{K} with Mackey obstruction

$$3 \in \mathbb{R} \cong H^2(\mathbb{R}^2, U(1)).$$

Then there is a \hat{G} -equivariant Morita equivalence

$$\left(\mathcal{K} \otimes \mathbb{C} \rtimes_{\frac{2}{3}} N, \hat{\mu}_3 \otimes \text{inf} \right) \sim \left(\mathcal{K} \otimes \mathbb{C} \rtimes_{\frac{1}{3}} N, \text{id} \otimes \text{inf} \right).$$

Proof.

Step 1. The symmetry groups $S \subset N$ of the two classes $\frac{1}{3}, \frac{2}{3}$ agree, i.e., the maps $h_{\frac{1}{3}}, h_{\frac{2}{3}} : N \rightarrow \widehat{N}$ have the same kernel $S = (3\mathbb{Z})^2$. The annihilator of S in $\widehat{G} = \mathbb{R}^2$ is $S^\perp = (\frac{1}{3}\mathbb{Z})^2$, and the annihilator of S in \widehat{N} is $S_N^\perp = (\frac{1}{3}\mathbb{Z}/\mathbb{Z})^2$. We are in the following situation:

$$\begin{array}{ccccc}
 & & N^\perp & & \\
 & & \downarrow & & \\
 & & S^\perp & \hookrightarrow & \widehat{G} \twoheadrightarrow \widehat{G}/S^\perp \\
 & & \downarrow & & \downarrow \\
 \widehat{N}/S & \cong & S^\perp/N^\perp & \cong & S_N^\perp \hookrightarrow \widehat{N} \twoheadrightarrow \widehat{N}/S_N^\perp \\
 & & \downarrow & & \downarrow \\
 & & \widehat{S} & & \widehat{S}
 \end{array}$$

The quotient map $N = \mathbb{Z}^2 \rightarrow (\mathbb{Z}/3\mathbb{Z})^2 = N/S$ induces a map in cohomology, and let us denote by $\frac{1}{3}$ also the pre-image of $\frac{1}{3}$ in

$$\begin{array}{ccc}
 \frac{1}{3} \in \frac{1}{3}\mathbb{Z}/\mathbb{Z} & \xrightarrow{\cong} & H^2(N/S, \mathbb{U}(1)) \\
 \downarrow & \text{inclusion} & \downarrow \text{induced map} \\
 \frac{1}{3} \in \mathbb{T} & \xrightarrow{\cong} & H^2(N, \mathbb{U}(1)).
 \end{array}$$

The action of \widehat{G} on $\widehat{S} \cong \text{Prim}(\mathbb{C} \rtimes_{\frac{1}{3}} N) \cong \text{Prim}(\mathbb{C} \rtimes_{\frac{2}{3}} N)$ has stabiliser S^\perp which acts via the quotient $S^\perp \rightarrow S_N^\perp = \widehat{N}/S$ and the dual actions on $\mathbb{C} \rtimes_{\frac{1}{3}} N/S$ and $\mathbb{C} \rtimes_{\frac{2}{3}} N/S$ on these two algebras. Let us denote these actions by σ and τ , respectively. Then there are \widehat{G} -equivariant isomorphisms

$$\begin{aligned}
 & \left(\mathcal{K} \otimes (\mathbb{C} \rtimes_{\frac{2}{3}} N), \hat{\mu}_3 \otimes \text{inf} \right) \\
 & \cong \left(\mathcal{K} \otimes \text{Ind}_{S^\perp}^{\widehat{G}} (\mathbb{C} \rtimes_{\frac{2}{3}} N/S, \tau), \hat{\mu}_3 \otimes \text{Ind}(\tau) \right) \\
 & \cong \left(\text{Ind}_{S^\perp}^{\widehat{G}} (\mathcal{K} \otimes (\mathbb{C} \rtimes_{\frac{2}{3}} N/S), \hat{\mu}_3|_{S^\perp} \otimes \tau), \text{Ind}(\hat{\mu}_3|_{S^\perp} \otimes \tau) \right)
 \end{aligned}$$

and

$$\begin{aligned}
 & \left(\mathcal{K} \otimes (\mathbb{C} \rtimes_{\frac{1}{3}} N), \text{id} \otimes \text{inf} \right) \\
 & \cong \left(\mathcal{K} \otimes \text{Ind}_{S^\perp}^{\widehat{G}} (\mathbb{C} \rtimes_{\frac{1}{3}} N/S, \sigma), \text{id} \otimes \text{Ind}(\sigma) \right) \\
 & \cong \left(\text{Ind}_{S^\perp}^{\widehat{G}} (\mathcal{K} \otimes (\mathbb{C} \rtimes_{\frac{1}{3}} N/S), \text{id} \otimes \sigma), \text{Ind}(\text{id} \otimes \sigma) \right).
 \end{aligned}$$

Step 2. The product in $\mathbb{C} \rtimes_{\frac{1}{3}} N/S$ is given by

$$(f * f')(n) = \sum_{m \in N/S} f(m)f'(n - m) \exp\left(2\pi i \frac{1}{3} m_2(n_1 - m_1)\right).$$

It is a straight forward computation to see that the mapping which assigns to a function $f : N/S \rightarrow \mathbb{C}$ the matrix \tilde{f} with entries

$$\tilde{f}(a, b) := \sum_{c=0,1,2} f(c, b - a) \exp\left(2\pi i \frac{1}{3} ca\right), \quad a, b = 0, 1, 2,$$

defines an isomorphism $\tilde{\cdot} : \mathbb{C} \rtimes_{\frac{1}{3}} N/S \cong M_3(\mathbb{C}) = \mathcal{K}(L^2(\mathbb{Z}/3\mathbb{Z}))$. The action σ transforms under this isomorphism to a conjugation action

$$\tilde{\sigma} : S^\perp \rightarrow \text{Aut}(\mathcal{K}(L^2(\mathbb{Z}/3\mathbb{Z}))),$$

i.e., $\tilde{\sigma}_s = \text{Ad}(V(s))$, where the unitary $V(s) \in U(L^2(\mathbb{Z}/3\mathbb{Z}))$ is given by

$$(V(s)\psi)(a) = \exp\left(2\pi i \frac{1}{3} c(a + d)\right)^{-1} \psi(a + d),$$

for $s = (\frac{1}{3}c, \frac{1}{3}d) \in S^\perp = (\frac{1}{3}\mathbb{Z})^2, \psi \in L^2(\mathbb{Z}/3\mathbb{Z})$. One can then start calculating

$$(\partial V)(s, s') = V(s')V(s + s')^{-1}V(s) = \exp\left(2\pi i \frac{1}{3} dc'\right) = \exp\left(2\pi i \frac{2}{3} dc'\right)^{-1}.$$

So after identifying $\frac{1}{3} \cdot - : \mathbb{Z}^2 \cong (\frac{1}{3}\mathbb{Z})^2 = S^\perp$, we find the Mackey obstruction of $\tilde{\sigma}$ satisfying

$$\begin{array}{ccc} H^2(S^\perp, U(1)) & \xrightarrow[\left(\frac{1}{3}\cdot-\right)^*]{\cong} & H^2(\mathbb{Z}^2, U(1)) \xrightarrow{\cong} \mathbb{T} \\ \Psi & & \Psi \\ \text{Ma}(\tilde{\sigma}) \vdash & \xrightarrow{\hspace{10em}} & \frac{2}{3}. \end{array}$$

Step 3. The Mackey obstruction of $\hat{\mu}_3|_{S^\perp}$ is given by the cocycle

$$\omega(s, s') = \exp\left(2\pi i \frac{1}{3} s_2 s'_1\right) = \exp\left(2\pi i \frac{1}{3} dc'\right),$$

for $s = (s_1, s_2) = (\frac{1}{3}c, \frac{1}{3}d), s' = (s'_1, s'_2) = (\frac{1}{3}c', \frac{1}{3}d') \in S^\perp$. So here we find

$$\begin{array}{ccc} H^2(S^\perp, U(1)) & \xrightarrow[\left(\frac{1}{3}\cdot-\right)^*]{\cong} & H^2(\mathbb{Z}^2, U(1)) \xrightarrow{\cong} \mathbb{T} \\ \Psi & & \Psi \\ \text{Ma}(\hat{\mu}|_{S^\perp}) \vdash & \xrightarrow{\hspace{10em}} & \frac{1}{3}. \end{array}$$

Step 4. A similar isomorphism to that found in Step 2 can be used to identify $\mathbb{C} \rtimes_{\frac{2}{3}} N/S$ also with $\mathcal{K}(L^2(\mathbb{Z}/3\mathbb{Z}))$. In fact, the product in $\mathbb{C} \rtimes_{\frac{2}{3}} N/S$ agrees with the product in $\mathbb{C} \rtimes_{\frac{1}{3}} N/S$ up to a sign:

$$\begin{aligned} (f * f')(n) &= \sum_{m \in N/S} f(m)f'(n - m) \exp\left(2\pi i \frac{2}{3} m_2(n_1 - m_1)\right) \\ &= \sum_{m \in N/S} f(m)f'(n - m) \exp\left(2\pi i \frac{1}{3} m_2(n_1 - m_1)\right)^{-1}. \end{aligned}$$

Following this sign in the construction made in Step 2 one finally finds that the Mackey obstruction of τ is exactly the inverse of σ :

$$\begin{array}{ccc} H^2(S^\perp, U(1)) & \xrightarrow[\left(\frac{1}{3} \cdot \right)^*]{\cong} & H^2(\mathbb{Z}^2, U(1)) \xrightarrow{\cong} \mathbb{T} \\ \Downarrow & & \Downarrow \\ \text{Ma}(\tau) & \longmapsto & \frac{1}{3}. \end{array}$$

Step 5. Summing up all the Mackey obstructions, we see that the two C^* -dynamical systems

$$(\mathcal{K} \otimes \mathbb{C} \rtimes_{\frac{2}{3}} N/S, S^\perp, \hat{\mu}_3|_{S^\perp} \otimes \tau) \text{ and } (\mathcal{K} \otimes \mathbb{C} \rtimes_{\frac{1}{3}} N/S, S^\perp, \text{id} \otimes \sigma)$$

are equivalent. This implies that also their induced systems are equivalent which proves the claim. \square

Appendix B. Lemma 5.2

The canonical isomorphism $H^2(G, L^\infty(G/N, U(1))) \rightarrow H^2(N, U(1))$ of ([Mo76, Thm. 6]) makes the diagram

$$(37) \quad \begin{array}{ccc} H^2(G, U(1)) & & \\ \downarrow & \searrow & \\ H^2(N, U(1)) & \xleftarrow{\cong} & H^2(G, L^\infty(G/N, U(1))) \end{array}$$

commute, where the vertical map is restriction and the diagonal map is induced by the inclusion of coefficients.

Proof. Let us introduce some notation used in ([Mo76]). Let

$$I(X) := \{N \rightarrow X\}$$

for any X . It has the structure of an N -module by left translation:

$$n \cdot f := f(- - n) \in I(N).$$

Let us denote by A the quotient of $I(U(1))$ by the constants $U(1)$, i.e., we have an N -equivariant short exact sequence

$$(38) \quad 1 \rightarrow U(1) \rightarrow I(U(1)) \rightarrow A \rightarrow 1,$$

where $U(1)$ has the trivial N -structure and A has the quotient structure. We obtain an N -equivariant embedding $i : A \rightarrow I(A)$ by $(ia)(n) := (-n) \cdot a$, for $a \in A, n \in N$. The quotient of $I(A)$ by $i(A)$ is denoted by $U(A)$, i.e., we have another N -equivariant sequence

$$1 \rightarrow A \rightarrow I(A) \rightarrow U(A) \rightarrow 1.$$

By definition (cp. the axioms of group cohomology as a derived functor ([Mo76, Sec. 4])) we have an exact sequence

$$1 \rightarrow A^N \rightarrow I(A)^N \rightarrow U(A)^N \rightarrow H^1(N, A) \rightarrow 0,$$

where the exponent denotes taking invariants. Let $I_N^G(Y)$ denote the induced G -module for any Y -module, i.e., equivalence classes of functions $f : G \rightarrow Y$, such that for all $n \in N$ ($f(g - n) = n \cdot (f(g))$) holds for almost all $g \in G$. Two functions are identified if they agree almost everywhere. $I_N^G(Y)$ is a G -module by left translation. Note that $I_N^G(U(1)) = L^\infty(G/N, U(1))$ as G -modules. The functor $I_N^G(-)$ from N -modules to G -modules is exact ([Mo76, Proposition 19]) and again by the axioms and ([Mo76, Proposition 18]) we have an exact sequence

$$1 \rightarrow I_N^G(A)^G \rightarrow I_N^G(I(A))^G \rightarrow I_N^G(U(A))^G \rightarrow H^1(G, I_N^G(A)) \rightarrow 0.$$

The canonical isomorphism $H^2(N, U(1)) \rightarrow H^2(G, I_N^G(U(1)))$ is defined by the diagram:

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \\
 & & A^N & \xrightarrow{\cong} & I_N^G(A)^G & & \\
 & & \downarrow & & \downarrow & & \\
 & & I(A)^N & \xrightarrow{\cong} & I_N^G(I(A))^G & & \\
 & & \downarrow & & \downarrow & & \\
 & & U(A)^N & \xrightarrow{\cong} & I_N^G(U(A))^G & & \\
 & & \downarrow & & \downarrow & & \\
 H^2(N, U(1)) & \xleftarrow{\cong} & H^1(N, A) & \xrightarrow{\dots \cong} & H^1(G, I_N^G(A)) & \xrightarrow{\cong} & H^2(G, I_N^G(U(1))) \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Here the top three isomorphisms are given by sending an invariant element ν to the constant function $\bar{\nu} : g \mapsto \nu$ (cp. [Mo76, Proposition 19]). The dotted isomorphism is induced by the one above and the two isomorphisms to the left and right are the connecting morphisms in the long exact sequence induced by (38).

Let us describe the dotted arrow. To do so, start with some element $x \in H^1(N, A)$ and choose some $\nu \in U(A)^N$ which maps to x . Following the isomorphism to the right we get $\bar{\nu} \in I_N^G(U(A))^G \subset I_N^G(U(A))$. Let us denote by $\tilde{\nu} \in I_N^G(I(A))$ an element that maps to $\bar{\nu}$. Then the image y of x under the dotted arrow is (the class of) $d_G \tilde{\nu} : G \rightarrow I_N^G(A)$ defined by

$$G \ni g \rightarrow \tilde{\nu}(- - g) \tilde{\nu}(-)^{-1} \in I_N^G(i(A)) \cong I_N^G(A).$$

Let us now describe the element x in terms of $\tilde{\nu}$: We consider for each $n \in N$ a co-null set $S_n \subset G$ such that $\tilde{\nu}(g - n) = n \cdot \tilde{\nu}(g)$ holds for all $g \in S_n$. Then let $S^1 := \bigcap_{n \in N} S_n$. So $\tilde{\nu}(g - n) = n \cdot \tilde{\nu}(g)$ holds for all $n \in N$ and all $g \in S^1$. Moreover, there is another co-null set $S^2 \subset G$ such that $\tilde{\nu}(g) \bmod A = \nu$ for all $g \in S^2$. Let $s \in S^1 \cap S^2$, then $\tilde{\nu}(s) \in I(A)$ is a lift of $\nu \in U(A)^N$, and x is represented by the class of $d_N(\tilde{\nu}(s)) : N \rightarrow A$ defined by

$$N \ni n \rightarrow n \cdot (\tilde{\nu}(s)) \tilde{\nu}(s)^{-1} = \tilde{\nu}(s - n) \tilde{\nu}(s)^{-1} \in i(A) \cong A.$$

Note that that the operation restriction to N and evaluation at s transforms $d_G \tilde{\nu}$ into $d_N(\tilde{\nu}(s))$: $(d_G \tilde{\nu})(n)(s) = d_N(\tilde{\nu}(s))(n)$.

Let $\sigma : A \rightarrow I(U(1))$ be a Borel section of the quotient map. Then the element $[\omega_{N,\nu,s}]$ in $H^2(N, A)$ corresponding to $x = [d_n(\tilde{\nu}(s))]$ is given by

$$\omega_{N,\tilde{\nu},s}(n, m) = \sigma(d_N(\tilde{\nu}(s))(m)) \sigma(d_N(\tilde{\nu}(s))(n + m))^{-1} \sigma(d_N(\tilde{\nu}(s))(n)).$$

The element $[\omega_{G,\tilde{\nu}}]$ in $H^2(G, I_N^G(U(1)))$ corresponding to $y = [d_G \tilde{\nu}]$ is given by

$$\omega_{G,\tilde{\nu}}(g, h) = \sigma_*((d_G \tilde{\nu})(h)(- - g)) \sigma_*((d_G \tilde{\nu})(g + h)(-))^{-1} \sigma_*((d_G \tilde{\nu})(g)(-))$$

where $\sigma_* : I_N^G(A) \rightarrow I_N^G(I(U(1)))$ is just composition with σ . Note again that that the operation restriction to N and evaluation at s transforms $\omega_{G,\tilde{\nu}}$ into $\omega_{N,\tilde{\nu},s}$: $\omega_{G,\tilde{\nu}}(n, m)(s) = \omega_{N,\tilde{\nu},s}(n, m)$.

Now let us turn to the commutativity of (37). Let $[\omega] \in H^2(G, U(1))$, and consider its image y (also given by ω) in $H^2(G, I_N^G(U(1)))$. We can represent y as above by finding some $\nu, \tilde{\nu}$ such that for all g, h

$$\omega(g, h) (d_G c)(g, h) = \omega_{G,\tilde{\nu}}(g, h)$$

holds in $I_N^G(U(1))$, for some cochain $c : G \rightarrow I_N^G(U(1))$. So there is another co-null set S^3 such that

$$\omega(n, m) (dc)(n, m)(s) = \omega_{G,\tilde{\nu}}(n, m)(s)$$

holds for all $n, m \in N$ and all $s \in S^3$. If we choose $s_0 \in S^1 \cap S^2 \cap S^3$, then by the above construction we have that the image of $[\omega]$ along the composition in diagram (37) is given by the cocycle

$$\begin{aligned} \omega_{N,\tilde{\nu},s_0}(n, m) &= \omega_{G,\tilde{\nu}}(n, m)(s_0) \\ &= \omega(n, m) c(m)(s_0) c(n + m)(s_0)^{-1} c(n)(s_0). \end{aligned}$$

It follows that $\omega_{N,\tilde{\nu},s_0} \sim \omega|_{N \times N}$ by the cochain $n \mapsto c(n)(s_0)$. This proves the lemma. □

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