

# $\infty$ -categorical monadicity and descent

Yuri J. F. Sulyma

ABSTRACT. Riehl and Verity have introduced an “ $\infty$ -cosmic” framework in which they redevelop the category theory of  $\infty$ -categories using 2-categorical arguments. In this paper, we begin with a self-contained review of the parts of their theory needed to discuss adjunctions and monadicity. This is applied in order to extend to the  $\infty$ -categorical context the classical criterion for fully faithfulness of the comparison functor induced by an adjunction. We discuss the relation with previous work in the literature—which primarily uses model-categorical techniques—and indicate applications to descent theory.

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## 1. Introduction

Descent theory plays an important role in algebraic geometry, as well as in the plethora of fields which draw upon its technology. Motivated by the problem of assembling local data into global data, it is profitably reinterpreted in terms of co/monads. For example, if  $E = \bigcup U_i \rightarrow B$  is a cover of a topological space  $B$ , and  $F \rightarrow B$  is a presheaf on  $B$ , then  $E \times_B F$  consists of the values of  $F$  on the open cover  $\{U_i\}$ ,  $E \times_B E \times_B F$  consists of the values of  $F$  on intersections  $\{U_i \cap U_j\}$ , and so on. The condition for  $F$  to be a *sheaf* is evidently equivalent to demanding an equivalence

$$F \xrightarrow{\sim} (E \times_B F \rightrightarrows E \times_B E \times_B F \rightrightarrows \cdots)$$

of  $F$  with its simplicial resolution given by the comonad  $E \times_B (-)$ . We refer to [H, §2] for a review of this formalism (and some examples) in the classical setting.

**Example 1.1** ([GAGA]). Let  $X$  be a complex algebraic variety. We can consider  $X$  in the analytic topology  $X^{\text{an}}$  with the sheaf  $\mathcal{H}$  of holomorphic functions, or in the Zariski topology<sup>1</sup>  $X^{\text{Zar}}$  with the sheaf  $\mathcal{O}$  of regular functions. It is not too difficult to show that the forgetful function

$$(1) \quad (X^{\text{an}}, \mathcal{H}) \rightarrow (X^{\text{Zar}}, \mathcal{O})$$

is a faithfully flat map of locally ringed spaces, which is a good notion of “cover”.

The map (1) induces a functor

$$\text{Coh}(\mathcal{O}) \rightarrow \text{Coh}(\mathcal{H})$$

between categories of coherent sheaves of modules. The main theorem of [GAGA] is that this is an equivalence of categories when  $X$  is projective. We can interpret this as saying that coherent sheaves descend along the cover (1) when  $X$  is projective. This is *false* for general  $X$ , even for  $X$  affine.

Descent theory can be formulated using only elementary category theory, and so it is easy to *ask* descent questions. The preceding example shows that *answering* descent questions can involve deep mathematics. It is thus desirable to have very general theorems on when descent holds, which in particular applications may be further simplified to explicit, easily-checkable criteria. The general formalism involves a “comparison functor”  $k$ , and the two basic theorems of general monadic descent theory concern when this functor is fully faithful or an equivalence of categories (we say that *descent* is satisfied in the first case and *effective descent* in the second).

So far, all this is classical. The rise of derived algebraic geometry and derived stacks has contributed to growing consumer demand for higher descent

<sup>1</sup>Here we mean the classical Zariski topology, with no schemey generic points.

theory. Following the classical case, this should be formulated in terms of  $\infty$ -monads. We indicate the challenges in doing so, then explain our approach to surmounting them.

Mathematical theories frequently admit both an *extrinsic* (“choosing coordinates”) as well an *intrinsic* (“coordinate-free”) approach. Typically, the extrinsic approach is useful for carrying out calculations but awkward for developing general theory, while the reverse is true of the intrinsic approach. In abstract homotopy theory/higher category theory<sup>2</sup>, the “extrinsic” approach is to “model” an  $\infty$ -category via an ordinary category equipped with additional data specifying the homotopical structure (ideally a simplicial model category). One can then work with the familiar strict morphisms, co/limits, . . . , as long as one makes homotopical corrections along the way (co/fibrant replacements, deriving functors, . . . ). In contrast, the “intrinsic” approach is to work in a environment where everything is “fully derived”; as we shall see, an  $\infty$ -*cosmos* is an extremely robust such environment.

The strategy of working strictly and making homotopical adjustments along the way is extremely effective for a great deal of  $\infty$ -categorical work (as evidenced by the ubiquity of model categories in the literature). It becomes problematic when working with  $\infty$ -monads: the equations defining a point-set monad will rarely continue to hold after we make homotopical corrections, thus destroying the strictness which is the point of model categories in the first place. This is compounded when we take iterated composites of a monad. Obviously, this presents a problem for higher descent theory. In particular, while papers such as [H] and [AC] have had some success in treating  $\infty$ -monads and higher descent model-categorically, they must demand fairly stringent hypotheses on the model categories and/or monads involved in order to do so. Although Blumberg–Riehl were able to remove these hypotheses in [BIR], using the theory of algebraic model categories, control over the  $\infty$ -category of algebras remained elusive. In view of the preceding discussion, it is natural to move to a fully derived environment in order to treat the foundations of higher descent.

At present, the most comprehensive such environment is that of quasicategories, developed by Joyal and Lurie. Lurie has indeed proven a Barr–Beck theorem in this context [HA, 4.7.4.5]. Subsequently, Riehl and Verity gave a new proof [RV2, 7.2.7], working in the more general context of  $\infty$ -*cosmoi*. However, the Barr–Beck theorem only addresses the question of when the monadic comparison functor is an equivalence. As mentioned above, it is also important to know when it is merely fully faithful. The purpose of this paper is to establish this criterion in the  $\infty$ -categorical setting. We shall deploy the Riehl–Verity framework in order to prove:

**Theorem.** *Let  $X \begin{array}{c} \xrightarrow{f} \\ \lrcorner \perp \rrightarrow \\ \xleftarrow{u} \end{array} A$  be a homotopy coherent adjunction between  $\infty$ -categories, inducing a homotopy coherent monad  $t = uf$  on  $X$  and homotopy coherent comonad  $g = fu$  on  $A$ . Then the comparison functor  $A \xrightarrow{k} X^t$  to*

<sup>2</sup>Opinion is divided on whether or not these terms are synonymous.

*the  $\infty$ -category of homotopy coherent  $t$ -algebras is fully faithful if and only if every object of  $A$  is  $g$ -cocomplete, i.e., weakly equivalent to the geometric realization of the simplicial resolution given by  $g$ .*

We now extol the virtues of the Riehl–Verity framework. Classically, it has proven fruitful to develop category theory by working in a nice (behaving like **Cat**) 2-category. Thus one trades explicit constructions for 2-universal properties. The advantage of this method is that it applies simultaneously to develop the theory of more general categorical structures, such as enriched, internal, or parametrized categories. This approach is often referred to as *formal category theory*, e.g., in [Gr]; one could succinctly describe the Riehl–Verity project as *formal  $\infty$ -category theory*, and an  $\infty$ -cosmos as an  $(\infty, 2)$ -category behaving like that of  $(\infty, 1)$ -categories (or even  $(\infty, n)$ -categories).

One thus expects to characterize  $\infty$ -categorical constructions via  $(\infty, 2)$ - (modelled as simplicially enriched) universal properties. But remarkably, the majority of the theory takes place in the *homotopy 2-category*, and so these universal properties are close or identical to those we’d find in the classical case. Sufficiently slick classical proofs<sup>3</sup> can thus be transported nearly word-for-word into the  $\infty$ -categorical context. Indeed, once we get the definitions out of the way, the reader will note we make scarce explicit reference in §3 to the definitions of  $\infty$ -cosmoi.

We now turn to the outline of the paper.

In §2 we review the definitional framework and results of Riehl–Verity that we need; this section is expository and discursive, and only sketches of proofs are to be found therein. Readers familiar with their work may skip to §3, which begins with a notational review for the convenience of those who do so. Our results are contained in §3; we explain how to interpret Theorem 3.14 in an  $\infty$ -cosmic environment, and prove it. We then indicate some applications to descent problems, including descent spectral sequences.

Finally, we state our position on the most controversial question in the whole of  $\infty$ -cosmology: how to spell the plural of  $\infty$ -cosmos. The reader will already have observed that we adhere to the convention of the pioneering  $\infty$ -cosmologists. We have nothing further to say on the matter, except to affirm that, when we go out for a ramble on a cold day, we do indeed carry supplies of hot coffee with us in *thermoi* [J].

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<sup>3</sup>The task of converting a down-to-earth classical proof into a sophisticated (2-categorical) one—suitable for interpretation in an  $\infty$ -cosmos—is not necessarily a trivial one.

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## 2. Background

Here we review the necessary parts of the prior work of Riehl and Verity. Full details are available in [RV1], [RV2], and [RV4]; we recommend [RV0] for a rapid overview. In §2.1 we introduce the fundamental notions of an  $\infty$ -cosmos and its homotopy 2-category; this is the setting in which the rest of our work takes place. In §2.2 we define homotopy coherent/commutative adjunctions and monads, and recall the descriptions of the categories **Adj** and **Mnd** which corepresent these. Comma  $\infty$ -categories, which are key to the “model independence” of Riehl–Verity’s results, are reviewed in §2.3. Limits and colimits inside  $\infty$ -categories are discussed in §2.4. In §2.5, we review the enriched-categorical notion of *weighted limits* and discuss their use in the  $\infty$ -cosmic context, which is simplicially enriched. Finally, §2.6 shows how to construct the various  $\infty$ -categories and functors relevant to discussions of monadicity and descent.

**2.1.  $\infty$ -cosmoi.** Informally speaking, an  $\infty$ -cosmos is a presentation of an  $(\infty, 2)$ -category which is sufficiently well-behaved for us do “formal  $\infty$ -category theory” (à la [Gr]) inside it. (The name is meant to evoke Street, not Bénabou, cosmoi.) The definition is reminiscent of the properties enjoyed by fibrant objects in any model category enriched (cf. [HTT, §A.3.2]) over the Joyal model structure on **sSet**, and indeed these are examples [RV4, 2.2.1]. The reference for this section is [RV4, §2].

**Definition 2.1** ( $\infty$ -cosmos). Let  $\mathcal{E}$  be a simplicially enriched category, equipped with two distinguished classes of 1-cells: the *equivalences*, denoted  $\xrightarrow{\sim}$ , and the *isofibrations*, denoted  $\rightarrow$ . For psychological reasons, we refer to the objects of  $\mathcal{E}$  as  *$\infty$ -categories* and its arrows as *functors*. A functor which is both an equivalence and an isofibration will be called an *acyclic fibration* and denoted  $\xrightarrow{\sim}$ . We assume that equivalences satisfy the 2-of-6 property, that isofibrations are closed under composition, and that all isomorphisms are acyclic fibrations.

We shall say that  $\mathcal{E}$  is an  *$\infty$ -cosmos* if it satisfies the following axioms:

- (1) (Completeness). As a simplicially enriched category,  $\mathcal{E}$  possesses a terminal object 1, cotensors  $E^J$  of objects  $E$  by all simplicial sets  $J$ , and pullbacks of isofibrations along any functor.
- (2) (Fibrancy). All of the maps  $E \rightarrow 1$  are isofibrations.
- (3) (Pullback stability). Isofibrations and acyclic fibrations are stable under pullback along any functor.

- (4) (SM7). If  $E \xrightarrow{p} B$  is an isofibration in  $\mathcal{E}$  and  $I \xrightarrow{i} J$  is an inclusion of simplicial sets then the Leibniz cotensor  $i \hat{\pitchfork} p: E^J \rightarrow E^I \times_{B^I} B^J$  is an isofibration, and further an acyclic fibration whenever  $p$  is an acyclic fibration in  $\mathcal{E}$  or  $i$  is an acyclic cofibration in  $\mathbf{sSet}_{\text{Joyal}}$ .
- (5) (Cofibrancy). All objects enjoy the left lifting property with respect to all acyclic fibrations in  $\mathcal{E}$ .

We will also require an  $\infty$ -cosmos to have limits of transfinite towers of isofibrations, and for isofibrations to be stable by retracts. We write  $\underline{\text{map}}(A, B) \in \mathbf{sSet}$  for the mapping space between two objects  $A, B$  of  $\mathcal{E}$ .

**Remark 2.2.** The axioms can be made stronger or weaker, depending on what one wants to prove. More fundamental is the *style* of arguing about  $\infty$ -categories: one can imagine working with quasicategories throughout, and the axioms record those properties of quasicategories we use (which turn out to be satisfied much more generally). For example, in [RV4] the axioms only require cotensors by simplicial sets with finitely many nondegenerate simplices; and the ability to take limits of transfinite towers of isofibrations is absent altogether. Our “infinitary” assumptions are necessary for the constructions in §2.6 and §3.2.

**Remark 2.3.** Our assumption that all objects are cofibrant has the crucial consequence that the mapping spaces  $\underline{\text{map}}(A, B)$  are actually *quasicategories*, and that  $\underline{\text{map}}(A, -)$  takes isofibrations (resp. acyclic fibrations) in  $\mathcal{E}$  to isofibrations (resp. acyclic fibrations) of quasicategories [RV4, 2.1.8]. We refer to [RV1, §2.2] for a review of quasicategories. One can get by by merely assuming that every object of  $\mathcal{E}$  has a cofibrant replacement (as in [RV4], for example); in this case one should speak of *weak equivalences* of  $\mathcal{E}$  rather than equivalences. We have chosen to assume all objects cofibrant in order to simplify the exposition.

**Definition 2.4.** If  $A$  is an  $\infty$ -category in an  $\infty$ -cosmos  $\mathcal{E}$ , the *underlying quasicategory of  $A$*  is  $\underline{\text{map}}_{\mathcal{E}}(1, A)$ . We define *objects* and *maps* in abstract  $\infty$ -categories in terms of their underlying quasicategories.

**Example 2.5.** In [RV4, §2.2], Riehl and Verity present several ways to easily produce examples of  $\infty$ -cosmoi. Chief among these examples are:

- **Cat**, the  $\infty$ -cosmos of ordinary categories. Equivalences are equivalences of categories, and isofibrations are functors with the right lifting property with respect to  $\{\bullet\} \hookrightarrow \{\bullet \cong \star\}$ .
- **qCat**, the  $\infty$ -cosmos of quasicategories. Equivalences and isofibrations are as usual.
- The  $\infty$ -cosmos of  $\theta_n$ -spaces, a model of  $(\infty, n)$ -categories.

Simplicial categories do *not* form an  $\infty$ -cosmos: although there is a Quillen adjunction

$$\mathbf{sSet}_{\text{Joyal}} \begin{array}{c} \xrightarrow{\mathfrak{C}} \\ \perp \\ \xleftarrow{N} \end{array} \mathbf{sSetCat},$$

the Bergner model structure is not cartesian closed, and so [RV4, 2.2.3] is inapplicable.

**Example 2.6** ([RV4, 2.1.11]). If  $A$  is an  $\infty$ -category in an  $\infty$ -cosmos  $\mathcal{E}$ , we let  $\mathcal{E}/A$  denote the full simplicial subcategory of the usual simplicial slice category spanned by the *isofibrations*  $B \rightarrow A$ . This is again an  $\infty$ -cosmos, called the *slice  $\infty$ -cosmos* over  $A$ . Thus the  $\infty$ -cosmic framework captures parametrized  $\infty$ -category theory.

With the  $\infty$ -cosmic framework in hand, Riehl and Verity are able to re-derive a great deal of the theory of  $\infty$ -categories. Their proofs are “formal” in nature—in contrast to the combinatorial arguments of [HTT]—and thus permit arguments very close to the classical case. Moreover, as the above examples indicate, their work is not limited to developing the category theory of  $(\infty, 1)$ -categories: it simultaneously applies to develop the category theory of  $(\infty, n)$ -categories and recapture that of ordinary categories.

However, the import of their work is not merely that  $\infty$ -cosmoi provide a robust environment in which to develop the category theory of  $\infty$ -categories. They also show (somewhat unexpectedly) that a much simpler structure suffices for much of this development.

**Definition 2.7.** The *homotopy 2-category* of an  $\infty$ -cosmos  $\mathcal{E}$  is the (strict) 2-category  $\mathbf{Ho}(\mathcal{E})$  with the same underlying category as  $\mathcal{E}$ , but with hom-categories  $\underline{\text{hom}}(E, F)$  given by

$$\underline{\text{hom}}(E, F) := h(\underline{\text{map}}(E, F))$$

for  $E, F \in \mathcal{E}$ . Here  $h$  sends a quasicategory (or simplicial set) to its homotopy category.

**Remark 2.8.** When we drop the assumption that all objects in  $\mathcal{E}$  are cofibrant, the 2-category just defined is notated  $h_*\mathcal{E}$ , and the correct definition of  $\mathbf{Ho}(\mathcal{E})$  is the full subcategory of  $h_*\mathcal{E}$  spanned by the (images of) cofibrant objects of  $\mathcal{E}$ .

Recall that a 1-cell  $A \xrightarrow{f} B$  in a 2-category  $\mathcal{C}$  is an *equivalence* if there is a 1-cell  $B \xrightarrow{g} A$  and isomorphic 2-cells  $1_A \xrightarrow{\sim} gf$  and  $fg \xrightarrow{\sim} 1_B$ . The following proposition is one of the first indications that  $\mathbf{Ho}(\mathcal{E})$  remembers enough information about  $\mathcal{E}$  to develop the category theory of its objects.

**Proposition 2.9** ([RV4, 3.1.8]). *A functor  $A \rightarrow B$  is an equivalence in the  $\infty$ -cosmos  $\mathcal{E}$  if and only if it is an equivalence in the homotopy 2-category  $\mathbf{Ho}(\mathcal{E})$ .*

For this reason, we will sometimes write  $A = B$  to mean that there exists an equivalence  $A \xrightarrow{\sim} B$  in  $\mathcal{E}$  (or, if there is an obvious map  $A \rightarrow B$  in play, that this particular map is an equivalence).

**2.2. Homotopy coherent adjunctions.** The reference for this section is [RV2, §3].

**Definition 2.10.** Let  $\mathcal{C}$  be a 2-category. An *adjunction* in  $\mathcal{C}$  consists of

- a pair of objects  $X, A$  of  $\mathcal{C}$ ;
- maps  $X \xrightarrow{f} A$  and  $A \xrightarrow{u} X$ ;
- two-cells  $1_X \xrightarrow{\eta} uf$  and  $fu \xrightarrow{\epsilon} 1_A$ ;
- satisfying the triangle identities  $\epsilon f \cdot f\eta = 1_f$  and  $u\epsilon \cdot \eta u = 1_u$ .

We call  $f$  the *left adjoint*,  $u$  the *right adjoint*,  $\eta$  the *unit*, and  $\epsilon$  the *counit* of the adjunction. We indicate an adjunction by writing  $f \dashv u$ ,  $X \xleftarrow[u]{f} A$  or  $f: X \xrightarrow[u]{\dashv} A : u$ .

**Definition 2.11.** Let  $\mathcal{C}$  be a 2-category. A *monad* in  $\mathcal{C}$  consists of an object  $X \in \mathcal{C}$  and a monoid  $t$  in the monoidal category  $\underline{\text{hom}}(X, X)$ .

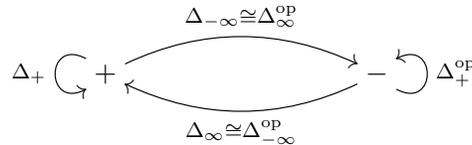
When  $\mathcal{C} = \mathbf{Cat}$ , these specialize to the usual notions. Since these notions are equationally defined, they are corepresentable, i.e., there is a 2-category  $\mathbf{Adj}$  (resp.  $\mathbf{Mnd}$ ) such that adjunctions (resp. monads) in  $\mathcal{C}$  are the same thing as 2-functors  $\mathbf{Adj} \rightarrow \mathcal{C}$  (resp.  $\mathbf{Mnd} \rightarrow \mathcal{C}$ ). The explicit description of  $\mathbf{Adj}$  is due to Schanuel and Street [SS], of  $\mathbf{Mnd}$  to Lawvere [L]. Before giving the definition, we set some notation.

**Definition 2.12.** As usual,  $\Delta_+$  and  $\Delta$  will denote the category of finite linearly ordered sets and the full subcategory of nonempty sets. We shall use the notation  $\Delta_\infty$  (respectively  $\Delta_{-\infty}$ ) to denote the subcategory of  $\Delta$  consisting of those maps which preserve top (respectively bottom) elements.

**Definition 2.13.** The free adjunction is the small 2-category  $\mathbf{Adj}$  with two objects  $+$  and  $-$ , with hom-categories given by

$$\begin{aligned} \mathbf{Adj}(+, +) &= \Delta_+ & \mathbf{Adj}(-, -) &= \Delta_+^{\text{op}} \\ \mathbf{Adj}(-, +) &= \Delta_\infty \cong \Delta_{-\infty}^{\text{op}} & \mathbf{Adj}(+, -) &= \Delta_{-\infty} \cong \Delta_\infty^{\text{op}} \end{aligned}$$

as summarized in the following picture:



We write  $+\xrightarrow{f}-$  (resp.  $-\xrightarrow{u}+$ ) for the map corresponding to  $[0] \in \Delta_{-\infty}$  (resp. to  $[0] \in \Delta_\infty$ ).

**Definition 2.14.** The free monad is the small 2-category **Mnd** which is the full subcategory of **Adj** on the object  $+$ . We write  $\underline{t}$  for the endomorphism corresponding to  $[0] \in \Delta_+$ .

**Definition 2.15.** Any 2-category gives rise to a simplicially enriched (in fact, quasicategorically enriched) category by identifying the hom-categories with their nerves (this uses the fact that the nerve preserves products). This process is the right adjoint  $N_*$  in a 2-adjunction

$$(\text{simplicial categories}) \begin{array}{c} \xrightarrow{h_*} \\ \perp \\ \xleftarrow{N_*} \end{array} (2\text{-categories})$$

arising from the Quillen adjunction  $\mathbf{sSet}_{\text{Joyal}} \begin{array}{c} \xrightarrow{h} \\ \perp \\ \xleftarrow{N} \end{array} \mathbf{Cat}$ ; we have already made use of  $h_*$  in defining the homotopy 2-category  $\mathbf{Ho}(\mathcal{E})$  of an  $\infty$ -cosmos  $\mathcal{E}$ .

Applying this to the 2-categories **Adj** and **Mnd**, we obtain simplicially enriched categories which we continue to notate **Adj** and **Mnd**. Since  $N$  is fully faithful, this conflation is anodyne.

**Remark 2.16.** The calculus of string diagrams for 2-categories extends naturally to describe the  $n$ -arrows of simplicial categories which arise in this way. Riehl and Verity show in [RV2] that when specialized to **Adj**, this graphical calculus admits a variation—the calculus of “strictly undulating squiggles”—enabling a simple combinatorial description of the  $n$ -arrows of **Adj** which behaves well with respect to both vertical and horizontal composition. Strikingly, they use this to show that **Adj** is cofibrant in the Bergner model structure on simplicial categories [Be], and to work with explicit cellular presentations of **Adj**.

**Notation 2.17.** The symbol  $-$  is often used as a placeholder symbol in category theory. To avoid confusion with the object  $-$  of **Adj**, we will use  $\blacksquare$  instead. Thus  $\mathbf{Adj}(\blacksquare, +)$  is a functor  $\mathbf{Adj}^{\text{op}} \rightarrow \mathbf{sSet}$ , but  $\mathbf{Adj}(-, +)$  is an object of  $\mathbf{sSet}$ .

**Definition 2.18.** Let  $\mathcal{E}$  be an  $\infty$ -cosmos with homotopy 2-category  $\mathbf{Ho}(\mathcal{E})$ .

- A *homotopy coherent adjunction*, or  $\infty$ -adjunction, in  $\mathcal{E}$  is a simplicial functor  $\mathbf{Adj} \rightarrow \mathcal{E}$ .
- A *homotopy commutative adjunction*, or 1-adjunction, in  $\mathcal{E}$  is a 2-functor  $\mathbf{Adj} \rightarrow \mathbf{Ho}(\mathcal{E})$ .
- A *homotopy coherent monad*, or  $\infty$ -monad, in  $\mathcal{E}$  is a simplicial functor  $\mathbf{Mnd} \rightarrow \mathcal{E}$ .
- A *homotopy commutative monad*, or 1-monad, in  $\mathcal{E}$  is a 2-functor  $\mathbf{Adj} \rightarrow \mathbf{Ho}(\mathcal{E})$ .

**Warning 2.19.** When an  $\infty$ -category  $X$  is presented by a 1-category (e.g., a simplicial model category)  $\mathcal{X}$ , 1-monads on  $X$  as defined above must not be confused with “point-set” monads on  $\mathcal{X}$ . The former are monads on

$\mathbf{Ho}(\mathcal{X})$ ; the latter sometimes induce  $\infty$ -monads on  $X$ , but we shall make no pre-derived use of them. It does not appear to be possible to give a simple definition of  $\infty$ -monads on  $X$  purely in terms of  $\mathcal{X}$  unless  $\mathcal{X}$  is very special, e.g., a simplicial model category in which everything is bifibrant.

**Parallel 2.20.** Let  $\mathcal{X} \begin{array}{c} \xrightarrow{F} \\ \leftarrow \perp \\ \xrightarrow{U} \end{array} \mathcal{A}$  be a simplicial Quillen adjunction between simplicial model categories  $\mathcal{X}$  and  $\mathcal{A}$ .  $\mathcal{X}$  and  $\mathcal{A}$  model  $(\infty, 1)$ -categories  $X$  and  $A$ . For example, to obtain quasicategorical models, we would take homotopy coherent nerves of the subcategories of bifibrant objects:

$$X := N(\mathcal{X}_{\text{cf}}) \quad \text{and} \quad A := N(\mathcal{A}_{\text{cf}}).$$

By [RV1, 6.2.1], there is an induced  $\infty$ -adjunction  $X \begin{array}{c} \xrightarrow{f} \\ \leftarrow \perp \\ \xrightarrow{u} \end{array} A$ . These functors are obtained by correcting  $F$  and  $U$  to land in bifibrant objects. For example, if every object of  $\mathcal{X}$  is cofibrant and every object of  $\mathcal{A}$  is fibrant, then no correction is needed.

Work of Dugger, Rezk, Schwede, and Shipley shows that a Quillen adjunction between left proper combinatorial model categories is functorially equivalent to a simplicial Quillen adjunction as above; see [BIR, §A] for discussion of this. Thus we again get an induced  $\infty$ -adjunction between  $\infty$ -categories.

Even more generally, Mazel-Gee [MG] has shown that an arbitrary Quillen adjunction between model categories induces an  $\infty$ -adjunction between  $\infty$ -categories. In this case  $X$  is modelled by the homotopy coherent nerve of a Bergner-fibrant replacement of the Dwyer–Kan hammock localization of  $\mathcal{X}$ . However, at this level of generality we cannot necessarily obtain the Blumberg–Riehl resolutions (cf. Parallel 3.1) which are crucial for maintaining control in  $\mathcal{X}$  over the constructions we do with  $X$ .

**Parallel 2.21.** Let  $X$  be an  $(\infty, 1)$ -category modelled by a simplicial model category  $\mathcal{X}$ , and let  $\mathbb{T}$  be a simplicial monad on  $\mathcal{X}$ . Under reasonable conditions, the category  $\mathcal{X}^{\mathbb{T}}$  of  $\mathbb{T}$ -algebras is a simplicial model category in such a way that the monadic adjunction  $F^{\mathbb{T}}: \mathcal{X} \begin{array}{c} \xrightarrow{\quad} \\ \leftarrow \perp \\ \xrightarrow{\quad} \end{array} \mathcal{X}^{\mathbb{T}} : U^{\mathbb{T}}$  is simplicial Quillen [H, §C]. We thus obtain an  $\infty$ -adjunction out of  $X$ , and hence an  $\infty$ -monad  $t$  on  $X$ .

Theorems 4.3.9, 4.3.11, 4.4.11, and 4.4.18 of [RV2] show that every 1-adjunction  $\mathbf{Adj} \rightarrow \mathbf{Ho}(\mathcal{E})$  lifts to an  $\infty$ -adjunction  $\mathbf{Adj} \rightarrow \mathcal{E}$ , and moreover that such lifts are unique in a suitable homotopical sense. The proof proceeds by explicit analysis of the combinatorics of such lifting problems, made possible by the squiggle calculus mentioned above. In contrast, it is *not* possible in general to lift 1-monads to  $\infty$ -monads.

### 2.3. Comma $\infty$ -categories.

**Definition 2.22.** Let  $B \xrightarrow{f} A \xleftarrow{g} C$  be a pair of functors in the  $\infty$ -cosmos  $\mathcal{E}$ . The *comma  $\infty$ -category*  $(f \downarrow g)$  is defined to be the pullback in the

diagram below:

$$\begin{array}{ccc}
 (f \downarrow g) & \longrightarrow & A^{\Delta^1} \\
 \downarrow \lrcorner & & \downarrow \langle \text{cod}, \text{dom} \rangle \\
 C \times B & \xrightarrow{g \times f} & A \times A
 \end{array}$$

In the case of an identity functor, we write  $(f \downarrow A)$  instead of  $(f \downarrow \text{id}_A)$ .

**Remark 2.23.**  $(f \downarrow g)$  should be thought of as having objects triples  $\langle b \in B, c \in C, fb \xrightarrow{\phi} gc \in A \rangle$ . The reason for writing  $C \times B$  and  $\langle \text{cod}, \text{dom} \rangle$  instead of the seemingly more natural  $B \times C$  and  $\langle \text{dom}, \text{cod} \rangle$  in the above diagram is that  $(f \downarrow g)$  is a “(left  $C$ , right  $B$ )-bimodule”. By this we mean that  $(f \downarrow g)$  carries a covariant action by  $C$  and a contravariant action by  $B$ , and these commute. This perspective is extremely useful, and is the subject of [RV5]. Some of the proofs in §3.3 use the 1-categorical (but simplicially enriched) version of this “calculus of modules”, for which a good reference is [R, §§4.1 and 4.3].

**Remark 2.24.** Comma ∞-categories are important for (at least) two reasons. First, general 2-category theory would have us define many notions representably, carrying around “generalized objects”  $Z \rightarrow A$  (since we can’t “look inside” our ∞-categories). Comma categories frequently (although perhaps not always) allow us to dispense with this artifice and work more directly with the ∞-category  $A$ , thus keeping our intuition close to the classical case. Secondly, as we shall see, all the basic notions of category theory can be expressed in terms of commas. Once it is shown that functors of ∞-cosmoi preserve commas [RV5, 2.3.10], it follows that ∞-category theory developed in the ∞-cosmic framework is “model independent”. See [RV0, §§3.6 and 4.5] for further discussion.

**Example 2.25** ([RV1, 4.4.2 and 4.4.3]). Let  $X \xrightarrow{f} A$  and  $A \xrightarrow{u} X$  be a pair of functors between ∞-categories. Then  $f \dashv u$  if and only if  $(f \downarrow A) = (X \downarrow u)$  in the slice ∞-cosmos  $\mathcal{E}/(A \times X)$ :

$$\begin{array}{ccc}
 (f \downarrow A) & \xrightarrow{\sim} & (X \downarrow u) \\
 \searrow & & \swarrow \\
 & A \times X &
 \end{array}$$

**Example 2.26.** If  $a$  and  $b$  are objects of an ∞-category  $A$ , the comma ∞-category  $(a \downarrow b)$  is a model of the mapping space between  $a$  and  $b$  inside  $\mathcal{E}$ .

**Notation 2.27.** If  $a$  and  $b$  are objects of an ∞-category  $A$ , we write  $A(a, b) := \underline{\text{map}}(1, (a \downarrow b))$  for the underlying quasicategory of  $(a \downarrow b)$ ; by [RV0, §3.2], this is a Kan complex. If a map  $a \xrightarrow{f} b$  is given, we write  $A(a, b)_f$  to denote  $A(a, b)$  equipped with the basepoint  $f$ .

**Lemma 2.28.** *If  $J \in \mathbf{sSet}$ , then  $(f \downarrow g)^J = (f^J \downarrow g^J)$ .*

**2.4. Limits and colimits in  $\infty$ -categories.** The reference for this section is [RV1, §5].

**Definition 2.29.** An *absolute left lifting diagram* in a 2-category consists of the data

$$(2) \quad \begin{array}{ccc} & & C \\ & \nearrow \ell & \downarrow \psi \\ A & \xrightarrow{\phi} & B \end{array} \quad \begin{array}{c} \uparrow \lambda \\ \uparrow \end{array}$$

inducing unique factorizations of 2-cells:

$$\begin{array}{ccc} Z & \xrightarrow{q} & C \\ p \downarrow & \uparrow \zeta & \downarrow \psi \\ A & \xrightarrow{\phi} & B \end{array} = \begin{array}{ccc} Z & \xrightarrow{q} & C \\ p \downarrow & \exists! \uparrow \ell & \downarrow \psi \\ A & \xrightarrow{\phi} & B \end{array}$$

The pasting operation can be broken down as

$$\begin{array}{ccc} \underline{\mathbf{hom}}(Z, C)(\ell p, q) & \xrightarrow{\text{paste with } \lambda} & \underline{\mathbf{hom}}(Z, B)(\phi p, \psi q) \\ & \searrow \text{whisker with } \psi & \swarrow \text{precompose vertically with } \lambda \\ & \underline{\mathbf{hom}}(Z, B)(\psi \ell p, \psi q) & \end{array}$$

and the definition is demanding that “paste with  $\lambda$ ” be a bijection for all spans  $A \xleftarrow{p} Z \xrightarrow{q} C$ .

It will be useful to characterize absolute lifting diagrams in terms of comma categories rather than a test object  $Z$ . In fact, (2) is an absolute left lifting diagram if and only if the map  $(\ell \downarrow C) \rightarrow (\phi \downarrow \psi)$  induced by  $\lambda$  is an equivalence [RV1, 5.1.3].

**Definition 2.30.** Let  $J \in \mathbf{sSet}$ . We say that an  $\infty$ -category  $E$  *admits colimits of a family of diagrams  $D \xrightarrow{d} E^J$  of shape  $J$*  if there is an absolute left lifting diagram

$$\begin{array}{ccc} & & E \\ \lim \nearrow & & \downarrow c \\ D & \xrightarrow{d} & E^J \end{array} \quad \begin{array}{c} \uparrow \lambda \\ \uparrow \end{array}$$

in  $\mathbf{Ho}(\mathcal{E})$ ; here  $c$  is the constant map. In this case we call  $\lambda$  a *colimiting cone*.

The definition asks for the existence of a functor  $D \rightarrow E$  and a 2-cell  $\lambda$  satisfying certain universal properties. In our work in §3.2, the functor and 2-cell will always exist: the question will be *whether* they define an absolute

left lifting diagram. As mentioned above, this is the case if and only if the map  $(\lim_{\rightarrow} \downarrow E) \rightarrow (d \downarrow c)$  induced by  $\lambda$  is an equivalence.

**Remark 2.31.** When the  $\infty$ -cosmos  $\mathcal{E}$  is cartesian closed, one can give a completely analogous definition of colimits for shapes  $J \in \mathcal{E}$ . If  $\mathcal{E}$  is not cartesian closed, a different approach must be used; see [RV5]. We shall only require diagram shapes given by simplicial sets.

**2.5. Weighted limits.** The last section discussed limits *in*  $\infty$ -categories; we will also require limits *of*  $\infty$ -categories. In §2.6 this will be employed to tame the zoo of  $\infty$ -categories unleashed by an adjunction, by characterizing them by universal properties. The first half of this section is our telling of a standard story; the reference for the second half is [RV2, §5.2].

Being in the context of simplicially enriched categories imposes enriched category theory on us. Limits are one area where very different behavior arises in the enriched world than for ordinary categories: ordinary limits still make sense in the enriched case, but are woefully inadequate. The enriched context demands we consider *weighted limits*, a notion we suggest some intuition for before giving the precise definition.

Let  $\mathcal{V}$  be a *Bénabou* cosmos: a bicomplete closed symmetric monoidal category (“a category suitable for enriching over”). We shall only need  $\mathcal{V} = \mathbf{sSet}$ , but the theory is perfectly general. Let  $\mathcal{A} \xrightarrow{T} \mathcal{C}$  be a  $\mathcal{V}$ -functor between  $\mathcal{V}$ -categories  $\mathcal{A}$  and  $\mathcal{C}$ , with  $\mathcal{A}$  small. Remember that the *limit*  $\lim_{\leftarrow} T$  of  $T$  is defined by requiring it to represent cones over  $T$ ; that is, we have a natural correspondence between

$$Z \xrightarrow{\phi} \lim_{\leftarrow} T \quad \text{and} \quad \begin{array}{ccc} & Z & \\ \phi_a \swarrow & & \searrow \phi_{a'} \\ Ta & \xrightarrow{\quad} \dots \xrightarrow{\quad} & Ta' \end{array}$$

In the enriched context, we can demand richer structure

$$W_a \xrightarrow{\phi_a} \mathcal{C}(Z, Ta) \in \mathcal{V}$$

than just specifying a single map  $\phi_a$ , and we define  $\{W, T\}_{\mathcal{A}}$ , the *limit of T weighted by W*, by demanding a natural correspondence between

$$Z \xrightarrow{\phi} \{W, T\}_{\mathcal{A}} \quad \text{and} \quad \begin{array}{ccc} & Z & \\ \phi_a(W_a) \swarrow & & \searrow \phi_{a'}(W_{a'}) \\ Ta & \xrightarrow{\quad} \dots \xrightarrow{\quad} & Ta' \end{array}$$

More precisely, let  $\mathcal{A} \xrightarrow{W} \mathcal{V}$  be a  $\mathcal{V}$ -functor, which we call the *weight*. The weighted limit  $\{W, T\}_{\mathcal{A}}$  is defined by the universal property

$$(3) \quad \mathcal{C}(Z, \{W, T\}_{\mathcal{A}}) = \mathcal{V}^{\mathcal{A}}(W, \mathcal{C}(Z, T(\blacksquare))).$$

Important cases include

$$\{(\text{constant at object } V \in \mathcal{V}), T\}_{\mathcal{A}} = \left(\lim_{\leftarrow} T\right)^V = \lim_{\leftarrow} T(\blacksquare)^V,$$

a cotensor of the ordinary limit, and  $\{\mathcal{A}(a, -), T\}_{\mathcal{A}} = T(a)$  (the latter, as usual, is more or less the Yoneda lemma). Importantly,

$$\{\blacksquare, T\}_{\mathcal{A}}: (\mathcal{V}^{\mathcal{A}})^{\text{op}} \rightarrow \mathcal{C}$$

is a right adjoint, and so takes colimits of weights to limits of weighted limits. Combined with the two cases just mentioned, this gives the end formula

$$\{W, T\}_{\mathcal{A}} = \int_{a \in \mathcal{A}} T a^{W a}$$

which in particular shows that having all weighted limits is equivalent to having all  $\mathcal{V}$ -enriched ends and all cotensors over  $\mathcal{V}$ . Since  $\mathcal{V}$ -enriched ends can be expressed in terms of cotensors over  $\mathcal{V}$  and ordinary limits, we see that having all weighted limits is equivalent to having all ordinary limits and all cotensors over  $\mathcal{V}$ .

In general, an  $\infty$ -cosmos will not have all weighted limits. However, there is a conceptually elegant description of the weighted limits which do exist. If  $\mathcal{A}$  is a small simplicial category, say that a natural transformation in  $\mathbf{sSet}^{\mathcal{A}}$  is a *projective cofibration* if it has the left lifting property with respect to level acyclic fibrations. The projective cofibrations are evidently the closure of the set

$$\{\partial \Delta^n \times \mathcal{A}(a, \blacksquare) \hookrightarrow \Delta^n \times \mathcal{A}(a, \blacksquare) \mid n \geq 0, a \in \mathcal{A}\}$$

of *projective cells* in the Galois correspondence defined by left/right lifting properties. In particular, a natural transformation is a projective cofibration if and only if it is a retract of a transfinite composite of pushouts of projective cells.

**Proposition 2.32** ([RV2, 5.2.4]). *An  $\infty$ -cosmos has all limits weighted by projective cofibrant weights.*

Indeed,  $\{\blacksquare, T\}_{\mathcal{E}}$  turns all the types of colimits used to build projective cofibrations from projective cells into types of limits which are guaranteed to exist by the  $\infty$ -cosmos axioms (we added some axioms for precisely this purpose). We are thus reduced to showing that limits weighted by projective cells exist and are isofibrations; but this follows immediately from the completeness and SM7 axioms for an  $\infty$ -cosmos.

**Warning 2.33.** If  $L$  is an object of  $\mathcal{E}$  is defined by a weighted limit, and thus satisfying a  $\mathbf{sSet}$ -enriched universal property, it is generally not true that the image of  $L$  in  $\mathbf{Ho}(\mathcal{E})$  will have the analogous  $\mathbf{Cat}$ -enriched universal property. However, it may satisfy a weaker uniqueness condition guaranteeing its uniqueness up to isomorphism, although not up to automorphisms.

Let  $L \in \mathcal{E}$ , and let  $W \xrightarrow{\phi} \mathcal{C}(L, T(\blacksquare))$  be a weighted cone. We say that  $\phi$  displays  $L$  as a *weak 2-limit of  $T$  weighted by  $W$*  if the induced functors

$$\mathcal{C}(Z, L) \longrightarrow \mathcal{V}^A(W, \mathcal{C}(Z, T(\blacksquare))),$$

rather than being equalities as in (3), are *smothering*: surjective on objects, full, and conservative. For example, comma categories are weak 2-limits in this sense [RV1, 3.3.18], a fact which we will use in the proof of Lemma 3.19.

Since the above properties can be given by right lifting properties, it follows that fibres of a smothering functor, while not necessarily *contractible*, are at least (nonempty) connected groupoids (i.e., classifying spaces of discrete groups). For further details, see [RV1, §3.3].

**2.6. Monadic adjunction.** Suppose given a homotopy coherent monad  $t$  on an  $\infty$ -category  $X$ . In this section we construct the  $\infty$ -category  $X^t$  of homotopy coherent  $t$ -algebras, as well as the monadic adjunction  $X \xrightleftharpoons[u^t]{f^t} X^t$ .

When  $t$  arises from a homotopy coherent adjunction  $X \xrightleftharpoons[u]{f} A$ , we construct

the comparison functor  $A \xrightarrow{k} X^t$  which will be the subject of §3. The reference for this section is [RV2, §§6 and 7].

Denote the corepresentable functors by  $\mathbf{Adj}^\pm = \mathbf{Adj}(\pm, \blacksquare)$ . Denote the left Kan extension  $\dashv$  restriction adjunction arising from the inclusion of  $\mathbf{Mnd}$  into  $\mathbf{Adj}$  by

$$\mathbf{sSet}^{\mathbf{Mnd}} \xrightleftharpoons[\text{res}]{\text{lan}} \mathbf{sSet}^{\mathbf{Adj}}.$$

**Definition 2.34.** Let  $t$  be a homotopy coherent monad on  $X$ , given by a simplicial functor  $\mathbf{Mnd} \xrightarrow{H} \mathcal{E}$ . The  $\infty$ -category  $X^t$  of homotopy coherent  $t$ -algebras (or  $\infty$ - $t$ -algebras) is defined by the weighted limit

$$X^t := \{\text{res } \mathbf{Adj}^-, H\}_{\mathbf{Mnd}}.$$

This is legitimate by Proposition 2.32 and [RV2, 6.1.8]. Since

$$\mathbf{Mnd} \hookrightarrow \mathbf{Adj}$$

is fully faithful, we have  $\{\text{res } \mathbf{Adj}^+, H\}_{\mathbf{Mnd}} = H(+)= X$ . The monadic adjunction  $f^t: X \xrightleftharpoons[u^t]{f^t} X^t$  is defined to be cclassified by

$$\{\text{res } \mathbf{Adj}^\blacksquare, H\}_{\mathbf{Mnd}}: \mathbf{Adj} \rightarrow \mathcal{E}.$$

**Parallel 2.35.** Our homotopy coherent  $t$ -algebras correspond to the strictly  $\mathbb{T}$ -complete objects of [H, 4.14]. See also [H, 4.20].

Now suppose that  $t$  comes from a homotopy coherent adjunction  $X \xrightleftharpoons[u]{f} A$

in  $\mathcal{E}$ , cclassified by a simplicial functor  $\mathbf{Adj} \xrightarrow{T} \mathcal{E}$ . An inspection of universal properties shows that

$$\{\text{lan res } \mathbf{Adj}^-, T\}_{\mathbf{Adj}} = \{\text{res } \mathbf{Adj}^-, \text{res } T\}_{\mathbf{Mnd}} = X^t,$$

so we may take all weighted limits over  $\mathbf{Adj}$ .

**Definition 2.36.** The *comparison functor*  $A \xrightarrow{k} X^t$  is defined by requiring the diagram of  $\infty$ -categories on the right to be induced by the diagram of weights on the left.

$$\begin{array}{ccc} \text{lan res } \mathbf{Adj}^- & \xrightarrow{\perp} & \mathbf{Adj}^+ \\ \text{counit} \downarrow & & \parallel \\ \mathbf{Adj}^- & \xrightarrow[\text{Adj}^\#]{\text{Adj}^f} & \mathbf{Adj}^+ \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} X & \xrightarrow[\text{u}]{f} & A \\ \parallel & & \downarrow k \\ X & \xrightarrow[\text{u}^t]{f^t} & X^t \end{array}$$

That is,  $k$  is induced by the counit of the  $\text{lan } \dashv \text{res}$  adjunction, valued at  $\mathbf{Adj}^-$ . [RV2, 7.1.5] shows that  $\text{lan res } \mathbf{Adj}^-$  is the subfunctor of  $\mathbf{Adj}^-$  consisting of maps which factor through  $+$  (and the counit is the inclusion); in particular,  $\text{lan res } \mathbf{Adj}^-(+) = \Delta_\infty$  and  $\text{lan res } \mathbf{Adj}^-(-) = \Delta^{\text{op}}$ .

### 3. Comparison and cocompletion

In this section we prove the main theorem. Background on fully faithful functors appears in §3.1. In §3.2 we state and prove our main result, characterizing when the comparison functor induced by a monad is fully faithful in terms of a “cocomplete” criterion. Applications to descent, including descent spectral sequences, are discussed in §3.3.

We begin by establishing the notation to be used throughout this section, and reviewing that which was introduced in §2. Recall that the “walking adjunction” is denoted by  $\mathbf{Adj}$ , the “walking monad” by  $\mathbf{Mnd}$ ; we write

$$\mathbf{sSet}^{\mathbf{Mnd}} \xrightleftharpoons[\text{res}]{\text{lan } \perp} \mathbf{sSet}^{\mathbf{Adj}}$$

for the resulting left Kan extension  $\dashv$  restriction adjunction. Corepresentable functors are written  $\mathbf{Adj}^\pm := \mathbf{Adj}(\pm, \blacksquare)$ .

Fix once and for all an  $\infty$ -cosmos  $\mathcal{E}$  and homotopy coherent adjunction

$\mathbf{Adj} \xrightarrow{T} \mathcal{E}$ . We write  $X \xrightarrow[\text{u}]{f} A$  for the image of  $+\xrightarrow[\text{u}]{f}-$  in  $\mathcal{E}$ . Let  $t = uf$  (resp.  $g = fu$ ) be the (co)monad induced on  $X$  (resp.  $A$ ). The monadic adjunction is denoted  $X \xrightarrow[\text{u}^t]{f^t} X^t$ , the comparison functor  $A \xrightarrow{k} X^t$ , and the descent comonad  $g^t = f^t u^t$ . All this is summarized in the following

picture.

$$\begin{array}{ccc}
 \begin{array}{c} \curvearrowright \\ X \end{array} & \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{\perp} \\ \xrightarrow{u} \end{array} & \begin{array}{c} \curvearrowright \\ A \end{array} \\
 \parallel & & \downarrow k \\
 X & \begin{array}{c} \xrightarrow{f^t} \\ \xleftarrow{\perp} \\ \xrightarrow{u^t} \end{array} & \begin{array}{c} \curvearrowright \\ X^t \end{array}
 \end{array}$$

Recall that this diagram is obtained by applying  $\{\blacksquare, T\}_{\mathbf{Adj}}$  to the following diagram of weights:

$$\begin{array}{ccc}
 \text{lan res } \mathbf{Adj}^- & \begin{array}{c} \xrightarrow{\perp} \\ \xleftarrow{\perp} \end{array} & \mathbf{Adj}^+ \\
 \text{counit} \downarrow & & \parallel \\
 \mathbf{Adj}^- & \begin{array}{c} \xrightarrow{\text{Adj}^f} \\ \xleftarrow{\perp} \\ \xrightarrow{\text{Adj}^u} \end{array} & \mathbf{Adj}^+
 \end{array}$$

**Parallel 3.1.** The algebraic model category approach of [BIR] provides perhaps the closest link between model-categorical input and ∞-categorical output.

Let  $\mathcal{X} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{\perp} \\ \xrightarrow{U} \end{array} \mathcal{A}$  be a simplicial Quillen adjunction between cofibrantly

generated<sup>4</sup> simplicial model categories, inducing an ∞-adjunction  $X \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{\perp} \\ \xrightarrow{u} \end{array} A$

between the  $(\infty, 1)$ -categories  $X = N(\mathcal{X}_{\text{cf}})$  and  $A = N(\mathcal{A}_{\text{cf}})$ . By [BIR, 6.1], there is a simplicially enriched fibrant replacement monad  $\mathbb{R} = (R, r, \mu)$  on  $\mathcal{A}$  and a simplicially enriched cofibrant replacement comonad  $\mathbb{Q} = (Q, q, \nu)$  on  $\mathcal{X}$ ; thus  $f = N(RF|_{\mathcal{X}_{\text{cf}}})$  and  $u = N(QU|_{\mathcal{A}_{\text{cf}}})$ . Let  $T = QURF$  and  $G = RFQU$ , which model the ∞-monad  $t = uf = N(T|_{\mathcal{X}_{\text{cf}}})$  on  $X$  and the ∞-comonad  $g = fu = N(G|_{\mathcal{A}_{\text{cf}}})$  on  $A$ . By [BIR, 6.3], there are *point-set level* simplicially enriched resolutions

$$\begin{array}{l}
 Q \xrightarrow{\zeta} TQ \begin{array}{c} \xleftarrow{\zeta} \\ \xrightarrow{\zeta} \end{array} T^2Q \begin{array}{c} \xleftarrow{\zeta} \\ \xrightarrow{\zeta} \end{array} \dots \\
 R \xleftarrow{\xi} GR \begin{array}{c} \xleftarrow{\xi} \\ \xrightarrow{\xi} \end{array} G^2R \begin{array}{c} \xleftarrow{\xi} \\ \xrightarrow{\xi} \end{array} \dots \\
 RFQ \begin{array}{c} \xleftarrow{\zeta} \\ \xrightarrow{\zeta} \end{array} GRFQ \begin{array}{c} \xleftarrow{\zeta} \\ \xrightarrow{\zeta} \end{array} G^2RFQ \begin{array}{c} \xleftarrow{\zeta} \\ \xrightarrow{\zeta} \end{array} \dots \\
 QUR \begin{array}{c} \xleftarrow{\zeta} \\ \xrightarrow{\zeta} \end{array} TQUR \begin{array}{c} \xleftarrow{\zeta} \\ \xrightarrow{\zeta} \end{array} T^2QUR \begin{array}{c} \xleftarrow{\zeta} \\ \xrightarrow{\zeta} \end{array} \dots
 \end{array}$$

<sup>4</sup>Consult Aside 3.4 and the remarks after Theorem 3.3 of [BIR] for precisely what we mean by this term here.

presenting

$$\begin{array}{c}
 1 \xrightarrow{\eta} t \xrightleftharpoons{\eta} t^2 \xrightleftharpoons{\eta} \dots \\
 1 \xleftarrow{\epsilon} g \xrightleftharpoons{\epsilon} g^2 \xrightleftharpoons{\epsilon} \dots \\
 f \xrightleftharpoons{\quad} gf \xrightleftharpoons{\quad} g^2f \xrightleftharpoons{\quad} \dots \\
 u \xrightleftharpoons{\quad} tu \xrightleftharpoons{\quad} t^2u \xrightleftharpoons{\quad} \dots
 \end{array}$$

at the level of  $\infty$ -categories. Here the unit  $\eta$  and counit  $\epsilon$  of the  $\infty$ -adjunction  $f \dashv u$  are modelled on the point-set level, in terms of the unit  $\hat{\eta}$  and counit  $\hat{\epsilon}$  of the point-set adjunction  $F \dashv U$ , by

$$\zeta: Q \xrightarrow{\nu} Q^2 \xrightarrow{Q\hat{\eta}} QUFQ \xrightarrow{QUr} QURFQ$$

and

$$\xi: RFQUR \xrightarrow{RFq} RFUR \xrightarrow{R\hat{\epsilon}} R^2 \xrightarrow{\mu} R.$$

We will summon these assumptions and notations with the phrase, “suppose given model-categorical input”.

**3.1. Fully faithful functors.** We recall the  $\infty$ -cosmic definition of “fully faithful”, and demonstrate some elementary facts about it.

**Definition 3.2.** A functor  $P \xrightarrow{\phi} Q$  is *fully faithful* if the induced functor  $(P \downarrow P) \rightarrow (\phi \downarrow \phi)$  is an equivalence. If  $E \xrightarrow{e} P$  is a generalized object of  $P$ , we say that  $\phi$  is *fully faithful on maps out of  $e$*  if  $(e \downarrow P) \rightarrow (\phi e \downarrow \phi)$  is an equivalence.

These are equivalent to asking that

$$\begin{array}{ccc}
 \begin{array}{c} P \\ \parallel \\ P \xrightarrow{\phi} Q \\ \downarrow \phi \end{array} & \text{or} & \begin{array}{c} P \\ \nearrow e \\ E \xrightarrow{\phi e} Q \\ \downarrow \phi \end{array}
 \end{array}$$

be absolute left lifting diagrams. Explicitly,  $\phi$  is fully faithful if and only if it induces a bijection between 2-cells  $Z \xrightarrow[p]{q} P$  and  $Z \xrightarrow[\phi q]{\phi p} Q$  for every parallel pair  $Z \xrightarrow[p]{q} P$  in  $\mathbf{Ho}(\mathcal{E})$  (i.e.,  $\phi$  is “representably fully faithful”).

**Remark 3.3.** When  $\mathcal{E} = \mathbf{qCat}$  it suffices to consider only ordinary objects ( $E = 1$ ) by [RV1, 6.1.8]. In particular, this is fine when given model-categorical input. For more general  $\mathcal{E}$  (such as a slice  $\infty$ -cosmos), however,

$\phi$  could be fully faithful on maps out of  $e$  for all ordinary (or “global”) objects  $1 \xrightarrow{e} P$ , but fail to be fully faithful.

**Parallel 3.4.** In the  $\infty$ -cosmos of quasicategories, we recapture Lurie’s definition [HTT, 1.2.10.1]. If  $\mathcal{P} \xrightarrow{\Phi} \mathcal{Q}$  is a simplicial functor between simplicial categories  $\mathcal{P}$  and  $\mathcal{Q}$ , then the induced functor  $N(\mathcal{P}) \xrightarrow{N(\Phi)} N(\mathcal{Q})$  between quasicategories is fully faithful if and only if  $\Phi$  is a DK-embedding (locally a weak equivalence of simplicial sets).

**Example 3.5.** The comparison functor  $A \xrightarrow{k} X^t$  is fully faithful on maps out of the comonad  $g$ , viewed as a generalized object  $A \xrightarrow{g} A$ . Indeed,  $(g \downarrow A) = (fu \downarrow A) = (u \downarrow u) = (u^t k \downarrow u^t k) = (f^t u^t k \downarrow k) = (kfu \downarrow k) = (kg \downarrow k)$ . This is implicit in Theorem 3.14: everything has a resolution by objects out of which  $k$  is fully faithful, so we can reduce to asking if this resolution is a presentation.

The results in the remainder of this subsection are not needed in the sequel, but may be of independent interest. (They appeared in earlier attempts to prove Theorem 3.14.)

**Lemma 3.6.** *If  $\phi$  is fully faithful, then so is  $\phi^J$  for any  $J \in \mathbf{sSet}$ .*

**Proof.** Apply Lemma 2.28 and [HTT, 1.2.7.3]. □

**Lemma 3.7.** *A fully faithful functor reflects colimits.*

**Proof.** Consider a diagram of the form

$$(4) \quad \begin{array}{ccc} & P & \xrightarrow{\phi} Q \\ & \uparrow L & \downarrow c \\ K & \xrightarrow{D} P^J & \xrightarrow{\phi^J} Q^J \\ & \uparrow \lambda & \downarrow c \end{array}$$

in which  $\phi$  is fully faithful and  $J \in \mathbf{sSet}$ . The statement means that the triangle is an absolute left lifting diagram whenever the composite diagram is. Consider the commutative diagram

$$\begin{array}{ccc} (L \downarrow P) & \xrightarrow{\lambda} & (D \downarrow c) \\ \phi \downarrow & & \downarrow \phi^J \\ (\phi L \downarrow \phi) & \xrightarrow{\phi^J \lambda} & (\phi^J D \downarrow c\phi). \end{array}$$

The vertical arrows are equivalences by assumption and Lemma 3.6. If the composite diagram in (4) is an absolute left lifting diagram, then the bottom horizontal arrow is an equivalence. In this case the top horizontal arrow must be an equivalence, which is precisely the condition for the triangle of (4) to be an absolute left lifting diagram. □

**3.2. Proof of main result.** In this section we prove the main result. In order to state it, we first require some more notation.

Consider the natural transformation

$$\begin{array}{ccc}
 \Delta^{\text{op}} & \xrightarrow{\quad} & \Delta_+^{\text{op}} \\
 & \searrow \text{!} & \downarrow \\
 & & \mathbb{1}
 \end{array}
 \begin{array}{c}
 \nearrow \\
 \nearrow [-1]
 \end{array}$$

Multiplying with  $\mathbf{Adj}^-$  and applying the composition map

$$\mathbf{Adj}^- \times \Delta_+^{\text{op}} \longrightarrow \mathbf{Adj}^-$$

yields a natural transformation

$$\mathbf{Adj}^- \times \Delta^{\text{op}} \begin{array}{c} \xrightarrow{\text{compose}} \\ \downarrow \\ \xrightarrow{\text{project}} \end{array} \mathbf{Adj}^-$$

which is evidently compatible with the counit  $\text{lan res } \mathbf{Adj}^- \rightarrow \mathbf{Adj}^-$  (recall that this is the inclusion of maps out of  $-$  which factor through  $+$ ). We thus get a commutative diagram of weights

$$(5) \quad \begin{array}{ccc}
 \mathbf{Adj}^- & \begin{array}{c} \xleftarrow{\text{compose}} \\ \downarrow \\ \xleftarrow{\text{project}} \end{array} & \mathbf{Adj}^- \times \Delta^{\text{op}} \\
 \uparrow \text{include} & & \uparrow \text{include} \\
 \text{lan res } \mathbf{Adj}^- & \begin{array}{c} \xleftarrow{\text{compose}} \\ \downarrow \\ \xleftarrow{\text{project}} \end{array} & \text{lan res } \mathbf{Adj}^- \times \Delta^{\text{op}}
 \end{array}$$

where the dotted arrow commutes with the composition maps, not the projection maps. The existence of the dotted arrow is crucial for the proof of the main theorem, and follows from the fact that every element of  $\Delta^{\text{op}} \subset \mathbf{Adj}(-, -)$  factors through  $+$ . (In contrast, the composition map

$$\mathbf{Adj}^- \times \Delta_+^{\text{op}} \longrightarrow \mathbf{Adj}^-$$

does *not* factor through  $\text{lan res } \mathbf{Adj}^-$ .)

**Notation 3.8.** We define the maps in the diagram to the right by requiring it to come from applying  $\{\blacksquare, T\}_{\mathbf{Adj}}$  to (5):

$$\begin{array}{ccc}
 \mathbf{Adj}^- & \begin{array}{c} \xleftarrow{\text{compose}} \\ \downarrow \\ \xleftarrow{\text{project}} \end{array} & \mathbf{Adj}^- \times \Delta^{\text{op}} \\
 \uparrow \text{include} & & \uparrow \text{include} \\
 \text{lan res } \mathbf{Adj}^- & \begin{array}{c} \xleftarrow{\text{compose}} \\ \downarrow \\ \xleftarrow{\text{project}} \end{array} & \text{lan res } \mathbf{Adj}^- \times \Delta^{\text{op}}
 \end{array}
 \rightsquigarrow
 \begin{array}{ccc}
 A & \begin{array}{c} \xrightarrow{g^\bullet} \\ \downarrow \alpha \\ \xrightarrow{c} \end{array} & A^{\Delta^{\text{op}}} \\
 \downarrow k & & \downarrow k^{\Delta^{\text{op}}} \\
 X^t & \begin{array}{c} \xrightarrow{g^t} \\ \downarrow \alpha^t \\ \xrightarrow{c} \end{array} & (X^t)^{\Delta^{\text{op}}}
 \end{array}$$

where  $\tilde{\ell}$  commutes with the resolution maps  $g_\bullet$  and  $g_\bullet^t$ , not the constant maps  $c$ .

**Remark 3.9.** Let's unpack what all these maps mean by looking at the familiar case  $\mathcal{E} = \mathbf{Cat}$ . The comparison functor  $k$  sends an object  $a \in A$  to the object  $ua$  with its canonical  $t$ -algebra structure  $tua = ufa \xrightarrow{u\epsilon} ua$ . The comonads  $g$  and  $g^t$  give, for any  $a \in A$  or  $x \in X^t$ , augmented simplicial objects

$$(6) \quad a \leftarrow \epsilon \leftarrow ga \begin{array}{c} \xleftarrow{\epsilon} \\ \xrightarrow{\epsilon} \\ \xleftarrow{\epsilon} \\ \xrightarrow{\epsilon} \end{array} g^2a \begin{array}{c} \xleftarrow{\epsilon} \\ \xrightarrow{\epsilon} \\ \xleftarrow{\epsilon} \\ \xrightarrow{\epsilon} \end{array} \dots$$

and

$$(7) \quad x \leftarrow \epsilon^t \leftarrow g^t x \begin{array}{c} \xleftarrow{\epsilon^t} \\ \xrightarrow{\epsilon^t} \\ \xleftarrow{\epsilon^t} \\ \xrightarrow{\epsilon^t} \end{array} (g^t)^2 x \begin{array}{c} \xleftarrow{\epsilon^t} \\ \xrightarrow{\epsilon^t} \\ \xleftarrow{\epsilon^t} \\ \xrightarrow{\epsilon^t} \end{array} \dots ;$$

the maps  $g_\bullet$  and  $g_\bullet^t$  pick out the simplicial part of these, and then  $\alpha$  and  $\alpha^t$  come from the counit augmentation maps  $\epsilon$  and  $\epsilon^t$ . The simplicial part of (6) *almost* factors through  $f$ : all the objects  $g^n a$  are in the image of  $f$ , and so are all the maps between them *except* the top  $\epsilon$ s. This is what the map  $\tilde{\ell}$  is for<sup>5</sup>. A  $t$ -algebra  $x$  with structure map  $tx \xrightarrow{\phi} x$  has a resolution

$$(8) \quad x \begin{array}{c} \xrightarrow{\eta} \\ \xleftarrow{\phi} \end{array} tx \begin{array}{c} \xrightarrow{\eta} \\ \xleftarrow{\phi} \end{array} t^2 x \begin{array}{c} \xrightarrow{\eta} \\ \xleftarrow{\phi} \end{array} \dots ,$$

and then  $\tilde{\ell}x$  is the simplicial object in  $A$  obtained by applying  $f$  to (8) and throwing in the maps  $ft^{n-1}x \xleftarrow{\epsilon} gft^{n-1}x = ft^n x$ . We then indeed recover the simplicial part of (6) by taking  $x = ka$ , or the simplicial part of (7) by applying  $k$  levelwise.

**Definition 3.10.** Let  $g$  be a homotopy coherent comonad on an  $\infty$ -category  $A$ . A generalized object  $E \xrightarrow{a} A$  is  $g$ -cocomplete if  $\alpha_a$  is a colimiting cone. It is equivalent to ask that

$$\begin{array}{ccc} & & A \\ & \nearrow a & \downarrow c \\ E & \xrightarrow{g_\bullet a} & A^{\Delta^{op}} \end{array}$$

be an absolute left lifting diagram, or that  $(a \downarrow A) = (g_\bullet a \downarrow c)$ .

**Definition 3.11.** If the colimit of  $g_\bullet a$  exists, we call it the  $g$ -cocompletion of  $a$  and notate it  $a_g^\vee$ . It is characterized by  $(g_\bullet a \downarrow c) = (a_g^\vee \downarrow A)$ . Thus  $a$  is  $g$ -cocomplete if and only if  $a_g^\vee$  exists and is equal to  $a$ .

**Parallel 3.12.** Our  $g$ -cocomplete objects correspond to the strongly  $\mathbb{K}$ -cocomplete objects of [H, 4.33].

<sup>5</sup>This is how we originally arrived at  $\tilde{\ell}$ . Another interpretation is given in the comments preceding Lemma 3.19.

**Parallel 3.13.** Suppose given model-categorical input. The *derived G-cocompletion*  $a_G^\vee$  of  $a \in \mathcal{A}$  is defined as the geometric realization of a Reedy cofibrant replacement of the simplicial object  $G_\bullet Ra$  given by the Blumberg–Riehl homotopical resolution. This is evidently compatible with our definition. The coaugmentation  $a_G^\vee \rightarrow a$  is modelled by the zig-zag

$$a_G^\vee \longrightarrow Ra \xleftarrow{\sim} a$$

at the point-set level.

We can now state the main theorem.

**Theorem 3.14.** *The comparison functor  $k$  is fully faithful if and only if  $g$ -cocompletion is the identity, in the sense that  $(g_\bullet \downarrow c) = (A \downarrow A)$ . More generally,  $k$  is fully faithful on maps out of a generalized object  $E \xrightarrow{a} A$  if and only if  $a$  is  $g$ -cocomplete.*

**Corollary 3.15.** *Suppose we are given model-categorical input. Then the comparison functor  $A \xrightarrow{k} X^t$  (in the world of  $\infty$ -categories) is fully faithful on maps out of  $a \in \mathcal{A}$  if and only if  $a$  is weakly equivalent to its derived  $G$ -cocompletion  $a_G^\vee$  (in the world of model categories).*

**Warning 3.16.** One might colloquially pronounce  $(g_\bullet \downarrow c) = (A \downarrow A)$  as “every object of  $A$  is  $g$ -cocomplete”. This is potentially misleading: another natural interpretation of that phrase is that  $(g_\bullet a \downarrow c) = (a \downarrow A)$  for every object  $1 \xrightarrow{a} A$ . When  $\mathcal{E}$  is **qCat** (or biequivalent to it), these are equivalent by [RV1, 6.1.8]; in particular, this interpretation is fine when given model-categorical input. However, as in Remark 3.3, the pointwise statement is in general strictly weaker.

When  $A$  has enough colimits, there is an easy proof of Theorem 3.14 not requiring any new results. First, a definition:

**Definition 3.17** ([RV2, 7.2.1]). Define the diagram of  $\infty$ -categories on the right to be induced by the diagram of weights on the left.

$$\begin{array}{ccc} \mathbf{Adj}^+ \times \Delta^{\text{op}} & \xrightarrow{(u \circ \blacksquare) \times \Delta^{\text{op}}} & \mathbf{Adj}^- \times \Delta^{\text{op}} & & S(u) \longrightarrow X^{\Delta_\infty} \\ \mathbf{Adj}^+ \times (u \circ \blacksquare) \downarrow & & \downarrow & \rightsquigarrow & \downarrow \lrcorner \downarrow X^{u \circ \blacksquare} \\ \mathbf{Adj}^+ \times \Delta_\infty & \longrightarrow & W_s & & A^{\Delta^{\text{op}}} \xrightarrow{u_{\Delta^{\text{op}}}} X^{\Delta^{\text{op}}} \end{array}$$

We call  $S(u)$  the  $\infty$ -category of  $u$ -split simplicial objects. We say that  $A$  *admits colimits of  $u$ -split simplicial objects* if there exists a functor  $S(u) \xrightarrow{\lim} A$  and a 2-cell defining an absolute left lifting diagram

$$\begin{array}{ccc} & & A \\ & \xrightarrow{\lim} & \downarrow c \\ S(u) & \xrightarrow{k} & A^{\Delta^{\text{op}}} \end{array}$$

**Proposition 3.18.** *Theorem 3.14 holds if  $A$  admits colimits of  $u$ -split simplicial objects.*

**Proof.** By [RV2, 7.2.4],  $k$  admits a left adjoint  $X^t \xrightarrow{\ell} A$ , given by

$$\ell := \lim_{\rightarrow} \circ \tilde{\ell},$$

and the comonad  $\ell k$  so induced on  $A$  is nothing but  $g$ -cocompletion  $\blacksquare_g^{\vee}$ . Thus  $g$ -cocompletion is the identity if and only if the counit of  $\ell \dashv k$ , which is the coagumentation from the cocompletion, is an isomorphism. But the counit of an adjunction is an isomorphism if and only if the right adjoint is fully faithful [RV5, 5.2.9].  $\square$

In general, we may not be able to define  $\ell$  on objects of  $X^t$  not in the image of  $k$ . However, the “nonrepresentable left adjoint”  $\tilde{\ell}$  turns out to suffice for the argument. In place of an adjunction  $\ell \dashv k$ , which would yield  $(\ell \downarrow A) = (X^t \downarrow k)$ , we get

**Lemma 3.19.** *We have  $(\tilde{\ell} \downarrow c) = (X^t \downarrow k)$ .*

**Proof.** Functors in each direction are given by

$$\begin{array}{ccc} (\tilde{\ell} \downarrow c) & \xrightarrow{k^{\Delta^{\text{op}}}} & (g_{\bullet}^t \downarrow ck) \\ \alpha \uparrow & & \parallel \\ (\tilde{\ell} \downarrow g_{\bullet}) & \xleftarrow{\tilde{\ell}} & (X^t \downarrow k) \end{array}$$

where the identification comes from  $(g_{\bullet}^t \downarrow c) = (X^t \downarrow X^t)$  [RV2, 6.3.17]. Since  $\alpha\lambda, k^{\Delta^{\text{op}}}$ , and the identification all commute with the projections to  $X^t$  and  $A$ , it follows from the 2-cell induction and 2-cell conservativity properties of commas (cf. [RV1, 3.3.20]) that these define inverse equivalences between  $(\tilde{\ell} \downarrow c)$  and  $(X^t \downarrow k)$ .  $\square$

**Proof of Theorem 3.14.** By Lemma 3.19, we have

$$(k \downarrow k) = (X^t \downarrow k) \circ (k \downarrow A) = (\tilde{\ell} \downarrow c) \circ (k \downarrow A) = (g_{\bullet} \downarrow c)$$

which fits into a commutative diagram

$$\begin{array}{ccc} \begin{array}{ccc} (A \downarrow A) & & \\ k \swarrow & & \searrow \alpha \\ (k \downarrow k) & \equiv & (g_{\bullet} \downarrow c) \end{array} & \text{or more generally} & \begin{array}{ccc} (a \downarrow A) & & \\ k \swarrow & & \searrow \alpha \\ (ka \downarrow k) & \equiv & (g_{\bullet} a \downarrow c) \end{array} \end{array}$$

It follows that  $k$  is an equivalence if and only if  $\alpha$  is so; but this is precisely the statement of the theorem.  $\square$

**3.3. Applications to descent.** We will apply the results from the previous section to the monadic formulation of descent.

The previous section discussed monads. As is usual in category theory, we would like to obtain corresponding results for comonads “by duality”, without having to repeat the arguments. The following construction achieves this in the  $\infty$ -cosmic setting.

**Definition 3.20.** Let  $\mathcal{E}$  be an  $\infty$ -cosmos. We define  $\mathcal{E}^{\text{co}}$  to be the simplicially enriched category with:

- the same objects as  $\mathcal{E}$ ,
- mapping spaces given by  $\underline{\text{map}}_{\mathcal{E}^{\text{co}}}(A, B) = \underline{\text{map}}_{\mathcal{E}}(A, B)^{\text{op}}$ .

Let  $\mathcal{C}$  be a 2-category. We define  $\mathcal{C}^{\text{co}}$  to be the 2-category with:

- the same objects as  $\mathcal{C}$ ,
- mapping categories  $\underline{\text{hom}}_{\mathcal{C}^{\text{co}}}(A, B) = \underline{\text{hom}}_{\mathcal{C}}(A, B)^{\text{op}}$ .

Observe that  $\mathcal{E}^{\text{co}}$  is again an  $\infty$ -cosmos, with  $\mathbf{Ho}(\mathcal{E}^{\text{co}}) = \mathbf{Ho}(\mathcal{E})^{\text{co}}$ .

We quickly summarize what this means for us. The “walking comonad”  $\mathbf{Cmd}$  is the full subcategory of  $\mathbf{Adj}$  on the object  $-$ . We now add  $\pm$  subscripts to distinguish between our extension/restriction operations, writing them as

$$\mathbf{sSet}^{\mathbf{Mnd}} \begin{array}{c} \xrightarrow{\text{lan}_+} \\ \leftarrow \frac{\perp}{\text{res}_+} \end{array} \mathbf{sSet}^{\mathbf{Adj}} \quad \text{and} \quad \mathbf{sSet}^{\mathbf{Cmd}} \begin{array}{c} \xrightarrow{\text{lan}_-} \\ \leftarrow \frac{\perp}{\text{res}_-} \end{array} \mathbf{sSet}^{\mathbf{Adj}}.$$

Given a homotopy coherent comonad  $A \xrightarrow{g} A$  coming from  $\mathbf{Cmd} \xrightarrow{H} \mathcal{E}$  we have an  $\infty$ -category  $A_g$  of homotopy coherent  $g$ -coalgebras, defined through weights by

$$A_g = \{\text{res}_- \mathbf{Adj}^+, H\}_{\mathbf{Cmd}},$$

producing a comonadic adjunction  $u_g: A_g \xrightarrow{\perp} A : f_g$ , with  $u_g f_g = g$ . If the comonad is induced from an adjunction  $X \xrightarrow{\perp} A$  associated to  $\mathbf{Adj} \xrightarrow{T} \mathcal{E}$ ,

then  $A_g = \{\text{lan}_- \text{res}_- \mathbf{Adj}^+, T\}_{\mathbf{Adj}}$ , and we have a cocomparison functor  $X \xrightarrow{\kappa} A_g$  fitting into a commutative diagram

$$\begin{array}{ccc} \begin{array}{c} \curvearrowright \\ X \end{array} \xrightarrow{f} \begin{array}{c} \curvearrowright \\ A \end{array} & & \\ \downarrow \kappa & \begin{array}{c} \leftarrow \frac{\perp}{u} \\ \rightarrow \end{array} & \parallel \\ \begin{array}{c} \curvearrowright \\ A_g \end{array} \xrightarrow{u_g} \begin{array}{c} \curvearrowright \\ A \end{array} & & \\ \downarrow & \begin{array}{c} \leftarrow \frac{\perp}{f_g} \\ \rightarrow \end{array} & \\ \begin{array}{c} \curvearrowright \\ X^t \end{array} & & \end{array}$$

Let  $X \xrightarrow{t} X$  be a homotopy coherent monad in  $\mathcal{E}$  coming from a functor  $\mathbf{Mnd} \xrightarrow{H} \mathcal{E}$ . The monadic adjunction  $f^t: X \xrightarrow{\perp} X^t : u^t$  is classified by the

simplicial functor  $\{\text{res}_+ \mathbf{Adj}^\blacksquare, H\}_{\mathbf{Mnd}} : \mathbf{Adj} \rightarrow \mathcal{E}$ , and induces a comonad  $g^t = f^t u^t$  on  $X^t$ .

**Definition 3.21.** The  $\infty$ -category  $\mathfrak{D}_X^t$  of descent data for the monad  $t$  is the  $\infty$ -category of  $g^t$ -coalgebras in  $X^t$ ,  $\mathfrak{D}_X^t := (X^t)_{g^t}$ .

Keeping in the spirit of the previous section, we would like a description of  $\mathfrak{D}_X^t$  as a weighted limit. (We shan't need this, but the weight to use is not immediately obvious, and may be of use to posterity.) Let  $\mathfrak{W}$  denote the subfunctor of  $\mathbf{Mnd}^+ = \text{res}_+ \mathbf{Adj}^+$  consisting of endomorphisms which factor through  $-$  in  $\mathbf{Adj}$ ; thus  $\mathfrak{W}(+) = \Delta$ . We can express this definition cleanly in terms of functor tensor products (which will be used in the proof) by  $\mathfrak{W} = (\mathbf{Cmd} \downarrow \mathbf{Mnd}) \otimes_{\mathbf{Cmd}} (+ \downarrow \mathbf{Cmd})$ ; we refer to [R, §§4.1 and 4.3] for an introduction to functor co/tensor products.

**Proposition 3.22.**  $\mathfrak{D}_X^t = \{\mathfrak{W}, H\}_{\mathbf{Mnd}}$ .

**Proof.** Expanding the definitions,

$$\mathfrak{D}_X^t := (X^t)_{g^t} = \{\text{lan}_- \text{res}_- \mathbf{Adj}^+, \{\text{res}_+ \mathbf{Adj}^\blacksquare, H\}_{\mathbf{Mnd}}\}_{\mathbf{Adj}}$$

Applying the tensor hom-adjunction, this becomes

$$\mathfrak{D}_X^t = \{(\text{res}_+ \mathbf{Adj}^\blacksquare) \otimes_{\mathbf{Adj}} (\text{lan}_- \text{res}_- \mathbf{Adj}^+), H\}_{\mathbf{Mnd}},$$

which is a description of  $\mathfrak{D}_X^t$  as a single weighted limit over  $\mathbf{Mnd}$ . All that remains is to identify the weight; for this, we apply the tensor product formula for left Kan extension followed by the co-Yoneda lemma and get

$$\begin{aligned} \mathfrak{D}_X^t &= \{(\text{res}_+ \mathbf{Adj}^\blacksquare) \otimes_{\mathbf{Adj}} (\text{lan}_- \text{res}_- \mathbf{Adj}^+), H\}_{\mathbf{Mnd}} \\ &= \{(\mathbf{Adj} \downarrow \mathbf{Mnd}) \otimes_{\mathbf{Adj}} (\mathbf{Cmd} \downarrow \mathbf{Adj}) \otimes_{\mathbf{Cmd}} (+ \downarrow \mathbf{Cmd}), H\}_{\mathbf{Mnd}} \\ &= \{(\mathbf{Cmd} \downarrow \mathbf{Mnd}) \otimes_{\mathbf{Cmd}} (+ \downarrow \mathbf{Cmd}), H\}_{\mathbf{Mnd}} \\ &= \{\mathfrak{W}, H\}_{\mathbf{Mnd}}. \end{aligned} \quad \square$$

Thus the diagram of weights on the left induces the diagram of  $\infty$ -categories on the right.

$$\begin{array}{ccc} \text{res}_+ \mathbf{Adj}^+ & \xleftarrow{\perp} & \text{res}_+ \mathbf{Adj}^- \\ \uparrow & & \parallel \\ \mathfrak{W} & \xleftarrow{\perp} & \text{res}_+ \mathbf{Adj}^- \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} X & \xleftarrow{\perp} & X^t \\ \delta \downarrow & & \parallel \\ \mathfrak{D}_X^t & \xleftarrow{\perp} & X^t \end{array}$$

*(Note: The top row of the right diagram has  $f^t$  above  $X \rightarrow X^t$  and  $u^t$  below  $X \leftarrow X^t$ . The bottom row has  $u_{g^t}$  above  $\mathfrak{D}_X^t \rightarrow X^t$  and  $f_{g^t}$  below  $\mathfrak{D}_X^t \leftarrow X^t$ .)*

**Definition 3.23.** The  $\infty$ -monad  $t$  satisfies *descent* if  $X \xrightarrow{\delta} \mathfrak{D}_X^t$  is fully faithful. More generally, if  $E \xrightarrow{x} X$  is a generalized object then  $t$  satisfies *descent for maps into  $x$*  if  $\delta$  is fully faithful on maps into  $x$ . It satisfies *effective descent* if  $\delta$  is an equivalence.

The dual of Theorem 3.14 immediately gives

**Proposition 3.24.** *A  $\infty$ -monad  $t$  on an  $\infty$ -category  $X$  satisfies descent if and only if  $t$ -completion is the identity on  $X$ . More generally,  $t$  satisfies descent for maps into  $E \xrightarrow{x} X$  if and only if  $x$  is  $t$ -complete.*

**Corollary 3.25.** *Suppose given model-categorical input. Then  $t = N(T|_{\mathcal{X}_{\text{cf}}})$  satisfies descent (in the world of  $\infty$ -categories) if and only if every  $x \in \mathcal{X}$  is weakly equivalent to its derived  $T$ -completion  $x_T^\wedge$  (in the world of model categories).*

The remainder of this section is dual. Let  $g$  be an  $\infty$ -comonad on the  $\infty$ -category  $A$ , coming from a functor  $\mathbf{Cmd} \xrightarrow{H} \mathcal{E}$ . The comonadic adjunction  $u_g : A_{g \leftarrow \perp} \rightarrow A : f_g$  induces a homotopy coherent monad  $t_g = f_g u_g$  on  $A_g$ .

**Definition 3.26.** The  $\infty$ -category  $\mathfrak{C}_g^A$  of codescent data for the comonad  $g$  is the  $\infty$ -category of  $t_g$ -algebras in  $A_g$ ,  $\mathfrak{C}_g^A := (A_g)^{t_g}$ .

Let  $\mathfrak{V}$  denote the subfunctor of  $\mathbf{Cmd}^- = \text{res}_- \mathbf{Adj}^-$  consisting of endomorphisms which factor through  $+$  in  $\mathbf{Adj}$ ; thus

$$\mathfrak{V} = (\mathbf{Mnd} \downarrow \mathbf{Cmd}) \otimes_{\mathbf{Mnd}} (- \downarrow \mathbf{Mnd})$$

and  $\mathfrak{V}(-) = \Delta^{\text{op}}$ .

**Proposition 3.27.**  $\mathfrak{C}_g^A = \{\mathfrak{V}, H\}_{\mathbf{Cmd}}$ .

Thus the diagram of weights on the left induces the diagram of  $\infty$ -categories on the right.

$$\begin{array}{ccc} \text{res}_- \mathbf{Adj}^+ \xleftarrow{\perp} \text{res}_- \mathbf{Adj}^- & & A_g \xrightarrow{u_g} A \\ \parallel & \rightsquigarrow & \left\langle \begin{array}{c} \perp \\ f_g \\ \downarrow \gamma \end{array} \right\rangle \\ \text{res}_- \mathbf{Adj}^+ \xleftarrow{\perp} \mathfrak{V} & & A_g \xrightarrow{u^{t_g}} \mathfrak{C}_g^A \end{array}$$

**Definition 3.28.** The  $\infty$ -comonad  $g$  satisfies *codescent* if  $A \xrightarrow{\gamma} \mathfrak{C}_g^A$  is fully faithful. More generally, if  $E \xrightarrow{a} A$  is a generalized object then  $g$  satisfies *codescent for maps out of  $a$*  if  $\gamma$  is fully faithful on maps out of  $a$ . It satisfies *effective codescent* if  $\gamma$  is an equivalence.

**Proposition 3.29.** *A  $\infty$ -comonad  $g$  on an  $\infty$ -category  $A$  satisfies codescent if and only if  $g$ -cocompletion is the identity on  $A$ . More generally,  $g$  satisfies codescent for maps out of  $E \xrightarrow{a} A$  if and only if  $a$  is  $g$ -cocomplete.*

**Corollary 3.30.** *Suppose given model-categorical input. Then  $g = N(G|_{\mathcal{A}_{\text{cf}}})$  satisfies codescent (in the world of  $\infty$ -categories) if and only if every  $a \in \mathcal{A}$  is weakly equivalent to its derived  $G$ -cocompletion  $a_G^\vee$  (in the world of model categories).*

**3.4. Spectral sequences.** Descent spectral sequences fall out easily in our setting. Our discussion follows [H, §5.3] and is a trivial application of [BoK, §§X.6–7]. In particular, we refer to [BoK, §IX.5] for treatment of convergence issues.

We quickly recall Bousfield-Kan spectral sequences. A fibrant cosimplicial pointed space  $S^\bullet$  gives rise to a spectral sequence, which under favorable conditions converges to  $\pi_* \text{Tot } S^\bullet$ . For  $s \geq r \geq 0$ , the  $E_2$  term may be described by

$$E_2^{r,s} = \pi^r \pi_s S^\bullet$$

where for  $s \geq 2$ , the cohomotopy on the right hand side may be interpreted as the cohomology of the cochain complex corresponding to the cosimplicial abelian group  $\pi_s S^\bullet$  under the Dold-Kan correspondence. We caution the reader that this spectral sequence is *fringed* in general. Given a map  $x \xrightarrow{\phi} y$  in an  $\infty$ -category  $X$ , we write  $X(x, y)_\phi$  for the Kan complex  $X(x, y)$  equipped with the basepoint  $\phi$ .

Let  $t$  be an  $\infty$ -monad on an  $\infty$ -category  $X$  with unit  $\eta$ , and assume that  $X$  has all  $t$ -completions. Observe that a map  $x \xrightarrow{\phi} y$  gives rise to a cosimplicial pointed space  $X(x, t^\bullet y)_{\eta \bullet \phi}$  whose totalization is  $X(x, y_t^\wedge)_\phi$ .

**Proposition 3.31.** *A map  $x \xrightarrow{\phi} y$  in  $X$  gives rise to a spectral sequence, which for  $s \geq r \geq 0$  satisfies*

$$E_2^{r,s} = \pi^r \pi_s X(x, t^\bullet y)_{\eta \bullet \phi} = \pi^r \pi_s \mathfrak{D}_X^t(\delta x, \delta t^\bullet y)_{\delta \circ (\eta \bullet \phi)},$$

and which under suitable conditions converges to  $\pi_* X(x, y_t^\wedge)_\phi$ .

**Proof.** Only the identification  $X(x, t^\bullet y)_{\eta \bullet \phi} = \mathfrak{D}_X^t(\delta x, \delta t^\bullet y)_{\delta \circ (\eta \bullet \phi)}$  requires comment. This follows from Proposition 3.24 and the dual of Example 3.5. □

**Corollary 3.32.** *Suppose given model-categorical input. For cofibrant  $x$  and fibrant  $y$  in  $\mathcal{X}$ , a map  $x \xrightarrow{\phi} y$  gives rise to a spectral sequence, which for  $s \geq r \geq 0$  satisfies*

$$E_2^{r,s} = \pi^r \pi_s \text{Map}_{\mathcal{X}}(x, T^\bullet Qy)_{\zeta \bullet \phi}$$

and which under suitable conditions converges to  $\pi_* \text{Map}_{\mathcal{X}}(x, y_T^\wedge)_\phi$ .

Dually, let  $g$  be an  $\infty$ -comonad on an  $\infty$ -category  $A$  with counit  $\epsilon$ , and assume that  $A$  has all  $g$ -cocompletions. Observe that a map  $a \xrightarrow{\psi} b$  gives rise to a cosimplicial pointed space  $A(g_\bullet a, b)_{\psi \epsilon_\bullet}$  whose totalization is  $A(a_g^\vee, b)_\psi$ .

**Proposition 3.33.** *A map  $a \xrightarrow{\psi} b$  in  $A$  gives rise to a spectral sequence, which for  $s \geq r \geq 0$  satisfies*

$$E_2^{r,s} = \pi^r \pi_s A(g_\bullet a, b)_{\psi \epsilon_\bullet} = \pi^r \pi_s \mathfrak{C}_g^A(\gamma g_\bullet a, \gamma b)_{\gamma \circ (\psi \epsilon_\bullet)}$$

and which under suitable conditions converges to  $\pi_* A(a_g^\vee, b)_\psi$ .

**Corollary 3.34.** *Suppose given model-categorical input. For cofibrant  $a$  and fibrant  $b$  in  $\mathcal{A}$ , a map  $a \xrightarrow{\psi} b$  gives rise to a spectral sequence, which for  $s \geq r \geq 0$  satisfies*

$$E_2^{r,s} = \pi^r \pi_s \operatorname{Map}_{\mathcal{A}}(G_{\bullet} Ra, b)_{\psi \xi_{\bullet}}$$

*and which under suitable conditions converges to  $\pi_* \operatorname{Map}_{\mathcal{A}}(a_G^{\vee}, b)_{\psi}$ .*

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(Yuri J. F. Sulyma) UNIVERSITY OF TEXAS, AUSTIN, TX 78712  
ysulyma@math.utexas.edu

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