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The automorphism group of the hyperelliptic Torelli group

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Abstract. The *hyperelliptic Torelli group*, $\mathcal{SI}(S_g)$, is the subgroup of the mapping class group consisting of those elements that commute with a fixed hyperelliptic involution ι and act trivially on the homology of the surface S_g. The group $\mathcal{SI}(S_g)$ appears in a variety of contexts, e.g., as a kernel of a Burau representation and as the fundamental group of the branch locus of the period mapping on Torelli space. The main result of this paper is that, for $g \geq 3$, we have $\mathrm{Aut}(\mathcal{SI}(S_g)) \cong \mathrm{SMod}^{\pm}(S_g)/\langle\iota\rangle$, where $\mathrm{SMod}^{\pm}(S_g)$ is the extended hyperelliptic mapping class group. Our main tool is the *symmetric separating curve complex*, $C_{\mathrm{ssep}}(S_g)$, and we show that if $g \geq 3$, $\mathrm{Aut}(C_{\mathrm{ssep}}(S_g)) \cong \mathrm{SMod}^{\pm}(S_g)/\langle\iota\rangle$. Another key ingredient is an algebraic characterization of Dehn twists about symmetric separating curves. These results are analogous to results of Ivanov, Farb–Ivanov, and Brendle–Margalit for the mapping class group, the Torelli group, and the Johnson kernel.

Contents

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1. Introduction

Let $S_{g,b,p}$ be an oriented surface of genus g with b boundary components and p punctures. We will often omit b and p when they are equal to 0. We define the *mapping class group of a surface* S_g, $\mathrm{Mod}(S_g)$, to be

$$\mathrm{Mod}(S_g) := \mathrm{Homeo}^+(S_g)/\mathrm{Homeo}_0^+(S_g)$$

where $\mathrm{Homeo}^+(S_g)$ is the group of orientation-preserving homeomorphisms of S_g and $\mathrm{Homeo}_0^+(S_g)$ is the normal subgroup consisting of elements isotopic to the identity.

For a surface with boundary, we restrict to homeomorphisms that fix boundary components pointwise. For a surface with punctures (or marked points), we restrict to homeomorphisms that leave the set of punctures invariant. Let $\mathrm{Mod}^{\pm}(S_g)$ denote the *extended mapping class group*, the group of isotopy classes of all homeomorphisms of S_g. See [1], [13], and [18] for background information.

Mapping class groups act naturally on the first homology group of the surface and preserve the intersection form, giving rise to a surjective map onto $\mathrm{Sp}(2g, \mathbb{Z})$ called the *symplectic representation*. A subgroup of the mapping class group of primary importance is the Torelli group, $\mathcal{I}(S_g)$, the kernel of this representation (see Chapter 6 of [13]).

Let ι be a fixed hyperelliptic involution of S_g. That is, ι is a homeomorphism of order two that acts by $-I$ on the homology of S_g, or equivalently has $2g + 2$ fixed points. The hyperelliptic involution is unique up to conjugacy. The *hyperelliptic mapping class group*, $\mathrm{SMod}(S_g)$, is the centralizer in $\mathrm{Mod}(S_g)$ of ι, $C_{\mathrm{Mod}(S_g)}(\iota)$. Define the *hyperelliptic Torelli group*, $\mathcal{SI}(S_g)$ as follows:

$$\mathcal{SI}(S_g) := \mathrm{SMod}(S_g) \cap \mathcal{I}(S_g).$$

The hyperelliptic Torelli group is closely related to the kernel of the Burau representation evaluated at $t = -1$ and also the fundamental group of the branch locus of the period mapping. As such, $\mathcal{SI}(S_g)$ appears in many areas of mathematics including algebraic geometry, number theory and topology: see the introduction of [9] for more details. Other work, greatly motivated by Hain's problem list [14], includes [5], [6], [8], and [10]. In this paper we will characterize $\mathrm{Aut}(\mathcal{SI}(S_g))$.

Main Theorem 1. *For S_g a surface with genus $g \geq 3$, we have:*

$$\mathrm{Aut}(\mathcal{SI}(S_g)) \cong \mathrm{Mod}^{\pm}(S_{0,0,2g+2}) \cong \mathrm{SMod}^{\pm}(S_g)/\langle\iota\rangle.$$

The second congruence is due to Birman–Hilden [2]. This work is inspired by similar results for $\mathrm{Mod}(S_g)$, $\mathcal{I}(S_g)$, and $\mathcal{K}(S_g)$, the group generated by Dehn twists about separating curves. In particular Ivanov [16], Farb–Ivanov [12], and Brendle–Margalit [7], proved that for high enough genus

$$\mathrm{Aut}(\mathrm{Mod}(S_g)) \cong \mathrm{Aut}(\mathcal{I}(S_g)) \cong \mathrm{Aut}(\mathcal{K}(S_g)) \cong \mathrm{Mod}^{\pm}(S_g).$$

The key step in each of their arguments was to find an appropriate complex for the group and consider the automorphism group of that complex. Motivated by this, we define an abstract simplicial flag complex, the *symmetric separating curve complex* $C_{\text{ssep}}(S_g)$, whose 1-skeleton is given by vertices corresponding to isotopy classes of symmetric separating curves and edges are between vertices with disjoint representatives. A simple closed curve is *symmetric*, if it is fixed by ι.

Main Theorem 2. *Let $C_{\text{ssep}}(S_g)$ be the symmetric separating curve complex and $g \geq 3$. Then*

$$\text{Aut}(C_{\text{ssep}}(S_g)) \cong \text{Mod}^{\pm}(S_{0,0,2g+2}) \cong \text{SMod}^{\pm}(S_g)/\langle\iota\rangle.$$

As above the second congruence is due to Birman–Hilden [2]. Ivanov proved the analogous result that $\text{Aut}(\text{Mod}(S_g))$ is isomorphic to the automorphism group of the classical curve complex (Theorem 1, [19]). Similar results were proven by Farb–Ivanov for $\mathcal{I}(S_g)$ and the so-called *Torelli geometry* (Theorem 3.1, [12]) and Brendle–Margalit for $\mathcal{K}(S_g)$ and the separating curve complex (Theorem 1.5, [7]). Ivanov's proof relies on an algebraic characterization of nonseparating Dehn twists which are known to generate $\text{Mod}(S_g)$. Farb–Ivanov and Brendle–Margalit use an algebraic characterization of separating twists and *BP-maps*, known generators of $\mathcal{I}(S_g)$ (see Birman and Powell [4], [22]). In Theorem 3.4 we provide an algebraic characterization of Dehn twists about symmetric separating curves.

Outline of the argument.

Step 1. First let $\phi \in \text{Aut}(\mathcal{SI}(S_g))$. Our goal is to find $f \in \text{SMod}^{\pm}(S_g)$ so that $\phi(h) = fhf^{-1}$ for all $h \in \mathcal{SI}(S_g)$. In Section 2 we prove basic algebraic properties of $\mathcal{SI}(S_g)$ and basic facts of the symmetric separating curve complex.

Step 2. We show in Section 3 that ϕ induces a map $\phi^* \in \text{Aut}(C_{\text{ssep}}(S_g))$ because ϕ takes powers of Dehn twists about symmetric separating curves to powers of Dehn twists about symmetric separating curves. This relies on Theorem 3.4 which gives an algebraic characterization of Dehn twists about symmetric separating curves and we show ϕ preserves this characterization. We defer the somewhat technical proof of Theorem 3.4 until Section 7.

Step 3. In Section 4 we show that $\phi^* \in \text{Aut}(C_{\text{ssep}}(S_g))$ detects topological properties of curves on S_g.

Step 4. In Section 5 we extend ϕ^* to a map on all symmetric curves, thus showing ϕ^* induces $\phi^{**} \in \text{Aut}(C_{sym}(S_g))$, where $C_{sym}(S_g)$ is the *symmetric curve complex*. Any nonseparating symmetric curve c maps to an arc, $\bar{c}$, connecting two marked points in $S_{0,0,2g+2}$. Note that $\bar{c}$ is uniquely determined up to isotopy by any two genus one symmetric separating curves which both contain the marked endpoints of $\bar{c}$ in $S_{0,0,2g+2}$. Since we know where ϕ^* maps symmetric separating curves, this determines $\phi^{**}(c)$ up to isotopy. Furthermore this extension preserves disjointness between symmetric curves, making ϕ^{**} a simplicial map.

Step 5. In Section 6 we extend $\phi^{**} \in \mathrm{Aut}(C_{sym}(S_g))$ to "presymmetric curves," which correspond to a boundary component of a regular neighborhood of a chain of symmetric curves in $S_{0,0,2g+2}$. Hence ϕ^{**} induces $\hat{\phi} \in \mathrm{Aut}(C(S_{0,0,2g+2}))$. Disjointness properties are preserved in $S_{0,0,2g+2}$, so we have now shown $\hat{\phi} \in \mathrm{Aut}(C(S_{0,0,2g+2}))$.

Step 6. Using results of Korkmaz [21] and Birman–Hilden [2] we show $\hat{\phi} \in \mathrm{Aut}(C(S_{0,0,2g+2}))$ induces a map in $\mathrm{SMod}^{\pm}(S_g)/\langle\iota\rangle$. To recap, we started with $\phi \in \mathrm{Aut}(\mathcal{SI}(S_g))$ and extended the map as follows:

$$\phi \mapsto \phi^* \mapsto \phi^{**} \mapsto \hat{\phi}. \tag{1}$$

Step 7. To show this map is an isomorphism note that every element in $\mathrm{SMod}^{\pm}(S_g)$ restricts to an element of $\mathrm{Aut}(\mathcal{SI}(S_g))$ as follows. If $f \in \mathrm{SMod}^{\pm}(S_g)$, then $f \mapsto \phi$, where $\phi(h) = fhf^{-1}$ for $h \in \mathcal{SI}(S_g)$ and it is clear ι is in the kernel of this map. Surjectivity follows directly from the Alexander method because symmetric separating curves fill a surface, see Chapter 2 of [13]. This completes the proof of Main Theorems 1 and 2.

Our argument can be simplified in Step 5 if we apply the main result of [9] which states that $\mathcal{SI}(S_g)$ is generated by Dehn twists about symmetric separating curves. This immediately implies (1) is surjective.

2. Background

In this section we prove basic algebraic properties of $\mathcal{SI}(S_g)$ and basic facts about the symmetric separating curve complex.

2.1. Curves. We refer to a simple closed curve as a *curve* unless stated otherwise and we will often not distinguish between a curve and its isotopy class unless necessary.

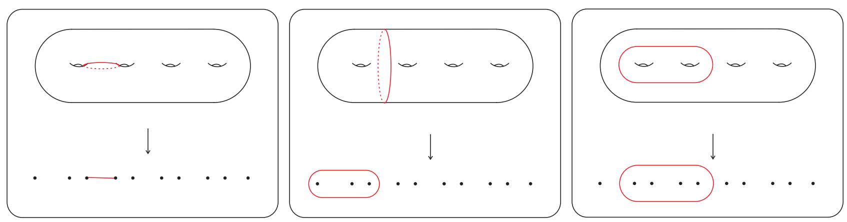
Classification of curves. Recall a curve is *symmetric* if it is fixed by the hyperelliptic involution ι. We say an isotopy class of curves is symmetric if it has a symmetric representative. A curve c is *presymmetric* if c and $\iota(c)$ are disjoint. Similarly, an isotopy class of curves is presymmetric if it has a presymmetric representative. Birman–Hilden show the following relating the symmetric mapping class group to the mapping class group of a $2g+2$ punctured sphere.

Theorem 2.1 (Birman–Hilden, Theorem 1, [2])**.** *Let S_g be a surface of genus g, then* $\mathrm{SMod}(S_g)/\langle\iota\rangle \cong \mathrm{Mod}(S_{0,0,2g+2})$.

Birman–Hilden use the 2-fold branched cover of S_g with $2g+2$ cone (or marked) points to classify curves in S_g by looking at their projection in

$S_{0,0,2g+2}$. We call a curve *odd (or even)* if its projection in $S_{0,0,2g+2}$ partitions the marked points into an odd (or even) collection of points. Birman–Hilden created the following dictionary relating curves in S_g to their projection in $S_{0,0,2g+2}$. We will denote the projection of a curve, c, in $S_{0,0,2g+2}$ by $\bar{c}$.

Curve in S_g	Curve/arc in $S_{0,0,2g+2}$
Symmetric nonseparating curve	Arc between marked points
Symmetric separating curve	Odd curve
Pre-symmetric (nonseparating) curve	Even curve



(a) Symmetric nonseparating curve (b) Symmetric separating curve (c) Pre-symmetric (nonseparating) curve

Rank. The *rank* of a group G, $\operatorname{rk} G$, is the rank of a largest maximal free abelian subgroup contained in G. We will find the rank of $\mathcal{SI}(S)$. This will be a key fact used later to classify twists about symmetric separating curves as well as in Section 4. In order to find the rank of $\mathcal{SI}(S)$ we will use the following theorem about the generation of $\mathcal{SI}(S)$. This is the only place that depends on the generators of $\mathcal{SI}(S)$.

Theorem 2.2 (Brendle–Margalit–Putman, Theorem A, [9])**.** *For $g \geq 0$, the group $\mathcal{SI}(S_g)$ is generated by Dehn twists about symmetric separating curves.*

Proposition 2.3. *For any surface S with genus $g \geq 3$ and b boundary components (where $b = 0$ or 1), then $\operatorname{rk} \mathcal{SI}(S_{g,b}) = g - 1 + b$.*

Proof. When $b = 0$ (or $b = 1$), it suffices to show that the maximal number of disjoint symmetric separating curves in S_g is $g - 1$ (or g). Suppose $g = 3$ and $b = 0$. It is clear a maximal collection of symmetric separating curves contains at least 2 curves as shown in $S_{0,0,8}$ here:

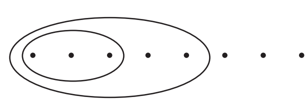


FIGURE 1. The projection of disjoint symmetric separating curves is $S_{0,0,2g+2}$ when $g = 3$ and $b = 0$.

It is also clear that there cannot be a distinct third such curve else the curves would no longer be disjoint. Furthermore if $g = 3$ and $b = 1$, a maximal collection of symmetric separating curves contains 3 curves as shown below in $S_{0,0,8}$.

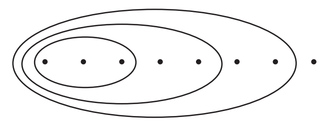


FIGURE 2. The projection of disjoint symmetric separating curves is $S_{0,0,2g+2}$ when $g = 3$ and $b = 1$.

Assume the proposition is true for $g \leq n$ (when $b = 0$ or 1). Suppose $g = n + 1$ and $b = 0$. Consider a maximal collection of disjoint symmetric separating curves and their projections in $S_{0,0,2n+4}$. We know this collection contains at least n curves because Figure 3 is a collection of n such curves projected in $S_{0,0,2n+4}$.

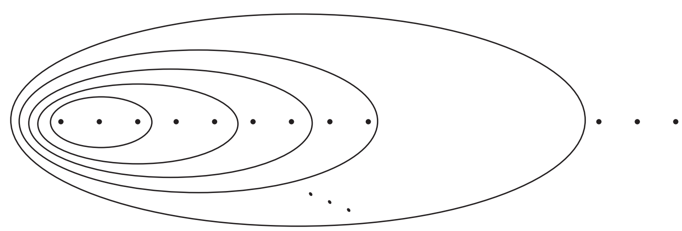


FIGURE 3. The projection of disjoint symmetric separating curves is $S_{0,0,2g+2}$ when $g = n + 1$ and $b = 0$.

What remains to be shown is that this collection contains no more than n curves. Let c be a curve in this collection whose projection in $S_{0,0,2n+4}$ partitions the marked points into two regions of say x and y marked points so that $|x - y|$ is maximal. Without loss of generality, assume $x > y$ and $x = 2k + 1$ for some k.

By the inductive hypothesis, the side of $\bar{c}$ that has x marked points has at most the projection of k curves (including c). The remaining number of marked points (those on the side of $\bar{c}$ with y marked points) is $2(n - k) + 3$, and those contain the projection of at most $n-k$ curves. Thus the collection has at most n symmetric separating curves.

If $g = n + 1$ and $b = 1$ then again let c be a curve in the collection whose projection in $S_{0,0,2n+4}$ partitions the marked points into two regions of say x and y marked points so that $|x - y|$ is maximal. Without loss of generality, we assume $x > y$ and $x = 2k + 1$ for some k. If $k < n + 1$ the argument above shows the result. If $k = n + 1$, then let d be a curve whose projection is on the x side of $\bar{c}$ so that $\bar{d}$ partitions the x marked points into two regions of say w and z marked points and $|w - z|$ is maximal. Without loss of generality we can assume $w > z$ and $w = 2l+1$ for some $l < k$. So by the inductive hypothesis there are at most l curves on the w side of $\bar{d}$ and at most $n-l-1$ curves on the x side of $\bar{c}$ and the z side of $\bar{d}$. Note there are no curves on the y side of $\bar{c}$. Hence there are a total of at most $n+1$ curves as desired. □

2.2. Curve complexes. When studying $\mathrm{Mod}(S_g)$ and its subgroups it is natural to try to find a simplicial complex on which the group acts nicely. For $\mathrm{Mod}(S_g)$, Harvey introduced the *curve complex* $C(S_g)$ in [15]. It is the simplicial flag complex with vertices corresponding to simple closed curves on S_g and edges between vertices which can be realized as disjoint curves in S_g.

Symmetric separating curve complex. For $\mathcal{SI}(S_g)$, the natural subcomplex of $C(S_g)$ to consider is the one spanned by symmetric separating curves, called the *symmetric separating curve complex*, $C_{\mathrm{ssep}}(S_g)$. In order to prove that $C_{\mathrm{ssep}}(S_g)$ is connected we will use the following lemma by Putman that will also be used in Section 5.

Lemma 2.4 (Putman, Lemma 2.1, [23])**.** *Consider a group G acting upon a simplicial complex X. Fix a basepoint $v \in X^{(0)}$, and a set S of generators for G. If:*

(1) *For all $v' \in X^{(0)}$, the orbit $G \cdot v$ intersects the connected component of X containing v',*
(2) *for all $s \in S^{\pm 1}$, there is some path P_s in X from v to $s \cdot v$,*

then X is connected.

We now apply the previous lemma to the complex of symmetric separating curves.

Lemma 2.5. *If $g \geq 3$, then $C_{\mathrm{ssep}}(S_g)$ is connected.*

Proof. Given a symmetric separating curve, c, on S_g, we know its image $\bar{c}$ in $S_{0,0,2g+2}$ is an odd curve, so $\bar{c}$ will partition the marked points in $S_{0,0,2g+2}$ into two regions, one of which will have more than three marked points. Thus we can find a genus 1 symmetric separating curve disjoint from c. Now it suffices to show that the subcomplex of all genus one symmetric separating curves is connected. Let the genus one symmetric separating curve v shown in Figure 4 act as the base point for the genus one subcomplex of $C_{\mathrm{ssep}}(S_g)$.

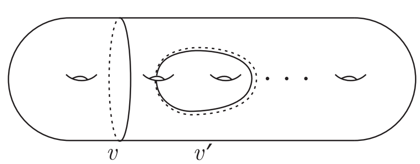


FIGURE 4. The curves needed for the proof of Lemma 2.5.

The group $\mathrm{SMod}(S_g)$ acts transitively on the set of genus one symmetric separating curves, so Condition (1) of Lemma 2.4 is satisfied. Birman–Hilden showed that the group $\mathrm{SMod}(S_g)$ is generated by Dehn twists about the $2g+1$ curves $b_1, b_2, \cdots, b_{2g+1}$ shown in Figure 5 (Theorem 3, [2]).

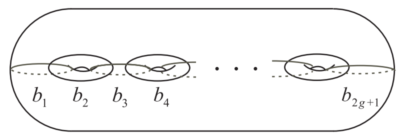


FIGURE 5. Dehn twists about the curves $b_1, b_2, \cdots, b_{2g+1}$ generate $\mathrm{SMod}(S_g)$.

Call this set of generators S. To show Condition (2) is satisfied we will use the curve v' shown in Figure 4. Consider $s \in S^{\pm}$. If $s = T_{b_3}^{\pm}$, then $v - v' - s \cdot v$ is the desired path. Otherwise, we have $s \in S^{\pm}$ but $s \neq T_{b_3}^{\pm}$, so $s \cdot v = v$. Thus by Lemma 2.4 the genus one subcomplex of $C_{\mathrm{ssep}}(S_g)$ is connected. □

Note that if X is a flag complex, then $\mathrm{Aut}(X) = \mathrm{Aut}(X^{(1)})$, where $X^{(1)}$ is the 1-skeleton of X. Hence we only need to focus on vertices and edges when looking at automorphisms of $C(S_g)$ and $C_{\mathrm{ssep}}(S_g)$. In order to prove Main Theorem 2 we will use the result of Birman–Hilden in Theorem 2.1 and the following result of Korkmaz.

Theorem 2.6 (Korkmaz, Theorem 1, [21])**.** *Let $C(S_{0,0,2g+2})$ be the curve complex associated to $S_{0,0,2g+2}$ with $g > 1$. Then*

$$\mathrm{Aut}(C(S_{0,0,2g+2})) \cong \mathrm{Mod}^{\pm}(S_{0,0,2g+2}).$$

For the remainder of this paper let $\phi \in \mathrm{Aut}(\mathcal{SI}(S_g))$. In Section 3 we show that ϕ induces an element $\phi^* \in \mathrm{Aut}(C_{\mathrm{ssep}}(S_g))$. We will proceed to show that ϕ^* induces an element of $\mathrm{SMod}^{\pm}(S_g)/\langle \iota \rangle$ by extending ϕ^* to a simplicial map on all symmetric and presymmetric curves.

3. Characterizing symmetric separating curves

The goal of this section is to give an algebraic characterization in terms of centers and centralizers that classifies when a mapping class is a Dehn twist about a symmetric separating curve. We will show that this characterization is preserved by automorphisms of $\mathcal{SI}(S_g)$. First we will need a few definitions and lemmas about the structure of elements in $\mathcal{I}(S_g)$ and in $\mathcal{SI}(S_g)$.

3.1. Classification of mapping class groups. Mapping classes are often classified according to whether or not they fix any curves in the surface as follows. A curve, c, is called a *reducing curve* for a mapping class f, if $f^n(c) = c$ for some n.

Nielson–Thurston trichotomy. We are able to classify any mapping class, f, into one of the following categories: f is either a *finite order* element that is, there exists an n such that $f^n = id$, *reducible* if it has a reducing curve, or *pseudo-Anosov* if it is not finite order or reducible. As stated, there is nontrivial overlap between the finite order and reducible elements. In order to make this a true trichotomy, we can replace the condition of having a reducing curve with that of having an essential reducing curve, where we call a reducing curve c *essential* for a mapping class h if for each simple close curve b on the surface such that $i(c, b) \neq 0$, and for each integer $m \neq 0$, the classes $h^m(b)$ and b are distinct.

Theorem 3.1 (Birman–Lubotzky–McCarthy, [3])**.** *For every mapping class h there exists a system of essential reducing curves. Moreover, the system is unique up to isotopy, and cutting along it, the restriction of h to each component of the cut-open surface is either pseudo-Anosov or finite order.*

A fixed curve of a finite order mapping class is never essential. The *canonical reduction system* for a mapping class, f, is the collection of all essential reducing curves for f. We call a mapping class f *pure* if f contains a homeomorphism f' which satisfies the following conditions on some closed one-dimensional submanifold C of S: the components of C are nontrivial and are pairwise disjoint, the homeomorphism f' is fixed on C and does not rearrange the components of $S - C$, and f' induces the identity or a pseudo-Anosov homeomorphism on each component of $S - C$. It is well known that all elements of $\mathcal{I}(S_g)$ are pure (Ivanov, Theorem 3, [17]).

A *curve system* C is a one-dimensional submanifold of S, where no component of C is homotopically trivial or boundary parallel. A curve system, C, determines a surface $S - C$ which is obtained from S by cutting along C. Observe that each component of C determines two boundary components of the surface $S - C$. Any homeomorphism $f : S \longrightarrow S$ with $f(C) = C$ induces a homeomorphism $f_C : S - C \longrightarrow S - C$. If Q is a component of $S - C$ and $f_C(Q) = Q$ then f_C induces a homeomorphism $f_Q : Q \longrightarrow Q$.

For a group Γ and $f \in \Gamma$, let $C_\Gamma(f)$ be the subgroup of elements $g \in \Gamma$ commuting with f, and let $Z(\Gamma)$ be the center of Γ. We will need the following result of Ivanov–McCarthy in order to complete the proof of our characterization.

Lemma 3.2 (Ivanov–McCarthy, Lemma 5.6, [20])**.** *Let Γ be any subgroup of finite index in $\mathrm{Mod}(S)$ consisting entirely of pure elements. Let $f, h \in \Gamma$ and let C_f denoted a representative on S of a canonical reduction system for f. Then $h \in C_\Gamma(f)$ if and only if h_Q commutes with f_Q for every component Q of $S - C_f$.*

Note that h_Q makes sense in this context because the canonical reduction system for h, C_h, equals C_f because $h(C_f) = C_{hfh^{-1}}$ (see Lemma 2.6 of [3]).

Characterizing Dehn twists about symmetric separating curves. Ivanov in Section 2 of [16] characterized Dehn twists about nonseparating curves based on purely algebraic properties. In a similar fashion, Farb–Ivanov characterized powers of a Dehn twist about a separating curve or a BP-map. Where given two disjoint, nonseparating, homologous simple closed curves c and d, a *bounding pair map* (BP-map) is the product $T_cT_d^{-1}$.

Proposition 3.3 (Farb–Ivanov, Proposition 8, [12])**.** *Let S_g be a closed, oriented surface of genus $g \geq 3$ and $f \in \mathcal{I}(S_g)$ is nontrivial. Then f is a power of a Dehn twist about a separating curve or a power of a BP-map if and only if:*

(1) $Z(C_{\mathcal{I}}(f)) \cong \mathbb{Z}$.
(2) $C_{\mathcal{I}}(f) \ncong \mathbb{Z}$.
(3) $\max_{\mathcal{I}}(f) \cong 2g - 3$.

Where $\max_{\mathcal{I}}(f)$ is the rank of a maximal abelian subgroup of $\mathcal{I}$ that has f as a generator. Modifying the arguments of Farb–Ivanov [11], we characterize Dehn twists about symmetric separating curves denoted T_α using the additional characterization called a *parity changing pseudo-Anosov, PCPA*. We define *PCPA* as follows:

Let $f \in \mathcal{SI}(S_g)$, then f is a *PCPA* if and only if:

- f is pseudo-Anosov on at most one component of $S - C_f$.
- There exists a nontrivial element $g \in \mathcal{SI}(S_g)$, $g \neq f^k$, such that given any maximal abelian subgroup of $\mathcal{SI}(S_g)$ $\langle f, x_1, x_2, \ldots, x_{g-2}\rangle$, where $Z(C_{\mathcal{SI}}(x_i)) = \mathbb{Z}$ for each i, the group $\langle fg, x_1, x_2, \ldots, x_{g-2}\rangle$ is also a maximal abelian subgroup of $\mathcal{SI}(S_g)$.

Theorem 3.4. *Let S be a closed, oriented surface of genus $g \geq 3$. For nontrivial $f \in \mathcal{SI}(S_g)$, then f is a power of a Dehn twist about a symmetric separating curve if and only if:*

(1) $Z(C_{\mathcal{SI}}(f)) \cong \mathbb{Z}$.
(2) *f is not a $PCPA$.*
(3) $C_{\mathcal{SI}}(f) \ncong \mathbb{Z}$.
(4) $\max_{\mathcal{SI}}(f) \cong g - 1$.

A straightforward consequence of Theorem 3.4 is the following key corollary.

Corollary 3.5. *Let $\phi \in Aut(\mathcal{SI}(S_g))$ and $f = T_\gamma^k$ where γ is a symmetric separating curve and $k \neq 0$. Then $\phi(f) = T_\alpha^l$ for some symmetric symmetric separating curve α and nonzero l.*

The proofs of Theorem 3.4 and Corollary 3.5 can be found in Section 7. It is perhaps surprising that Condition (2) in Theorem 3.4 is necessary. We will show the necessity of this condition by the following example. Let $f = T_aT_b^{-1}$, where a and b are as shown.

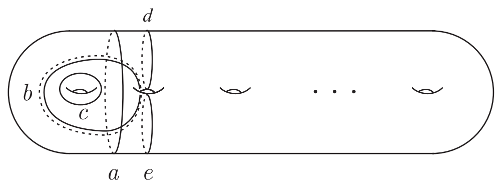


FIGURE 6. Curves a, b, c, d and e.

Condition (1): $Z(C_{\mathcal{SI}}(f)) \cong \mathbb{Z}$. We know $f \in Z(C_{\mathcal{SI}}(f))$. We claim only powers of f are in $Z(C_{\mathcal{SI}}(f))$. First note that the canonical reduction system for f, C_f, is the following collection of curves:

$$C_f = \{c, d, e\}.$$

Because f is a pure mapping class, we know the restriction of f to any component of $S - C_f$ is either the identity or a pseudo-Anosov element. We will let Q_1 be the component of $S - C_f$ that contains the curves a and b and Q_2 be the other component. Let $h \in Z(C_{\mathcal{SI}}(f))$, so $h \in C_{\mathcal{SI}}(f)$. Hence by Lemma 3.2 h_Q commutes with f_Q for every component Q of $S - C_f$. Because no two independent pseudo-Anosov elements commute and h is a pure mapping class, we know h_{Q_1} is either the identity or a power of f. Moreover h_{Q_2} must be the identity, otherwise h_{Q_2} would be a pseudo-Anosov element and we know there exist other mapping classes in $C_{\mathcal{SI}}(f)$ which are independent pseudo-Anosov elements when restricted to Q_2. Hence the only type of mapping class h can be, besides a power of f, is a multitwist about curves in C_f, but no such multitwist is in $\mathcal{SI}(S_g)$.

Condition (2): f is a $PCPA$. Let $g = T_b$, then g satisfies the conditions needed to be $PCPA$.

Condition (3): $C_{\mathcal{SI}}(f) \ncong \mathbb{Z}$. This follows because there exists a symmetric separating curve disjoint from a and b.

Condition (4): $\max_{\mathcal{SI}}(f) \cong g - 1$. Such a group is generated by f and Dehn twists about the curves pictured in Figure 7.

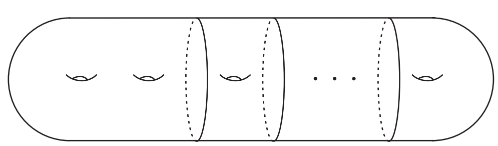


FIGURE 7. A collection of disjoint symmetric separating curves that with f form a basis for a maximal rank abelian subgroup of $\mathcal{SI}(S_g)$.

4. Basic topology

In this section we will see in what ways $\phi^* \in \mathrm{Aut}(C_{\mathrm{ssep}}(S_g))$ is able to detect the topological properties of curves on S_g. This section has the same results as those shown by Brendle–Margalit in [7] with the added condition that all curves are symmetric. We show their proofs can be realized symmetrically except for that of the genus result which differs quite significantly. The following are two facts which will be necessary to show the key lemmas.

Fact 4.1 (See Chapter 3 of [13]). *Let f and h be Dehn twists about separating curves. Then the commutator $[f^j, h^k] = 1$ if and only if the intersection number between the corresponding curves is zero.*

Fact 4.2. *For any surface S_g with genus $g \geq 3$ and no boundary, a maximal collection of disjoint symmetric genus 1 separating curves contains $\lfloor \frac{2g+2}{3} \rfloor$ curves.*

The following lemma is a direct consequence of Fact 4.1 and the fact that the simplicial map ϕ^* is injective.

Lemma 4.3 (Disjointness). *If a and b are symmetric separating curves in S, then $i(a, b) \neq 0$ if and only if $i(\phi^*(a), \phi^*(b)) \neq 0$.*

We define a *side* of a separating curve, z, to be one of the components of $S - z$.

Lemma 4.4 (Sides). *If a and b are symmetric separating curves on the same side of a symmetric separating curve, z, then $\phi^*(a)$ and $\phi^*(b)$ are symmetric separating curves on the same side of $\phi^*(z)$.*

Proof. Symmetric separating curves a and b are on the same side of z if and only if there exists a symmetric separating curve c such that $i(a, c) \neq 0$, $i(b, c) \neq 0$, and $i(z, c) = 0$. Thus by Lemma 4.3 we can conclude that $\phi^*(a)$ and $\phi^*(b)$ are symmetric separating curves on the same side of $\phi^*(z)$. □

Proposition 4.5 (Genus). *Suppose S_g is a surface with genus $g \geq 3$ and no boundary. If z is a genus m symmetric separating curve, then $\phi^*(z)$ is a genus m symmetric separating curve. Moreover, if a is on a genus m side of z, then $\phi^*(a)$ is on a genus m side of $\phi^*(z)$.*

Proof. Suppose z is a genus m symmetric separating curve, then by Proposition 2.3 any maximal collection of disjoint symmetric separating curves in S which contain z is of the form:

$$\{a_1, \ldots, a_{m-1}, z, b_1, \ldots, b_{(g-m)-1}\}$$

where the $a_i's$ are disjoint symmetric separating curves on one side of z and the $b_i's$ are disjoint symmetric separating curves on the other. By disjointness, Proposition 2.3, and the fact that $\phi^* \in \mathrm{Aut}(C_{\mathrm{ssep}}(S_g))$ we have that the set

$$\{\phi^*(a_1), \ldots, \phi^*(a_{m-1}), \phi^*(z), \phi^*(b_1), \ldots, \phi^*(b_{(g-m)-1})\}$$

is a maximal collections of mutually disjoint symmetric separating curves on S. By Lemma 4.4 and Proposition 2.3, either $\phi^*(z)$ is a genus 1 symmetric separating curve and $\phi^*(a_i)$ and $\phi^*(b_i)$ are on the same side of $\phi^*(z)$, or

$\phi^*(z)$ is a genus m curve with $\phi^*(a_i)$ on one side of $\phi^*(z)$ and $\phi^*(b_i)$ on the other.

When $m = 1$ both cases are the same. Hence ϕ^* maps genus 1 symmetric separating curves to genus 1 symmetric separating curves. Now we will show if $m \geq 2$, then $\phi^*(z)$ cannot be a genus 1 curve, which will prove that $\phi^*(z)$ is a genus m symmetric separating curve as desired.

Here is where this proof differs extensively from [7] because in a maximal collection of disjoint symmetric separating curves there is not a fixed number of genus 1 curves. For example here are two maximal collections (viewed in $S_{0,0,2g+2}$ with a different number of genus 1 curves).

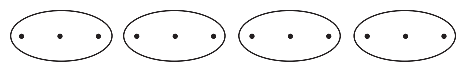


FIGURE 8. A maximal collection of disjoint symmetric separating curves with four genus 1 curves.

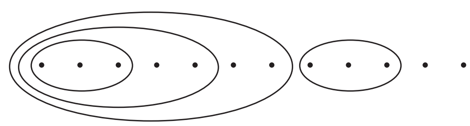


FIGURE 9. A maximal collection of disjoint symmetric separating curves with two genus 1 curves.

Choose a maximal collection of disjoint genus 1 symmetric separating curves on each side of z. Note that there will be $\lfloor \frac{2m+1}{3} \rfloor$ such curves on the "inside" of z and $\lfloor \frac{2(g-m)+1}{3} \rfloor$ on the "outside" of z.

If $\lfloor \frac{2m+1}{3} \rfloor + \lfloor \frac{2(g-m)+1}{3} \rfloor = \lfloor \frac{2g+2}{3} \rfloor$ then by Fact 4.2 the union of the maximal collections of genus 1 symmetric separating curves on each side of z is actually a maximal disjoint collection of genus 1 symmetric separating curves for S, which maps to a maximal collection of disjoint genus 1 curves. Hence $\phi^*(z)$ cannot be a genus 1 curve.

If $\lfloor \frac{2m+1}{3} \rfloor + \lfloor \frac{2(g-m)+1}{3} \rfloor \neq \lfloor \frac{2g+2}{3} \rfloor$ then we will choose a second maximal collection of disjoint symmetric genus 1 separating curves and sometimes one additional genus 1 curve on each side of z which will force $\phi^*(z)$ to be a genus m curve.

Since $m \geq 2$ there are at least 5 marked points contained on both sides of $\bar{z}$. Without loss of generality suppose the inside of $\bar{z}$ has at least 5 marked points; that is, $2m+1 \geq 5$. First we will consider the case where $2m+1 > 5$.

If $2m + 1 \equiv 2 \bmod 3$, then let $\{c_1, \ldots, c_k\}$ for some $k \in \mathbb{Z}^+$, be one collection of maximal genus 1 curves on the inside of z. Then choose a second disjoint maximal collection of symmetric genus 1 curves $\{d_1, \ldots, d_k\}$ so that $i(c_j, d_i) = 0$ if and only if $j \neq 1$ and $j = i + 1$, and one additional genus 1 symmetric separating curve e can be chosen so that:

- $i(e, c_j) = 0$ if and only if $j \neq 1$.
- $i(e, d_i) = 0$ if and only if $i \neq k$.
- $i(e, z) = 0$.

Now we will consider the image of these curves under ϕ^*. That is,

$$\{\phi^*(c_i), \phi^*(d_j), \phi^*(e), \phi^*(z) \mid 1 \leq i, j \leq k\}.$$

First note that no $\overline{\phi^*(c_i)}$ and $\overline{\phi^*(d_j)}$ can contain the same three marked points, otherwise $\overline{\phi^*(c_i)} \cup \overline{\phi^*(d_j)}$ would separate $S_{0,0,2g+2}$ which would not allow for disjointness among the curves to be preserved by ϕ^*. The collection $\{\overline{\phi^*(c_i)}, \overline{\phi^*(d_j)}, \overline{\phi^*(e)} \mid 1 \leq i, j \leq k\}$ must "contain" the same number of marked points as $\{c_i, d_j, e \mid 1 \leq i, j \leq k\}$ because reducing the number of marked points would force two of the curves to contain the same three marked points.

Note if $2m+1 \equiv 1 \bmod 3$ then the above argument works except there is no need for the curve e. The case $2m+1 \equiv 0 \bmod 3$ was shown when $\lfloor \frac{2m+1}{3} \rfloor + \lfloor \frac{2(g-m)+1}{3} \rfloor = \lfloor \frac{2g+2}{3} \rfloor$.

Similarly, if the outside of $\bar{z}$ contains more than 5 marked points, that is $2(g-m)+1 > 5$, then the same argument holds showing the image of the two maximal disjoint collection of genus 1 curves must contain the same number of marked points. This means the genus of z must be m else $\phi^*(z)$ would be a genus 1 symmetric separating curve, forcing $\overline{\phi^*(z)}$ to contain 3 additional marked points. Because ϕ^* preserves disjointness, this does not leave enough room for the maximal collections on each of side of z.

Now if one side of $\bar{z}$ contains exactly 5 marked points, that is $2m+1 = 5$, then the above argument gives the desired result by choosing one genus 1 symmetric separating curve on the side of $\bar{z}$ with 5 marked points, call this curve f. If $\phi^*(z)$ is a genus 1 curve, then by disjointness $\overline{\phi^*(z)}$ and $\overline{\phi^*(f)}$ must contain 6 marked points. We use the previous methods to obtain a maximal collection of curves on the other side of $\bar{z}$ whose images contain at least the same number of marked points, but this cannot happen; we would need additional marked points. □

5. Symmetric curves

We show in this section how to extend $\phi^* \in \mathrm{Aut}(C_{\mathrm{ssep}}(S_g))$ to a map on all symmetric curves thus showing ϕ^* induces $\phi^{**} \in \mathrm{Aut}(C_{sym}(S_{0,0,2g+2}))$. In order to do this, we use the idea of "sharing pairs," previously defined by Brendle–Margalit in [7], with the added condition that all curves are symmetric. Many of the proofs in [7] hold with the added symmetric condition, we will use a more recent result of Putman [23] to prove well-definedness.

Sharing pairs. A nonseparating symmetric curve β is uniquely determined by a pair of distinct genus 1 symmetric separating curves, which bound subsurfaces that intersect in an annulus, with the condition that β lies on both of the corresponding genus 1 subsurfaces.

Let a and b be genus 1 symmetric separating curves bounding genus 1 subsurfaces S_a and S_b of S respectively. We say a and b *share* a symmetric nonseparating curve β if $S_a \cap S_b$ is an annulus containing β as its core and $S - (S_a \cap S_b)$ is connected. We say that a and b form a *sharing pair* for β. See Figure 10 for an example of a sharing pair.

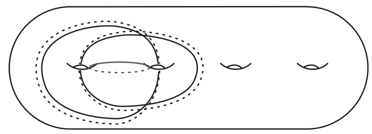


FIGURE 10. A sharing pair in the surface S.

For much of this paper it will be useful to consider the projection of sharing pairs in $S_{0,0,2g+2}$. Figure 11 shows the projection of Figure 10.

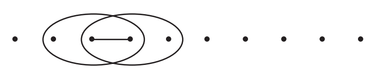


FIGURE 11. The projection of a sharing pair in $S_{0,0,2g+2}$.

Hence we see a sharing pair viewed in $S_{0,0,2g+2}$ is simply two 3-curves which "share" two marked points, or equivalently an arc.

The map ϕ^{**} is defined on symmetric nonseparating curves as follows. If $\mathcal{P}(\beta) = \{a, b\}$ is a sharing pair for a symmetric nonseparating curve β, then $\phi^{**}(\beta)$ is the curve shared by $\phi^*(\mathcal{P}(\beta))$, that is, the curve shared by $\{\phi^*(a), \phi^*(b)\}$. In order to show that this extension of ϕ^* is well-defined on symmetric nonseparating curves, we need to show that $\phi^*(\mathcal{P}(\beta))$ is a sharing pair and that $\phi^{**}(\beta)$ is independent of the choice of $\mathcal{P}(\beta)$.

There is a useful characterization of sharing pairs introduced by Brendle–Margalit that we will show can also be realized symmetrically.

Lemma 5.1 (Brendle–Margalit, Lemma 4.1, [7])**.** *Let a and b be genus 1 separating curves in S. Then a and b are a sharing pair if and only if there exist separating curves w, x, y, and z in S with the following properties:*

- *z is a genus 2 curve bounding a genus 2 subsurface S_z.*
- *a and b are in S_z so that $i(a, b) \neq 0$.*
- *x and y are disjoint.*
- *w intersects z, but not a and not b.*
- *x intersects a and z, but not b.*
- *y intersects b and z, but not a.*

We show this configuration of curves can be realized symmetrically in Figure 12.

The following lemma is a special case of Proposition 4.2 in [7]. The proof is included here for completeness.

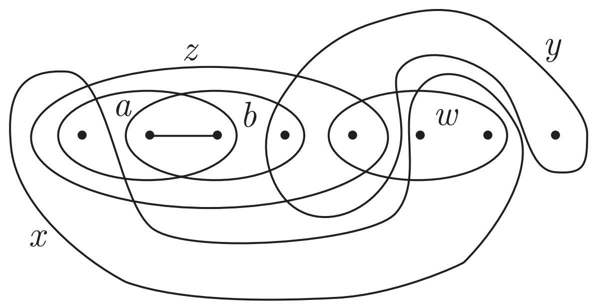


FIGURE 12. The projection of symmetric curves characterizing a sharing pair in $S_{0,0,2g+2}$.

Lemma 5.2. *If two genus one symmetric curves a and b in S form a sharing pair, then so do $\phi^*(a)$ and $\phi^*(b)$.*

Proof. Since a and b share a curve, there are characterizing curves w, x, y, and z as in Lemma 5.1. Each property of this collection of curves (disjointness, sides, genus) is preserved by ϕ^*, by Lemmas 4.3, 4.4, and 4.5, thus Lemma 5.1 implies $\phi^*(a)$ and $\phi^*(b)$ share a curve. □

We now have a function from $\mathrm{Aut}(C_{\mathrm{ssep}}(S_g))$ to the set of functions on symmetric curves. In order to show that this function is well-defined with respect to the choice of sharing pairs, we will use Lemma 2.4.

Let α be a symmetric nonseparating curve in S_g. Collapse the arc $\bar{\alpha}$ in $S_{0,0,2g+2}$ to a point p. Let Σ_α denote the resulting surface $S_{0,0,2g+1}$. Sharing pairs of α descend bijectively to arcs between marked points (or 2-curves) intersecting only at p in Σ_α. Let X_α be a *sharing pair graph* where the vertices are pairs of arcs (a, b) in Σ_α that lift in S_g to a sharing pair of α. Two vertices $(a.b)$ and (c, d) share an edge if precisely two of the four arcs a, b, c, and d are equal, and the three distinct arcs pairwise intersect at p. So the lifts of any two of the three arcs share α in S_g. In order to show ϕ^{**} is well defined we only need to show that X_α is connected.

Lemma 5.3. *The graph X_α is connected.*

Proof. To apply Lemma 2.4 let $X = X_\alpha$, let G be the subgroup of $\mathrm{Mod}(\Sigma_\alpha)$ fixing p, and fix a base point $v = (a, b)$. The group G acts transitively on the set of all sharing pairs of p, so the first condition holds. Let $c_1, c_2, \ldots, c_{2g+1}$ be the curves shown in Figure 13.

Denote by H the set consisting of half-twists

$$H_1, H_2, \ldots, H_n \quad \text{about} \quad c_1, c_2, \ldots, c_{2g+1}$$

together with the Dehn twists T_a and T_b. By symmetry and disjointness, it suffices to show that there is a path in X_α between (a, b) and its images under $H_1^\pm$ and $T_a^\pm$. The half-twists $H_1^\pm$ sends (a, b) to the adjacent vertices $(b, H_1^\pm(a))$ since $a, b, H_1^\pm$ share an edge. For the Dehn twist T_a there is a

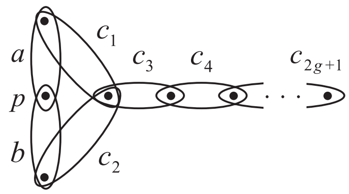


FIGURE 13. The curves needed for the proof of Lemma 5.3.

path $(a, T_a(b))$ to $(a, H_2(b))$ to (a, b), and for T_a^{-1} there is a path $(a, T_a^{-1}(b))$ to $(a, H_1(a))$ to (a, b). Thus by Lemma 2.4 the graph X_α is connected. □

Now that we have extended $\phi \in \mathrm{Aut}(\mathcal{SI}(S_g))$ to a function on all symmetric curves, we observe that ϕ^{**} preserves disjointness between symmetric curves making $\phi^{**} \in \mathrm{Aut}(C_{sym}(S_g))$.

Lemma 5.4. *Suppose a and b are symmetric curves in S. Then*

$$i(\phi^{**}(a), \phi^{**}(b)) = 0 \quad \textit{if and only if} \quad i(a, b) = 0.$$

Proof. This proof is, in essence, a restriction of the result by Brendle–Margalit in [7] (Section 4.3). Their argument fails for the superinjective case they were trying to prove, but holds for automorphisms and is outlined here. The argument breaks down into three cases:

(1) If a and b are both separating the result is Lemma 4.3.
(2) If a and b are both nonseparating, then the result follows from that fact that a and b are disjoint if and only if there are disjoint sharing pairs representing a and b.
(3) If a is separating, and b is nonseparating, then a and b are disjoint if and only if either a is a part of a sharing pair for b or b has a sharing pair whose curves are disjoint from a. □

An immediate consequence of Lemma 5.4 is that ϕ^{**} is a simplicial map on all symmetric curves.

6. Presymmetric curves

The goal of this section is to extend $\phi^{**} \in \mathrm{Aut}(C_{sep}(S_g))$ to a function including presymmetric curves. We will use the following result of Ivanov to show that ϕ^{**} preserves certain topological properties of symmetric curves.

Lemma 6.1 (Ivanov, Lemma 8.2A, [18])**.** *Suppose that the genus of S is at least 2. Let α_1 and α_2 be isotopy classes of two nontrivial curves on S. Then the geometric intersection number $i(\alpha_1, \alpha_2) = 1$ if and only if there exist isotopy classes α_3, α_4, and α_5 of nontrivial curves having the following two properties:*

(1) *$i(\alpha_i, \alpha_j) = 0$ if and only if the i-th and j-th curves in Figure 14 are disjoint.*

(2) *If* α_4 *is the isotopy class of a curve* C_4*, then* C_4 *divides* S *into two parts, one of which is a torus with one hole containing some representatives of the isotopy classes* α_1 *and* α_2*.*

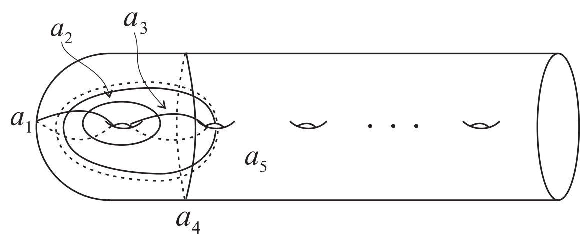


FIGURE 14. Curves characterizing geometric intersection 1.

Lemma 6.2. *Suppose* a *and* b *are symmetric curves in* S*, then*

$$i(\phi^{**}(a), \phi^{**}(b)) = 1 \quad \text{if and only if} \quad i(a, b) = 1.$$

Proof. This follows from Lemma 6.1 since this characterization can be realized symmetrically (see Figure 15) and only depends on preserving disjointness and genus 1 symmetric separating curves. □

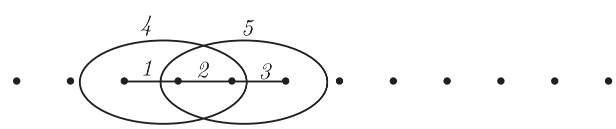


FIGURE 15. A collection of curves characterizing intersection one.

We will extend ϕ^{**} to presymmetric curves via a "symmetric spine."

Symmetric spines. Let a be a presymmetric curve in S, so that $\bar{a}$ is an even curve with $2k$ marked points on one side of $\bar{a}$ in $S_{0,0,2g+2}$. A *symmetric spine* is a collection of symmetric curves $\{c_1, \ldots, c_{2k-1}\}$ on S_g, so that $i(c_i, c_j) = 1$ if $j = i + 1$ and 0 otherwise. Because $\{c_i\}$ is a collection of an odd number of curves, the boundary of a regular neighborhood of $\cup c_i$ will have two components. We say $\{c_i\}$ is a *symmetric spine for* a, if a is one of the boundary components of a regular neighborhood of $\cup c_i$.

Lemma 6.3. *If* $\{c_i\}$ *is a symmetric spine, then* $\{\phi^{**}(c_i)\}$ *is a symmetric spine.*

Proof. Follows directly from Lemma 5.4 and Lemma 6.2. □

In order to extend ϕ^{**} to a presymmetric curve a, we will choose a symmetric spine for a, and then we will set $\overline{\hat{\phi}(a)}$ equal to the boundary component of a regular neighborhood of $\cup\overline{\phi^{**}(c_i)}$ in $S_{0,0,2g+2}$.

Next we will need to show this extension of ϕ^{**} does not depend on the choice of symmetric spine of a. Let $\{c_i\}$ and $\{d_i\}$ be two symmetric spines for a presymmetric curve a in S. By Lemma 6.3 we know $\{\phi^{**}(c_i)\}$ and $\{\phi^{**}(d_i)\}$ are also symmetric spines. It suffices to show regular neighborhoods of $\cup\overline{\phi^{**}(c_i)}$ and $\cup\overline{\phi^{**}(d_i)}$ in $S_{0,0,2g+2}$ are isotopic.

First consider $\{\bar{c_i}\}$ and $\{\bar{d_i}\}$ in $S_{0,0,2g+2}$. Clearly the boundary of a regular neighborhood of both $\cup\bar{c_i}$ and $\cup\bar{d_i}$ is $\bar{a}$. The proof reduces to two cases.

Case 1. Suppose $\cup\bar{c_i}$ and $\cup\bar{d_i}$ share the same marked points in $S_{0,0,2g+2}$.

Lemma 6.4. *There exist symmetric curves e and f so that $\bar{e}\cup\bar{f}$ separate $S_{0,0,2g+2}$ into two subsurfaces where $\cup\bar{c_i}$ and $\cup\bar{d_i}$ are contained in the same subsurface, S', of $S_{0,0,2g+2}$, and S' only contains marked points that intersect $\cup\bar{c_i}$.*

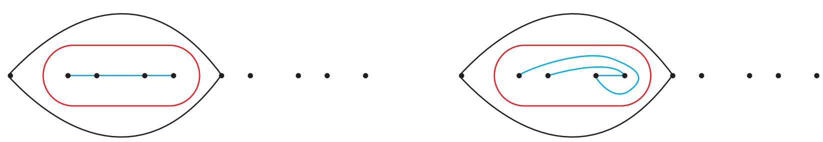


FIGURE 16. And example of the construction used in Lemma 6.4.

Proof. Choose two marked points not used by $\cup\bar{c_i}$ or $\cup\bar{d_i}$. Then connect the marked points by two arcs, $\bar{e}$ and $\bar{f}$, so that the subsurface S' is as desired. This construction can be done because $\{c_i\}$ and $\{d_i\}$ are symmetric spines for a, see Figure 16 for an example of this construction. □

Clearly $\hat{\phi}(e)$ and $\hat{\phi}(f)$ are defined because they are symmetric curves. Moreover, $\hat{\phi}(e)$ and $\hat{\phi}(f)$ intersect at least twice, specifically at their endpoints because of Lemma 5.4 and 6.2.

By Lemma 5.4 it is clear that $\overline{\hat{\phi}(e)}$ and $\overline{\hat{\phi}(f)}$ separate $S_{0,0,2g+2}$. In addition, $\cup\overline{\hat{\phi}(c_i)}$ and $\cup\overline{\hat{\phi}(d_i)}$ are in the same subsurface of $S_{0,0,2g+2}-(\overline{\hat{\phi}(e)}\cup\overline{\hat{\phi}(f)})$; we will denote this subsurface of $S_{0,0,2g+2}$ as S''.

Lemma 6.5. *The subsurface S'' contains only marked points used in $\{\overline{\hat{\phi}(c_i)}\}$.*

Proof. Suppose S'' contains a marked point, x, that is not used in $\{\overline{\hat{\phi}(c_i)}\}$. Pick any marked point used in $\{\overline{\hat{\phi}(c_i)}\}$, we will call is y. Then there exists an arc, k, from x to y contained in S'', this means $i(k,\overline{\hat{\phi}(e)}) = i(k,\overline{\hat{\phi}(f)}) = 0$. By Lemma 5.4, we can conclude $i(k,\bar{e}) = i(k,\bar{f}) = 0$. But this cannot happen by the choice of e and f. □

Now we are ready to show that the boundary of regular neighborhoods of $\cup\overline{\hat{\phi}(c_i)}$ and $\cup\overline{\hat{\phi}(d_i)}$ are isotopic, thus showing the extension of ϕ^{**} to all presymmetric curves $\hat{\phi}$ is well-defined.

Proposition 6.6. *Let $\{c_i\}$ and $\{d_i\}$ be two symmetric spines for a presymmetric curve a in S, then regular neighborhoods of $\cup\hat{\phi}(c_i)$ and $\cup\hat{\phi}(d_i)$ have an isotopic boundary component.*

Proof. If we consider the subsurface obtained by cutting $S_{0,0,2g+2}$ along $\cup\overline{\hat{\phi}(c_i)}$, $\overline{\hat{\phi}(e)}$, and $\overline{\hat{\phi}(f)}$, we will have an annulus. Up to homotopy there is only one curve on the annulus and it is isotopic to the boundary of $\cup\overline{\hat{\phi}(c_i)}$. Similarly if we consider the subsurface obtained by cutting $S_{0,0,2g+2}$ along $\cup\overline{\hat{\phi}(d_i)}$, $\overline{\hat{\phi}(e)}$, and $\overline{\hat{\phi}(f)}$, we will have an another annulus, containing one curve homotopic to the boundary of $\cup\overline{\hat{\phi}(d_i)}$. But because both of these annuli also share one boundary component, namely the one obtaining from $\overline{\hat{\phi}(e)}$ and $\overline{\hat{\phi}(f)}$, these two curves are isotopic. □

Case 2. Suppose $\cup\bar{c}_i$ and $\cup\bar{d}_i$ are disjoint, implying they do not share any of the same marked points in $S_{0,0,2g+2}$.

In this situation, $\cup\bar{c}_i$ and $\cup\bar{d}_i$ use all $2g+2$ marked points. By Lemma 5.4 $\{\hat{\phi}(c_i)\}$ and $\{\hat{\phi}(d_i)\}$ are disjoint, hence $\cup\overline{\hat{\phi}(c_i)}$ and $\cup\overline{\hat{\phi}(d_i)}$ are disjoint. Thus, if we cut $S_{0,0,2g+2}$ along $\cup\overline{\hat{\phi}(c_i)}$ and $\cup\overline{\hat{\phi}(d_i)}$ we will get an annulus, hence the boundary components of regular neighborhoods of $\cup\overline{\hat{\phi}(c_i)}$ and $\cup\overline{\hat{\phi}(d_i)}$ are isotopic.

Lemma 6.7. *Let a and b be presymmetric curves in S. Then the intersection number $i(\overline{\hat{\phi}(a)}, \overline{\hat{\phi}(b)}) = 0$ if and only if $i(\bar{a}, \bar{b}) = 0$.*

Proof. The projections of the presymmetric curves a and b, $\bar{a}$ and $\bar{b}$, are disjoint if and only if they have disjoint symmetric spines. Hence by Lemma 5.4, the result is shown. □

Thus ϕ^{**} induces a map $\hat{\phi} \in \mathrm{Aut}(C(S_{0,0,2g+2}))$. Hence we have shown our main theorems.

7. Proof of Theorem 3.4

This section is dedicated to the proof of Theorem 3.4. We will begin with several lemmas needed to prove this main result.

Lemma 7.1. *Let a and c be two disjoint symmetric separating curves. Then there exists a maximal abelian subgroup of $\mathcal{SI}(S_g)$ containing both T_a and T_c as generators.*

Proof. Let a and c be two disjoint symmetric separating curves. The projection of a in $S_{0,0,2g+2}$ partitions the $2g+2$ marked points into two collections each containing an odd number of marked points, say $2p+1$ and $2q+1$.

The curve c partitions one of these sets. Without loss of generality, we will assume c partitions the collection of $2p+1$ marked points into two sets of $2r+1$ and $2s$ marked points. Note that $(2r+1)+(2s)+(2q+1)=2g+2$.

We can find $r-1$ disjoint nontrivial odd curves on the $2r+1$ marked points, $s-1$ odd curves on the $2s$ marked points, and $q-1$ odd curves on the $2q+1$ marked points. Normally when there are an even number of marked points $2t$ for example we would only have $t-2$ disjoint odd curves because we do not allow curves that partition the marked points into sets of 1 and $2t-1$. In our situation though, partitions of 1 marked point are allowed because the partitions of the entire set of marked points will include more than 1 marked point.

So we have a total of $(r-1)+(s-1)+(q-1)+2=g-1$ disjoint odd curves (which include a and c). Hence the Dehn twists about these curves will generate a maximal abelian subgroup of $\mathcal{SI}(S_g)$ as desired. □

Proposition 7.2. *Let a be a symmetric separating curve in S_g. Then T_a^l, where $l \neq 0$, is not a $PCPA$.*

Proof. Suppose T_a^l is a $PCPA$. Then we know there is a nontrivial element $g \in \mathcal{SI}(S)$ such that $g \neq (T_a^l)^k$ which satisfies the conditions for a $PCPA$. We know $T_a^l g$ is not supported on an annulus, hence there is a symmetric separating curve, c, such that $i(a,c)=0$ and $[T_c, g] \neq 1$. Then by Lemma 7.1, there exists a maximal abelian subgroup containing T_a^l and T_c as generators. But clearly $T_a^l g$ and T_c do not commute, hence the original statement must be true. □

Lemma 7.3. *Let $f \in \mathcal{SI}(S)$ be pseudo-Anosov on at most one component of $S - C_f$, call it Q. Given any maximal abelian subgroup $\langle f, x_1, x_2, \ldots, x_{g-2}\rangle$, if $Z(C_{\mathcal{SI}}(x_i)) = \mathbb{Z}$ for each i, then each x_i is supported on $S - Q$.*

Proof. Suppose x_i is not supported on $S-Q$. Since $[f, x_i]=1$, Lemma 3.2 implies that f_Q commutes with x_{iQ}. Hence $x_{iQ}{=}f_Q^k$ for some $k \neq 0$. But because $Z(C_{\mathcal{SI}}(x_i)) = \mathbb{Z}$, we can deduce that $x_i = f^k$. This contradicts the maximality of $\langle f, x_1, x_2, \ldots, x_{g-2}\rangle$. Thus x_i must be supported on $S-Q$. □

Lemma 7.4. *If f satisfies Conditions (1), (2), (3), and (4) of Theorem 3.4, then $Z(C_{\mathcal{I}}(f)) = \mathbb{Z}$.*

Proof. Suppose not, that is suppose $Z(C_{\mathcal{I}}(f)) \neq \mathbb{Z}$. If $Z(C_{\mathcal{I}}(f)) \neq \mathbb{Z}$ then we also know f is not a power of a Dehn twist by Proposition 3.3. This implies f must be pseudo-Anosov on some subsurface Q of S where Q not an annulus. Otherwise f would be a product of twists about curves in C_f, which would imply that there is a symmetric separating curve in C_f contradicting (1).

Moreover, we know that f is pseudo-Anosov on exactly one subsurface, Q, else (1) would be contradicted. In addition, we know there must be a symmetric separating curve c supported on Q. Hence by Lemma 7.3 we know f is a $PCPA$ (let $g = T_c$), which is a contradiction. Hence we can conclude $Z(C_{\mathcal{I}}(f)) = \mathbb{Z}$. □

Using the previous lemmas we are now ready to prove the main theorem which gives an algebraic characterization of Dehn twists about symmetric separating curves.

Proof of Theorem 3.4. Suppose $f = T_\gamma^k$ where γ is a symmetric separating curve and $k \neq 0$. Let Q_1 and Q_2 be the two components of $S - \gamma$ and let $\mathcal{SI}(Q_1)$ and $\mathcal{SI}(Q_2)$ denote the subgroups of $\mathcal{SI}(S)$ supported on Q_1 and Q_2 respectively, similarly $\mathcal{I}(Q_i)$. The group $\mathcal{SI}(Q_i)$ is the subgroup of the group generated by twists about symmetric separating curves on Q_i containing elements which fix the boundary of Q_i pointwise. Note that if $\mathcal{SI}'(Q_i)$ is the corresponding group where the boundary need not be fixed pointwise, we have the exact sequence:

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{SI}(Q_i) \longrightarrow \mathcal{SI}'(Q_i) \longrightarrow 1$$

Since $g \geq 3$ one of the Q_i's, say Q_1, has genus at least 2. Furthermore $\mathcal{SI}(Q_1)$ contains two independent pseudo-Anosov maps u_1 and v_1. See Chapter 13 of [13] for additional details about constructing pseudo-Anosov maps.

If the genus of $Q_2 = 1$, then $\mathcal{SI}(Q_2) = 1$ and we have the following exact sequence:

$$1 \longrightarrow \langle T_\gamma \rangle \longrightarrow C_{\mathcal{SI}(S)}(T_\gamma) \longrightarrow \mathcal{SI}(Q_1) \longrightarrow 1$$

Hence we can deduce that

$$Z(C_{\mathcal{SI}(S)}(T_\gamma)) = Z(\mathcal{SI}(Q_1)) = \mathbb{Z}$$

where $\mathbb{Z}$ is generated by T_γ. We use Lemma 3.2 and the fact that no nontrivial mapping class commutes with two independent pseudo-Anosov maps, in our case u_1 and v_1.

If the genus of $Q_2 \geq 2$, then $\mathcal{SI}(Q_2)$ contains two independent pseudo-Anosov maps u_2 and v_2. If $h \in Z(C_{\mathcal{SI}(S)}(T_\gamma))$, then h_{Q_i} must commute with both u_i and v_i for $i = 1$ or 2. Since u_i and v_i are independent, it is clear that $h_{Q_i} = Id$, $i = 1$ or 2. Thus $Z(C_{\mathcal{SI}(S)}(T_\gamma))$ can only contain powers of T_γ, and hence is infinite cyclic.

Proposition 7.2 gives Condition (2). Moreover because there exists a symmetric separating curve disjoint from γ, $C_{\mathcal{SI}}(f)$ contains a $\mathbb{Z}^2$ subgroup showing (3) is true. Lastly, Condition (4) follows from Proposition 2.3.

Suppose $f \in \mathcal{SI}$ satisfies Conditions (1), (2), (3) and (4). We know $f \in \mathcal{SI} \subset \mathcal{I}$ is pure (see [17]), so if f leaves a system C of mutually disjoint, nonhomotopic essential curves invariant, then f leaves each component of C invariant. Let E_f be the canonical reduction system for f. Let d denote the maximal rank of an abelian subgroup in $\mathcal{I}(S)$ generated by Dehn twists about separating curves or bounding pairs in E_f. We will look at cases according to d and argue that $d = 1$.

Suppose $d \geq 2$. Then E_f must contain two distinct elements T_α and T_β where each of α and β is a separating curve or a bounding pair. Note that any $h \in C_{\mathcal{I}(S)}(f)$ leaves E_f invariant, ([3], Lemma 2.6: $\sigma(E_f) = E_{\sigma f \sigma^{-1}}$) hence T_α and T_β commute with any such h. Thus

$$Z(C_{\mathcal{I}(S)}(f)) \supseteq \langle T_\alpha, T_\beta \rangle \cong \mathbb{Z}^2.$$

This contradicts Lemma 7.4, so we can assume $d \leq 1$.

We can also assume E_f is nonempty otherwise f is a pseudo-Anosov element contradicting Condition (3).

Next, as a step to proving that f is a power of a Dehn twist about a symmetric separating curve, we show that none of the maps f_Q, where Q is a component of $S - E$, is a pseudo-Anosov homeomorphism. Because f is pure, we know every such map, f_Q, is either pseudo-Anosov or the identity.

Now we will consider what a component Q of $S - E$ on which f_Q is pseudo-Anosov must look like. Since Q is a proper subsurface of S it is homeomorphic to $\Sigma_{g,r}$ for some $g \geq 0$ and $r \geq 1$.

Case 1. If $g = 0$ then for Q to admit any pseudo-Anosov element it must be that $r \geq 4$.

Case 2. If $g = 1$ and $r = 1$, then ∂Q is a genus 1 separating curve and so $f_Q \in \mathcal{I}(Q) = \mathcal{I}(\Sigma_{1,1})$. But $\mathcal{I}(\Sigma_{1,1})$ is generated by the twist about its boundary curve, hence does not contain a pseudo-Anosov element.

Case 3. Suppose $g \geq 2$ and $r = 1$ or 2, or that $g = 1$ and $r = 2$. Since Q is a component of $S - E$, it must be that $\partial Q = \alpha$ where

(1) α is a separating curve, or
(2) α is a bounding pair.

Otherwise ∂Q would consist of a separating curve and another curve (which must be separating too for Q to be a component of $S - E$), but this contradicts the fact that $d \leq 1$. Now let $\hat{f_Q}$ be the extension of f_Q to the whole surface where it is the identity on $S - Q$. Then we have

$$Z(C_{\mathcal{I}(S)}(f)) \supseteq \langle \hat{f_Q}, T_\alpha \rangle \cong \mathbb{Z}^2$$

which contradicts Lemma 7.4.

Remaining cases. We are left with the cases $g = 0$ and $r \geq 4$, or $g \geq 1$ and $r \geq 3$.

Claim. *There exists a $k \geq 3$ so that there is a union $C = \beta_1 \cup \cdots \cup \beta_k$ of components of ∂Q so that C bounds in S but no subcollection of the β_i's bound.*

Proof. Since $r \geq 3$, if ∂Q has no bounding pairs or separating curves, then we can take C to be any minimal subset of components of ∂Q which separate S.

Suppose ∂Q contains a bounding pair α. Now $r \neq 3$ and $r \neq 4$ for otherwise $\partial Q \backslash \alpha$ being homologous to α, would be separating (or a bounding pair) in S contradicting the fact that $d \leq 1$. Hence we have $r \geq 5$. Since $d = 1$ in this case, a minimal subcollection of components of ∂Q which bound and which has at least 3 elements exists.

Suppose ∂Q contains a separating curve α. Now $r \neq 3$ for otherwise $\partial Q \backslash \alpha$ being homologous to α, would be a bounding pair in S contradicting the fact that $d \leq 1$. Hence we have $r \geq 4$, so we can conclude there is a minimal bounding subcollection with at least 3 elements. □

Now choose such a collection $C = \beta_1 \cup \cdots \cup \beta_k$. Let γ be a separating curve in S which lies in $S \backslash Q$ and which together with C bounds a genus 0 subsurface of S.

By Condition (4) we know $\max_{\mathcal{SI}}(f) = g - 1$, so there exists some free abelian group $A < \mathcal{SI}(S)$ of rank $g-1$ which contains f. In the following arguments we will argue that certain situations cannot happen by contradicting the maximality of A. Note that any $h \in \mathcal{I}(S)$ which leaves the components of Q invariant must also leave γ invariant.

Let U_1 be the component of S_γ which does not contain Q and let $U_2 = S_\gamma \backslash U_1$. Since every $a \in A$ commutes with f, it leaves E_f, the canonical reduction system of f, invariant. Hence $a(Q) = Q$ and so $a(\partial Q) = \partial Q$, and so $a(C) = C$ and $a(\gamma) = \gamma$. Let A_i with $i = 1, 2$ be the image of A under the reduction homomorphism $\pi_{U_i} : \mathcal{SI}(S_\gamma) \to \mathcal{SI}(U_i)$. Since γ is a separating curve, the natural inclusion $U_i \to S$ induces an injection

$$\mathcal{SI}(U_i) \to P_\gamma := \{p \in \mathcal{SI}(S) : p(\gamma) = \gamma\}$$

and we have a homomorphism

$$\psi : \mathcal{SI}(U_1) \times \mathcal{SI}(U_2) \to P_\gamma$$

sending x_1 and x_2 to the mapping class with is x_i on U_i. Note that this is well-defined since elements of $\mathcal{SI}(U_i)$ fix ∂U_i pointwise.

If γ is a symmetric separating curve then let γ_i with $i = 1, 2$ be a curve in U_i isotopic to γ. Then it is easy to see that the kernel of ψ is generated by $T_{\gamma_1} T_{\gamma_2}^{-1}$. Thus we have the following exact sequence:

$$1 \to \mathbb{Z} \to \mathcal{SI}(U_I) \times \mathcal{SI}(U_2) \to P_\gamma \to 1.$$

Restricting ψ to A and noting that by maximality $T_\gamma \in A$, we have the following restriction:

$$1 \to \mathbb{Z} \to A_1 \times A_2 \to A \to 1.$$

So we have the following:

$$\begin{aligned} \operatorname{rank}(A) &\leq \operatorname{rank}(A_1) + \operatorname{rank}(A_2) - 1 \\ &\leq \max(\mathcal{SI}(U_1)) + \max(\mathcal{SI}(U_2)) - 1 \\ &\leq g_1 + g_2 - 1. \end{aligned}$$

Since $r \geq 3$ we have that

$$g_1 + g_2 \leq genus(S) - 2$$

so

$$\begin{aligned} rank(A) &\leq (g_1 + g_2) - 1 \\ &\leq genus(S) - 3 \end{aligned}$$

which contradicts the fact that A is maximal.

If γ is not a symmetric separating curve then the kernel of ψ is empty. Thus we have

$$\mathcal{SI}(U_I) \times \mathcal{SI}(U_2) \cong P_\gamma$$

Restricting ψ to A, we have the following:

$$A_1 \times A_2 \cong A.$$

Hence we have the following:

$$\begin{aligned} \mathrm{rank}(A) &= \mathrm{rank}(A_1) + \mathrm{rank}(A_2) \\ &\leq \max(\mathcal{SI}(U_i)) + \max(\mathcal{SI}(U_2)) \\ &\leq g_1 + g_2. \end{aligned}$$

Since $r \geq 3$ we have that

$$g_1 + g_2 \leq \mathrm{genus}(S) - 2$$

so

$$\begin{aligned} \mathrm{rank}(A) &\leq (g_1 + g_2) \\ &\leq \mathrm{genus}(S) - 2 \end{aligned}$$

which contradicts the fact that A is maximal. Hence we have proven that f acts by the identity on every component of $S - E$.

We now have that f must be a multitwist about curves in E_f. Since $f \in \mathcal{I}(S)$, f must be a multitwist about a union of separating curves and bounding pairs. Since $d \leq 1$ there is only one such curve or pair. Note, $d \neq 0$, otherwise f would not be in $\mathcal{I}$. Since $f \in \mathcal{SI}(S)$, f must be a power of a Dehn twist about a symmetric separating curve. □

Proof of Corollary 3.5. It is clear that if $\phi \in \mathrm{Aut}(\mathcal{SI}(S))$. Then f satisfies Conditions (1), (3) and (4) of Theorem 3.4 if and only if $\phi(f)$ does. What remains to be shown is that f satisfies Condition (2) if and only if $\phi(f)$ does.

Suppose f is a $PCPA$. Thus f is pseudo-Anosov on at most one component Q of $S - C_f$. We know there exists a $g \in \mathcal{SI}(S)$ such that given any maximal abelian subgroup $(f) = \langle f, x_1, x_2, \ldots, x_{g-2} \rangle$, where $Z(C_{\mathcal{SI}}(x_i)) = \mathbb{Z}$ for each i, the group $\langle fg, x_1, x_2, \ldots, x_{g-2} \rangle$ is also a maximal abelian subgroup of $\mathcal{SI}(S_g)$.

We will show $\phi(f)$ is a $PCPA$. We know $\phi(f)$ is pseudo-Anosov on at most one component, Q_ϕ, of $S - C_{\phi(f)}$, else $Z(C_{\mathcal{SI}}(\phi(f))) \neq \mathbb{Z}$. We claim

$\phi(g)$ is the desired element of $\mathcal{SI}(S)$. Given any maximal abelian subgroup $\langle \phi(f), y_1, y_2, \ldots, y_{g-2} \rangle$, where $Z(C_{\mathcal{SI}}(y_i)) = \mathbb{Z}$ for each i, then

$$\langle f, \phi^{-1}(y_1), \phi^{-1}(y_2), \ldots, \phi^{-1}(y_{g-2}) \rangle$$

is a maximal abelian subgroup for f where $Z(C_{\mathcal{SI}}(\phi^{-1}(y_i))) = \mathbb{Z}$ for each i. Since f is a $PCPA$, we know $\langle fg, \phi^{-1}(y_1), \phi^{-1}(y_2), \ldots, \phi^{-1}(y_{g-2}) \rangle$ is also a maximal abelian subgroup. Thus

$$\langle \phi(f)\phi(g), y_1, y_2, \ldots, y_{g-2} \rangle$$

is a maximal abelian subgroup as desired. Thus $\phi(f)$ is a $PCPA$. □

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