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Everywhere divergence of the one-sided ergodic Hilbert transform for circle rotations by Liouville numbers

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Abstract. We prove some results on the behavior of infinite sums of the form $\sum f \circ T^n(x)\frac{1}{n}$, where $T : S^1 \to S^1$ is an irrational circle rotation and f is a mean-zero function on S^1. In particular, we show that for a certain class of functions f, there are Liouville α for which this sum diverges everywhere and Liouville α for which the sum converges everywhere.

Contents

1. Introduction

Let $(X, \mathcal{B}, \mu)$ be a probability measure space. Let T be an invertible, measure-preserving, ergodic transformation on $(X, \mathcal{B}, \mu)$. Let $\sum b_n$ be a

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positive, divergent series. Under what conditions do sums of the form

$$\sum_{n=1}^{\infty} f \circ T^n(x) b_n \tag{1}$$

converge or diverge?

In the specific case $b_n = \frac{1}{n}$, the sum (1) is known as the *one-sided ergodic Hilbert transform* (EHT) of f. Its convergence properties are of interest in part because convergence of Equation (1) ensures convergence of the Birkhoff averages, and so results on the convergence or divergence of Equation (1) provide a stronger version of Birkhoff's theorem, or indicate that no strengthening in this direction is possible.

Conditions under which the one-sided ergodic Hilbert transform and the more general Equation (1) converge or diverge are very well studied. The first study of convergence was by Izumi in [20]. Following Izumi's work, Halmos showed in [12] that if the measure is nonatomic, then there are L^2 functions f for which L^2-convergence of the EHT fails. Dowker and Erdös showed that there are L^∞ functions for which the more general sum (1) diverges for almost every x ([9]).

Kakutani and Petersen in [15] extended the results of Dowker and Erdös to show that mean zero functions for which the supremum of the norms of the partial sums of (1) is infinite for a.e. x always exist in L^∞ (see, similarly, [18]). In [8], del Junco and Rosenblatt further extend this work. They work in very general settings, allowing a range of transformations T, sequences b_n, function spaces from which f is chosen, and a number of different summation processes. They show that a.e.-divergence is generic (a dense G_δ subset) in the function f in these settings. Their methods also show that a.e. divergence occurs for a generic mean-zero continuous function.

The case of continuous functions is also studied in [10] where the authors prove that given any irrational α, a continuous mean-zero function exists such that the EHT diverges for all x and that, under certain conditions on its Fourier series, given any nonpolynomial continuous function there are irrational α for which the EHT diverges at all x. Their paper presents a number of other results on the interplay of conditions on the Fourier series of f, the diophantine properties of α, and the convergence in various norms of the EHT.

Divergent sum behavior has subsequently been investigated in very general contexts. [2] provides a monograph-length treatment of the subject, and [1] provides a good overview of work in the general setting of contracting operators on Banach spaces. Further results can also be found in [7], [6], [5], and [21].

In contrast to these general settings and the nonconstructive proofs that appear, we will demonstrate divergence in the case of the simple and well-understood dynamics of circle rotations, with specific random variables f that are quite simple — essentially indicator functions of intervals.

Let $\alpha \in (0,1)$ be an irrational number and let $T := R_\alpha$ be the rotation by α on $S^1 = \mathbb{R}/\mathbb{Z}$. That is, $Tx = x + \alpha \pmod 1$.

In Section 3 below we prove our first main theorem:

Theorem 1. *Let $b_n = \frac{1}{n}$ and let $f = \sum_{i=1}^{B} v_i \chi_{U_i}$ where $\{U_i\}$ is a partition of $[0,1]$ by intervals, satisfying the following three conditions:*

(i) $\int f = 0$.
(ii) *The values $\{v_i\}$ generate a discrete additive subgroup of $\mathbb{R}$.*
(iii) *There exists some integer $d > 1$ such that whenever $\sum_{j=1}^{l} v_{i_j} = 0$, d divides l.*

Then there are irrational α such that (1) diverges for $T = R_\alpha$ and all points x. Such α can be provided explicitly in terms of the continued fraction expansion.

The second and third conditions on f say that a sum of some number of f's values which is not divisible by d will be uniformly bounded away from zero. We note that these functions are Riemann-integrable, and form a dense subset of $L^1([0,1])$. Another way to think of condition (iii) is as follows. Let $e : \mathbb{Z}^B \to \mathbb{R}$ by $(m_1, \dots, m_B) \mapsto \sum m_i v_i$. Then consider $s : \ker(e) \to \mathbb{Z}$ by $(m_1, \dots, m_B) \mapsto \sum m_i$. Condition (iii) is that s is not surjective.

A simple case arises when f takes only the values ± 1, in which case $d = 2$. We will call such functions f *mean-zero indicator functions* for a finite union of intervals. More generally, conditions (ii) and (iii) are satisfied when U is a union of finitely many intervals with measure $m(U) \in \mathbb{Q}$ and $f = \chi_U - m(U)$ (see Lemma 2.2). Finiteness of the set of intervals $\{U_i\}$ is important — given α, the reader can easily construct a mean-zero indicator function for a *countable* union of intervals for which divergence everywhere will fail.

Previous work on this problem has mainly used tools from functional analysis, and has produced results for almost every x. Some exceptions to this can be found in [10] and [2], but in each case some regularity of f or additional assumptions on its Fourier series are required. A key difference in Theorem 1 — and our subsequent theorems — is that we prove divergence for all x.

There are several straightforward consequences of the proof of Theorem 1, which are proved in Corollary 3.3. First, the set of divergent α is dense. Second, α can be taken to depend on f only through basic data: the number B of intervals in U, an upper bound on the set $\{|v_i|\}$ and a lower bound on the positive elements of the additive subgroup generated by $\{v_i\}$. Third, one can replace $\frac{1}{n}$ with any sequence b_n such that $\sum n(b_{n+1} - b_n)$ diverges.

Fourth, by a careful choice of α, one may further ensure that

$$\sup_N \Big| \sum_{n=1}^{N} f \circ T^n(x) b_n \Big| = \infty \text{ for all } x.$$

The unbounded partial sums we find here are also present in the divergence result [10, Theorem 2.1].

In Section 4, we investigate the set of divergent α, for mean-zero indicator functions. As noted previously, this is a Lebesgue measure zero set; the following stronger result is true for $b_n = \frac{1}{n}$:

Theorem 2. *Let f be a mean-zero indicator function for a finite union of intervals. If α is not a Liouville number, then the ergodic Hilbert transform of f converges at all points. Hence the set of α for which the EHT of f diverges for any x has Hausdorff dimension zero.*

This theorem was previously known. In [15], Kakutani and Petersen note that convergence holds for non-Liouville α when f is the (mean-zero) indicator function of an interval. They remark that this result follows from number-theoretic results on the discrepancy for a non-Liouville number. As they do not detail the proof and as we have been unable to find it elsewhere in the literature, we include it in Section 4.1.

Theorem 2 leads to the question of whether the ergodic Hilbert transform diverges for all Liouville numbers. The answer is no, and we prove the following theorem in Section 4.2, which does not seem to follow from the type of arguments used for the proof of Theorem 2:

Theorem 3. *Let f be a mean-zero indicator function for a finite union of intervals. Then there are Liouville numbers α for which the ergodic Hilbert transform of f converges for all x.*

Our proof shows specifically how to produce such numbers using the continued fraction expansion. The technology of this proof also provides an alternate way to prove Theorem 2. The proof relies only on the mechanics of the continued fraction expansion, but it is considerably longer than Kakutani and Petersen's. We provide a brief discussion of how to use the elements of this proof to prove Theorem 2 in Section 4.4.

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2. Setup

We fix the following notation throughout the paper:

- $\alpha \in (0,1)$ is an irrational number.
- We write $\alpha = [a_1 a_2 a_3 \ldots]$ for the continued fraction expansion of α.
- $S[a_1 \ldots a_n]$ is the set of all irrational α with a continued fraction expansion beginning with $[a_1 \ldots a_n]$.
- $\frac{p_n}{q_n} = [a_1 \ldots a_n]$ is the n^{th} convergent to α. The q_n can be determined from the a_n via the recurrence relation

$$q_n = a_n q_{n-1} + q_{n-2}, \quad q_1 := a_1, \quad q_0 := 1.$$

- $T = R_\alpha$.
- $\langle\langle x \rangle\rangle$ denotes the distance from x to 0 in S^1.
- $U = \cup_{l=1}^{B} I_l$ is a finite union of intervals in S^1.
- For a fixed $x \in S^1$ and for integers $j_1 \le j_2$, let

$$\mathcal{O}_\alpha[j_1, j_2] = \left\{ R_\alpha^i(x) : j_1 \le i \le j_2 \right\}.$$

  We call $\mathcal{O}_\alpha[j_1, j_2]$ an *orbit segment* with *length* $j_2 - j_1 + 1$.
- If $\mathcal{O}_\alpha[j_1, j_2]$ is an orbit segment, let

$$s(\mathcal{O}_\alpha[j_1, j_2]) = \sum_{i=j_1}^{j_2} f \circ T^i(x).$$

The following basic facts relate the continued fraction expansion of α to the dynamics of T:

- For n odd, $T^{q_n}(0) = q_n\alpha \ (mod\ 1)$ is closer to 1 than to 0, and for n even, $T^{q_n}(0)$ is closer to 0 than to 1. In other words, the n^{th} convergent to α is an overestimate for n odd and an underestimate for n even (see, e.g., [17, Thm 8]).
- For irrational α, $\langle\langle T^{q_n} 0 \rangle\rangle < \langle\langle T^m 0 \rangle\rangle$ for any $m < q_{n+1}$. In other words, the convergents are precisely the best approximations of the second kind to α, i.e., $0 < m \le q_n$ and $\frac{l}{m} \ne \frac{p_n}{q_n}$ imply

$$|m\alpha - l| > |q_n\alpha - p_n|$$

  (see, e.g., [17, Thms 16 & 17]).

We also state here a pair of simple lemmas. The first will be used in the proof of Theorem 1:

Lemma 2.1. *For a fixed integer $d > 1$ and any irrational α, at least one of each successive pair q_n, q_{n+1} is not divisible by d.*

Proof. Using the recurrence relation $q_{m+1} = a_{m+1} q_m + q_{m-1}$, if d divides both q_m and q_{m+1}, it divides q_{m-1}. Inducting downward in the index, we would find that d divides $q_0 = 1$, a contradiction. □

We note that an easy way to arrange that d never divides q_m (for large m) is to pick the a_m so that some pair q_m, q_{m-1} are both not divisible by d (this is easy to arrange) and then choosing all subsequent a_m to be multiples of d.

Our second lemma justifies a remark in the Introduction on functions that satisfy the conditions of Theorem 1:

Lemma 2.2. *Let U be a finite union of intervals such that $m(U) \in \mathbb{Q}$. Then $f = \chi_U - m(U)$ satisfies the conditions of Theorem 1.*

Proof. That conditions (i) and (ii) are satisfied is obvious. For condition (iii), write $m(U) = p/q$ with p and q relatively prime. Let $d = q$. The function f takes values $v_1 = 1 - \frac{p}{q} = \frac{q-p}{p}$ and $v_2 = -\frac{p}{q}$. If $m_1 v_1 + m_2 v_2 = 0$, multiplying through by q we have $m_1(q-p) - m_2 p = 0$. Reducing mod q we have $-(m_1 + m_2)p \equiv 0 \ (mod\ q)$. Since p and q are relatively prime, $d = q$ must divide $m_1 + m_2$, as desired. $\square$

3. Divergent α exist

3.1. The basic idea. The proof of Theorem 1 is driven by a simple idea. The convergent $\frac{p_n}{q_n}$ is the best rational approximation of the second kind for α with denominator less than $q_{n+1} = a_{n+1}q_n + q_{n-1}$. If a_{n+1} is quite large, then this approximation must be quite good (i.e., $\langle\langle q_n\alpha\rangle\rangle$ is quite small) and the orbits of x under T and R_{p_n/q_n} track closely for a long time. The orbit of x under R_{p_n/q_n} is periodic, hitting q_n points. Suppose that q_n is large and not divisible by d. Then, by assumptions (ii) and (iii) of the theorem, the sum of values of f over the orbit of x under the rational rotation is bounded away from zero. This constant rate of accumulation of positive or negative values causes the sum for the rational rotation to diverge. With a_{n+1} quite large, orbit of the irrational rotation tracks that of the rational rotation closely for a long time, and we will show that it must also accumulate extra positive or negative values at a constant rate for a long stretch of orbit. This will drive divergence of the sum.

3.2. Lemmas. Consider the orbit segment $\mathcal{O}_\alpha[1, q_{n+1}]$. We decompose it into segments $\sigma_0 = \mathcal{O}_\alpha[1, q_{n-1}]$ and $\sigma_l = \mathcal{O}_\alpha[q_{n-1} + (l-1)q_n + 1, q_{n-1} + lq_n]$ for $l = 1, 2, \ldots, a_{n+1}$. Note that the cardinality of the orbit segment σ_l depends on α only through the value of q_n, i.e., only through the values of the first n terms in the continued fraction expansion of α.

Lemma 3.1. *Let $C = \{l \in [1, a_{n+1} - 1] : s(\sigma_l) \neq s(\sigma_{l+1})\}$. That is, C is the set of l at which $s(\sigma_l)$ changes. Then $|C| \leq 2B$, where B is the number of intervals U_i in the definition of f.*

Proof. First, $\sigma_{l+1} = R_{q_n\alpha}\sigma_l = R_{\pm\langle\langle q_n\alpha\rangle\rangle}\sigma_l$ with sign depending on whether $\frac{p_n}{q_n}$ over- or underestimates α. As σ_1 is an orbit segment of length q_n, the minimum distance between its points is $\langle\langle q_{n-1}\alpha\rangle\rangle > a_{n+1}\langle\langle q_n\alpha\rangle\rangle$. Thus, if z is an endpoint of some interval in $\{U_i\}$, over the a_{n+1} rotations of σ_1 by $R_{\pm\langle\langle q_n\alpha\rangle\rangle}$, each of which moves points by $\langle\langle q_n\alpha\rangle\rangle$, at most one orbit point can cross z. Moreover, $s(\sigma_l) \neq s(\sigma_{l+1})$ only if an orbit point crosses the endpoint of an interval under the rotation $R_{\pm\langle\langle q_n\alpha\rangle\rangle}$ of σ_l. As there are $2B$ endpoints, and each is crossed at most once, $|C| \leq 2B$. $\square$

We also need the following technical lemma:

Lemma 3.2. *Fix a nonzero $L \in \mathbb{R}$ and $\kappa, N_1 \in \mathbb{N}$. Suppose that $\{c_m\}$ satisfies $|c_m| < B_0$ for all m, and that the partial sums $s_n = \sum_{m=1}^{n} c_m$ satisfy $\frac{s_n}{n+\kappa} \geq \frac{L}{3}$ for all $n > N_1$ if $L > 0$, or satisfy $\frac{s_n}{n+\kappa} \leq \frac{L}{3}$ for all $n > N_1$ if $L < 0$. Then for any real number $A > 0$ there is an $N^* \in \mathbb{N}$ such that $|\sum_{m=1}^{N^*} c_m \frac{1}{m+\kappa}| > A$, where N^* depends only on L, κ, N_1, B_0 and A.*

The proof of this lemma is similar to the proof of Kronecker's Lemma. The crucial fact here is that N^* can be determined by the values of L, κ, N_1, B_0 and A, without dependence on the specific terms of the sequence $\{c_m\}$. This will play a key role in the proof of Theorem 1.

Proof. For any N, the summation by parts formula yields

$$\sum_{m=1}^{N} c_m \frac{1}{m+\kappa} = \frac{s_N}{N+\kappa+1} - \sum_{m=1}^{N} s_m \left(\frac{1}{m+\kappa+1} - \frac{1}{m+\kappa} \right) \tag{2}$$
$$= \frac{s_N}{N+\kappa+1} + \sum_{m=1}^{N} \frac{s_m}{m+\kappa} \frac{1}{m+\kappa+1}.$$

If $N > N_1$, then the terms $\frac{S_N}{N+\kappa+1}$ and $\sum_{m=N_1+1}^{N} \frac{s_m}{m+\kappa} \frac{1}{m+\kappa+1}$ have the same sign as L, by our hypothesis on the sequence $\{c_m\}$. On the other hand, the middle terms $\sum_{m=1}^{N_1} \frac{s_m}{m+\kappa} \frac{1}{m+\kappa+1}$ might have a different sign. Since $|s_m| \leq m B_0$ for any such sequence $\{c_m\}$, we can bound this potential cancellation without dependence on the specific sequence. By our hypothesis on the sequence $\{c_m\}$, we can choose $n > N_1$ such that the magnitude of $\sum_{m=N_1+1}^{n} \frac{s_m}{m+\kappa} \frac{1}{m+\kappa+1}$ becomes arbitrarily large, without further assumptions on the sequence.

To be more concrete in our choices, let $E = \sum_{m=1}^{N_1} \frac{m B_0}{m+\kappa} \frac{1}{m+\kappa+1}$. Since $|s_m| \leq m B_0$, we have the bound $|\sum_{m=1}^{N_1} \frac{s_m}{m+\kappa} \frac{1}{m+\kappa+1}| \leq E$. Choose $N_2 \in \mathbb{N}$ such that $\sum_{m=1}^{N_2} \frac{1}{m+\kappa+1} > (A+E)|3/L|$. Take $N^* \geq \max\{N_1, N_2\}$. Since $N^* \geq N_1$, the term $\frac{s_{N^*}}{N^*+\kappa+1}$ has the same sign and is larger in magnitude than $L/3$. Since $N^* \geq N_2$, we have

$$\left| \sum_{m=1}^{N^*} \frac{s_m}{m+\kappa} \frac{1}{m+\kappa+1} \right| \geq \left| \sum_{m=1}^{N^*} \frac{L}{3} \frac{1}{m+\kappa+1} \right|$$
$$\geq A + E.$$

Additionally, these two terms on the right side of Equation (2) have the same sign. Hence, $|\sum_{m=1}^{N^*} c_m \frac{1}{m+\kappa}| > |L/3| + A > A$. □

3.3. Proof of Theorem 1. Suppose q_n is not divisible by d (possible by Lemma 2.1). Let δ_0 be the minimum nonzero value in the additive subgroup of $\mathbb{R}$ generated by $\{v_i\}$, and let $\Delta_0 = \max\{|v_i|\}$. By Lemma 3.1, if a_{n+1} is much larger than B, then there will be a long stretch of indices l during which $s(\sigma_l)$ does not change. Over this stretch, the sum accumulates values of at least δ_0 or at most $-\delta_0$ at a rate of at least $1/q_n$. This, via Lemma 3.2, will drive the divergence of the sum.

We will find α satisfying the requirements of Theorem 1 by inductively defining its continued fraction expansion. This requires some care in the sort of arguments we can make. Recall that $S[a_1 \dots a_n]$ is the set of all irrational α with a continued fraction expansion beginning with $[a_1 \dots a_n]$. To define a_{n+1} in our inductive scheme, we must make arguments which rely only on fixed data (such as the values of B and δ_0) and on statements which are true for all $\alpha \in S[a_1 \dots a_n]$. For example, since α is not yet known, we do not know the exact sequence $(f \circ T^n x)_n$. We must instead rely on information about it gleaned only from the first n terms of the continued fraction expansion, such as the values of $q_1, \dots q_n$.

Proof of Theorem 1. We define α by producing its continued fraction expansion inductively. We prove divergence via the Cauchy criterion, showing that for any q_n, there are $n_2 > n_1 > q_n$ such that $|\sum_{n=n_1}^{n_2} f \circ T^n(x)\frac{1}{n}| > 1$.

Let $a_1 = 1$. Then $q_1 = 1$. By choosing all $a_n \equiv 0 \pmod{d}$ for $n \geq 2$, we can ensure that all q_n are not divisible by d, slightly simplifying the proof below (otherwise we invoke Lemma 2.1). Since the argument below proceeds by choosing a_{n+1} sufficiently large, this causes no problems.

For some $n \geq 1$, suppose that we have chosen $a_1, \dots, a_n$ with $a_1 = 1$ and $a_n \equiv 0 \pmod{d}$ for $n \geq 2$. Then $q_n \not\equiv 0 \pmod{d}$. Moreover, note that all $\alpha \in S[a_1 \dots a_n]$ have the same q_n and q_{n-1}.

Let $\kappa = q_{n-1}$. Pick any $\alpha_n \in S[a_1 \dots a_n]$ and let

$$c'_m(\alpha_n) = f \circ T^{\kappa+m}(x) = f \circ R_{\alpha_n}^{\kappa+m}(x).$$

The sequence $\{c'_m(\alpha_n)\}$ depends, of course, on the value of α_n. However, for all choices of $\alpha_n \in S[a_1 \dots a_n]$, each orbit segment σ_l accumulates values uniformly bounded away from zero:

$$\left| \sum_{m=1}^{q_n} f \circ T^{\kappa+(l-1)q_n+m}(x) \right| \geq \delta_0$$

for any $l = 1, 2, \dots a_{n+1}$. Now let

$$c_m(\alpha_n) = \begin{cases} c'_m(\alpha_n) & \text{if } m \in \sigma_l \text{ with } s(\sigma_l) > 0 \\ -c'_m(\alpha_n) & \text{if } m \in \sigma_l \text{ with } s(\sigma_l) < 0. \end{cases}$$

That is, we switch the signs of $c'_m(\alpha_n)$ on an orbit segment σ_l precisely when necessary to ensure the sum of $c_m(\alpha_n)$ over the segment is positive. Now, for any $\alpha_n \in S[a_1 \dots a_n]$, the sequence $\{c_m(\alpha_n)\}$ accumulates values of at least δ_0 over each segment σ_l.

Take $B_0 = \Delta_0$, $L = \delta_0/q_n$ and $N_1 = 4\frac{\Delta_0}{\delta_0}q_n^2 > q_n^2$. As noted, $\kappa = q_{n-1}$. Let $s_m = \sum_{i=1}^m c_i$. First, we check that the hypotheses of Lemma 3.2 hold at multiples of q_n greater than N_1. Then, we check that the hypotheses of Lemma 3.2 hold between multiples of q_n. First, at each $m = lq_n$,

$$\frac{s_m}{m+\kappa} \geq \frac{l\delta_0}{lq_n + q_{n-1}} \geq \frac{\delta_0}{q_n}\frac{l}{l+1}.$$

If $m > N_1 > q_n^2$, then $l > q_n$ and $\frac{l}{l+1} > \frac{1}{3}$. Thus, in this case $\frac{s_m}{m+\kappa} > \frac{L}{3}$.

Now suppose that $lq_n < m < (l+1)q_n$ with $l \geq 4\frac{\Delta_0}{\delta_0}q_n$. Note that for such l, $\frac{\Delta_0}{2(l+2)} < \frac{1}{6}\frac{\delta_0}{q_n}$. Then $s_m > \delta_0 l - \Delta_0\frac{q_n}{2}$ since the sum accumulates at least l extra δ_0's up to step lq_n and, in the extreme case, the next $\frac{q_n}{2}$ of the terms between lq_n and $(l+1)q_n$ are $-\Delta_0$'s. From this we compute:

$$\begin{aligned}
\frac{s_m}{m+\kappa} &> \frac{\delta_0 l - \Delta_0 q_n/2}{(l+1)q_n + q_{n-1}} \\
&> \frac{\delta_0 l - \Delta_0 q_n/2}{(l+2)q_n} \\
&= \frac{\delta_0 l}{(l+2)q_n} - \frac{\Delta_0 q_n/2}{(l+2)q_n} \\
&> \frac{1}{2}\frac{\delta_0}{q_n} - \frac{\Delta_0}{2(l+2)} \\
&> \frac{L}{3}
\end{aligned}$$

by the choice of $l \geq 4\frac{\Delta_0}{\delta_0}q_n$. Therefore this choice of κ, L, B_0 and N_1 ensures that Lemma 3.2 holds for the sequence $\{c_m\}$ defined by any $\alpha_n \in S[a_1 \dots a_n]$. Let $A = 2B+1$, where B is the number of intervals in U. For this $A > 0$, Lemma 3.2 provides N^*. Choose an even a_{n+1} so that $a_{n+1}q_n > N^*$.

For any $\alpha_{n+1} \in S[a_1 \dots a_n a_{n+1}]$, we now return to $\{c'_m(\alpha_{n+1})\}$, the unadjusted sequence from the sum $\sum f \circ T^m(x)\frac{1}{m}$. By Lemma 3.1, there are at most $2B$ values of l at which $s(\sigma_l) \neq s(\sigma_{l+1})$. Let $l_1^* < l_2^* < \cdots < l_b^*$ be the values of l where $s(\sigma_l)$ changes. Let

$$\begin{aligned}
\tau_0 &= [1, q_{n-1} + l_1^* q_n], \\
\tau_i &= [q_{n-1} + l_i^* q_n + 1, q_{n-1} + l_{i+1}^* q_n] \qquad \text{for } 1 \leq i < b, \\
\tau_b &= [q_{n-1} + l_b^* q_n + 1, q_{n+1}].
\end{aligned}$$

If for all $0 \leq i \leq b$, $|\sum_{m\in\tau_i} f \circ T^m(x)\frac{1}{m}| < 1$, then we arrive at a contradiction to the fact, established above, that $\sum_{m=q_{n-1}+1}^{q_{n+1}} c_m\frac{1}{m} > A = 2B+1$. So there must be at least one τ_i such that $|\sum_{m\in\tau_i} f \circ T^m(x)\frac{1}{m}| \geq 1$. Taking n_1 and n_2 as the first and last integers in τ_i, we note that $n_2 > n_1 > q_{n-1}$.

Since $S[a_1 \dots a_n]$ is a nested sequence of closed subsets of the circle, its intersection is nonempty, and is in fact a single point since the continued fraction expansion defines a number uniquely. Take $\{\alpha\} = \cap_n S[a_1 \dots a_n]$. Let q_n be the denominator of the n^{th} convergent to α; it is clear that $q_n \to \infty$

as $n \to \infty$. For each n, $\alpha \in S[a_1 \cdots a_{n+2}]$, so there exist $n_2 > n_1 > q_n$ such that $|\sum_{m=n_1}^{n_2} f \circ T^m(x)\frac{1}{m}| > 1$. Hence, the series $\sum f \circ T^n(x)\frac{1}{n}$ diverges by the Cauchy criterion. □

As mentioned in the introduction, our argument actually proves a stronger result:

Corollary 3.3. *Let $\mathcal{F}_{B,\delta_0,\Delta_0,d}$ be the set of all functions satisfying the conditions of Theorem 1 with $\max|f| \leq \Delta_0$ and $\min\langle v_1, \ldots, v_B\rangle - \{0\} \geq \delta_0$. Let b_n be any positive sequence such that $\sum n(b_{n+1} - b_n)$ diverges. Then there is a dense, uncountable set of irrational α such that*

$$\sup_{N\geq 1}\left|\sum_{n=1}^{N} f \circ T^n(x) b_n\right| = \infty$$

for any $f \in \mathcal{F}_{B,\delta_0,\Delta_0,d}$ and any x.

Proof. First, our construction of α depends on f only through the number of intervals used to write U and the bounds δ_0 and Δ_0, so we get the result for all $f \in \mathcal{F}_{B,\delta_0,\Delta_0,d}$.

Second, the condition $\sum n(b_{n+1} - b_n)$ divergent is sufficient to run the argument of Lemma 3.2, which drives divergence throughout the rest of the proof.

Third, in the proof of Theorem 1, one is free to choose the initial terms of the continued fraction expansion of α, taking up the argument given there only after this initial segment. Note that in doing so, we must take up the argument at some n^* where $q_{n^*} \not\equiv 0 \pmod{d}$; this is possible by Lemma 2.1. This allows us to find a dense set of irrational α which are divergent.

Fourth, at each stage in the construction of α, we use $A = 2B + 1$ in our application of Lemma 3.2 and argue from there that there exist $n_1, n_2 > q_n$ with $|\sum_{n=n_1}^{n_2} f \circ T^n(x)b_n| > 1$. If instead we take at the nth stage of the construction $A = (2B + 1)n$ we will find that for some $n_1, n_2 > q_n$ we must have $|\sum_{m=n_1}^{n_2} f \circ T^m(x)b_m| > n$. This implies that the partial sums $\sum_{n=1}^{N} f \circ T^n(x)b_n$ are unbounded.

Finally, note that our only requirement for divergence is that a_n are sufficiently large. Since there are always infinitely many choices for each a_n, we can construct uncountably many divergent irrational α. □

4. Liouville numbers and convergence

4.1. Proof of Theorem 2. In this section we give a proof of Theorem 2 which is alluded to by Kakutani and Petersen in [15].

Definition 4.1. An irrational real number α is Liouville if for all $v \geq 1$, there exists a rational number $\frac{p}{q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < q^{-(v+1)}.$$

Note that the Liouville condition is equivalent to $\langle\langle q\alpha\rangle\rangle < q^{-v}$. Let

$$\mathcal{K}_v = \{\alpha : \langle\langle q\alpha\rangle\rangle < q^{-v} \text{ infinitely often}\}.$$

By Dirichlet's theorem, $\mathcal{K}_1 = \mathbb{R}$ and by a result of Khintchine, for $v > 1$, $\mathcal{K}_v$ is a null set (see, e.g., [4]). The Hausdorff dimension of the set of Liouville numbers is zero. This follows from results of Jarník [14] and Besicovitch [3]; see [22, Ch. 2] for a self-contained exposition.

Liouville numbers are very well approximated by rational numbers whose denominators are not too large. Similarly, the proof of Theorem 1 relies on constructing α which are very closely approximated by their convergents. So it is not too surprising that there is a connection between the two, and this is the content of Theorem 2.

Let $\omega = (x_1, x_2, \dots)$ be a sequence of elements in $[0, 1]$. For our work we will take $\omega = (n\alpha + x \ (mod\ 1))_n$. The failure of equidistribution of this sequence is measured by the discrepancy function:

Definition 4.2 (See, e.g., [19, §2.1])**.** The *discrepancy* of ω is

$$D_N = D_N(\omega) = \sup_{0\leq\alpha<\beta\leq 1}\left|\frac{\#([\alpha,\beta)\cap\omega|_{[1,N]})}{N} - (\beta-\alpha)\right|.$$

By the *discrepancy of* α, or $D_N(\alpha)$ we will mean $D_N((n\alpha + x\ (mod\ 1)))$. It is immediate from Definition 4.2 that this function of N is independent of x.

We recall the following definition and its connection to the Liouville property.

Definition 4.3 (See, e.g., [19, §2.3])**.** Let $\eta > 0$. We say α *is of type* η if $\eta = \sup \gamma$ such that

$$\liminf_{q\to\infty} q^{\gamma}\langle\langle q\alpha\rangle\rangle = 0 \quad \text{ where } q \in \mathbb{N}.$$

As before $\langle\langle - \rangle\rangle$ denotes distance from the nearest integer.

Remark 4.4. The usual notion of type compares $q\langle\langle q\alpha\rangle\rangle$ to $1/\psi(q)$ for a nondecreasing function ψ. These are related; α is of type η in the sense of Definition 4.3 if η is the infimum of all τ for which α is of type $< Cq^{\tau-1}$ for some $C > 0$ (see [19, Lemma 2.3.1]).

Liouville numbers are those of type ∞. By Dirichlet's approximation theorem, all numbers are of type at least 1, and by the Thue–Siegel–Roth theorem, all irrational algebraic numbers are of type 1.

The following result on the discrepancy of non-Liouville α is the key tool we need:

Theorem 4.5 (See, e.g., [19, Thm 3.2])**.** *Let α be of type η. Then for all $\epsilon > 0$, $D_N(\alpha) = O(N^{-1/\eta+\epsilon})$.*

We now prove Theorem 2 using this result.

Proof of Theorem 2. Let α be non-Liouville; suppose it is of type η, where $1 \leq \eta < \infty$. Fix any $x \in [0, 1)$. Let $S_n f = \sum_{i=1}^{n} f(n\alpha + x)$. Using summation by parts,

$$\sum_{n=1}^{N} \frac{f(n\alpha + x)}{n} = \frac{S_N f}{N} + \sum_{n=1}^{N-1} \frac{S_n f}{n(n+1)}.$$

Using Theorem 4.5 and picking $\epsilon > 0$ so small that $-1/\eta + \epsilon < 0$, one can easily show that $|S_n f| = O(n^{1-1/\eta+\epsilon}) = O(n^{1-t})$ for some $t > 0$. This argument uses that f is a mean-zero indicator function for a finite union of intervals. From this we immediately have that $\frac{S_N f}{N} \to 0$ as $N \to \infty$ and that $\sum_{n=1}^{\infty} \frac{S_n f}{n(n+1)}$ converges, completing the proof. $\square$

4.2. A decomposition scheme. We are left with the question of whether all Liouville α are divergent. The proof in the previous section does not address this question. We remark that the argument for divergence in Section 3 uses the fact that infinitely many q_n's are not divisible by d for the α we constructed. This certainly does not hold for all Liouville α. We do not know if some sort of requirement on residues of the q_n's is necessary to ensure divergence, but we will show here that, independent of this concern, there are convergent Liouville numbers. Specifically we will construct Liouville numbers for which the arguments of the previous section can still be used to prove convergence.

Theorem 3. *There exist Liouville numbers α which are convergent for any $f = 2\chi_U - 1$, where U is a union of finitely many intervals with $m(U) = \frac{1}{2}$, and any $x \in S^1$. The set of such α is dense.*

The idea of the proof is to decompose the sequence $(f \circ T^n(0)/n)$ into a countable number of (nearly) alternating subsequences whose sums we can bound individually. If there were only finitely many such subsequences, we would be done. That will not be the case here, but careful choice of α using the continued fraction expansions provides enough control on the sums of these individual sequences to enable us to prove convergence of the full series.

First, we want a scheme for decomposing the sequence $(f \circ T^n(0)/n)$ into alternating sequences. Throughout the following, we write $[a, b]$ for $\{a, a + 1, \ldots, b\}$ and will refer to such subsets of the integers as *intervals*. In the decomposition, we use nested intervals with lengths related to the denominators of the continued fraction expansion, q_i.

We write $(c_n) = (f \circ T^n(0))$ and $(\gamma_n) = (f \circ T^n(0)/n)$. We will use Roman letters (d_n, b_n) to denote subsequences of (c_n) and Greek letters (δ_n, β_n) to denote the corresponding subsequences of (γ_n).

We recall the Denjoy–Koksma Lemma:

Lemma 4.6 ([13, VI Thm 3.1])**.** *Let f be any mean zero function on S^1. Let $[a, b]$ be any interval of length q_k. Then for any $x \in S^1$,*

$$\left|\sum_{j\in[a,b]} f\circ T^j(x)\right| < \mathrm{Var}(f).$$

Corollary 4.7. *Let $f = 2\chi_U - 1$, where U is the union of B intervals and $m(U) = \frac{1}{2}$. Then, for any interval $[a,b]$ of length q_k and any $x \in S^1$,*

$$\left|\sum_{j\in[a,b]} f\circ T^j(x)\right| < 4B.$$

Definition 4.8. Sequence (x_n) is a *pair-permutation* of (y_n) if for all $k \in \mathbb{N}$, $\{x_{2k-1}, x_{2k}\}$ and $\{y_{2k-1}, y_{2k}\}$ are equal as sets.

In other words, (y_n) is obtained from (x_n) by permuting some pairs of adjacent terms.

Definition 4.9. We call a sequence *near-alternating* if it is a pair-permutation of an alternating sequence.

Let $Q_i = \prod_{j=1}^{i} q_j$. Then $Q_1 = q_1$ and $Q_{i+1} = Q_i q_{i+1}$ for all $i \geq 1$. To avoid the use of floor functions and to remove indices as efficiently as possible, we recursively define a sequence of 0's and 1's. First, ξ_1 is 0 if $q_1 - 4B$ is even and 1 if odd. For $i > 1$, ξ_i is 0 if $(4BQ_{i-2} + \xi_{i-1})q_i - 4BQ_{i-1}$ is even and 1 if odd. Then

$$\left\lfloor \frac{(4BQ_{i-2} + \xi_{i-1})q_i - 4BQ_{i-1}}{2} \right\rfloor = \frac{(4BQ_{i-2} + \xi_{i-1})q_i - 4BQ_{i-1} - \xi_i}{2}.$$

Without loss of generality, assume that $q_1 > 4B$. If this is not the case, the proof can be modified by shifting all of the indices.

Proposition 4.10. *There is a decomposition $(c_n) = \sqcup_{i=1}^{\infty}(d_n^{(i)})$ such that $(d_n^{(1)})$ is union of $(q_1 - 4B - \xi_1)/2$ subsequences, $(d_n^{(2)})$ is a union of*

$$((4B + \xi_1)q_2 - 4Bq_1 - \xi_2)/2$$

subsequences, and $(d_n^{(i)})$ is the union of

$$((4BQ_{i-2} + \xi_{i-1})q_i - 4BQ_{i-1} - \xi_i)/2$$

subsequences for $i > 2$. For each $i \in \mathbb{N}$, the subsequence $(d_n^{(i)})$ is near-alternating.

Proof. Throughout this proof, we will take "a length Q_i interval" to mean an interval of the form $[(j-1)Q_i + 1, jQ_i]$ for some integer $j \geq 1$.

Let $(c_n^{(0)}) = (c_n) = (f \circ T^n(0))$. By Corollary 4.7, each length $Q_1 = q_1$ interval contains at least $(q_1 - 4B - \xi_1)/2$ indices such that $c_n = +1$ and the same number of indices such that $c_n = -1$. For $l = 1, \ldots, (q_1 - 4B - \xi_1)/2$, let $b_{2j-1}^{(1,l)}$ be the l^{th} term of (c_n) that is equal to $+1$ with index in the j^{th}

length Q_1 interval. Similarly, let $b_{2j}^{(1,l)}$ be the l^{th} term of (c_n) that is equal to -1 with index in the j^{th} length Q_1 interval. Let $(d_n^{(1)})$ be the union of these $(q_1 - 4B - \xi_1)/2$ near-alternating sequences. Let X_1 be the indices of the terms of (c_n) that are not in $(d_n^{(1)})$. Then the intersection of each length Q_1 interval with X_1 has size $4B + \xi_1$.

Since each length Q_2 interval contains q_2 length Q_1 intervals, the intersection of Q_2 with X_1 has size $(4B+\xi_1)q_2$. Additionally, Corollary 4.7 implies that the sum over each length q_2 interval is at most $4B$. Since there are q_1 intervals of length q_2 in each length Q_2 interval, the sum over each length Q_2 interval is at most $4Bq_1$. Since $(d_n^{(1)})$ removes the same number of $+1$'s and -1's from each length $Q_2 = q_1 q_2$ interval, the sum over the remaining terms of each Q_2 interval (the terms also in X_1) is also at most $4Bq_1$. Thus, each length Q_2 interval contains at least $((4B+\xi_1)q_2 - 4Bq_1 - \xi_2)/2$ remaining indices (in X_1) such that $c_n = +1$, and similarly for -1. As above, let $b_{2j-1}^{(2,l)}$ be the l^{th} remaining $+1$ and $b_{2j}^{(2,l)}$ be the l^{th} remaining -1, where the indices as terms of (c_n) are in the j^{th} length Q_2 interval and in X_1. Let $(d_n^{(2)})$ be the union of these near-alternating sequences. Let X_2 be the indices of the terms of (c_n) that are not in $(d_n^{(1)})$ or in $(d_n^{(2)})$. Then the intersection of each length Q_2 interval with X_2 has size $4Bq_1 + \xi_2$.

For the induction hypothesis, let $i \geq 2$ be an integer. Suppose that we have chosen disjoint sequences $(d_n^{(j)})$ for all $1 \leq j \leq i$. Let X_i be the set of indices of the terms of (c_n) that are not in any of the $(d_n^{(j)})$ so far. Suppose that the intersection of each length Q_i interval with X_i has size $4BQ_{i-1} + \xi_i$.

Since each length Q_{i+1} interval contains q_{i+1} length Q_i intervals, the intersection of Q_{i+1} with X_i has size $(4BQ_{i-1} + \xi_i)q_{i+1}$. Additionally, Corollary 4.7 implies that the sum over each length q_{i+1} interval is at most $4B$. Since there are Q_i intervals of length q_{i+1} in each length Q_{i+1} interval, the sum over each length Q_{i+1} interval is at most $4BQ_i$. Since each $(d_n^{(j)})$ for $1 \leq j \leq i$ removes the same number of $+1$'s and -1's from each length Q_{i+1} interval, the sum over the remaining terms of each Q_{i+1} interval (the terms also in X_i) is also at most $4BQ_i$. Thus, each length Q_{i+1} interval contains at least $((4BQ_{i-1} + \xi_i)q_{i+1} - 4BQ_i - \xi_{i+1})/2$ remaining indices (in X_i) such that $c_n = +1$, and similarly for -1. As above, let $b_{2j-1}^{(i+1,l)}$ be the l^{th} remaining $+1$ and $b_{2j}^{(i+1,l)}$ be the l^{th} remaining -1, where the indices as terms of (c_n) are in the j^{th} length Q_{i+1} interval and in X_i. Let $(d_n^{(i+1)})$ be the union of these near-alternating sequences. Let X_{i+1} be the indices of the terms of (c_n) that are not in $(d_n^{(j)})$ for $1 \leq j \leq i+1$. Then the intersection of each length Q_{i+1} interval with X_{i+1} has size $4BQ_i + \xi_{i+1}$. Hence, the proof follows by induction. $\square$

Let $(b_n^{(i,l)})$ be the near-alternating subsequences of $(d_n^{(i)})$ obtained in Proposition 4.10; n indexes within each individual sequence, and l indexes the sequences themselves. Let l index them so that $b_1^{(i,l)}$ always comes before $b_1^{(i,l+1)}$ as elements of (c_n).

Definition 4.11. Let $\operatorname{ind}(b_1^{(i,l)})$ denote the index of $b_1^{(i,l)}$ as an element of (c_n).

We have the following control on the first elements of the near-alternating sequences.

Proposition 4.12. *For all l, and for $i > 5$,*

$$\operatorname{ind}(b_1^{(i,l)}) \geq \left\lfloor \frac{l}{4BQ_{i-4} + \xi_{i-3}} \right\rfloor Q_{i-3},$$
$$\operatorname{ind}(b_1^{(i,l)}) \geq \frac{(4BQ_{i-3} + \xi_{i-2})q_{i-1} - 4BQ_{i-2} - \xi_{i-1}}{2}.$$

Remark 4.13. The first bound is stronger for large l; the second bound's purpose is to give a nontrivial lower bound when $l < 4BQ_{i-4} + \xi_{i-3}$.

Proof. For the first bound, we have the following argument. Examining the proof of Proposition 4.10, we see that the terms of $(b_n^{(i,l)})$ have indices in X_{i-3}, i.e., among those indices which have not been used for $(d_n^{(j)})$ for $1 \leq j \leq i-3$. As noted in that proof, the intersection of X_{i-3} with each length Q_{i-3} interval has size $4BQ_{i-4}+\xi_{i-3}$. Therefore, the l^{th} index in X_{i-3} is at least $\lfloor \frac{l}{4BQ_{i-4}+\xi_{i-3}} \rfloor Q_{i-3}$.

For the second bound, we note that in Proposition 4.10 the first

$$\frac{(4BQ_{i-3} + \xi_{i-2})q_{i-1} - 4BQ_{i-2} - \xi_{i-1}}{2}$$

$+1$'s and -1's have been removed from c_n at the $i-1^{st}$ step of the process, leaving behind X_{i-1}. The terms of $(b_n^{(i,l)})$ are drawn from X_{i-1}, hence the index of any remaining term has this lower bound. $\square$

Now that we have carefully extracted our near-alternating sequences from (c_n) and carefully bounded the number of such sequences and the index of the first terms, we prove two lemmas on the growth rate of these quantities. These will simplify our convergence estimates in the next subsection.

Lemma 4.14. *Fix α. There exists a constant E (uniform in $i > 2$) such that*

$$\frac{(4BQ_{i-2} + \xi_{i-1})q_i - 4BQ_{i-1} - \xi_i}{2} \leq EQ_{i-2}q_i.$$

Proof. It is easy to check that

$$\frac{(4BQ_{i-2} + \xi_{i-1})q_i - 4BQ_{i-1} - \xi_i}{2} \frac{1}{Q_{i-2}q_i}$$

is uniformly bounded in i, since $Q_{i-2}q_i$ grows at least as fast as the first term. $\square$

Lemma 4.15. *Fix α. There exists a constant F (uniform in $i > 5$) such that*

$$\operatorname{ind}(b_1^{(i,l)}) \geq F q_{i-3} l$$

for all l.

Proof. For $1 \leq l \leq 4BQ_{i-4} + \xi_{i-3}$, we use Proposition 4.12:

$$\begin{aligned}
\operatorname{ind}(b_1^{(i,l)}) &\geq \frac{(4BQ_{i-3} + \xi_{i-2})q_{i-1} - 4BQ_{i-2} - \xi_{i-1}}{2} \\
&\geq \frac{4BQ_{i-3}q_{i-1} - 4BQ_{i-2} - \xi_{i-1}}{2} \\
&\geq BQ_{i-3}q_{i-1} - BQ_{i-2} \\
&= BQ_{i-3}(q_{i-1} - q_{i-2}) \\
&\geq BQ_{i-3}.
\end{aligned}$$

Note that $Q_{i-3} = q_{i-3}Q_{i-4} \geq \frac{1}{2}q_{i-3}Q_{i-4} + q_{i-3} \geq \frac{1}{2}q_{i-3}Q_{i-4} + \xi_{i-3}q_{i-3} \geq q_{i-3}\frac{4BQ_{i-4}+\xi_{i-3}}{8B}$, so

$$\begin{aligned}
\operatorname{ind}(b_1^{(i,l)}) &\geq Bq_{i-3}(4BQ_{i-4} + \xi_{i-3})/(8B) \\
&\geq \frac{1}{8}q_{i-3}l
\end{aligned}$$

as desired.

For $l \geq 4BQ_{i-4} + \xi_{i-3}$, again by Proposition 4.12

$$\begin{aligned}
\operatorname{ind}(b_1^{(i,l)}) &\geq \left\lfloor \frac{l}{4BQ_{i-4} + \xi_{i-3}} \right\rfloor Q_{i-3} \\
&\geq \frac{l}{16BQ_{i-4}} Q_{i-3} \\
&= \frac{1}{16B} l q_{i-3}.
\end{aligned}$$

Taking $F = \frac{1}{16B}$ finishes the proof. $\square$

4.3. Proof of Theorem 3. We need the following pair of straightforward lemmas on sums involving near-alternating series and decompositions of series:

Lemma 4.16. *Let (β_n) be a decreasing sequence, with $|\beta_n| \to 0$, and such that exactly one of $\{\beta_{2n-1}, \beta_{2n}\}$ is positive for each integer n. Then $\sum_n \beta_n$ converges and*

$$\left| \sum_{n=1}^{\infty} \beta_n \right| \leq |\beta_1|.$$

Furthermore, for any interval $[a, b]$,

$$\left|\sum_{n\in[a,b]}\beta_n\right| \leq 2|\beta_1|.$$

Proof. It is easy to verify that the sign pattern giving the largest value of the full sum is $(-1)^{n+1}$ and the pattern giving the smallest value is $(-1)^n$. The first statement is then a standard fact about alternating series.

The second statement follows from the first, after noting that it is possible that the a and $a+1^{st}$ terms of β_n may have the same sign. $\square$

Remark 4.17. The proof actually gives

$$\left|\sum_{n\in[a,b]}\beta_n\right| \leq 2|\beta_a|$$

but we don't need this below.

Lemma 4.18. *Suppose that we have a decomposition of a sequence* $(\gamma_n) = \sqcup_i(\delta_n^{(i)})$ *satisfying:*

- *For all* i, $\sum_n \delta_n^{(i)}$ *converges.*
- *For all* i, *there exist* $D^{(i)}$ *such that* $|\sum_{n\in[a,b]}\delta_n^{(i)}| \leq D^{(i)}$ *for all* $[a,b]$, *and* $\sum_i D^{(i)} < \infty$.

Then $\sum_n \gamma_n$ *converges.*

Proof. We prove convergence by the Cauchy Criterion. Let $\epsilon > 0$ be given. For each $i \in \mathbb{N}$, let X_i be the set of indices of the terms from (γ_n) that are in $(\delta_n^{(i)})$. Let $I \in \mathbb{N}$ be such that $\sum_{i>I} D^{(i)} < \epsilon/2$. For each $i \leq I$, let $N_i \in \mathbb{N}$ be such that for any $m_1, m_2 \geq N_i$ the terms of $(\delta_n^{(i)})$ whose indices $\mathrm{ind}(\delta_n^{(i)})$ as elements of the original sequence (γ_n) lie in $[m_1, m_2]$ satisfy

$$\left|\sum_{n:\mathrm{ind}(\delta_n^{(i)})\in[m_1,m_2]}\delta_n^{(i)}\right| = \left|\sum_{[m_1,m_2]\cap X_i}\delta_n\right| < \frac{\epsilon}{2^{i+1}}.$$

Let $N = \max_{i\leq I}\{N_i\}$. Then, for any $m_1, m_2 \geq N$,

$$\begin{aligned}
\left|\sum_{n\in[m_1,m_2]}\gamma_n\right| &= \left|\sum_{i=1}^{I}\sum_{[m_1,m_2]\cap X_i}\delta_n + \sum_{i>I}\sum_{[m_1,m_2]\cap X_i}\delta_n\right| \\
&\leq \sum_{i=1}^{I}\left|\sum_{[m_1,m_2]\cap X_i}\delta_n\right| + \sum_{i>I}\left|\sum_{[m_1,m_2]\cap X_i}\delta_n\right| \\
&\leq \frac{\epsilon}{2} + \sum_{i>I} D^{(i)} \\
&\leq \epsilon.
\end{aligned}$$

$\square$

We are now ready to complete our proof of Theorem 3.

Proof of Theorem 3. First, we specify how to choose a Liouville number which will prove the convergence behavior we want using the continued fraction expansion.

It is standard that

$$\|q_k\alpha\| < \frac{1}{q_{k+1}}. \tag{3}$$

Now suppose that we define α by choosing $a_{k+1} = q_k^{k-1}$. Note that as q_k is defined only in terms of $a_1, \ldots, a_k$, this defines α inductively. In addition, we can take up this inductive definition after any initial sequence $[a_1 a_2 \ldots a_n]$, producing a dense set of α. Then,

$$q_{k+1} = a_{k+1}q_k + q_{k-1} \geq q_k^k.$$

With (3), this implies that for all k,

$$\|q_k\alpha\| < \frac{1}{q_k^k}.$$

Thus, for any $v \geq 1$, all $\frac{p_k}{q_k}$ with $k \geq v$ satisfy the approximation condition in the definition of a Liouville number, so α is Liouville.

On the other hand,

$$q_{k+1} \leq 2q_k^k \quad \text{and so } q_k \leq 2^{k^2-2k+2} q_{k-3}^{(k-1)(k-2)(k-3)} \tag{4}$$

at least for $k > 3$.

Finally recall the further standard fact ([17, Thm 12]) that the q_k grow exponentially fast for any α:

$$q_k \geq C\varphi^k \quad \text{for some } C > 0 \text{ and } \varphi > 1. \tag{5}$$

Defining α this way gives the following facts. By Equation (4),

$$\begin{aligned} &\frac{(k-1)\log q_k}{q_{k-3}} \\ &\leq \frac{((k-1)\log 2^{k^2-2k+2} q_{k-3}^{(k-1)(k-2)(k-3)}}{q_{k-3}} \\ &= \frac{(k-1)(k^2-2k+2)\log 2 + (k-1)^2(k-2)(k-3)\log q_{k-3}}{q_{k-3}}. \end{aligned} \tag{6}$$

Clearly $\sum_k \frac{(k-1)(k^2-2k+2)\log 2}{q_{k-3}}$ converges by Equation (5). For the second summand, note that $q_3 > 3$ and that $\frac{\log x}{x}$ is decreasing for $x \geq 3$. Therefore, again using Equation (5),

$$
\begin{aligned}
(7)\qquad &\sum_{k=3}^{\infty} \frac{(k-1)^2(k-2)(k-3)\log q_{k-3}}{q_{k-3}} \\
&= \sum_{k=0}^{\infty} \frac{(k+2)^2(k+1)k\log q_k}{q_k} \\
&\le \frac{18\log q_1}{q_1} + \frac{96\log q_2}{q_2} + \sum_{k=3}^{\infty} \frac{(k+2)^2(k+1)k\log C\varphi^k}{C\varphi^k} \\
&= \frac{18\log q_1}{q_1} + \frac{96\log q_2}{q_2} + \sum_{k=3}^{\infty} \frac{(k+2)^2(k+1)k\log C}{C\varphi^k} \\
&\quad + \sum_{k=3}^{\infty} \frac{(k+2)^2(k+1)k^2\log \varphi}{C\varphi^k},
\end{aligned}
$$

which converges. This estimate will be key below.

We now use the decomposition from Proposition 4.10 to decompose the sequence $(\gamma_n) = (f \circ T^n(0)/n)$ into subsequences $(\delta_n^{(i)})$. Using the decomposition $(c_n) = \sqcup_i (d_n^{(i)})$ and the associated index function, let $\delta_n^{(i)} = \gamma_{\mathrm{ind}(d_n^{(i)})}$ and $\beta_n^{(i,l)} = \gamma_{\mathrm{ind}(b_n^{(i,l)})}$. Then the decompositions $(\gamma_n) = \sqcup_i(\delta_n^{(i)})$ and $(\delta_n^{(i)}) = \sqcup_l(\beta_n^{(i,l)})$ also satisfy Propositions 4.10 and 4.12.

To prove convergence using the mechanism of Lemma 4.18, we need only obtain estimates for i sufficiently large, so we restrict our attention to $i > 5$. Then, using Lemma 4.14 the sequence $(\delta_n^{(i)})$ consists of at most $EQ_{i-2}q_i$ near-alternating sequences $(\beta_n^{(i,l)})$. As before, they are indexed so that $\beta_1^{(i,l)}$ always comes before $\beta_1^{(i,l+1)}$ as elements of (γ_n). The individual series $\sum_n \beta_n^{(i,l)}$ converge by Lemma 4.16, with

$$
\left| \sum_{n\in[a,b]} \beta_n^{(i,l)} \right| \le 2|\beta_1^{(i,l)}|.
$$

Using Lemma 4.15

$$
\left|\beta_1^{(i,l)}\right| \le \frac{1}{Fq_{i-3}l}.
$$

Applying these bounds and Lemma 4.16, we get a bound on partial sums of $\delta_n^{(i)}$ as follows: for any interval $[a, b]$,

$$\begin{aligned}\left|\sum_{n\in[a,b]} \delta_n^{(i)}\right| &\le 2 \sum_{l=1}^{EQ_{i-2}q_i} |\beta_1^{(i,l)}| \\ &\le 2 \sum_{l=1}^{EQ_{i-2}q_i} \frac{1}{Fq_{i-3}l}.\end{aligned}$$

This can be bounded above by

$$\begin{aligned}\frac{2}{Fq_{i-3}}(1+\log(EQ_{i-2}q_i)) &\le \frac{2}{Fq_{i-3}}(1+\log(Eq_i^{i-1})) \\ &= \frac{2}{Fq_{i-3}}(1+\log E + (i-1)\log q_i).\end{aligned}$$

By the exponential growth of q_i, $\sum_i \frac{1}{q_i} < \infty$. We established using Equations (6) and (7) above that $\sum_i \frac{(i-1)\log q_i}{q_{i-3}} < \infty$.

We then have that $\sum_{i=1}^{\infty} \left|D^{(i)}\right|$ converges. By Lemma 4.18, this implies that $\left|\sum_{n=1}^{\infty} \gamma_n\right|$ converges, establishing the theorem. □

Of course, the proof of Theorem 3 presented here allows many other constructions of Liouville numbers for which the EHT for functions of the type we are considering will converge. Examining the proof above, we see that a sufficient condition for convergence is having $\frac{(k-1)\log q_k}{q_{k-3}}$ summable. This leaves plenty of leeway to choose a_k sufficiently large to produce a Liouville number.

4.4. An alternate proof of Theorem 2. As we remarked in the introduction, the technology which we have developed to prove Theorem 3 provides an alternate way to prove Theorem 2 which utilizes only standard results about the continued fraction expansion. To adapt the proof of Theorem 3 it is sufficient to replace the bound on q_{k+1} which appears in Equation (4) by $q_{k+1} < q_k^v$ for all k, which is satisfied for some $v > 1$ whenever α is not Liouville. The subsequent calculations in the proof of Theorem 3 need to be adjusted accordingly, but summability of $\frac{(k-1)\log q_k}{q_{k-3}}$ follows in much the same fashion.

5. Some questions

We collect here some questions related to the work above. These were suggested to us by an unnamed referee.

Question 1. Does Theorem 1 hold for $f = \chi_U - m(U)$ when U is a finite union of intervals but $m(U) \notin \mathbb{Q}$?

Question 2. The α produced via Theorem 1 form a dense set. Can one say more about the size of the set of α that satisfy this theorem? For instance, do they form a residual set, as do the corresponding α in Theorem 2.1 of [10]?

Question 3. Fix two mean-zero indicator functions f_1 and f_2 for finite unions of intervals. Is there a (necessarily Liouville) number α for which the EHT of f_1 diverges at some (or all) points, but the EHT of f_2 converges at all points? Corollary 3.3 suggests one might first investigate the situation where f_2 is the mean-zero indicator for a union of a larger number of disjoint intervals than f_1.

Question 4. Let α be a Liouville number such that the EHT of a mean-zero indicator function f for a finite union of intervals does not converge at some point. Can it converge at some other point? What can be said about the size of the set of nonconvergence? Note that such α could not be among those constructed by Theorems 1 or 3.

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