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Isomorphisms of Cayley graphs on nilpotent groups

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Abstract. Let S be a finite generating set of a torsion-free, nilpotent group G. We show that every automorphism of the Cayley graph $\mathrm{Cay}(G;S)$ is affine. (That is, every automorphism of the graph is obtained by composing a group automorphism with multiplication by an element of the group.) More generally, we show that if $\mathrm{Cay}(G_1;S_1)$ and $\mathrm{Cay}(G_2;S_2)$ are connected Cayley graphs of finite valency on two nilpotent groups G_1 and G_2, then every isomorphism from $\mathrm{Cay}(G_1;S_1)$ to $\mathrm{Cay}(G_2;S_2)$ factors through to a well-defined affine map from G_1/N_1 to G_2/N_2, where N_i is the torsion subgroup of G_i. For the special case where the groups are abelian, these results were previously proved by A. A. Ryabchenko and C. Löh, respectively.

Contents

1. Introduction

Definition 1.1 ([7, §1])**.** Let S be a subset of a group G.

- S is *symmetric* if it is closed under inverses; that is, $s^{-1} \in S$ for all $s \in S$.
- If S is symmetric, then the corresponding *Cayley graph* on G is the graph $\mathrm{Cay}(G;S)$ whose vertices are the elements of G, and with an edge $g — gs$, for all $g \in g$ and $s \in S$.

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It is easy to construct examples of nonisomorphic groups that have isomorphic Cayley graphs, even if the Cayley graphs are required to be connected and have finite valency. (For example, see Remark 1.15(4) below.) We show that this is not possible when the groups are torsion-free and nilpotent:

Theorem 1.2. *Suppose G_1 and G_2 are torsion-free, nilpotent groups. If G_1 has a connected Cayley graph of finite valency that is isomorphic to a Cayley graph on G_2, then $G_1 \cong G_2$.*

In fact, the next theorem establishes the stronger conclusion that every isomorphism of the Cayley graphs is obtained from an isomorphism of the groups.

Definition 1.3. Suppose $\varphi\colon G_1 \to G_2$, where G_1 and G_2 are groups. We say that φ is an *affine bijection* if it is the composition of a group isomorphism and a translation. That is, there exist a group isomorphism $\alpha\colon G_1 \to G_2$ and $h \in G_2$, such that $\varphi(x) = h \cdot \alpha(x)$, for all $x \in G_1$.

Theorem 1.4. *Assume:*
- *G_1 and G_2 are torsion-free, nilpotent groups.*
- *S_i is a finite, symmetric generating set of G_i, for $i = 1, 2$.*

Then every isomorphism from $\mathrm{Cay}(G_1; S_1)$ to $\mathrm{Cay}(G_2; S_2)$ is an affine bijection.

Remark 1.5. In the special case where G_1 and G_2 are abelian, Theorem 1.4 was proved by A. A. Ryabchenko [12].

Definition 1.6. [7, §6.4] Let G be a group. A Cayley graph $\mathrm{Cay}(G; S)$ is said to be *normal* if the left-regular representation of G is a normal subgroup of $\mathrm{Aut}\big(\mathrm{Cay}(G; S)\big)$ or, equivalently [2, Lem. 2.2(b)], if every automorphism of $\mathrm{Cay}(G; S)$ is an affine bijection.

Remark 1.7. It is easy to see that the left-regular representation of G is a subgroup of the automorphism group of every Cayley graph on G. Definition 1.6 requires this subgroup to be normal.

With this terminology, the special case of Theorem 1.4 in which $G_1 = G_2$ has the following known result as an immediate consequence.

Corollary 1.8 (Möller–Seifter [9, Thm. 4.1(1)])**.** *If G is a torsion-free, nilpotent group, then every connected Cayley graph of finite valency on G is normal.*

In the statement of Theorem 1.4, the word "nilpotent" cannot be replaced with "solvable" (or even "polycyclic"):

Example 1.9. Let G be the unique nonabelian semidirect product of the form $\mathbb{Z} \rtimes \mathbb{Z}$. More precisely,

$$G = \langle\, a, b \mid b^{-1}ab = a^{-1} \,\rangle = \langle a \rangle \rtimes \langle b \rangle.$$

(In other words, G is the fundamental group of the Klein bottle.) Then G is obviously polycyclic (so it is solvable), but it is not difficult to see that $\mathrm{Cay}\big(G; \{a^{\pm 1}, b^{\pm 1}\}\big)$ is not normal. (Namely, the map $\varphi(a^i b^j) = b^i a^j$ is a graph automorphism that is not an affine bijection.)

We will also show that if G is not torsion-free, then the conclusion of Corollary 1.8 fails:

Proposition 1.10. *Let G be a finitely generated, infinite group. If G is not torsion-free, then G has a connected Cayley graph of finite valency that is not normal.*

However, the next theorem shows that if the torsion-free hypothesis is removed from Theorem 1.4, then the conclusion still holds modulo the elements of finite order.

Definition 1.11 ([6, 1.2.13, p. 11])**.** Suppose G is a finitely generated, nilpotent group. The set of all elements of finite order in G is called the *torsion subgroup* of G. (This is a finite, normal subgroup of G.)

Theorem 1.12. *Assume:*

- *S_i is a symmetric, finite generating set of the nilpotent group G_i, for $i = 1, 2$.*
- *φ is an isomorphism from $\mathrm{Cay}(G_1; S_1)$ to $\mathrm{Cay}(G_2; S_2)$.*
- *N_i is the torsion subgroup of G_i, for $i = 1, 2$.*

Then φ induces a well-defined affine bijection $\overline{\varphi}\colon G_1/N_1 \to G_2/N_2$.

Corollary 1.13. *For $i = 1, 2$, assume N_i is the torsion subgroup of the finitely generated, nilpotent group G_i. Then there is a connected Cayley graph of finite valency on G_1 that is isomorphic to a Cayley graph on G_2 if and only if $G_1/N_1 \cong G_2/N_2$ and $|N_1| = |N_2|$.*

Corollary 1.14. *If $\mathrm{Cay}(G; S)$ is any connected Cayley graph of finite valency on a torsion-free, nilpotent group G, then the left-regular representation of G is the only nilpotent subgroup of $\mathrm{Aut}\big(\mathrm{Cay}(G; S)\big)$ that acts sharply transitively on the vertices of the Cayley graph.*

Remarks 1.15.

(1) In the special case where G_1 and G_2 are abelian, Theorem 1.12 and Corollary 1.13 were proved by C. Löh [8].
(2) Theorem 1.4 is the special case of Theorem 1.12 in which the torsion subgroups N_1 and N_2 are trivial.
(3) Although Theorems 1.4 and 1.12 are stated only for graphs, they obviously remain true in the setting of Cayley digraphs. This is because any isomorphism of digraphs is also an isomorphism of the underlying graphs.
(4) Some nonnilpotent groups have some connected Cayley graphs of finite valency that are isomorphic to Cayley graphs on nilpotent

groups—or even abelian groups. (For example, the Cayley graph in Example 1.9 is isomorphic to $\mathrm{Cay}\big(\mathbb{Z} \times \mathbb{Z}, \{(\pm 1, 0), (0, \pm 1)\}\big)$.) Theorem 1.12 implies that any such group must have a subgroup of finite index that is nilpotent (see Corollary 4.4), but this fact is well known to be a consequence of Gromov's famous theorem that groups of polynomial growth are virtually nilpotent [3]. Indeed, in order to conclude from Gromov's Theorem that G has a nilpotent subgroup of finite index, it suffices to know that G has a connected Cayley graph of finite valency that is *quasi-isometric* (not necessarily isomorphic) to a Cayley graph on a nilpotent group.

Theorem 1.4 is proved in Section 3, and this result is used to prove Theorem 1.12 (and its corollaries) in Section 4. (The arguments are based on techniques of A. A. Ryabchenko [12] and C. Löh [8].) Proposition 1.10 is proved in Section 5.

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2. Preliminaries

The following result is the special case of Theorem 1.4 in which G_1 and G_2 are abelian. (Although not stated in exactly this form in [12], the result follows from the proof that is given there and is reproduced in [10, Thm. 5.3]). This case is not covered by the proof in Section 3.

Proposition 2.1 (Ryabchenko [12, Thm. 2]). *Assume:*

- *G_1 and G_2 are torsion-free, abelian groups.*
- *S_i is a symmetric, finite generating set of G_i, for $i = 1, 2$.*
- *φ is an isomorphism from $\mathrm{Cay}(G_1; S_1)$ to $\mathrm{Cay}(G_2; S_2)$.*

Then φ is an affine bijection.

As in [8], we use geometric terminology, such as geodesics and convexity, instead of presenting our arguments in group-theoretic language.

Definition 2.2. Let S be a symmetric, finite generating set of a group G.

- For $g, h \in G$, the distance from g to h in the Cayley graph $\mathrm{Cay}(G; S)$ is denoted $\mathrm{dist}_S(g, h)$.
- A finite sequence $[g_i]_{i=m}^{n}$ of elements of G is a *geodesic segment* from g_m to g_n in $\mathrm{Cay}(G; S)$ if $\mathrm{dist}_S(g_i, g_j) = |i - j|$ for $m \le i, j \le n$.
- A bi-infinite sequence $[g_i]_{i=-\infty}^{\infty}$ of elements of G is a *geodesic line* in $\mathrm{Cay}(G; S)$ if $\mathrm{dist}_S(g_i, g_j) = |i - j|$ for all $i, j \in \mathbb{Z}$.
- A geodesic line $[g_i]_{i=-\infty}^{\infty}$ in $\mathrm{Cay}(G; S)$ is *convex* if $[g_i, g_{i+1}, \ldots, g_j]$ is the only path of length $j - i$ from g_i to g_j, for all $i, j \in \mathbb{Z}$ (with $i < j$).
- A geodesic line $[g_i]_{i=-\infty}^{\infty}$ in $\mathrm{Cay}(G; S)$ is *homogeneous* if there exists $\varphi \in \mathrm{Aut}\big(\mathrm{Cay}(G; S)\big)$, such that $\varphi(g_i) = g_{i+1}$ for all i.
- $\mathrm{Aut}_e\big(\mathrm{Cay}(G; S)\big) = \big\{\, \varphi \in \mathrm{Aut}\big(\mathrm{Cay}(G; S)\big) \bigm| \varphi(e) = e \,\big\}$.

- Each oriented edge of $\mathrm{Cay}(G;S)$ has a natural label, which is an element of S. Namely, each edge of the form $g \,—\, gs$ is labelled s. (Note that the same edge with the opposite orientation is labelled s^{-1}.) Each edge in a geodesic segment (or geodesic line) comes with a natural orientation, and therefore has a label.

Lemma 2.3. *For $i = 1, 2$, assume:*

- *S_i is a symmetric, finite generating set of a group G_i.*
- *φ_i is an isomorphism from $\mathrm{Cay}(G_1;S_1)$ to $\mathrm{Cay}(G_2;S_2)$, such that $\varphi_i(e) = e$.*
- *$g_i \in G_i$.*
- *$S_i^* = \{\, \rho(g_i) \mid \rho \in \mathrm{Aut}_e\big(\mathrm{Cay}(G_i;S_i)\big) \,\}$.*
- *$G_i^* = \langle S_i^* \rangle$.*

If $\varphi_1(g_1) = g_2$, then the restriction of φ_2 to G_1^* is an isomorphism from $\mathrm{Cay}\big(G_1^*; S_1^* \cup (S_1^*)^{-1}\big)$ to $\mathrm{Cay}\big(G_2^*; S_2^* \cup (S_2^*)^{-1}\big)$.

Proof. For $i = 1, 2$, let $A_i = \mathrm{Aut}\big(\mathrm{Cay}(G_i;S_i)\big)$, $A_i^e = \mathrm{Aut}_e\big(\mathrm{Cay}(G_i;S_i)\big)$, and $\Gamma_i = \mathrm{Cay}\big(G_i; S_i^* \cup (S_i^*)^{-1}\big)$. For $\rho \in A_i$ and $g \in G_i$, define $\rho_g \in A_i^e$ by $\rho_g(x) = \rho(g)^{-1}\,\rho(gx)$. Then, since S_i^* is A_i^e-invariant, we have

$$\rho(gS_i^*) = \rho(g)\,\rho_g(S_i^*) = \rho(g)S_i^*,$$

so ρ is an automorphism of Γ_i. Since A_i^e is transitive on S_i^*, and the left-regular representation of G_i is transitive on G_i, this implies that the set of edges of Γ_i is the A_i-orbit of the edge $e \,—\, g_i$.

Since φ_1 is a graph isomorphism, it maps the A_1-orbit of g_1 to the A_2-orbit of $\varphi_1(g_1) = g_2$. So φ_1 is an isomorphism from Γ_1 to Γ_2. Since the composition $\varphi_2 \circ \varphi_1^{-1}$ is in A_2, and is therefore an automorphism of Γ_2, we conclude that φ_2 is an isomorphism from Γ_1 to Γ_2. Since the graph $\mathrm{Cay}\big(G_i^*; S_i^* \cup (S_i^*)^{-1}\big)$ is the component of Γ_i that contains e, and $\varphi_2(e) = e$, the desired conclusion follows. □

Lemma 2.4 ([8, Prop. 2.5(3)])**.** *Let $s \in S$ be the label of some edge of a convex geodesic line in $\mathrm{Cay}(G;S)$. If $s \in Z(G)$, then every edge of the geodesic line is labelled s.*

Proof. Suppose $g_i \,—\, g_{i+1}$ is labelled s. Let t be the label of $g_{i+1} \,—\, g_{i+2}$. Then $g_{i+2} = g_i st = (g_i t)s$, so $[g_i, g_i t, g_{i+2}]$ is a path of length 2 from g_i to g_{i+2}. Therefore, convexity implies $[g_i, g_i t, g_{i+2}] = [g_i, g_{i+1}, g_{i+2}]$, so

$$g_i t = g_{i+1} = g_i s,$$

so $t = s$. This means the label of $g_{i+1} \,—\, g_{i+2}$ is s. By induction, we see that every edge is labelled s. □

In the remainder of this section, we recall some basic facts about nilpotent groups.

Definition 2.5 ([6, p. 38] or [1, Notn. 3.4])**.** For a subgroup H of a group G, we let

$$\sqrt{H} = \{\, g \in G \mid g^k \in H \text{ for some } k \in \mathbb{Z}^+ \,\}.$$

This is called the *isolator* of H in G.

Any finitely generated, abelian group A is isomorphic to $\mathbb{Z}^r \times F$, for some $r \in \mathbb{Z}^{\geq 0}$ and finite, abelian group F. The number r is called the *rank* of A, and is denoted $\operatorname{rank} A$. The following definition generalizes this notion from abelian groups to nilpotent groups.

Definition 2.6 ([6, 1.3.3 and p. 85 (1)])**.** Assume G is a nilpotent group. Then G is solvable, which means there is a series

$$\{e\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{r-1} \triangleleft G_r = G,$$

of subgroups of G, such that each quotient G_i/G_{i-1} is abelian. If G is finitely generated, then the *Hirsch rank* of G is the sum of the ranks of these (finitely generated) abelian groups. That is,

$$\operatorname{rank} G = \sum_{i=1}^{r} \operatorname{rank}(G_i/G_{i-1}).$$

It is not difficult to see that this is independent of the choice of the subgroups $G_1, \ldots, G_{r-1}$.

Lemma 2.7. *Assume G is a finitely generated, nilpotent group, and H is a subgroup of G. Then:*

1. [6, 1.2.16, p. 11]. *H is finitely generated.*
2. [6, 2.3.1(ii), p. 39]. *$\sqrt{H}$ is a subgroup of G that contains H, and $\left|\sqrt{H} : H\right| < \infty$.*
3. *If $N \trianglelefteq G$, then $\sqrt{N} \trianglelefteq G$ and $G/\sqrt{N}$ is torsion-free.*
4. [6, 2.3.8(ii), p. 42]. *If G is torsion-free, then $\sqrt{Z(G)} = Z(G)$.*
5. [6, 2.3.9(iv), p. 43]. $\left[\sqrt{H}, \sqrt{H}\right] \subseteq \sqrt{[H,H]}$.
6. *If N is a normal subgroup of G, then* $\operatorname{rank} G = \operatorname{rank} N + \operatorname{rank}(G/N)$. *Therefore,* $\operatorname{rank}(G/N) \leq \operatorname{rank} G$*, with equality if and only if N is finite.*
7. (*cf.* [4, Lem. 2.6, p. 9]). *We have* $\operatorname{rank} H \leq \operatorname{rank} G$*, with equality if and only if $|G : H| < \infty$.*
8. [11, 5.2.1, p. 129]. *If N is a nontrivial normal subgroup of G, then $N \cap Z(G)$ is nontrivial.*

Remark 2.8. Lemma 2.7(5) corrects a typographical error. It is stated in [6, 2.3.9(iv), p. 43] that equality holds, but a counterexample to this is provided by any finite-index subgroup G of the discrete Heisenberg group, such that $[G,G]$ is a proper subgroup of $Z(G)$: letting $H = G$, we have

$$\left[\sqrt{G}, \sqrt{G}\right] = [G,G] \neq Z(G) = \sqrt{[G,G]}.$$

Lemma 2.9. *Assume S is a symmetric, finite generating set of a finitely generated, nilpotent group G. Then:*

1. (*cf.* [6, 2.3.8(i), p. 42]). *If G is torsion-free, then the elements of $Z(G)$ are the only elements of G that have only finitely many conjugates.*
2. (*cf.* [6, 2.1.2, p. 30]). *Assume G is torsion-free, and $a, b, g \in G$. If*
$$\sup_{k \in \mathbb{Z}^+} \operatorname{dist}_S(a^k, gb^k) < \infty,$$
then $b = g^{-1}ag$.
3. [1, Lem. 3.5(i,iii)]. *For $g \in G$, we have $g \in \sqrt{[G,G]}$ if and only if $\operatorname{dist}_S(e, g^k)/k \to 0$ as $k \to \infty$.*

Proof. (1) In the terminology of [6, first paragraph of §2.3, p. 38], to say that a subgroup H of G is *isolated* means that if $g^k \in H$ for some nonzero $k \in \mathbb{Z}$, then $g \in H$. (More generally, the notion of π-*isolated* is defined for every set π of prime numbers, and the above definition is the special case where π is the set of all prime numbers.) Since G is torsion-free and nilpotent, [6, 2.3.8(i), p. 42] tells us that $C_G(H)$ is isolated, for every subgroup H of G.

Now, suppose h is an element of G that has only finitely many conjugates. This means $C_G(h)$ is a finite-index subgroup of G, so there is some nonzero $k \in \mathbb{Z}$, such that $g^k \in C_G(h)$ for all $g \in G$. From the preceding paragraph, we conclude that $g \in C_G(h)$. Since this is true for all $g \in G$, this means $h \in Z(G)$.

(2) Since $\operatorname{dist}_S(a^k, gb^k)$ is bounded as a function of k, we know that
$$\{\, a^{-k} g b^k \mid k \in \mathbb{Z} \,\} \text{ is finite.}$$
Hence, there exist $k \neq \ell$, such that $a^{-k}gb^k = a^{-\ell} g\, b^{\ell}$, so, letting $m = \ell - k \neq 0$, we have $g^{-1}a^m g = b^m$. In other words, $(g^{-1}ag)^m = b^m$. Since G is torsion-free nilpotent, this implies $g^{-1}ag = b$ [6, 2.1.2, p. 30].

(3) The paper [1] uses the following notation:

$\|x\| = \operatorname{dist}_S(e, x)$	[1, Defns. 2.2 and 2.3]
$\tau(x) = \lim_{n\to\infty} \|x^n\|/n$	[1, Lem. 2.43(i)]
$I(G) = \{\, g \in G \mid \tau(g) = 0 \,\}$	[1, Defn. 2.5]
$B(G) = \{\, g \in G \mid \tau(gx) = \tau(x),\ \forall x \in G \,\}$	[1, Defn. 3.1]
$G' = [G,G]$	[1, Notn. 3.2(ii)].

In this notation, we have
$$\operatorname{dist}_S(e, g^k)/k \to 0 \Leftrightarrow \|g^k\|/k \to 0 \Leftrightarrow \tau(g) = 0 \Leftrightarrow g \in I(G).$$
Therefore, the desired conclusion is immediate from [1, Lem. 3.5(i,iii)], which states that, since G is nilpotent (with finite generating set S), we have $I(G) = B(G) = \sqrt{G'}$. □

Definition 2.10. A group G is *bi-orderable* if it is has a total order $\prec$ that is invariant under both left-translations and right-translations. (That is, $x \prec y \Rightarrow axb \prec ayb$ for all $x, y, a, b \in G$.)

Lemma 2.11 ([5, Cor. 3.3.2, p. 57])**.** *Every torsion-free, nilpotent group is bi-orderable.*

Lemma 2.12 (cf. [12, 1st paragraph of §4] or [8, Prop. 2.9(1)])**.** *If S is a finite generating set of a nontrivial, bi-orderable group G, then there exists $s \in S$, such that $[s^i]_{i=-\infty}^{\infty}$ is a convex geodesic line in* $\mathrm{Cay}(G; S \cup S^{-1})$.

Proof. Let $\prec$ be a total order on G that is invariant under both left-translations and right-translations. Since the set $S \cup S^{-1}$ is finite, it has a maximal element s under this order. We may assume $s \in S$, by replacing $\prec$ with the order $\prec'$ defined by $x \prec' y \Leftrightarrow x^{-1} \prec y^{-1}$, if necessary.

For $a, b, c, d \in G$ with $a \preceq b$ and $c \preceq d$, the invariance under translations implies that $ac \preceq bd$ (and equality holds iff $a = b$ and $c = d$). By induction on k, we conclude that $s_1 s_2 \cdots s_k \preceq s^k$ for all $s_1, s_2, \ldots, s_k \in S \cup S^{-1}$, and that equality holds iff $s_1 = s_2 = \cdots = s_k = s$. This implies that $[s^i]_{i=-\infty}^{\infty}$ is a convex geodesic line. □

3. Torsion-free nilpotent groups

In this section, we prove Theorem 1.4. Let φ be an isomorphism from $\mathrm{Cay}(G_1; S_1)$ to $\mathrm{Cay}(G_2; S_2)$. By composing with a left translation, we may assume $\varphi(e) = e$. (Under this assumption, we will show that φ is a group homomorphism. Since φ is bijective, it must then be a group isomorphism.) The proof is by induction on $\operatorname{rank} G_1 + \operatorname{rank} G_2$.

Notation. Let $Z_i^\dagger = Z(G_i) \cap \sqrt{[G_i, G_i]}$ for $i = 1, 2$.

Step 1. *For every $g \in G_1$ and $z \in Z_1^\dagger$, there exists $\sigma_g(z) \in G_2$, such that $\varphi(gz^k) = \varphi(g)\, \sigma_g(z)^k$ for all $k \in \mathbb{Z}$.*

Proof. By composing with left translations in G_1 and G_2, we may assume $g = e$. Define S_1^*, S_2^*, G_1^*, and G_2^* as in Lemma 2.3, with $g_1 = z$ and $g_2 = \varphi(z)$. Combining Lemmas 2.11 and 2.12 yields $s \in S_2^*$, such that

$$[s^i]_{i=-\infty}^{\infty} \text{ is a convex geodesic line in } \mathrm{Cay}\big(G_2^*; S_2^* \cup (S_2^*)^{-1}\big).$$

We see from the definition of S_2^* that there is an isomorphism ψ from $\mathrm{Cay}(G_1; S_1)$ to $\mathrm{Cay}(G_2; S_2)$, such that $\psi(e) = e$ and $\psi(z) = s$. Since Lemma 2.3 implies ψ restricts to an isomorphism from $\mathrm{Cay}\big(G_1^*; S_1^* \cup (S_1^*)^{-1}\big)$ to $\mathrm{Cay}\big(G_2^*; S_2^* \cup (S_2^*)^{-1}\big)$, we know that $\psi^{-1}\big([s^i]_{i=-\infty}^{\infty}\big)$ is a convex geodesic line in $\mathrm{Cay}\big(G_1^*; S_1^* \cup (S_1^*)^{-1}\big)$. From the choice of ψ, this geodesic line contains the edge $e \,—\, z$, so Lemma 2.4 tells us that this geodesic line must be $[z^i]_{i=-\infty}^{\infty}$. This means $\mathrm{dist}_{S_1^*}(z^i, z^j) = |i - j|$ for all $i, j \in \mathbb{Z}$. We conclude from Lemma 2.9(3) that $z \notin \sqrt{[G_1^*, G_1^*]}$.

On the other hand, since $z \in Z_1^\dagger$, we know that $z \in \sqrt{[G_1, G_1]}$. Therefore $\sqrt{[G_1^*, G_1^*]} \neq \sqrt{[G_1, G_1]}$. This implies that $[G_1^*, G_1^*]$ has infinite index in $[G_1, G_1]$ (cf. Lemma 2.7(2)), so G_1^* must have infinite index in G_1 (cf. Lemma 2.7(5)). Therefore, $\operatorname{rank} G_1^* + \operatorname{rank} G_2^* < \operatorname{rank} G_1 + \operatorname{rank} G_2$ (see Lemma 2.7(7)), so our induction hypothesis tells us that the restriction of φ to G_1^* is a group isomorphism onto G_2^*. Hence, $\varphi(z^k) = \varphi(z)^k$ for all k, so we may let $\sigma_g(z) = \varphi(z)$. □

Step 2. *We have* $\varphi(xZ_1^\dagger) = \varphi(x)Z_2^\dagger$, *for all* $x \in G_1$.

Proof. By composing with left translations in G_1 and G_2, we may assume $x = e$. Then, since φ^{-1} is also an isomorphism, it suffices to show

$$\varphi(Z_1^\dagger) \subseteq Z_2^\dagger.$$

Fix $z \in Z_1^\dagger$. For all $k \in \mathbb{Z}$, we have

$$\operatorname{dist}_{S_1}(z^k, gz^k) = \operatorname{dist}_{S_1}(e, g)$$

(because $z \in Z(G_1)$). Since φ is a graph isomorphism, this implies

$$\operatorname{dist}_{S_2}\big(\varphi(z)^k, \varphi(g)\, \sigma_g(z)^k\big)$$

does not depend on k. So Lemma 2.9(2) tells us that

$$\varphi(g)^{-1}\varphi(z)\varphi(g) = \sigma_g(z).$$

From the definition of $\sigma_g(z)$, we see that $\operatorname{dist}_S\big(e, \sigma_g(z)\big) = \operatorname{dist}_S(e, z)$, so this implies that $\varphi(g)^{-1}\varphi(z)\varphi(g)$ is in a ball of fixed radius, independent of g. Since $\varphi(g)$ is an arbitrary element of G_2, we conclude that $\varphi(z)$ has only finitely many conjugates. Since G_2 is torsion-free nilpotent, this implies $\varphi(z) \in Z(G_2)$ (see Lemma 2.9(1)).

Also, we see from Lemma 2.9(3) that $\varphi\big(\sqrt{[G_1, G_1]}\,\big) = \sqrt{[G_2, G_2]}$ (since φ is a graph isomorphism). Therefore $\varphi(z) \in \sqrt{[G_2, G_2]}$. So $\varphi(z) \in Z_2^\dagger$. □

Step 3. *Completion of the proof of* Theorem 1.4.

Proof. We claim that we may assume $\operatorname{Cay}(Z_1^\dagger; S_1 \cap Z_1^\dagger)$ is connected. For a graph Γ and $r \in \mathbb{Z}^+$, the *rth power* of Γ is the graph Γ^r with the same vertex set as Γ, and with an edge from u to v iff $\operatorname{dist}_\Gamma(u, v) \le r$. It is clear that:

- Any isomorphism from Γ_1 to Γ_2 is also an isomorphism from Γ_1^r to Γ_2^r.
- $\operatorname{Cay}(G; S)^r = \operatorname{Cay}\big(G; S^{\le r}\big)$, where $S^{\le r}$ is the set of all elements of G that can be written as a product of $\le r$ elements of S.

We know that $Z_1^\dagger$ has a finite generating set (see Lemma 2.7(1)). For any sufficiently large r, this finite set is contained in $S_1^{\le r}$. Since φ is an isomorphism from $\operatorname{Cay}\big(G_1; S_1^{\le r}\big)$ to $\operatorname{Cay}\big(G_2; S_2^{\le r}\big)$, there is no harm in replacing S_1 and S_2 with $S_1^{\le r}$ and $S_2^{\le r}$. This completes the proof of the claim.

For $i = 1, 2$, we have

$$\begin{aligned}
\sqrt{Z_i^\dagger} &= \sqrt{Z(G_i) \cap \sqrt{[G_i, G_i]}} && \text{(definition of } Z_i^\dagger) \\
&= \sqrt{Z(G_i)} \cap \sqrt{\sqrt{[G_i, G_i]}} && (\sqrt{H \cap K} = \sqrt{H} \cap \sqrt{K}) \\
&= Z(G_i) \cap \sqrt{[G_i, G_i]} && \text{(Lemma 2.7(4) and } \sqrt{\sqrt{H}} = \sqrt{H}) \\
&= Z_i^\dagger && \text{(definition of } Z_i^\dagger),
\end{aligned}$$

so $G/Z_i^\dagger$ is torsion-free.

Let $\overline{G_i} = G_i/Z_i^\dagger$ for $i = 1, 2$. From Step 2, we know that φ induces a well-defined isomorphism $\overline{\varphi}$ from $\mathrm{Cay}(\overline{G_1}; S_1)$ to $\mathrm{Cay}(\overline{G_2}; S_2)$. We may assume that G_1 and G_2 are not both abelian (otherwise, Ryabchenko's Theorem (2.1) applies), so either $[G_1, G_1]$ or $[G_2, G_2]$ is nontrivial. This implies that either $Z_1^\dagger$ or $Z_2^\dagger$ is nontrivial (see Lemma 2.7(8)), and therefore infinite (since G_1 and G_2 are torsion-free). Hence, we have

$$\operatorname{rank} \overline{G_1} + \operatorname{rank} \overline{G_2} < \operatorname{rank} G_1 + \operatorname{rank} G_2$$

(see Lemma 2.7(6)), so, by induction on $\operatorname{rank} G_1 + \operatorname{rank} G_2$, we may assume that $\overline{\varphi}$ is a group isomorphism from $\overline{G_1}$ to $\overline{G_2}$.

For each $g \in G_1$ and $z \in Z_1^\dagger$, we have

$$\begin{aligned}
\mathrm{dist}_{S_2}\big(\sigma_e(z)^k, \varphi(g)\,\sigma_g(z)^k\big) &= \mathrm{dist}_{S_2}\big(\varphi(z^k), \varphi(gz^k)\big) \\
&= \mathrm{dist}_{S_1}(z^k, gz^k) \\
&= \mathrm{dist}_{S_1}(e, g),
\end{aligned}$$

since $z \in Z(G_1)$. Then, from Lemma 2.9(2) (and the fact that Step 2 tells us that $\sigma_e(z)$ is in $Z_2^\dagger$ and therefore commutes with $\varphi(g)$), we see that $\sigma_g(z) = \sigma_e(z)$. This means $\sigma_g(z)$ is independent of g (so we may drop the subscript).

Fix some $g \in G_1$ and $s \in S_1$. We have $\varphi(gs) = \varphi(g)\,\varphi(s)\,\sigma(z)$, for some $z \in Z_1^\dagger$ (because $\overline{\varphi}$ is a homomorphism and the surjectivity in Step 2 tells us $\sigma(Z_1^\dagger) = Z_2^\dagger$). Consider any $k \geq 0$ with $sz^k \in S_1$. Then

$$\varphi(gsz^k) = \varphi(gs)\,\sigma(z)^k = \varphi(g)\,\varphi(s)\,\sigma(z)\,\sigma(z)^k = \varphi(g)\,\varphi(sz^{k+1}).$$

Since φ is a graph homomorphism and, by assumption, $sz^k \in S_1$, we must have $\varphi(sz^{k+1}) \in S_2$. So $sz^{k+1} \in S_1$. By induction (with $k = 0$ as the base case), we conclude that $sz^k \in S_1$ for all $k \in \mathbb{Z}^+$. Since S_1 is finite (and G_1 is torsion-free), this implies $z = e$. So $\varphi(gs) = \varphi(g)\,\varphi(s)$. Since g is an arbitrary element of G_1 and s is an arbitrary element of the generating set S_1, this implies that φ is a group homomorphism. □

4. Nilpotent groups that may have torsion

Proposition 4.1. *Assume:*

- *S is a finite generating set of the group G.*
- *N is a finite, normal subgroup of G, such that G/N is bi-orderable.*

Then every automorphism of $\mathrm{Cay}(G;S)$ *induces a well-defined automorphism of* $\mathrm{Cay}(G/N;S)$.

Proof. Assume, without loss of generality, that $N \subseteq S$ (by passing to a power of $\mathrm{Cay}(G;S)$), and let

$$N^* = \{\, \varphi(n) \mid \varphi \in \mathrm{Aut}_e\big(\mathrm{Cay}(G;S)\big), n \in N \,\}.$$

It is important to note that, since N is contained in a ball of finite radius centred at e, and N^* must be contained in that same ball, the set N^* is finite. We wish to show $N^* \subseteq N$.

We claim that there is no harm in assuming $\langle N^* \rangle = G$. To this end, let φ be an automorphism of $\mathrm{Cay}(G;S)$ that fixes e. Since N^* is obviously invariant under $\mathrm{Aut}_e\big(\mathrm{Cay}(G;S)\big)$, we know that φ restricts to an automorphism φ^* of $\mathrm{Cay}(\langle N^* \rangle; N^*)$. Also, it is clear from the definition of N^* that N is contained in N^* (and $\langle N^* \rangle / N$ is bi-orderable, because it is a subgroup of G/N). Therefore, if we know the proposition is true for $\langle N^* \rangle$, then $\varphi^*(N)$ is contained in N. Since $\varphi^*(N) = \varphi(N)$, this means that $\varphi(N)$ is contained in N. This completes the proof of the claim.

Let $\overline{G} = G/N$, and let $\overline{N^*} = \{gN \mid g \in N^*\}$. We wish to show $\overline{G}$ is trivial. Suppose not. (This will lead to a contradiction.) Since, by assumption, $\overline{G}$ is bi-orderable, Lemma 2.12 provides $g \in N^*$, such that $[\overline{g}^i]_{i=-\infty}^{\infty}$ is a geodesic line in $\mathrm{Cay}(\overline{G}; \overline{N^*})$. Then, since the natural map $\mathrm{Cay}(G;N^*) \to \mathrm{Cay}(\overline{G}; \overline{N^*})$ decreases distances, it is clear that $\gamma = [g^i]_{i=-\infty}^{\infty}$ is a geodesic line in $\mathrm{Cay}(G;N^*)$. By the definition of N^*, there exists $\varphi \in \mathrm{Aut}_e\big(\mathrm{Cay}(G;S)\big)$, such that $\varphi(g) \in N$. Then $\varphi(\gamma)$ is a geodesic line that contains the edge $e - n$ for some $n \in N$.

To obtain the contradiction that completes the proof, we use an argument of C. Löh [8, first paragraph of page 105]. Write $\varphi(\gamma) = [h_i]_{i=-\infty}^{\infty}$. For each $k \in \mathbb{N}$, let $\#(k)$ be the number of geodesic segments from h_i to h_{i+k}. (Since $\gamma = [g^i]_{i=-\infty}^{\infty}$ is obviously homogeneous, we know that $\varphi(\gamma)$ is also homogeneous, so $\#(k)$ is independent of the choice of i.) We may assume $h_0 = e$ (so $h_1 = n$). Since N is a finite normal subgroup of G, it is easy to see that no geodesic segment can contain two edges that are labelled by elements of N. (Namely, if $(n, s_1, \ldots, s_k, n')$ is a path in $\mathrm{Cay}(G;N)$, then there exists $n'' \in N_1$, such that $n'' s_1 \cdots s_k = n s_1 \cdots s_k n'$, so $(n'', s_1, \ldots, s_k)$ is a shorter path with the same endpoints.) Hence, for all $k > 1$, no geodesic segment from h_1 to h_k has any edges that are labelled by elements of N. (Otherwise, concatenating (n) at the start would yield a geodesic segment from h_0 to h_k with more than one edge labelled by elements of N.)

For any geodesic segment $\gamma' = (s_1, \ldots, s_k)$ from h_1 to h_{k+1}, we can construct $k+1$ different geodesic segments $\gamma_1, \ldots, \gamma_{k+1}$ from h_0 to h_{k+1}, by inserting a single edge labelled by an element of N, as follows:

$$\gamma_i = (s_1, s_2, \ldots, s_{i-1}, n_i, s_i, \ldots, s_k),$$

where $n_i \in N$ is chosen so that $n s_1 s_2 \cdots s_{i-1} = s_1 s_2 \cdots s_{i-1} n_i$. (This is possible because the subgroup N is normal.) This implies

$$\#(k+1) \geq (k+1) \cdot \#(k),$$

for all k. Therefore $\#(k) \geq k!$. However, it is clear that $\#(k) \leq |S|^k$, so this contradicts the fact that factorials grow faster than exponentials. □

Combining this proposition with Theorem 1.4 yields the following modest generalization of Theorem 1.12 that allows G_1 and G_2 to be slightly nonnilpotent:

Theorem 4.2. *Assume:*

- S_i *is a symmetric, finite generating set of the group* G_i*, for* $i = 1, 2$.
- N_i *is a finite, normal subgroup of* G_i*, such that* G_i/N_i *is torsion-free nilpotent, for* $i = 1, 2$.
- φ *is an isomorphism from* $\mathrm{Cay}(G_1; S_1)$ *to* $\mathrm{Cay}(G_2; S_2)$.

Then φ *induces a well-defined affine bijection* $\overline{\varphi}\colon G_1/N_1 \to G_2/N_2$.

Proof. By using φ to identify $\mathrm{Cay}(G_1; S_1)$ with $\mathrm{Cay}(G_2; S_2)$, we can realize G_2 as a sharply transitive subgroup G_2' of $\mathrm{Aut}\big(\mathrm{Cay}(G_1; S_1)\big)$. (Namely, for $h \in G_2$, define $h'(x) = \varphi^{-1}\big(h\,\varphi(x)\big)$.)

For any $g \in G_1$ and $n \in N_1$, there exists $h \in G_2$, such that $\varphi(gn) = h\,\varphi(g)$. This means $h'g = gn \in gN_1$. From Proposition 4.1 (and Lemma 2.11), we know that G_2' factors through to a well-defined group of permutations on G_1/N_1, so this implies $h'(gN_1) = gN_1$. Since gN_1 is finite (and G_2' is sharply transitive), we conclude that h' has finite order, so h' is in the torsion subgroup N_2' of G_2'. This means $h \in N_2$, so

$$\varphi(gn) = h\,\varphi(g) \in N_2\,\varphi(g).$$

Therefore $\varphi(gN_1) \subseteq N_2\,\varphi(g)$. So φ induces a well-defined function

$$\overline{\varphi}\colon G_1/N_1 \to G_2/N_2.$$

□

Definition 4.3 ([7, p. 305])**.** The *wreath product* (or *lexicographic product*) of two graphs $X_1 = (V_1, E_1)$ and $X_2 = (V_2, E_2)$ is the graph $X_1[X_2]$ with vertex set $V_1 \times V_2$, such that (v_1, v_2) is adjacent to (v_1', v_2') if and only if either

- v_1 is adjacent to v_1' in X_1, or
- $v_1 = v_1'$ and v_2 is adjacent to v_2' in X_2.

Proof of Corollary 1.13. ($\Rightarrow$) Let S_1 and S_2 be finite, symmetric generating sets of G_1 and G_2, respectively, such that there is an isomorphism φ from $\mathrm{Cay}(G_1; S_1)$ to $\mathrm{Cay}(G_2; S_2)$. From Theorem 1.12, we know that φ

induces a well-defined affine bijection $\overline{\varphi}\colon G_1/N_1 \to G_2/N_2$. By composing with a left-translation, we may assume $\overline{\varphi}$ is a group isomorphism. Obviously, this implies $G_1/N_1 \cong G_2/N_2$. Also, since $\overline{\varphi}$ is a well-defined bijection, we must have $\varphi(N_1) = N_2$. Since φ is a bijection, this implies $|N_1| = |N_2|$.

($\Leftarrow$) Let

- $\overline{\varphi}$ be an isomorphism from G_1/N_1 to G_2/N_2,
- $\overline{S_1}$ be a finite generating set of G_1/N_1, with $e \notin \overline{S_1}$,
- $\overline{S_2} = \overline{\varphi}(\overline{S_1})$ be the corresponding generating set of G_2/N_2, and
- $S_i = \{\, s \in G_i \mid sN_i \in \overline{S_i} \,\}$, for $i = 1, 2$.

Let $n = |N_1| = |N_2|$, and let E_n be the edgeless graph on n vertices. Then, for $i = 1, 2$, it is easy to see that $\operatorname{Cay}(G_i; S_i)$ is isomorphic to the wreath product $\operatorname{Cay}\big(G_i/N_i; \overline{S_i}\big)[E_n]$. Since it is obvious that $\overline{\varphi}$ is an isomorphism from $\operatorname{Cay}\big(G_1/N_1; \overline{S_1}\big)$ to $\operatorname{Cay}\big(G_1/N_2; \overline{S_2}\big)$, we have $\operatorname{Cay}(G_1; S_1) \cong \operatorname{Cay}(G_2; S_2)$. □

Proof of Corollary 1.14. Let H be a sharply transitive, nilpotent subgroup of $\operatorname{Aut}\big(\operatorname{Cay}(G; S)\big)$. A well-known observation of G. Sabidussi [13, Thm. 2] tells us that a graph Γ is isomorphic to a Cayley graph on G if and only if $\operatorname{Aut}\Gamma$ contains a sharply transitive subgroup that is isomorphic to G. So $\operatorname{Cay}(G; S)$ is isomorphic to a Cayley graph on H. Therefore, if we let N be the torsion subgroup of H, then we see from Corollary 1.13 that $G/\{e\} \cong H/N$ and $|\{e\}| = |N|$ (because the torsion subgroup of G is trivial). So $G \cong H$.

From Theorem 1.4, we see that if S' is any symmetric, finite subset of G, such that $\operatorname{Cay}(G; S') \cong \operatorname{Cay}(G; S)$, then there is a group automorphism α of G with $\alpha(S) = S'$. Therefore, since H is a sharply transitive subgroup of $\operatorname{Aut}\big(\operatorname{Cay}(G; S)\big)$ that is isomorphic to G, a well-known theorem of L. Babai tells us that H is conjugate in $\operatorname{Aut}\big(\operatorname{Cay}(G; S)\big)$ to the left-regular representation of G [7, Thm. 4.1]. However, Corollary 1.8 states that the left-regular representation has no other conjugates in $\operatorname{Aut}\big(\operatorname{Cay}(G; S)\big)$, so we conclude that H is equal to the left-regular representation of G. □

As was mentioned in Remark 1.15(4), the following consequence of Theorem 1.12 is already known (as a corollary of Gromov's Theorem on groups of polynomial growth).

Corollary 4.4. *Assume:*

- *S_i is a finite, symmetric generating set of a group G_i, for $i = 1, 2$.*
- *$\operatorname{Cay}(G_1; S_1)$ is isomorphic to $\operatorname{Cay}(G_2; S_2)$.*

If G_1 is nilpotent, then G_2 has a nilpotent subgroup of finite index.

Proof. We know that G_2 is isomorphic to a group G_2' of automorphisms of $\operatorname{Cay}(G_1; S_1)$ (see the first paragraph of the proof of Theorem 4.2). Let N_1 be the torsion subgroup of G_1, and let $N_2' = \{\, g \in G_2' \mid g(e) \in N_1 \,\}$. Then Theorem 1.12 implies that G_2'/N_2' is isomorphic to a subgroup of

$\mathrm{Aut}\big(\mathrm{Cay}(G_1/N_1; S_1)\big)$. Since Theorem 1.4 implies that the left-regular representation of the nilpotent group G_1/N_1 has finite index in this automorphism group, we conclude that G_2'/N_2' has a nilpotent subgroup G_2''/N_2' of finite index. Then $C_{G_2''}(N_2')$ is a nilpotent subgroup of finite index in G_2'. $\square$

5. Other groups that have torsion

In this section, we prove Proposition 1.10. In fact, we prove a more specific version of Proposition 1.10:

Proposition 5.1. *Suppose F is a nontrivial, finite subgroup of a group G, and S is any finite, symmetric generating set for G. Then* $\mathrm{Cay}(G; FSF)$ *is a connected Cayley graph of finite valency that is not normal.*

Proof. It is straightforward to verify that FSF is a symmetric, finite generating set of G, so $\mathrm{Cay}(G; FSF)$ is a connected Cayley graph of finite valency. Furthermore, for all $g \in G$, it is straightforward to check that all vertices in the coset gF have the same neighbours. Therefore, if we choose some $h \in gF$ (with $h \neq g$), then there is an automorphism φ of $\mathrm{Cay}(G; FSF)$ that interchanges g and h, but fixes all other vertices of the Cayley graph. Since G is infinite, but FSF is finite, we may assume g has been chosen so that gF is disjoint from $FSF \cup \{e\}$. Then φ fixes e, but is obviously not a group automorphism, since it fixes every element of the generating set FSF, and is not the identity map (since it moves g to h). So φ is not an affine bijection. $\square$

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