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On the uniqueness of algebraic curves passing through n-independent nodes

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Abstract. A set of nodes in the plane is called n-independent if for arbitrary data at those nodes, there is a (not necessarily unique) polynomial of degree at most n that matches the given information. We proved in a previous paper (Hakopian–Toroyan, 2015) that the minimal number of n-independent nodes determining uniquely the curve of degree $k \leq n$ passing through them equals to $\mathcal{D} := (1/2)(k-1)(2n+4-k)+2$. In this paper we bring a characterization of the case when at least two curves of degree k pass through the nodes of an n-independent node set of cardinality $\mathcal{D}-1$. Namely, we prove that the latter set has a very special construction: All its nodes but one belong to a (maximal) curve of degree $k-1$. We show that this result readily yields the above cited one. At the end, an important application to the Gasca–Maeztu conjecture is presented.

Contents

1. Introduction 441
2. Some properties of n-independent nodes 444
3. Maximal curves 445
4. Main result 447
5. A corollary 449
6. An application to the Gasca–Maeztu conjecture 450

References 451

1. Introduction

Denote the space of all bivariate polynomials of total degree $\leq n$ by Π_n:

$$\Pi_n = \left\{ \sum_{i+j \leq n} a_{ij} x^i y^j \right\}.$$

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We have that

$$N := N_n := \dim \Pi_n = (1/2)(n+1)(n+2).$$

Consider a set of s distinct nodes

$$\mathcal{X}_s = \{(x_1, y_1), (x_2, y_2), \ldots, (x_s, y_s)\}.$$

The problem of finding a polynomial $p \in \Pi_n$ which satisfies the conditions

$$p(x_i, y_i) = c_i, \quad i = 1, \ldots, s, \tag{1.1}$$

is called interpolation problem.

A polynomial $p \in \Pi_n$ is called an n-fundamental polynomial for a node $A = (x_k, y_k) \in \mathcal{X}_s$ if

$$p(x_i, y_i) = \delta_{ik},\ i = 1, \ldots, s,$$

where δ is the Kronecker symbol. We denote this fundamental polynomial by $p_k^\star = p_A^\star = p_{A,\mathcal{X}_s}^\star$. Sometimes we call fundamental also a polynomial that vanishes at all nodes of $\mathcal{X}_s$ but one, since it is a nonzero constant times a fundamental polynomial.

Next, let us consider an important concept of n-independence (see [7], [11]).

Definition 1.1. A set of nodes $\mathcal{X}$ is called *n-independent* if all its nodes have n-fundamental polynomials. Otherwise, if a node has no n-fundamental polynomial, $\mathcal{X}$ is called *n-dependent*.

Fundamental polynomials are linearly independent. Therefore a necessary condition of n-independence of $\mathcal{X}_s$ is $s \le N$.

Suppose a node set $\mathcal{X}_s$ is n-independent. Then by the Lagrange formula we obtain a polynomial $p \in \Pi_n$ satisfying the interpolation conditions (1.1):

$$p = \sum_{i=1}^{s} c_i p_i^\star.$$

In view of this, we get readily that the node set $\mathcal{X}_s$ is n-independent if and only if the interpolating problem (1.1) is *solvable*, meaning that for any data $(c_1, \ldots, c_s)$ there is a polynomial $p \in \Pi_n$ (not necessarily unique) satisfying the interpolation conditions (1.1).

Definition 1.2. The interpolation problem with a set of nodes $\mathcal{X}_s$ and Π_n is called *n-poised* if for any data $(c_1, \ldots, c_s)$ there is a *unique* polynomial $p \in \Pi_n$ satisfying the interpolation conditions (1.1).

The conditions (1.1) give a system of s linear equations with N unknowns (the coefficients of the polynomial p). The poisedness means that this system has a unique solution for arbitrary right side values. Therefore a necessary condition of poisedness is $s = N$. If this condition holds then we obtain from the linear system:

Proposition 1.3. *A set of nodes $\mathcal{X}_N$ is n-poised if and only if*

$$p \in \Pi_n \quad and \quad p\big|_{\mathcal{X}_N} = 0 \quad \Longrightarrow \quad p = 0.$$

Thus, geometrically, the node set $\mathcal{X}_N$ is n-poised if and only if there is no curve of degree n passing through all its nodes.

It is worth mentioning:

Proposition 1.4. *For any set $\mathcal{X}_{N-1}$, i.e., set of cardinality $N-1$, there is a curve of degree n passing through all its nodes.*

Indeed, the existence of the curve reduces to a system of $N-1$ linear homogeneous equations with N unknowns – the coefficients of the polynomial of degree n.

It follows from Proposition 1.3 also that a node set of cardinality N is n-poised if and only if it is n-independent.

Next, let us describe the main result of this paper. Suppose we have an n-poised set $\mathcal{X}_N$. From what was said above we can conclude readily that through any $N-1$ nodes of $\mathcal{X}$ there pass a unique curve of degree n. Indeed, this curve is given by the fundamental polynomial of the missing node. Next, through any $N-2$ nodes of $\mathcal{X}$ there pass more than one curve of degree n, for example the curves given by the fundamental polynomials of two missing nodes. Thus we have that the minimal number of n-independent nodes determining uniquely the curve of degree n equals to $N-1$.

In [14] we considered this problem in the case of arbitrary degree $k,\ k \le n-1$. We proved that the minimal number of n-independent nodes determining uniquely the curve of degree $k \le n-1$ equals

$$\mathcal{D} := (1/2)(k-1)(2n+4-k)+2.$$

Or, more precisely, for any n-independent set of cardinality $\mathcal{D}$ there is at most one curve of degree $k \le n-1$ passing through its nodes, while there are n-independent node sets of cardinality $\mathcal{D}-1$ through which pass at least two such curves. Let us mention that the case $k = n-1$ of the above described problem is considered in [2].

In this paper we bring a characterization of the sets of cardinality $\mathcal{D}-1$ through which pass at least two curves of degree k. Namely, we prove that in this case all the nodes of $\mathcal{X}$ but one belong to a curve of degree $k-1$. Moreover, this latter curve is a maximal curve meaning that it passes through maximal possible number of n-independent nodes (see Section 3).

As we will see in Section 5, this result readily yields the above mentioned result of [14].

At the end let us bring a well-known Berzolari–Radon construction of n-poised sets (see [3], [15]).

Definition 1.5. A set of $N = 1 + \cdots + (n+1)$ nodes is called Berzolari–Radon set for degree n, or briefly BR_n set, if there exist lines $\ell_1, \ell_2, \ldots, \ell_{n+1}$, such that the sets $\ell_1,\ \ell_2 \setminus \ell_1,\ \ell_3 \setminus (\ell_1 \cup \ell_2), \ldots,\ \ell_{n+1} \setminus (\ell_1 \cup \cdots \cup \ell_n)$ contain exactly $(n+1), n, n-1, \ldots, 1$ nodes, respectively.

2. Some properties of n-independent nodes

Let us start with the following simple (see, e.g., [12], Lemma 2.3; [11] Lemma 2.2) lemma.

Lemma 2.1. *Suppose that a node set $\mathcal{X}$ is n-independent and a node $A \notin \mathcal{X}$ has n-fundamental polynomial with respect to the set $\mathcal{X} \cup \{A\}$. Then the latter node set is n-independent, too.*

Indeed, one can get readily the fundamental polynomial of any node $B \in \mathcal{X}$ with respect to the set $\mathcal{Y} := \mathcal{X} \cup \{A\}$ by using a linear combination of the given fundamental polynomial $p_A^\star$ and the fundamental polynomial of B with respect to the set $\mathcal{X}$.

Evidently, any subset of n-poised set is n-independent. According to the next lemma any n-independent set is a subset of some n-poised set:

Lemma 2.2 (e.g., [9], Lemma 2.1)**.** *Any n-independent set $\mathcal{X}$ with $\#\mathcal{X} < N$ can be enlarged to an n-poised set.*

Proof. It suffices to show that there is a node A such that the set $\mathcal{X} \cup \{A\}$ is n-independent. By Proposition 1.4 there is a nonzero polynomial $q \in \Pi_n$ such that $q\big|_{\mathcal{X}} = 0$. Now, in view of Lemma 2.1, we may choose a desirable node A by requiring only that $q(A) \neq 0$. Indeed, then q is a fundamental polynomial of A with respect to the set $\mathcal{X} \cup \{A\}$. □

Denote the linear space of polynomials of total degree at most n vanishing on $\mathcal{X}$ by

$$\mathcal{P}_{n,\mathcal{X}} = \left\{p \in \Pi_n : p\big|_{\mathcal{X}} = 0\right\}.$$

The following is well-known.

Proposition 2.3 (e.g., [9], [11])**.** *For any node set $\mathcal{X}$ we have that*

$$\dim \mathcal{P}_{n,\mathcal{X}} \geq N - \#\mathcal{X}.$$

Moreover, equality takes place here if and only if the set $\mathcal{X}$ is n-independent.

From Lemma 2.1 one gets readily:

Corollary 2.4 (e.g., [12], Corollary 2.4)**.** *Let $\mathcal{Y}$ be a maximal n-independent subset of $\mathcal{X}$, i.e., $\mathcal{Y} \subset \mathcal{X}$ is n-independent and $\mathcal{Y} \cup \{A\}$ is n-dependent for any $A \in \mathcal{X} \setminus \mathcal{Y}$. Then we have that*

$$\mathcal{P}_{n,\mathcal{Y}} = \mathcal{P}_{n,\mathcal{X}}. \tag{2.1}$$

Proof. We have that $\mathcal{P}_{n,\mathcal{X}} \subset \mathcal{P}_{n,\mathcal{Y}}$, since $\mathcal{Y} \subset \mathcal{X}$. Now, suppose that $p \in \Pi_n$, $p\big|_{\mathcal{Y}} = 0$ and A is any node of $\mathcal{X}$. Then $\mathcal{Y} \cup \{A\}$ is dependent and therefore, in view of Lemma 2.1, $p(A) = 0$. □

From (2.1) and Proposition 2.3 (part "moreover") we have that

$$\dim \mathcal{P}_{n,\mathcal{X}} = N - \#\mathcal{Y}, \tag{2.2}$$

where $\mathcal{Y}$ is any maximal n-independent subset of $\mathcal{X}$. Thus, all the maximal n-independent subsets of $\mathcal{X}$ have the same cardinality, which is denoted by $\mathcal{H}_n(\mathcal{X})$ – *the Hilbert n-function* of $\mathcal{X}$. Hence, according to (2.2), we have that

$$\dim \mathcal{P}_{n,\mathcal{X}} = N - \mathcal{H}_n(\mathcal{X}).$$

3. Maximal curves

An algebraic curve in the plane is the zero set of some bivariate polynomial of degree at least 1. We use the same letter, say p, to denote the polynomial $p \in \Pi_k \setminus \Pi_{k-1}$ and the corresponding curve p of degree k defined by equation $p(x, y) = 0$.

According to the following well-known statement there are no more than $n + 1$ n-independent points in any line:

Proposition 3.1. *Assume that ℓ is a line and $\mathcal{X}_{n+1}$ is any subset of ℓ containing $n + 1$ points. Then we have that*

$$p \in \Pi_n \quad \textit{and} \quad p|_{\mathcal{X}_{n+1}} = 0 \quad \Longrightarrow \quad p = \ell r,$$

where $r \in \Pi_{n-1}$.

Denote

$$d := d(n, k) := N_n - N_{n-k} = (1/2)k(2n + 3 - k).$$

The following is a generalization of Proposition 3.1.

Proposition 3.2 ([16], Prop. 3.1)**.** *Let q be an algebraic curve of degree $k \leq n$ without multiple components. Then the following hold:*

(i) *Any subset of q containing more than $d(n, k)$ nodes is n-dependent.*
(ii) *Any subset $\mathcal{X}_d$ of q containing exactly $d = d(n, k)$ nodes is n-independent if and only if the following condition holds:*

$$p \in \Pi_n \quad \textit{and} \quad p|_{\mathcal{X}_d} = 0 \quad \Longrightarrow \quad p = qr, \tag{3.1}$$

where $r \in \Pi_{n-k}$.

Suppose that $\mathcal{X}$ is an n-poised set of nodes and q is an algebraic curve of degree $k \leq n$. Then of course any subset of $\mathcal{X}$ is n-independent too. Therefore, according to Proposition 3.2(i), at most $d(n, k)$ nodes of $\mathcal{X}$ can lie in the curve q. Let us mention that a special case of this when q is a set of k lines is proved in [6].

This motivates the following definition.

Definition 3.3 ([16], Def. 3.1)**.** Given an n-independent set of nodes $\mathcal{X}_s$, with $s \geq d(n, k)$. A curve of degree $k \leq n$ passing through $d(n, k)$ points of $\mathcal{X}_s$, is called *maximal*.

Note that the maximal line, as a line passing through $n+1$ nodes, is defined in [4]. Let us mention that $q = \ell_1 \cdots \ell_k$ is a maximal curve of degree k of the node set BR_n (see Def. 1.5), where $k = 1, \ldots, n$.

We say that a node $A \in \mathcal{X}$ *uses* a polynomial $q \in \Pi_k$ if the latter divides the fundamental polynomial $p = p_A^\star$, i.e., $p = qr$, for some $r \in \Pi_{n-k}$.

Next, we bring a characterization of maximal curves:

Proposition 3.4 ([16], Prop. 3.3)**.** *Let a node set $\mathcal{X}$ be n-poised. Then a polynomial μ of degree $k,\ k \le n$, is a maximal curve if and only if it is used by any node in $\mathcal{X} \setminus \mu$.*

Note that one side of this statement follows from Proposition 3.2(ii). In the case of degree one this was proved in [4].

For other properties of maximal curves we refer reader to [16], [13].

Proposition 3.5. *Assume that σ is an algebraic curve of degree k, without multiple components, and $\mathcal{X}_s \subset \sigma$ is any n-independent node set of cardinality $s,\ s < d(n,k)$. Then the set $\mathcal{X}_s$ can be extended to a maximal n-independent set $\mathcal{X}_d \subset \sigma$ of cardinality $d = d(n,k)$.*

Proof. It suffices to show that there is a point $A \in \sigma \setminus \mathcal{X}_s$ such that the set $\mathcal{X}_{s+1} := \mathcal{X}_s \cup \{A\}$ is n-independent. Assume to the contrary that there is no such point, i.e., the set $\mathcal{X}_{s+1} := \mathcal{X}_s \cup \{A\}$ is n-dependent for any $A \in \sigma$. Then, in view of Lemma 2.1, A has no fundamental polynomial with respect to the set $\mathcal{X}_{s+1}$. In other words we have

$$p \in \Pi_n \ \text{ and } \ p\big|_{\mathcal{X}_s} = 0 \quad \Longrightarrow \quad p(A) = 0 \ \text{ for any } \ A \in \sigma.$$

From here we obtain that

$$\mathcal{P}_{n,\mathcal{X}_s} \subset \mathcal{P}_{n,\sigma} := \{q\sigma : q \in \Pi_{n-k}\}.$$

Now, in view of Proposition 2.3, we get from here

$$N - s = \dim \mathcal{P}_{n,\mathcal{X}_s} \le \dim \mathcal{P}_{n,\sigma} = N_{n-k}.$$

Therefore $s \ge d(n,k)$, which contradicts the hypothesis. □

Let us mention that, as it follows from the above proof, the condition (3.1) does not hold if $d < d(n,k)$.

The following lemma follows readily from the fact that the Vandermonde determinant, i.e., the main determinant of the linear system described just after Definition 1.2, is a continuous function of the nodes of $\mathcal{X}_N$.

Lemma 3.6 (e.g., [8], Remark 1.14)**.** *Suppose that $\mathcal{X}_N = \{(x_i, y_i)\}_{i=1}^N$ is an n-poised set. Then there is a positive number ϵ such that any set*

$$\mathcal{X}_N' = \{(x_i', y_i')\}_{i=1}^N,$$

with the property that the distance between (x_i', y_i') and (x_i, y_i) is less than ϵ, for each i, is n-poised too.

From here, in view of Lemma 2.2 we get readily:

Corollary 3.7. *Suppose that $\mathcal{X}_s = \{(x_i, y_i)\}_{i=1}^s$ is an n-independent set. Then there is a positive number ϵ such that any set $\mathcal{X}'_s = \{(x'_i, y'_i)\}_{i=1}^s$, with the property that the distance between (x'_i, y'_i) and (x_i, y_i) is less than ϵ, for each i, is n-independent too.*

Finally, let us bring a well-known lemma:

Lemma 3.8. *Suppose that two different curves of degree at most k pass through all the nodes of $\mathcal{X}$. Then for any node $A \notin \mathcal{X}$ there is a curve of degree at most k passing through A and all the nodes of $\mathcal{X}$.*

Indeed, if the given curves are σ_1 and σ_2 then the desired curve can be found easily in the form of linear combination $c_1\sigma_1 + c_2\sigma_2$.

4. Main result

In a previous paper we determined the minimal number of n-independent nodes that uniquely determine the curve of degree $k,\ k \le n$, passing through them:

Theorem 4.1 ([14], Thm. 1)**.** *Assume that $\mathcal{X}$ is an n-independent set of $d(n, k-1)+2$ nodes lying in a curve of degree k with $k \le n$. Then the curve is determined uniquely by these nodes. Moreover, there is an n-independent set of $d(n, k-1)+1$ nodes such that more than one curves of degree k pass through all its nodes.*

Let us mention that this result is obvious in the case $k = n$, while in the case $k = n-1$ it was established in [2].

In this section we characterize the case when more than one curve of degree $k,\ k \le n-1$, passes through the nodes of an n-independent set $\mathcal{X}$ of cardinality $d(n, k-1) + 1$.

As we will see later in Section 5 this result yields readily Theorem 4.1.

Theorem 4.2. *Assume that $\mathcal{X}$ is an n-independent set of $d(n, k-1)+1$ nodes with $k \le n-1$. Then two different curves of degree k pass through all the nodes of $\mathcal{X}$ if and only if all the nodes of $\mathcal{X}$ but one lie in a maximal curve of degree $k-1$.*

Proof. Let us start with the inverse implication. Assume that $d(n, k-1)$ nodes of $\mathcal{X}$ are located in a curve μ of degree $k-1$. Therefore the curve μ is maximal and the remaining node of $\mathcal{X}$, which we denote by A, is outside of it: $A \notin \mu$.

Now assume that ℓ_1 and ℓ_2 are two different lines passing through A. Then it is easily seen that $\ell_1\mu$ and $\ell_2\mu$ are two different curves of degree k passing through all the nodes of $\mathcal{X}$.

Now let us prove the direct implication. Assume that there are two curves of degree $k:\ \sigma_1$ and σ_2 that pass through all the nodes of the n-independent set $\mathcal{X}$ with $\#\mathcal{X} = d(n, k-1) + 1$. Let us start by choosing a node $B \notin \mathcal{X}$ such that the following three conditions are satisfied:

(i) The set $\mathcal{X} \cup \{B\}$ is n-independent.
(ii) B does not lie in any line passing through two nodes of $\mathcal{X}$.
(iii) B does not lie in the curves σ_1 and σ_2.

Let us verify that one can find a such node. Indeed, in view of Lemma 2.2, we can start by choosing a node B' satisfying the condition (i). Then, according to Corollary 3.7, for some positive ϵ all the nodes in ϵ neighborhood of B' satisfy the condition (i). Finally, from this neighborhood we can choose a node B satisfying the conditions (ii) and (iii), too.

Next, in view of Lemma 3.8, there is a curve σ of degree at most k passing through all the nodes of $\mathcal{X}' := \mathcal{X} \cup \{B\}$. According to the condition (iii) σ is different from σ_1 and σ_2. Then notice that the curve σ passes through more than $d(n, k-1)$ nodes and therefore its degree equals to k and it has no multiple component.

Now, by using Proposition 3.5, let us extend the set $\mathcal{X}'$ till a maximal n-independent set $\mathcal{X}'' \subset \sigma$. Notice that, since $\#\mathcal{X}'' = d(n, k)$, we need to add $d(n,k) - (d(n,k-1)+2) = n-k$ nodes to $\mathcal{X}'$, denoted by $C_1, \ldots, C_{n-k}$:

$$\mathcal{X}'' := \mathcal{X} \cup \{B\} \cup \{C_i\}_{i=1}^{n-k}.$$

Thus the curve σ becomes maximal with respect to this set.

Then let us consider $n-k-1$ lines $\ell_1, \ell_2, \ldots, \ell_{n-k-1}$ passing through the nodes $C_1, C_2, \ldots, C_{n-k-1}$, respectively. We require that each line passes through only one of the mentioned nodes and therefore the lines are distinct. We require also that none of these lines is a component (factor) of σ. Finally let us denote by $\tilde{\ell}$ the line passing through the nodes B and C_{n-k}.

Now notice that the following polynomial

$$\sigma_1 \, \tilde{\ell} \, \ell_1 \, \ell_2 \ldots \ell_{n-k-1}$$

of degree n vanishes at all the $d(n, k)$ nodes of $\mathcal{X}'' \subset \sigma$. Consequently, according to Proposition 3.2, σ divides this polynomial:

$$\sigma_1 \, \tilde{\ell} \, \ell_1 \, \ell_2 \ldots \ell_{n-k-1} = \sigma \, q, \quad q \in \Pi_{n-k}. \tag{4.1}$$

The distinct lines $\ell_1, \ell_2, \ldots, \ell_{n-k-1}$ do not divide the polynomial $\sigma \in \Pi_k$, therefore all they have to divide $q \in \Pi_{n-k}$. Thus $q = \ell_1 \ldots \ell_{n-k-1}\ell'$, where $\ell' \in \Pi_1$. Therefore, we get from (4.1):

$$\sigma_1 \, \tilde{\ell} = \sigma \, \ell'.$$

If the lines $\tilde{\ell}, \ell'$ coincide then the curves σ_1, σ coincide, which is impossible. Therefore the line $\tilde{\ell}$ has to divide $\sigma \in \Pi_k$:

$$\sigma = \tilde{\ell} \, r, \quad r \in \Pi_{k-1}.$$

Now, we are going to derive from this relation that the curve r passes through all the nodes of the set $\mathcal{X}$ but one. Indeed, σ passes through all the nodes of $\mathcal{X}$. Therefore these nodes are either in the curve r or in the line $\tilde{\ell}$. But the latter line passes through B, and according to the condition (ii), it passes through at most one node of $\mathcal{X}$. Thus r passes through at least $d(n, k-1)$

nodes of $\mathcal{X}$. Since r is a curve of degree $k-1$ we conclude that r is a maximal curve and passes through exactly $d(n, k-1)$ nodes of $\mathcal{X}$. □

It is worth mentioning that for any n-independent node set $\mathcal{X}$ of cardinality $d(n, k-1) + 1$, where $k \le n-1$, we have that

$$\dim \mathcal{P}_{k,\mathcal{X}} \le 2,$$

where an equality takes place if only if all the nodes of $\mathcal{X}$ but one lie in a maximal curve of degree $k-1$.

Indeed, if

$$\dim \mathcal{P}_{k,\mathcal{X}} \ge 2$$

then according to Theorem 4.2 we have that all the nodes of $\mathcal{X}$ but one lie in a maximal curve μ of degree $k-1$. Now, according to Proposition 3.2, we have that

$$\mathcal{P}_{k,\mathcal{X}} = \{\alpha\mu | \alpha \in \Pi_1,\ \alpha(A) = 0\},$$

where $A \in \mathcal{X}$ is the node outside of μ. Therefore we get readily

$$\dim \mathcal{P}_{k,\mathcal{X}} = \dim \{\alpha | \alpha \in \Pi_1,\ \alpha(A) = 0\} = 2.$$

5. A corollary

Here we verify that our main result yields Theorem 4.1, which in view of Theorem 4.2, states that for any n-independent set $\mathcal{X}$ of cardinality

$$d(n, k-1) + 2$$

there is at most one curve of degree $k,\ k \le n$, passing through all its nodes.

Proof of Theorem 4.1. Note that the case $k = n$ is evident, since

$$d(n, n-1) + 2 = N - 1.$$

Now assume that $k \le n-1$. Choose a node $A \in \mathcal{X}$ and consider the set $\mathcal{Y} := \mathcal{X} \setminus \{A\}$. If there is at most one curve of degree which passes through all the nodes of $\mathcal{Y}$ then the same is true also for the set $\mathcal{X}$ and we are done. Thus assume that there are at least two curves of degree k which pass through all the nodes of the set $\mathcal{Y}$. Then, according to Theorem 4.2, all the nodes of $\mathcal{Y}$ but one, denoted by B, lie in a maximal curve μ of degree $k-1$. Therefore, all the nodes of $\mathcal{X}$ but A and B lie in the curve μ. Now, in view of Proposition 3.2, any curve of degree k passing through all the nodes of $\mathcal{X}$ has the following form

$$p = \ell\mu,$$

where $\ell \in \Pi_1$. Finally notice that the line ℓ passes through the nodes A and B and therefore is determined in a unique way. Hence p is determined uniquelly, too. □

6. An application to the Gasca–Maeztu conjecture

Let us recall that a node $A \in \mathcal{X}$ uses a line ℓ means that ℓ is a factor of the fundamental polynomial $p = p^{\star}_A$, i.e., $p = \ell r$, for some $r \in \Pi_{n-1}$.

A GC_n-set in plane is an n-poised set of nodes where the fundamental polynomial of each node is a product of n linear factors. Note that this always takes place in the univariate case.

The Gasca–Maeztu conjecture states that any GC_n-set possesses a subset of $n+1$ collinear nodes.

It was proved in [5] that any line passing through exactly 2 nodes of a GC_n-set $\mathcal{X}$ can be used at most by one node from $\mathcal{X}$, provided that the Gasca–Maeztu conjecture is true for all degrees not exceeding n.

Recently, it was announced in [1], that this result holds for any poised set $\mathcal{X}$, without other restrictions. By the way it follows readily also from Theorem 4.2.

Below we consider the case of lines passing through exactly 3 nodes.

Corollary 6.1. *Let $\mathcal{X}$ be an n-poised set of nodes and ℓ be a used line which passes through exactly* 3 *nodes. Then ℓ is used either by exactly one or by exactly three nodes from $\mathcal{X}$. Moreover, if it is used by three nodes, then they are noncollinear.*

Proof. Assume that $\ell \cap \mathcal{X} = \{A_1, A_2, A_3\}$. Assume also that there are two nodes $B, C \in \mathcal{X}$ using the line ℓ :

$$p^{\star}_B = \ell\, q_1, \quad p^{\star}_C = \ell\, q_2,$$

where $q_1, q_2 \in \Pi_{n-1}$.

Both the polynomials q_1, q_2 vanish at $N-5$ nodes of the set

$$\mathcal{X}' := \mathcal{X} \setminus \{A_1, A_2, A_3, B, C\}.$$

Hence these $N - 5 = d(n, n-2) + 1$ nodes do not uniquelly determine the curve of degree $n-1$ passing through them. By Theorem 4.2 there exists a maximal curve μ of degree $n-2$ passing through $N-6$ nodes of $\mathcal{X}'$ and the remaining node denoted by D is outside of it. Now, according to Proposition 3.4, the node D uses μ :

$$p^{\star}_D = \mu q, \quad q \in \Pi_2.$$

This quadratic polynomial q has to vanish at the three nodes $A_1, A_2, A_3 \in \ell$. Therefore, in view of Proposition 3.1, we have that $q = \ell\ell'$ with $\ell' \in \Pi_1$. Hence the node D uses the line ℓ :

$$p^{\star}_D = \mu\ell\ell', \quad \ell' \in \Pi_1.$$

Thus if two nodes $B, C \in \mathcal{X}$ use the line ℓ then there exists a third node $D \in \mathcal{X}$ using it and all the nodes of $\mathcal{Y} := \mathcal{X} \setminus \{A_1, A_2, A_3, B, C, D\}$ lie in a maximal curve μ of degree $n-2$:

$$\mathcal{Y} \subset \mu. \tag{6.1}$$

Next, let us show that there is no fourth node using ℓ. Assume by way of contradiction that except of the nodes B, C, D, there is a fourth node E using ℓ. Of course we have that $E \in \mathcal{Y}$.

Then B and E are using ℓ therefore, as was proved above, there exists a third node $F \in \mathcal{X}$ (which may coincide or not with C or D) using it and all the nodes of $\tilde{\mathcal{Y}} := \mathcal{X} \setminus \{A_1, A_2, A_3, B, E, F\}$ are located in a maximal curve $\tilde{\mu}$ of degree $n-2$. We have also that

$$p^{\star}_E = \tilde{\mu}\tilde{q}, \quad \tilde{q} \in \Pi_2. \tag{6.2}$$

Now, notice that both μ and $\tilde{\mu}$ pass through all the nodes of the set $\mathcal{Z} := \mathcal{X} \setminus \{A_1, A_2, A_3, B, C, D, E, F\}$ with $\#\mathcal{Z} \geq N-8$.

Then, we get from Theorem 4.1, with $k = n-2$, that $N-8 = d(n, n-3)+2$ nodes determine the curve of degree $n-2$ passing through them uniquely. Thus μ and $\tilde{\mu}$ coincide.

Therefore, in view of (6.1) and (6.2), $p^{\star}_E$ vanishes at all the nodes of $\mathcal{Y}$, which is a contradiction since $E \in \mathcal{Y}$.

Now, let us verify the last "moreover" statement. Suppose three nodes $B, C, D \in \mathcal{X}$ use the line ℓ. Then, as we obtained earlier, the nodes

$$\mathcal{Y} := \mathcal{X} \setminus \{A_1, A_2, A_3, B, C, D\}$$

are located in a maximal curve μ of degree $n-2$. Suppose conversely that the nodes B, C and D are lying in a line ℓ_1. Then we have that all the nodes of the set $\mathcal{X}$ are lying in the curve $\mu\ell\ell_1$ of degree n. This, in view of Proposition 1.3, is a contradiction. □

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