

On the integrability of co-CR quaternionic structures

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ABSTRACT. We characterise the integrability of any co-CR quaternionic structure in terms of the curvature and a generalized torsion of the connection.

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Introduction

The *twistor spaces* emerged in differential geometry through the notions of *anti-self-dual structures* [1] and the *three-dimensional Einstein–Weyl spaces* [2]. Up to conjugations, the corresponding twistor spaces are characterised as complex manifolds endowed with locally-complete families [5] of embedded (Riemann) spheres with normal bundles $2\mathcal{O}(1)$ and $\mathcal{O}(2)$, respectively, where $\mathcal{O}(j)$ denotes the holomorphic line bundle of Chern number j ($\in \mathbb{Z}$) over the sphere. There are, also, the *quaternionic manifolds* [15] whose twistor spaces are complex manifolds endowed with locally-complete families of spheres with normal bundles $2k\mathcal{O}(1)$, ($k \in \mathbb{N}$). Another construction of twistor spaces is provided by [6] which works for any *three-dimensional conformal manifold* and which produces a CR manifold; moreover, the CR structure of the twistor space is obtained through an embedding as a real hypersurface into a complex manifold if and only if the given conformal structure is real-analytic.

In [9] and [10] (see, also, [12]) the unification of the above mentioned classical geometric structures was initiated, through the notions of *CR quaternionic* and *co-CR quaternionic manifolds*. For the reader's convenience we briefly describe these notions here, up to integrability. As this paper deals,

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mainly, with co-CR quaternionic manifolds, details about these will be given in Section 2, below. As for the CR quaternionic manifolds, we refer the reader to [9], for more information.

Firstly, recall that a *quaternionic vector bundle* is a vector bundle E whose structural group is the Lie group $\mathrm{Sp}(1) \cdot \mathrm{GL}(k, \mathbb{H})$ acting on the typical fibre \mathbb{H}^k by $(\pm(a, A), q) \mapsto aqA^{-1}$, for any $\pm(a, A) \in \mathrm{Sp}(1) \cdot \mathrm{GL}(k, \mathbb{H})$ and $q \in \mathbb{H}^k$. Then the morphism of Lie groups $\mathrm{Sp}(1) \cdot \mathrm{GL}(k, \mathbb{H}) \rightarrow \mathrm{SO}(3)$, $\pm(a, A) \mapsto \pm a$, induces an oriented Riemannian vector bundle of rank three whose sphere bundle Z is the bundle of *admissible linear complex structures* on E . Note that, any $J \in Z$ is a linear complex structure on $E_{\pi(J)}$, where $\pi : Z \rightarrow M$ is the projection.

Now, an *almost CR quaternionic structure* on a manifold M is a pair (E, ι) , where E is a quaternionic vector bundle over M , and $\iota : TM \rightarrow E$ is an injective morphism of vector bundles such that, for any admissible linear complex structure J on E , we have $\mathrm{im} \iota_{\pi(J)} + J(\mathrm{im} \iota_{\pi(J)}) = E_{\pi(J)}$, where, as above, $\pi : Z \rightarrow M$ is the projection of the bundle of admissible linear complex structures on E . If $\dim M = 3$ then an almost CR quaternionic structure on M is just a conformal structure (this comes from the fact that, for vector bundles of rank four, a linear quaternionic structure is just an oriented linear conformal structure). Also, an almost CR quaternionic structure (E, ι) with ι an isomorphism is just an almost quaternionic structure.

Dualizing, an *almost co-CR quaternionic structure* on M is a pair (E, ρ) , where E is a quaternionic vector bundle over M , and $\rho : E \rightarrow TM$ is a surjective morphism of vector bundles such that, for any admissible linear complex structure J on E , we have $\ker \rho_{\pi(J)} \cap J(\ker \rho_{\pi(J)}) = \{0\}$. Note that, an almost co-CR quaternionic structure (E, ρ) with ρ an isomorphism is just an almost quaternionic structure.

It may happen for a manifold M to be endowed with ‘compatible’ almost CR quaternionic and co-CR quaternionic structures. This happens precisely when M is endowed with a pair (E, V) , where E is a quaternionic vector bundle over M , such that $V, TM \subseteq E$ are vector subbundles with $E = V \oplus TM$ and $JV_{\pi(J)} \subseteq T_{\pi(J)}M$, for any admissible J on E . Note that, (E, ι) and (E, ρ) are almost CR quaternionic and almost co-CR quaternionic structures on M , where $\iota : TM \rightarrow E$ and $\rho : E \rightarrow TM$ are the inclusion and the projection, respectively. Moreover, any admissible J on E induces a linear f -structure on $T_{\pi(J)}M$, that is, a linear map F satisfying $F^3 + F = 0$, characterised by $\ker F = JV_{\pi(J)}$ and $\ker(F + i) = T_{\pi(J)}^{\mathbb{C}}M \cap (\ker(J + i))$. Therefore we call (V, E) an *almost f -quaternionic structure* (and, in this setting, Z can be, also, seen as the bundle of admissible linear f -structures on M).

The main source of CR quaternionic manifolds is provided by (certain) submanifolds of quaternionic manifolds (for example, any hypersurface of a

quaternionic manifold is canonically endowed with a CR quaternionic structure). Moreover, assuming real-analiticity, any CR quaternionic manifold is obtained this way [9] (in dimension three, this reduces to a result of [6], mentioned above).

As for the co-CR quaternionic manifolds, up to now, we have the following classes of examples:

- (a) The three-dimensional Einstein–Weyl spaces.
- (b) The quaternionic manifolds (in particular, the anti-self-dual manifolds).
- (c) The local orbit spaces of any nowhere zero quaternionic vector field on a quaternionic manifold.
- (d) A principal bundle built over any quaternionic manifold (e.g., S^{4n+3}); the corresponding twistor space is the product of the sphere with the twistor space of the quaternionic manifold ($\mathbb{C}P^1 \times \mathbb{C}P^{2n+1}$ for S^{4n+3}).
- (e) Vector bundles over any quaternionic manifold; the twistor spaces are holomorphic vector bundles over the twistor space of the quaternionic manifold.
- (f) The Grassmannian of oriented three-dimensional vector subspaces of the Euclidean space of dimension $n + 1$; the twistor space is the nondegenerate hyperquadric in the n -dimensional complex projective space, $n \geq 3$.
- (g) The complex manifold M_V formed of the isotropic two-dimensional vector subspaces of any complex symplectic vector space V ; the twistor space is M_V itself.
- (h) The space of holomorphic sections of $P(\mathcal{O} \oplus \mathcal{O}(n))$ induced by the holomorphic sections of $\mathcal{O}(m) \oplus \mathcal{O}(m + n)$ which intertwine the antipodal map and the conjugation; the twistor space is $P(\mathcal{O} \oplus \mathcal{O}(n))$, where $m, n \in \mathbb{N}$ are even, $n \neq 0$.
- (i) The space of holomorphic maps of a fixed odd degree from $\mathbb{C}P^1$ to $\mathbb{C}P^1$ which commute with the antipodal map; the twistor space is, now, $\mathbb{C}P^1 \times \mathbb{C}P^1$.

The details for (d), (f), (g) can be found in [10], for (c), (h), (i) in [12], whilst the details for (e) will be given in Section 2, below.

In this paper, we settle the problem of finding a useful characterisation for the integrability of the co-CR quaternionic structures (the corresponding problem for CR quaternionic structures was settled in [9]). For this, we use a seemingly new generalized torsion associated to any connection on a vector bundle E , over a manifold M , endowed with a morphism $\rho : E \rightarrow TM$ (if ρ is an isomorphism this reduces to the classical torsion of a connection on a manifold). This is studied in Section 1, where we show that it provides a necessary tool to handle the integrability of distributions defined on Grassmannian bundles.

In Section 2, we give the main integrability result (Theorem 2.2), and its first applications. For example, there we prove (Theorem 2.3) that the following holds for any holomorphic vector bundle \mathcal{Z} , over the twistor space Z of a quaternionic manifold M , endowed with a conjugation covering the conjugation of Z : *if the Birkhoff–Grothendieck decomposition of \mathcal{Z} restricted to each twistor line contains only terms of Chern number $m \geq 1$ then \mathcal{Z} is the twistor space of a co-CR quaternionic manifold, built on the total space of a vector bundle over M .* This gives example (e), above, and, in particular, for $m = 1$ it reduces to [15, Theorem 7.2].

As already mentioned, an important particular type of co-CR quaternionic manifolds is provided by the f -quaternionic manifolds. These are endowed with a CR quaternionic structure and a co-CR quaternionic structure, which are compatible. Furthermore, the twistor space of the latter has the property that the Birkhoff–Grothendieck decomposition of the normal bundle of each twistor sphere contains only terms whose Chern numbers belong to $\{1, 2\}$. As for examples, all of the (a), (b), (d), (f), and (g), above, give such manifolds. Also, the same applies to example (e) (Theorem 2.3) if $m = 2$.

In Section 3, we apply Theorem 2.2, to study the integrability of f -quaternionic structures. This leads to Corollary 3.5 which gives a new proof, and slightly corrects [10, Theorem 4.9]. Moreover, along the way, we obtain Theorem 3.7 by which, under generic dimensional conditions, any manifold endowed with an almost f -quaternionic structure and a compatible torsion free connection is, locally, a product of a hypercomplex manifold with $(\text{Im}\mathbb{H})^k$, for some $k \in \mathbb{N}$.

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1. A generalized torsion

We work in the smooth and the complex-analytic categories (in the latter case, by the tangent bundle we mean the holomorphic tangent bundle). For simplicity, sometimes, the bundle projections will be denoted in the same way, when the base manifold is the same.

Let E be a vector bundle, endowed with a connection ∇ , over a manifold M . Suppose that we are given a morphism of vector bundles $\rho : E \rightarrow TM$.

Then, firstly, note that there exists a unique section T of $TM \otimes \Lambda^2 E^*$ such that

$$(1.1) \quad T(s_1, s_2) = \rho \circ (\nabla_{\rho \circ s_1} s_2 - \nabla_{\rho \circ s_2} s_1) - [\rho \circ s_1, \rho \circ s_2]$$

for any (local) sections s_1 and s_2 of E ; we call T the *torsion (with respect to ρ)* of ∇ .

Let F and G be the typical fibre and structural group of E , respectively; assume ∇ compatible with G . Denote by (P, M, G) the frame bundle of E and let $\mathcal{H} \subseteq TP$ be the principal connection on P corresponding to ∇ .

On composing the projection $P \times F \rightarrow E$ with ρ , we obtain a morphism of vector bundles from $P \times F$ to TM which covers the projection $\pi : P \rightarrow M$. Consequently, this morphism factorises as a morphism of vector bundles, over P , from $P \times F$ to \mathcal{H} followed by the canonical morphism from

$$\mathcal{H} (= \pi^*(TM))$$

onto TM . Thus, if $\xi \in F$ the corresponding (constant) section of $P \times F$ determines a horizontal vector field $B(\xi)$ on P .

Note that, $B(\xi)$ is characterised by $d\pi(B(\xi)_u) = \rho(u\xi)$, for any $u \in P$, and the fact that it is horizontal (compare [4, p. 119]). However, unlike the classical case $B(\xi)$ may have zeros; indeed $B(\xi)$ is zero at $u \in P$ if and only if $\rho(u\xi) = 0$. Also, \mathcal{H} is generated as a vector bundle by all $B(\xi)$, $\xi \in F$, if and only if ρ is surjective.

Furthermore, $B : F \rightarrow \Gamma(TP)$ is G -equivariant. Indeed, if we denote by R_a (the differential of) the right translation by some $a \in G$ on P , we have

$$d\pi((R_a(B(\xi)))_u) = d\pi(B(\xi)_{ua^{-1}}) = \rho(u(a^{-1}\xi)).$$

Hence, $R_a(B(\xi)) = B(a^{-1}\xi)$, for any $a \in G$ and $\xi \in F$; in particular, $[A, B(\xi)] = B(A\xi)$ for any $A \in \mathfrak{g}$ and $\xi \in F$, where \mathfrak{g} is the Lie algebra of G and we denote in the same way its elements and the corresponding fundamental vector fields on P (compare [4, Proposition III.2.3]).

Remark 1.1. Let $\xi \in F$ and let $(u(t))_t$ be an integral curve of $B(\xi)$; denote $c = \pi \circ u$, $s = u\xi$. Then c is a curve in M , and s is a section of c^*E satisfying $\rho \circ s = \dot{c}$ and $(c^*\nabla)(s) = 0$. These curves s lead to a natural generalization of the notion of geodesic of a connection on a manifold. Note that, for any $e \in E$ there exists a unique germ of such a curve s with $s(0) = e$.

In this setting, Cartan's first structural equation is replaced by the following fact.

Proposition 1.2. *For any $u \in P$ and $\xi, \eta \in F$ we have*

$$(1.2) \quad T(u\xi, u\eta) = -d\pi([B(\xi), B(\eta)]_u).$$

Proof. Let $u_0 \in P$ and let u be a local section of P , defined on some open neighbourhood U of $x_0 = \pi(u_0)$, such that $u_{x_0} = u_0$ and the local connection form Γ of \mathcal{H} , with respect to u , is zero at x_0 .

If $\xi \in F$ then, under the isomorphism $P|_U = U \times G$ corresponding to u , we have $B(\xi)_{ua} = (\rho(ua\xi), -\Gamma(\rho(ua\xi))a)$, for any $a \in G$.

By using the fact that $\Gamma_{x_0} = 0$, we quickly obtain that, at u_0 , both sides of (1.2) are equal to $-[\rho(u\xi), \rho(u\eta)]_{x_0}$, for any $\xi, \eta \in F$. \square

Also, we obtain the following natural generalization of the first Bianchi identity.

Proposition 1.3. *Let E be a vector bundle, over M , and suppose that there exists a morphism of vector bundles $\rho : E \rightarrow TM$.*

Then the curvature form R of any torsion free connection on E satisfies

$$(1.3) \quad \rho \left(R(\rho(e_1), \rho(e_2))e_3 + R(\rho(e_2), \rho(e_3))e_1 + R(\rho(e_3), \rho(e_1))e_2 \right) = 0,$$

for any $e_1, e_2, e_3 \in E$.

Proof. Let Ω be the curvature form of the corresponding principal connection on the frame bundle P of E (we think of Ω as a two-form on P with values in the Lie algebra of the structural group of E ; see [4]). Equation (1.3) is equivalent to the following

$$(1.4) \quad B \left(\Omega_u(B(\xi), B(\eta))\mu + \Omega_u(B(\eta), B(\mu))\xi + \Omega_u(B(\mu), B(\xi))\eta \right)_u = 0,$$

for any $u \in P$ and ξ, η, μ in the typical fibre F of E .

By using the fact that the connection is torsion free, we obtain that, for any $u \in P$ and $\xi, \eta, \mu \in F$, the horizontal part of $[B(\mu), [B(\xi), B(\eta)]]_u$ is $B(\Omega_u(B(\xi), B(\eta))\mu)_u$. Therefore (1.4) is just the horizontal part, at u , of the Jacobi identity, for the usual bracket, applied to $B(\xi), B(\eta), B(\mu)$. \square

Let S be a submanifold of a Grassmannian of F on which G acts transitively. Then $Z = P \times_G S$ is a subbundle of a Grassmannian bundle of E on which ∇ induces a connection $\mathcal{H} \subseteq TZ$.

Suppose that for any $p \in Z$ the restriction of ρ to p is an isomorphism onto some vector subspace of $T_{\pi(p)}M$, where $\pi : Z \rightarrow M$ is the projection. Then we can construct a distribution \mathcal{C} on Z by requiring $\mathcal{C} \subseteq \mathcal{H}$ and $d\pi(\mathcal{C}_p) = \rho(p)$, for any $p \in Z$.

Proposition 1.4. *The following assertions are equivalent, where R and T are the curvature form and the torsion of ∇ , respectively:*

- (i) \mathcal{C} is integrable.
- (ii) $R(\Lambda^2(\rho(p)))(p) \subseteq p$ and $T(\Lambda^2 p) \subseteq \rho(p)$, for any $p \in Z$.

Consequently, if ∇ is torsion free and \mathcal{C} is integrable then

$$(1.5) \quad R(\rho(e_1), \rho(e_2))e_3 + R(\rho(e_2), \rho(e_3))e_1 + R(\rho(e_3), \rho(e_1))e_2 = 0,$$

for any $p \in Z$ and $e_1, e_2, e_3 \in p$.

Proof. Let (P, M, G) be the frame bundle of E , and let H be the isotropy subgroup of G at some $p_0 \in S$. Then $Z = P/H$, and let $\sigma : P \rightarrow Z$ be the projection.

If we denote $\mathcal{C}_P = (d\sigma)^{-1}(\mathcal{C})$ then, as σ is a surjective submersion, we also have $\mathcal{C} = (d\sigma)(\mathcal{C}_P)$. Therefore \mathcal{C} is integrable if and only if \mathcal{C}_P is integrable.

Now, note that \mathcal{C}_P is generated by all $B(\xi)$, with $\xi \in p_0$ and all (the fundamental vector fields) $A \in \mathfrak{h}$, where \mathfrak{h} is the Lie algebra of H . Hence, \mathcal{C}_P is integrable if and only if $[B(\xi), B(\eta)]$ is a section of \mathcal{C}_P , for any $\xi, \eta \in p_0$; equivalently, $\Omega_u(B(\xi), B(\eta)) \in \mathfrak{h}$ and $d\pi([B(\xi), B(\eta)]_u) \in \rho(up_0)$ for any $u \in P$ and $\xi, \eta \in p_0$. Together with Cartan's second structural equation and Proposition 1.2, this completes the proof. \square

Note that, the last statement of Proposition 1.4 could have been proved directly by observing that, under that hypothesis, the leaves of \mathcal{C} are, locally, projected by π onto submanifolds of M on which ∇ induces a torsion free connection.

In the following definition, the notations are as in Proposition 1.4.

Definition 1.5.

- (1) We say that ∇ *satisfies the first Bianchi identity* if it is torsion free and (1.5) holds for any $e_1, e_2, e_3 \in E$.
- (2) We say that ∇ *satisfies the first Bianchi identity, with respect to Z* , if it is torsion free and (1.5) holds for any $p \in Z$ and $e_1, e_2, e_3 \in p$.

We shall, also, need the following definition.

Definition 1.6 (see [8]). A *Lie algebroid* on M is a triple $(E, [\cdot, \cdot], \rho)$, where E is a vector bundle over M , $\rho : E \rightarrow TM$ is a morphism of vector bundles, and $[\cdot, \cdot]$ is a Lie bracket on the sheaf of sections of E , satisfying:

- (1) $[s_1, fs_2] = (\rho \circ s_1)(f) s_2 + f[s_1, s_2]$, for any (local) sections s_1 and s_2 of E , and any function f on M .
- (2) $\rho \circ [s_1, s_2] = [\rho \circ s_1, \rho \circ s_2]$, for any sections s_1 and s_2 of E .

Let E be a vector bundle, endowed with a connection ∇ , over a manifold M . Suppose that $\rho : E \rightarrow TM$ is a morphism of vector bundles and let $[s_1, s_2] = \nabla_{\rho \circ s_1} s_2 - \nabla_{\rho \circ s_2} s_1$, for any sections s_1 and s_2 of E .

Proposition 1.7. *The following assertions are equivalent:*

- (i) ∇ *satisfies the first Bianchi identity.*
- (ii) $(E, [\cdot, \cdot], \rho)$ *is a Lie algebroid.*

Proof. This is a straightforward computation. □

2. On the integrability of co-CR quaternionic structures

A *quaternionic vector bundle* is a vector bundle E whose structural group is $\text{Sp}(1) \cdot \text{GL}(k, \mathbb{H})$ acting on the typical fibre \mathbb{H}^k by $(\pm(a, A), q) \mapsto aqA^{-1}$, for any $\pm(a, A) \in \text{Sp}(1) \cdot \text{GL}(k, \mathbb{H})$ and $q \in \mathbb{H}^k$. Then the morphism of Lie groups $\text{Sp}(1) \cdot \text{GL}(k, \mathbb{H}) \rightarrow \text{SO}(3)$, $\pm(a, A) \mapsto \pm a$, induces an oriented Riemannian vector bundle of rank three whose sphere bundle Z is the bundle of *admissible linear complex structures* on E ; we shall denote by $\pi : Z \rightarrow M$ the projection of this bundle.

An *almost co-CR quaternionic structure* on M is a pair (E, ρ) where E is a quaternionic vector bundle over M and $\rho : E \rightarrow TM$ is a surjective morphism of vector bundles whose kernel contains no nonzero subspace preserved by some admissible linear complex structure of E .

By duality, we obtain the notion of *almost CR quaternionic structure*.

Let (M, E, ρ) be an almost co-CR quaternionic manifold and let ∇ be a compatible connection on E (that is, ∇ is compatible with the structural group of E). Then we can construct a complex distribution \mathcal{C} on the bundle

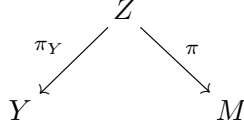
Z of admissible linear complex structures of E , as follows. Firstly, for any $J \in Z$, let \mathcal{B}_J be the horizontal lift of $\rho(\ker(J+i))$, with respect to ∇ . Then $\mathcal{C} = (\ker d\pi)^{0,1} \oplus \mathcal{B}$ is a complex distribution on Z such that $\mathcal{C} + \bar{\mathcal{C}} = T^{\mathbb{C}}Z$; that is, \mathcal{C} is an *almost co-CR structure* on Z .

Now, we can make the following [10]:

Definition 2.1. We say that (E, ρ) is *integrable, with respect to ∇* , if \mathcal{C} is integrable; then (E, ρ, ∇) is a (*integrable*) *co-CR quaternionic structure*.

A *co-CR quaternionic manifold* is a manifold endowed with a co-CR quaternionic structure.

Let (M, E, ρ, ∇) be a co-CR quaternionic manifold. Then $\mathcal{C} \cap \bar{\mathcal{C}}$ is the complexification of (the tangent bundle of) a foliation \mathcal{F} on Z ; moreover, with respect to it, \mathcal{C} is projectable onto complex structures on the local leaf spaces of \mathcal{F} . If there exists a surjective submersion $\pi_Z : Z \rightarrow Y$ such that $\ker d\pi_Y = \mathcal{F}$, whose restriction to each fibre of π is injective, and \mathcal{C} is projectable with respect to π_Y (the latter condition is unnecessary if the fibres of π_Y are connected) then the complex manifold $(Y, d\pi_Y(\mathcal{C}))$ is the *twistor space* of (M, E, ρ, ∇) .



Note that, if ρ is an isomorphism then we obtain the classical notion of *quaternionic manifold* [15] (see [3, Remark 2.10(2)]).

Also, each fibre of π is holomorphically embedded, through the restriction of π_Y , into Y . Then, for each $x \in M$, the embedded sphere $\pi_Y(\pi^{-1}(x))$ is the (*real*) *twistor sphere* corresponding to x . Further, the map $Z \rightarrow Y \times M$, $z \mapsto (\pi_Y(z), \pi(z))$, is an embedding.

In this context, a *twistorial map* ([13], [7]) between the co-CR quaternionic manifolds M and N , with twistor spaces Y_M and Y_N , respectively, is a map $\varphi : M \rightarrow N$ which corresponds to a holomorphic map $\Phi : Y_M \rightarrow Y_N$. By this we mean, that the map $\Phi \times \varphi$ gives by restriction a map between the corresponding bundles of admissible linear complex structures over M and N , respectively; in particular, $\varphi(x) = y$ if and only if Φ maps the twistor sphere corresponding to $x \in M$ diffeomorphically onto the twistor sphere corresponding to $y \in N$.

More information on (co-)CR quaternionic manifolds can be found in [9], [10], [11], [12].

Theorem 2.2. *Let (M, E, ρ) be an almost co-CR quaternionic manifold and let Z be the bundle of admissible linear complex structures on E . Let ∇ be a compatible connection on E .*

The following assertions are equivalent, where R and T are the curvature form and the torsion of ∇ , respectively:

- (i) (E, ρ, ∇) is integrable.
- (ii) $R(\Lambda^2(\rho(E^J)))(E^J) \subseteq E^J$ and $T(\Lambda^2(E^J)) \subseteq \rho(E^J)$, for any $J \in Z$, where $E^J = \ker(J + i)$.

Proof. Assuming real-analyticity, this follows quickly from Proposition 1.4, after a complexification. In the smooth category, this follows from an extension of Proposition 1.4, similar to [9, Theorem A.3]. \square

Let M be a quaternionic manifold, $\dim M = 4k$, endowed with a torsion free compatible connection. Denote by L the complexification of the line bundle over M characterised by the fact that its $4k$ tensorial power is $\Lambda^{4k}TM$ (we use the orientation on M compatible with all of the admissible linear complex structures on it).

Then, at least locally, we have $T^{\mathbb{C}}M = H \otimes W$, where H and W are complex vector bundles of rank 2 and $2k$, respectively, and the structural group of H is $\mathrm{SL}(2, \mathbb{C})$ (H and W exist globally if and only if the vector bundle generated by the admissible linear complex structures on M is spin).

Denote $H' = (L^*)^{k/k+1} \otimes H$. Then $H' \setminus 0$ is endowed with a natural hyper-complex structure ([15]; see [14]), such that the projection onto M is twistorial. In particular, on endowing $H' \setminus 0$ with one of the admissible complex structures (corresponding to some imaginary quaternion of length 1) then $H' \setminus 0$ is the total space of a holomorphic principal bundle over the twistor space Z of M , with group $\mathbb{C} \setminus \{0\}$. We shall denote by \mathcal{L} the dual of the corresponding holomorphic line bundle over Z ; note that, if m is even then \mathcal{L}^m is globally defined. For example, if $M = \mathbb{H}P^k$ then \mathcal{L} is just the hyperplane line bundle over $\mathbb{C}P^{2k+1}$.

Now, let $U_m = \odot^m(H')^*$, where \odot denotes the symmetric product, $m \in \mathbb{N}$.

If m is even then U_m is globally defined and is the complexification of a (real) vector bundle which will be denote in the same way (note that, $L^{k/k+1} \otimes U_2$ is just the oriented Riemannian vector bundle of rank three generated by the admissible linear complex structures on M). Let F be a vector bundle over M endowed with a connection whose $(0, 2)$ components of its curvature, with respect to any admissible linear complex structure on M , are zero. We endow $\mathcal{F} = (\pi^*F)^{\mathbb{C}}$ with the (Koszul–Malgrange) holomorphic structure determined by the pull back of the connection on F and the complex structure of Z .

If m is odd then U_m is a hypercomplex vector bundle. Therefore if F is a hypercomplex vector bundle over M then $U_m \otimes F$ is the complexification of a vector bundle which will be denoted in the same way; in the tensor product U_m and F are endowed with I_1 and J_1 , respectively, whilst the conjugation on $U_m \otimes F$ is $I_2 \otimes J_2$, where I_i and J_i , $i = 1, 2, 3$, give the linear hypercomplex structures of U_m and F , respectively. Suppose that F is endowed with a compatible connection whose $(0, 2)$ components of its curvature, with respect to any admissible linear complex structure on M , are zero. On endowing F with J_1 , let $\mathcal{F} = \pi^*F$ endowed with the holomorphic structure determined

by the pull back of the connection on F and the complex structure of Z .

Before giving our next result, recall the classical Birkhoff–Grothendieck decomposition, according to which any holomorphic vector bundle over the (Riemann) sphere is of the form $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$, where a_1, \dots, a_r are integers with $a_1 \leq \cdots \leq a_r$, and $\mathcal{O}(j)$ denotes the holomorphic line bundle of Chern number j over the sphere.

In the next result $m \in \mathbb{N} \setminus \{0\}$, and, note that, for $m = 1$ it gives [15, Theorem 7.2],

Theorem 2.3.

- (a) *There exists a natural co-CR quaternionic structure on the total space of $U_m \otimes F$ whose twistor space is $\mathcal{L}^m \otimes \mathcal{F}$.*
- (b) *Conversely, let \mathcal{Z} be a holomorphic vector bundle over Z such that:*
 - (i) *The Birkhoff–Grothendieck decomposition of \mathcal{Z} restricted to each twistor line contains only terms of Chern number m .*
 - (ii) *\mathcal{Z} is endowed with a conjugation covering the conjugation of Z . Then \mathcal{Z} is the twistor space of a co-CR quaternionic manifold, obtained as in (a).*

Proof. For simplicity, we work in the complex-analytic category. Thus, in particular, at least locally, a (complex-)quaternionic vector bundle is a bundle which is the tensor product of a vector bundle of rank 2 and another vector bundle; for example, on denoting $V = L^{k/k+1} \otimes W$, we have

$$TM = U_1^* \otimes V.$$

Also, let $E = (U_{m-2} \oplus U_m) \otimes F$, with U_{-1} the (trivial) zero bundle over M . As $U_{m-2} \oplus U_m = U_1 \otimes U_{m-1}$, we have $E = U_1 \otimes U_{m-1} \otimes F$, and, in particular, E is a quaternionic vector bundle. Furthermore, by using the fact that the structural group of $L^{k/k+1} \otimes U_1^*$ is $\mathrm{SL}(2, \mathbb{C})$, we obtain that $U_1 = U_1^* \otimes L^{2k/k+1}$. Hence, also, $E \oplus TM$ is a quaternionic vector bundle.

By using the induced connection on $U_m \otimes F$ we obtain

$$(2.1) \quad \begin{aligned} T(U_m \otimes F) &= \pi^*(U_m \otimes F) \oplus \pi^*(TM), \\ \pi^*(E \oplus TM) &= T(U_m \otimes F) \oplus \pi^*(U_{m-2} \otimes F), \end{aligned}$$

where $\pi : U_m \otimes F \rightarrow M$ is the projection.

Thus, $\pi^*(E \oplus TM)$ and the projection ρ from it onto $T(U_m \otimes F)$ provide an almost co-CR quaternionic structure on $U_m \otimes F$. Furthermore, the connections on M and F induce a compatible connection ∇ on $\pi^*(E \oplus TM)$, which preserves the decomposition given by the second relation of (2.1). Thus, ∇ is flat when restricted to the fibres of $U_m \otimes F$, whilst if $X \in \pi^*(TM)$ then ∇_X is given by the pull back of the connection on $E \oplus TM$; in particular, if X and Y are pull backs of local vector fields on M then $\nabla_X Y$ is the pull back of the covariant derivative of $d\pi(Y)$ along $d\pi(X)$.

We have $U_m = \odot^m U_1$, where \odot denotes the symmetric product. Also, each $e \in U_1$ may be extended to a local section of U_1 whose covariant

derivative is zero along $\{e \otimes f \mid f \in V\}$ (this is the reason for which the ‘tensorisation’ with $(L^*)^{k/k+1}$ is needed).

In this setting, the bundle of admissible linear complex structures on M is replaced by $P(U_1^*)$ so that if J ‘corresponds’ to $[e]$, for some $e \in U_1^*$, then $\ker(J + i)$ corresponds to the space $\{e \otimes f \mid f \in V\}$. Then, on denoting $\mathcal{E} = \pi^*(E \oplus TM)$, for any nonzero $e \in U_1$, we have that $\rho(\mathcal{E}^e)$ is isomorphic to the direct sum of $\{e \otimes f \mid f \in V\}$ and the tensor product of the corresponding fibre of F with the space of polynomials from $U_m = \odot^m U_1$ which are divisible by e .

To verify that condition (ii) of Theorem 2.2 is satisfied we shall, also, use the fact that ∇ restricted to each fibre of $U_m \otimes F$ is flat. This and the fact that M is quaternionic (and endowed with a torsion free compatible connection) quickly implies that the curvature form of ∇ satisfies (ii) of Theorem 2.2. For the torsion T , it is sufficient to check the condition on pairs of local sections A, X and X, Y from \mathcal{E}^e with A induced by a section of E and X, Y induced by sections of TM , where $e \in U_1$. Then we have $T(A, X) = -\rho(\nabla_X A) - [\rho(A), X]$ and $T(X, Y)$ is the ‘vertical’ component of $-[X, Y]$; in particular, $T(X, Y)$ is determined by the curvature form of $U_m \otimes F$, applied to (X, Y) .

Locally, we may assume L trivial so that $U_1 = U_1^*$ but, note that, this isomorphism does not preserve the connections (the connection on U_1 is just the dual of the connection on U_1^*). Then we may choose (e_1, e_2) a local frame for U_1^* such that it corresponds to $(e^2, -e^1)$, where (e^1, e^2) is the dual of (e_1, e_2) , and such that the covariant derivative of e_1 is zero along $\{e_1 \otimes f \mid f \in V\}$. Thus, we have to check that $T(A, X)$ and $T(X, Y)$ are contained by $\rho(\mathcal{E}^{e_1})$, where A is the pull back of the tensor product of a local section of F and a polynomial of degree m which is divisible by e^2 , whilst $X = \pi^*(e_1 \otimes u)$ and $Y = \pi^*(e_1 \otimes v)$, with u and v local sections of V . Now, the condition on the torsion follows quickly by using the fact that the covariant derivative of e_1 is zero along $\{e_1 \otimes f \mid f \in V\}$ and the fact that the curvature form of F is zero when restricted to spaces of the form $\{e \otimes f \mid f \in V\}$, with $e \in U_1^*$.

In the complex-analytic category, the twistor space of M is (locally) the leaf space of the foliation \mathcal{V} on $P(U_1)$ which, at each $[e] \in P(U_1)$, is the horizontal lift of the space $\{e \otimes f \mid f \in V\}$. Similarly, the twistor space of $U_m \otimes F$ is the leaf space of the foliation \mathcal{V}_m on $\pi^*(P(U_1))$ which at each $\pi^*[e]$ is the horizontal lift of $\rho(\mathcal{E}^e)$.

On the other hand, the pull back of $\mathcal{L}^* \setminus 0$ to $P(U_1)$ is the principal bundle whose projection is $U_1 \setminus 0 \rightarrow P(U_1)$; equivalently, the pull back of \mathcal{L}^* to $P(U_1)$ is the tautological line bundle over $P(U_1)$. This is, further, equivalent to the fact that the pull back of \mathcal{L} to $P(U_1)$ is (locally; globally, if $H^1(M, \mathbb{C} \setminus \{0\})$ is zero) isomorphic to the quotient of $\pi_1^*(U_1)$ through the tautological line bundle over $P(U_1)$, where $\pi_m : U_m \rightarrow M$ is the projection. Therefore \mathcal{L} is the leaf space of the foliation on $\pi_1^*(P(U_1))$ which at each

$\pi_1^*[e]$ is the horizontal lift of $[e] \oplus \{e \otimes f \mid f \in V\}$. Similarly, we obtain that the twistor space of U_m is \mathcal{L}^m . Together with $\pi_m^*(P(U_1)) = U_m + P(U_1)$, this gives a surjective submersion $\varphi_m : U_m + P(U_1) \rightarrow \mathcal{L}^m$ which is linear along the fibres of the projection from $U_m + P(U_1)$ onto $P(U_1)$; that is, φ_m is a surjective morphism of vector bundles, covering the surjective submersion $P(U_1) \rightarrow Z$.

Also, the condition on the connection of F is equivalent to the fact that its pull back to $P(U_1)$ is flat when restricted to the leaves of \mathcal{V} . Hence, the pull back of F to $P(U_1)$ is, also, the pull back of a vector bundle \mathcal{F} on Z . Thus, we, also, have a surjective morphism of vector bundles $\varphi : F + P(U_1) \rightarrow \mathcal{F}$, covering $P(U_1) \rightarrow Z$.

Therefore there exists a morphism of vector bundles ψ from

$$(U_m \otimes F) + P(U_1)$$

onto $\mathcal{L}^m \otimes \mathcal{F}$, covering $P(U_1) \rightarrow Z$. Moreover, $\ker d\psi = \mathcal{V}_m$ and, hence, the twistor space of $U_m \otimes F$ is $\mathcal{L}^m \otimes \mathcal{F}$.

Conversely, if \mathcal{Z} is a vector bundle over Z satisfying (i) then $(\mathcal{L}^*)^m \otimes \mathcal{Z}$ restricted to each twistor line is trivial. Thus, it corresponds (through the Ward transform) to a vector bundle F over M endowed with a connection whose curvature form is zero when restricted to spaces of the form

$$\{e \otimes f \mid f \in V\}$$

with $e \in U_1$. Similarly to above, we obtain that \mathcal{Z} is the twistor space of $U_m \otimes F$, and the proof is complete. \square

With the same notations as in Theorem 2.3, the projection from $U_m \otimes F$ onto M is the twistorial map corresponding to the projection from $\mathcal{L}^m \otimes \mathcal{F}$ onto Z . Also, further examples of co-CR quaternionic manifolds can be obtained by taking direct sums of bundles $U_m \otimes F$ (with different values for m).

Here is another application of Theorem 2.2.

Corollary 2.4. *Let (M, E, ρ) be an almost co-CR quaternionic manifold, $\text{rank } E > 4$, and let Z be the bundle of admissible linear complex structures of E .*

If there exists a compatible connection ∇ on E which satisfies the first Bianchi identity, with respect to Z , then (E, ρ, ∇) is integrable.

Proof. Locally, we may suppose $E^{\mathbb{C}} = H \otimes F$, where H and F are complex vector bundles with $\text{rank } H = 2$. Moreover, the following hold:

- (1) $Z = PH$ such that if $J \in Z$ corresponds to $[e] \in PH$ then

$$E^J = \{e \otimes f \mid f \in F_{\pi(e)}\},$$

where π is the projection.

- (2) $\nabla = \nabla^H \otimes \nabla^F$ for some connections ∇^H on H and ∇^F on F .

By Theorem 2.2, we have to show that, for any $e \in H$ and $f_1, f_2, f_3 \in F$, we have

$$R(\rho(e \otimes f_1), \rho(e \otimes f_2))(e \otimes f_3) \in \{e \otimes f \mid f \in F_{\pi(e)}\};$$

equivalently, $R^E(\rho(e \otimes f_1), \rho(e \otimes f_2))e$ is proportional to e , where R^E is the curvature form of ∇^E .

We know that, for any $e \in H$ and $f_1, f_2, f_3 \in F$, we have

$$R(\rho(e \otimes f_1), \rho(e \otimes f_2))(e \otimes f_3) + \text{circular permutations} = 0,$$

which implies

$$(2.2) \quad (R^E(\rho(e \otimes f_1), \rho(e \otimes f_2))e) \otimes f_3 + \text{circular permutations} \\ \in \{e \otimes f \mid f \in F_{\pi(e)}\}.$$

As $\text{rank } E > 4$, we have $\text{rank } F > 2$. Therefore (2.2) holds, for any $e \in H$ and $f_1, f_2, f_3 \in F$, if and only if each term of the left hand side of (2.2) is contained by $\{e \otimes f \mid f \in F_{\pi(e)}\}$, for any $e \in H$ and $f_1, f_2, f_3 \in F$. The proof is complete. \square

Note that, in the proof of Corollary 2.4 it is not used the fact that $\text{rank } H = 2$.

Proposition 2.5. *Let (M, E, ρ) be an almost co-CR quaternionic manifold such that $\text{rank } E > 4$ and there exists a compatible connection ∇ on E which satisfies the first Bianchi identity.*

Then, locally, $\ker \rho$ can be endowed with a quaternionic structure such that the projection from $\ker \rho$ onto M is a twistorial map.

Proof. From Proposition 1.7 and [8, Theorem 2.2] it follows that, locally, there exists a section $\iota : TM \rightarrow E$ of ρ such that for any sections s_1 and s_2 of the vector subbundle $\text{im } \iota \subseteq E$ we have that $\nabla_{\rho \circ s_1} s_2 - \nabla_{\rho \circ s_2} s_1$ is a section of $\text{im } \iota$. In particular, we have $E = \ker \rho \oplus TM$, where we have used the obvious isomorphism $TM = \text{im } \iota$.

Furthermore, Proposition 1.7 quickly implies that ∇ restricts to a flat connection ∇^v on $\ker \rho$. Locally, we may suppose that ∇^v is the trivial connection corresponding to some trivialization of $\ker \rho$; that is, $\ker \rho$ is generated by (global) sections which are covariantly constant, with respect to ∇ .

Let $\pi : \ker \rho \rightarrow M$ be the projection. Note that, we have two decompositions $\pi^* E = \pi^*(\ker \rho) \oplus \pi^*(TM)$ and $T(\ker \rho) = \pi^*(\ker \rho) \oplus \pi^*(TM)$, where the latter is induced by ∇^v . Therefore we have a vector bundle isomorphism $T(\ker \rho) = \pi^* E$ which depends only of ι (and the given co-CR quaternionic structure). Hence, $\ker \rho$ is endowed with an almost quaternionic structure.

To complete the proof it is sufficient to prove that $\pi^* \nabla$ is torsion free. Indeed, let U, V be sections of $\pi^*(\ker \rho)$ induced by sections of $\ker \rho$ which are covariantly constant, with respect to ∇ , and let X, Y be sections of $\pi^*(TM)$ induced by vector fields on M ; in particular, X, Y are projectable,

with respect to $d\pi$.

Then we have that all of $[U, V]$, $[U, X]$, $(\pi^*\nabla)_U V$, $(\pi^*\nabla)_V U$, $(\pi^*\nabla)_U X$, $(\pi^*\nabla)_X U$ are zero. Also, as ∇ is torsion free, we have

$$(\pi^*\nabla)_X Y - (\pi^*\nabla)_Y X - [X, Y] = 0,$$

thus, completing the proof. \square

Let N be a quaternionic-Kähler manifold and let ∇ be its Levi-Civita connection. If $M \subseteq N$ is a hypersurface or a CR quaternionic submanifold [11] then the following assertions are equivalent:

- (i) ∇ restricted to $TN|_M$ satisfies the first Bianchi identity.
- (ii) M is geodesic and the normal connection is flat.

3. On the integrability of f -quaternionic structures

An *almost f -quaternionic structure* [10] on a manifold M is a pair (E, V) , where E is a quaternionic vector bundle over M , with $V, TM \subseteq E$ vector subbundles such that $E = V \oplus TM$ and $JV_{\pi(J)} \subseteq T_{\pi(J)}M$ for any admissible linear complex structure J of E . Then (E, ι) and (E, ρ) are almost CR quaternionic and almost co-CR quaternionic structures on M , where $\iota : TM \rightarrow E$ and $\rho : E \rightarrow TM$ are the inclusion and the projection, respectively. If E is endowed with a compatible connection ∇ , we can make the following [10] (see, also, [9] for the integrability of CR quaternionic structures):

Definition 3.1. The almost f -quaternionic structure (E, V) is *integrable with respect to ∇* if both (E, ι, ∇) and (E, ρ, ∇) are integrable; then (E, V, ∇) is an (*integrable*) *f -quaternionic structure*.

A manifold endowed with an f -quaternionic structure is called an *f -quaternionic manifold*.

Any almost f -quaternionic structure on M corresponds to a reduction of its frame bundle to the group $G_{l,m}$ of *f -quaternionic linear isomorphisms* of $(\text{Im}\mathbb{H})^l \times \mathbb{H}^m$, in particular $\dim M = 3l + 4m$. More precisely, $G_{l,m} = \text{GL}(l, \mathbb{R}) \times (\text{Sp}(1) \cdot \text{GL}(m, \mathbb{H}))$, where $\text{Sp}(1) \cdot \text{GL}(m, \mathbb{H})$ acts canonically on \mathbb{H}^m , whilst the action of $G_{l,m}$ on $(\text{Im}\mathbb{H})^l = \mathbb{R}^l \otimes \text{Im}\mathbb{H}$ is given by the tensor product of the canonical representations of $\text{GL}(l, \mathbb{R})$ and $\text{SO}(3)$ on \mathbb{R}^l and $\text{Im}\mathbb{H}$, respectively, and the canonical morphisms of Lie groups from $G_{l,m}$ onto $\text{GL}(l, \mathbb{R})$ and $\text{SO}(3)$. Furthermore, $G_{l,m}$ is isomorphic to the group of quaternionic linear isomorphisms of \mathbb{H}^{l+m} which preserve both \mathbb{R}^l and $(\text{Im}\mathbb{H})^l \times \mathbb{H}^m$.

Consequently, any almost f -quaternionic structure on M , also, corresponds to a decomposition $TM = (V \otimes Q) \oplus W$, where V is a vector bundle, Q is an oriented Riemannian vector bundle of rank three, and W is a quaternionic vector bundle such that the frame bundle of Q is the principal bundle induced by the frame bundle of W through the canonical morphism of Lie groups $\text{Sp}(1) \cdot \text{GL}(m, \mathbb{H}) \rightarrow \text{SO}(3)$, where $\text{rank } W = 4m$.

Thus, any connection ∇ on E compatible with $G_{l,m}$ induces a connection D on M such that $\nabla = D^V \oplus D$, where D^V is the connection induced by D on V ; then we say that D is *compatible with* (E, V) . Moreover, if we denote by T^E and T the torsions of ∇ and D , respectively, then $T^E = \rho^*T$; in particular, ∇ is torsion free if and only if D is torsion free.

Furthermore, $D = (D^V \otimes D^Q) \oplus D^W$, where D^W is a compatible connection on the quaternionic vector bundle W , and D^Q is the connection induced by D^W on Q . In particular, if D is torsion free then $V \otimes Q$ and W are foliations on M , and the leaves of the latter are quaternionic manifolds.

Corollary 3.2. *Let (E, V) be an almost f -structure on M and let ∇ be the connection on E induced by some connection D on M , compatible with (E, V) . Let $\rho : E \rightarrow TM$ be the projection, T the torsion of D , and R^Q the curvature form of the connection induced on Q .*

Then (E, ρ, ∇) is integrable if and only if, for any $J \in Z$, we have

$$(3.1) \quad \begin{aligned} T(\Lambda^2(\rho(E^J))) &\subseteq \rho(E^J), \\ R^Q(\Lambda^2(\rho(E^J)))(J) &\subseteq (J' + iJ'')^\perp, \end{aligned}$$

where $E^J = \ker(J + i)$, and $J', J'' \in Z$ such that (J, J', J'') is a positive orthonormal frame.

Proof. Let D^V be the connection induced on V and T^E the torsion of ∇ . Because $T^E = \rho^*T$, from Theorem 2.2 we obtain that it is sufficient to prove that, for any $J \in Z$, the second relation of (3.1) holds if and only if

$$(3.2) \quad R^E(\Lambda^2(\rho(E^J)))(E^J) \subseteq E^J,$$

where R^E is the curvature form of ∇ .

We have $R^Q(X, Y)J = [R^E(X, Y), J]$, for any $J \in Z$ and $X, Y \in TM$. Therefore if $J \in Z$ and $A, B, C \in E^J$ then

$$(R^Q(\rho(A), \rho(B))J)C = -(J + i)(R^E(\rho(A), \rho(B))C).$$

As, up to a nonzero factor, $J + i$ is the projection from E^C onto $\overline{E^J}$, we have that (3.2) holds if and only if $(R^Q(\Lambda^2(\rho(E^J)))(J))(E^J) = 0$. But, for any $X, Y \in TM$, we have $R^Q(X, Y)J = \alpha(X, Y)(J' + iJ'') + \beta(X, Y)(J' - iJ'')$, for some two-forms α and β .

To complete the proof just note that the obvious relation

$$J' + iJ'' = J' \circ (1 - iJ)$$

and its conjugate imply that $(J' + iJ'')(E^J) = 0$ whilst $J' - iJ''$ maps E^J isomorphically onto $\overline{E^J}$. \square

Proposition 3.3. *Let (E, V) be an almost f -structure on M and let ∇ be the connection on E induced by some torsion free connection on M , compatible with (E, V) .*

If $\text{rank } E > 4 \text{rank } V \geq 8$ then the connection induced on Q is flat.

Proof. Let $TM = (V \otimes Q) \oplus W$ be the decomposition corresponding to (E, V) . Note that, $\text{rank } E > 4\text{rank } V$ if and only if $\text{rank } W > 0$. Also, for any $J \in Z (\subseteq Q)$ and $U \in V$, we have $JU = U \otimes J$.

Let D, D^V, D^Q, D^W be the connections induced on TM, V, Q, W , and let R^M, R^V, R^Q, R^W be their curvature forms, respectively; recall that, $D = (D^V \otimes D^Q) \oplus D^W$.

Now, firstly, let $J \in Z, U \in V$ and $X, Y \in W$. From the first Bianchi identity applied to D we obtain

$$R^M(X, Y)(U \otimes J) + R^M(Y, U \otimes J)X + R^M(U \otimes J, X)Y = 0;$$

hence, we, also, have

$$(3.3) \quad (R^V(X, Y)U) \otimes J + U \otimes (R^Q(X, Y)J) = 0.$$

As $R^Q(X, Y)J \in J^\perp$, from (3.3) we obtain $R^Q(X, Y)J = 0$.

Secondly, let $J, J' \in Z, S, U \in V$, and $X \in W$. Then we have

$$(3.4) \quad R^M(X, S \otimes J)(U \otimes J') + R^M(S \otimes J, U \otimes J')X \\ + R^M(U \otimes J', X)(S \otimes J) = 0.$$

Relation (3.4) implies $R^W(S \otimes J, U \otimes J')X = 0$, and, as this holds for any $X \in W$, we obtain $R^Q(S \otimes J, U \otimes J') = 0$.

Furthermore, with $J = J'$, relation (3.4), also, gives

$$(3.5) \quad (R^V(X, S \otimes J)U) \otimes J + U \otimes (R^Q(X, S \otimes J)J) \\ + (R^V(U \otimes J, X)S) \otimes J + S \otimes (R^Q(U \otimes J, X)J) = 0.$$

Thus, if in (3.5) we assume S, U linearly independent, we obtain

$$R^Q(X, S \otimes J)J = 0$$

for any $X \in W$; equivalently,

$$(3.6) \quad \langle R^Q(X, S \otimes J)J', J \rangle = 0,$$

for any $X \in W$ and $J' \in Z$, orthogonal on J , where $\langle \cdot, \cdot \rangle$ denotes the Riemannian structure on Q .

Finally, if (J, J', J'') is an orthonormal frame on Q , then (3.4) gives

$$(3.7) \quad (R^V(X, S \otimes J)U) \otimes J' + U \otimes (R^Q(X, S \otimes J)J') \\ + (R^V(U \otimes J', X)S) \otimes J + S \otimes (R^Q(U \otimes J', X)J) = 0,$$

for any $S, U \in V$ and $X \in W$. Hence, if S, U are linearly independent, we deduce $\langle R^Q(X, S \otimes J)J', J'' \rangle = 0$, for any $X \in W$. Together with (3.6), this shows that $R^Q(X, S \otimes J)J' = 0$, and the proof is complete. \square

Proposition 3.4. *Let (E, V) be an almost f -structure on M and let ∇ be the connection on E induced by some torsion free connection on M , compatible with (E, V) ; denote by $\rho: E \rightarrow TM$ the projection.*

If $\text{rank } E = 4\text{rank } V \geq 12$ then (E, ρ, ∇) is integrable.

Proof. We shall use the same notations as in the proof of Proposition 3.3. Note that, $\text{rank } E = 4 \text{rank } V \geq 12$ if and only if $W = 0$ and $\text{rank } V \geq 3$.

Firstly, we shall prove that, for any $S, U \in V$ and any orthonormal frame (J, J', J'') on Q , the following relations hold:

$$(3.8) \quad \begin{aligned} R^Q(S \otimes J, U \otimes J)J &= 0, \\ R^Q(S \otimes J, U \otimes J)J' &= 0, \\ R^Q(S \otimes J, U \otimes J')J'' &= 0. \end{aligned}$$

From the first Bianchi identity applied to D we obtain that, for any $J, J' \in Z$ and $S, T, U \in V$, we have:

$$(3.9) \quad \begin{aligned} (R^V(S \otimes J, T \otimes J)U) \otimes J' + U \otimes (R^Q(S \otimes J, T \otimes J)J') \\ + (R^V(T \otimes J, U \otimes J')S) \otimes J + S \otimes (R^Q(T \otimes J, U \otimes J')J) \\ + (R^V(U \otimes J', S \otimes J)T) \otimes J + T \otimes (R^Q(U \otimes J', S \otimes J)J) = 0. \end{aligned}$$

If in (3.9) we take $J = J'$ and S, T, U linearly independent, we obtain that the first relation of (3.8) holds, for any $J \in Z$ and $S, U \in V$ (note that, if S, U are linearly dependent then the first two relations of (3.8) are trivial).

If in (3.9) we take $J \perp J'$ and S, T, U linearly independent, or $S = U$ and S, T linearly independent, we obtain

$$(3.10) \quad \begin{aligned} \langle R^Q(S \otimes J, U \otimes J)J', J'' \rangle &= 0, \\ \langle R^Q(S \otimes J, U \otimes J')J, J'' \rangle &= 0, \end{aligned}$$

for any orthonormal frame (J, J', J'') on Q , and any $S, U \in V$.

On swapping J and J' , in the second relation of (3.10), we deduce

$$(3.11) \quad \langle R^Q(S \otimes J, U \otimes J')J', J'' \rangle = 0,$$

for any orthonormal frame (J, J', J'') on Q , and any $S, U \in V$.

Now, the second relation of (3.10) and (3.11) imply that the third relation of (3.8) holds, as claimed.

Further, the first relation of (3.10) implies that the second relation of (3.8) holds if and only if $\langle R^Q(S \otimes J, U \otimes J)J', J \rangle = 0$; but this is a consequence of the first relation of (3.8).

To complete the proof, we use Corollary 3.2. Thus, we have to prove that for any positive orthonormal frame (J, J', J'') on Q , and any $S, U \in V$, the following holds:

$$(3.12) \quad \begin{aligned} \langle R^Q(S \otimes J, U \otimes J)J, J' + iJ'' \rangle &= 0, \\ \langle R^Q(S \otimes J, U \otimes (J' + iJ''))J, J' + iJ'' \rangle &= 0, \\ \langle R^Q(S \otimes (J' + iJ''), U \otimes (J' + iJ''))J, J' + iJ'' \rangle &= 0. \end{aligned}$$

Obviously, the first relation of (3.12) is an immediate consequence of the first relation of (3.8).

Note that, the second relation of (3.10) implies that, for any $A \in J^\perp \setminus \{0\}$, we have

$$R^Q(S \otimes J, U \otimes A)J = \frac{\langle R^Q(S \otimes J, U \otimes A)J, A \rangle}{\langle A, A \rangle} A;$$

in particular, $\langle R^Q(S \otimes J, U \otimes A)J, A \rangle = c\langle A, A \rangle$ for any $A \in J^\perp$, where c does not depend of A . As $J' + iJ''$ is isotropic this shows that the second relation of (3.12) holds.

Finally, the last two relations of (3.8) (applied to suitable orthonormal frames) imply $R^Q(S \otimes (J' + iJ''), U \otimes (J' + iJ''))J = 0$. Hence, also, the third relation of (3.12) holds. \square

We can, now, give a new proof for [10, Theorem 4.9], where, note that, the condition $\text{rank } V \neq 1$ was discarded, due to a misprint.

Corollary 3.5. *Let (E, V) be an almost f -structure on M and let ∇ be the connection on E induced by some torsion free connection D on M , compatible with (E, V) .*

If either $\text{rank } E > 4\text{rank } V \geq 8$ or $\text{rank } E = 4\text{rank } V \geq 12$ then (E, V, ∇) is integrable.

Proof. The integrability of the underlying almost co-CR quaternionic structure is a consequence of Corollary 3.2, and Propositions 3.3 and 3.4.

The integrability of the underlying almost CR quaternionic structure is a consequence of Proposition 3.3 and [9, Proposition 4.5]. \square

Corollary 3.6. *Let (E, V) be an almost f -structure on M and let ∇ be the connection on E induced by some torsion free connection on M , compatible with (E, V) .*

If the connection induced on Q is flat then, also, the connection induced on V is flat.

The converse also holds if $\text{rank } V \geq 3$.

Proof. As in the proof of Proposition 3.3 we deduce that (3.3) and (3.7) hold. Hence, if $R^Q = 0$ then $R^V(X, Y)U = 0$ and $R^V(S \otimes J, X)U = 0$ for any $S, U \in V$, $X, Y \in W$ and $J \in Z$.

From the first Bianchi identity applied to D we obtain that, for any $J, J', J'' \in Z$ and $S, T, U \in V$, we have:

$$(3.13) \quad \begin{aligned} & (R^V(S \otimes J, T \otimes J')U) \otimes J'' + U \otimes (R^Q(S \otimes J, T \otimes J')J'') \\ & + (R^V(T \otimes J', U \otimes J'')S) \otimes J + S \otimes (R^Q(T \otimes J', U \otimes J'')J) \\ & + (R^V(U \otimes J'', S \otimes J)T) \otimes J' + T \otimes (R^Q(U \otimes J'', S \otimes J)J') = 0. \end{aligned}$$

If $R^Q = 0$ and J, J', J'' are linearly independent then, from (3.13) we obtain that $R^V(S \otimes A, T \otimes B)U = 0$ for any $S, T, U \in V$ and $A, B \in Q$ (here, we have used the continuity of the map $(A, B) \mapsto R^V(S \otimes A, T \otimes B)U$, to allow A, B linearly dependent).

Similarly, if $\text{rank } V \geq 3$ and $R^V = 0$, from Proposition 3.3 and (3.13) we obtain $R^Q = 0$. \square

We end with the following result.

Theorem 3.7. *Let (E, V) be an almost f -structure on M and let ∇ be the connection on E induced by some torsion free connection on M , compatible with (E, V) .*

If $\text{rank } E > 4\text{rank } V \geq 8$ then, locally, (M, E, V, ∇) is the product of $(\text{Im}\mathbb{H})^{\text{rank } V}$ with a hypercomplex manifold.

Proof. By Proposition 3.3 and Corollary 3.6 the connections induced on V and Q are flat. Furthermore, as in the proof of Proposition 2.5 (note that, here, we do not need [8, Theorem 2.2]) we obtain that, locally, V is a hypercomplex manifold such that the projection onto M is twistorial.

Moreover, we have that $\pi^*\nabla$ restricts to give a flat connection on the quaternionic distribution K on V generated by $\ker d\pi$; indeed, we have $K = \pi^*(V \oplus (V \otimes Q))$. Therefore K is integrable and, as $\pi^*\nabla$ is, also, torsion free, its leaves are, locally, quaternionic vector spaces, whose linear quaternionic structures are preserved by the parallel transport of $\pi^*\nabla$. Thus, if U is a covariantly constant section of K and X is a section of π^*W then $[U, X] = (\pi^*\nabla)_U X$ is a section of π^*W . Hence, the linear quaternionic structures on the leaves of K are (locally) projectable with respect to π^*W . Therefore, locally, there exists a quaternionic submersion φ from V onto $\mathbb{H}^{\text{rank } V}$ which, by [3] is twistorial. Thus, φ restricted to M , identified with the zero section of V , is a twistorial submersion onto $(\text{Im}\mathbb{H})^{\text{rank } V}$ whose fibres are the leaves of W .

Now, as Q is flat, V is locally a hypercomplex manifold and $\pi^*\nabla$ is its Obata connection. Let J be any covariantly constant admissible complex structure on V . Thus, $T^J V = \ker(J + i)$ is preserved by $\pi^*\nabla$. Hence, if X is a section of $T^J V$ and U is section of K we have that

$$[U, X] = (\pi^*\nabla)_U X - (\pi^*\nabla)_X U$$

is a section of $T^J + K$. Therefore $T^J V$ is projectable with respect to K . This shows that, locally there exists a triholomorphic submersion from V onto a hypercomplex manifold N , with $\dim N = \text{rank } W$, which factorises into π followed by a twistorial submersion ψ from M to N ; also, the latter is triholomorphic when restricted to the leaves of W .

Finally, the map

$$\begin{aligned} M &\rightarrow (\text{Im}\mathbb{H})^{\text{rank } V} \times N, \\ x &\mapsto (\varphi(x), \psi(x)) \end{aligned}$$

provides the claimed (twistorial) identification. \square

References

- [1] ATIYAH, M. F.; HITCHIN, N. J.; SINGER, I. M. Self-duality in four-dimensional Riemannian geometry. *Proc. Roy. Soc. London Ser. A* **362** (1978), no. 1711, 425–461. MR0506229 (80d:53023), Zbl 0389.53011, doi:10.1098/rspa.1978.0143.
- [2] HITCHIN, N. J. Complex manifolds and Einstein’s equations. *Twistor geometry and nonlinear systems* (Primorsko, 1980), 73–99, Lecture Notes in Math., 970. Springer, Berlin-New York, 1982. MR0699802 (84i:32041), Zbl 0507.53025, doi:10.1007/BFb0066025.
- [3] IANUȘ, STERE; MARCHIAFAVA, STEFANO; ORNEA, LIVIU; PANTILIE, RADU. Twistorial maps between quaternionic manifolds. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **9** (2010), no. 1, 47–67. MR2668873 (2011j:53077), Zbl 1193.53121, arXiv:0801.4587.
- [4] KOBAYASHI, SHOSHICHI; NOMIZU, KATSUMI. Foundations of differential geometry. I. Reprint of the 1963 original. Wiley Classics Library. A Wiley-Interscience Publication. *John Wiley & Sons, Inc., New York*, 1996. xii+329 pp. ISBN: 0-471-15733-3. MR1393940 (97c:53001a).
- [5] KODAIRA, K. A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds. *Ann. of Math.* (2) **75** (1962), 146–162. MR0133841 (24 #A3665b), Zbl 0112.38404, doi:10.2307/1970424.
- [6] LEBRUN, CLAUDE R. Twistor CR manifolds and three-dimensional conformal geometry. *Trans. Amer. Math. Soc.* **284** (1984), no. 2, 601–616. MR0743735 (86m:32033), Zbl 0513.53006, doi:10.2307/1999098.
- [7] LOUBEAU, ERIC; PANTILIE, RADU. Harmonic morphisms between Weyl spaces and twistorial maps. II. *Ann. Inst. Fourier* (Grenoble) **60** (2010), no. 2, 433–453. MR2667782 (2011f:53146), Zbl 1203.58005, arXiv:math/0610676, doi:10.5802/aif.2528.
- [8] MACKENZIE, KIRILL C. H. Lie algebroids and Lie pseudoalgebras. *Bull. London Math. Soc.* **27** (1995), no. 2, 97–147. MR1325261 (96i:58183), Zbl 0829.22001, doi:10.1112/blms/27.2.97.
- [9] MARCHIAFAVA, S.; ORNEA, L.; PANTILIE, R. Twistor theory for CR quaternionic manifolds and related structures. *Monatsh. Math.* **167** (2012), no. 3–4, 531–545. MR2961297, Zbl 1254.53080, arXiv:0905.1455, doi:10.1007/s00605-011-0326-0.
- [10] MARCHIAFAVA, STEFANO; PANTILIE, RADU. Twistor theory for co-CR quaternionic manifolds and related structures. *Israel J. Math.* **195** (2013), no. 1, 347–371. MR3101253, Zbl 1281.53051, arXiv:1106.5431, doi:10.1007/s11856-013-0001-3.
- [11] MARCHIAFAVA, S.; PANTILIE, R. A note on CR quaternionic maps. *Adv. Geom.* **13** (2013), no. 4, 605–612. MR3181537, Zbl 1277.32046, arXiv:1108.3199, doi:10.1515/advgeom-2013-0006.
- [12] PANTILIE, RADU. On the twistor space of a (co-)CR quaternionic manifold. *New York J. Math.* **20** (2014), 959–971. MR3272918, Zbl 1314.53090, arXiv:1201.3870.
- [13] PANTILIE, RADU; WOOD, JOHN C. Twistorial harmonic morphisms with one-dimensional fibres on self-dual four-manifolds. *Q. J. Math.* **57** (2006), no. 1, 105–132. MR2204263 (2007a:53123), Zbl 1117.53047, arXiv:math/0305110, doi:10.1093/qmath/hah063.
- [14] PEDERSEN, H.; POON, Y. S.; SWANN, A. F. Hypercomplex structures associated to quaternionic manifolds. *Differential Geom. Appl.* **9** (1998), no. 3, 273–292. MR1661193 (99k:53062), Zbl 0920.53018, doi:10.1016/S0926-2245(98)00026-6.
- [15] SALAMON, S. M. Differential geometry of quaternionic manifolds. *Ann. Sci. École Norm. Sup.* (4) **19** (1986), no. 1, 31–55. MR0860810 (87m:53079), Zbl 0616.53023.

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