

# Quotient quandles and the fundamental Latin Alexander quandle

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ABSTRACT. Defined in Joyce, 1982, and Matveev, 1984, the fundamental quandle is a complete invariant of oriented classical knots up to ambient homeomorphism. We consider invariants of knots defined from quotients of the fundamental quandle. In particular, we introduce a generalization of the Alexander quandle of a knot known as the *fundamental Latin Alexander quandle* and consider its Gröbner basis-valued invariants, which generalize the Alexander polynomial. We show via example that the invariant is not determined by the generalized Alexander polynomial for virtual knots.

## CONTENTS

|   |     |
|---|-----|
| 1. Introduction                                   | 251 |
| 2. Quandles                                       | 252 |
| 3. Quotients of the fundamental quandle           | 254 |
| 4. Fundamental Latin Alexander Gröbner invariants | 258 |
| 5. Questions                                      | 262 |
| References  | 263 |

## 1. Introduction

In [7, 11] Joyce and Matveev introduced the algebraic structures known as *quandles* or *distributive groupoids*. In particular, the *fundamental quandle* of an oriented classical knot was shown to determine the knot group and the peripheral subgroup and thus the knot complement up to homeomorphism, yielding a complete invariant of oriented classical knots. In [7, 12] quotients of the fundamental quandle, including the fundamental involutory quandle and the fundamental involutory abelian quandle were studied, including some connections to the Alexander invariant. In particular, Joyce showed that the fundamental involutory abelian quandle of a knot is always finite

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Received August 1, 2014.

2010 *Mathematics Subject Classification*. 57M27, 57M25.

*Key words and phrases*. Quandles, Alexander quandles, Latin quandles, knot invariants, Gröbner bases.

The first author was partially supported by Simons Foundation collaboration grant 316709.

with cardinality equal to the determinant of the knot, while Winker showed that some knots have infinite involutory (nonabelian) quandle.

In this paper we consider some quotients of the fundamental quandles of classical and virtual knots and describe an algorithm which can sometimes reveal when a quotient of the fundamental quandle of a knot is finite. Showing that a given quotient is infinite is harder for general quotient quandles, but is simpler for quotients of the fundamental Alexander quandle of a knot, which has a module structure. We introduce the *fundamental Latin Alexander quandle* of a knot, a generalization of the Alexander quandle with coefficients in an extension ring such that the resulting quandle is Latin. From this new structure we define Gröbner basis-valued invariants akin to those defined in [2]. We include an example which shows that the new invariant is not determined by the generalized Alexander polynomial for virtual knots.

The paper is organized as follows. In Section 2 we review the basics of quandles. In Section 3 we consider some quotients of the fundamental quandle. In Section 4 we define the Fundamental Latin Alexander quandle and the Fundamental Latin Alexander Gröbner (FLAG) invariants, including computations of the  $\text{FLAG}_1$  invariant for all classical knots with up to eight crossings. We end in Section 5 with some questions for future research.

## 2. Quandles

We begin with a definition (see [7, 11, 5]).

**Definition 1.** A *quandle* is a set  $Q$  with an operation

$$\triangleright : Q \times Q \rightarrow Q$$

satisfying for all  $x, y, z \in Q$ :

- (i)  $x \triangleright x = x$ .
- (ii) The map  $f_y : Q \rightarrow Q$  defined by  $f_y(x) = x \triangleright y$  is a bijection.
- (iii)  $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ .

The inverse of  $f_y$  defines another operation called the *dual quandle* operation  $f_y^{-1}(x) = x \triangleright^{-1} y$ .

It is a straightforward exercise to show that  $Q$  forms a quandle under the dual quandle operation and that the two operations mutually right-distribute, i.e., we have

$$\begin{aligned} (x \triangleright y) \triangleright^{-1} z &= (x \triangleright^{-1} z) \triangleright (y \triangleright^{-1} z) \\ (x \triangleright^{-1} y) \triangleright z &= (x \triangleright z) \triangleright^{-1} (y \triangleright z). \end{aligned}$$

**Example 1.** Any  $\mathbb{Z}$ -module  $A$  is a quandle under the operation

$$x \triangleright y = 2y - x.$$

In particular, the dual quandle operation is the same as the original quandle operation, i.e.,  $x \triangleright^{-1} y = x \triangleright y$ . Quandles with this property are called *involutory* since the maps  $f_y$  are involutions.

**Example 2.** Let  $G$  be any group. Then  $G$  is a quandle under  $n$ -fold conjugation

$$x \triangleright y = y^{-n}xy^n$$

and under the *core* operation

$$x \triangleright y = yx^{-1}y.$$

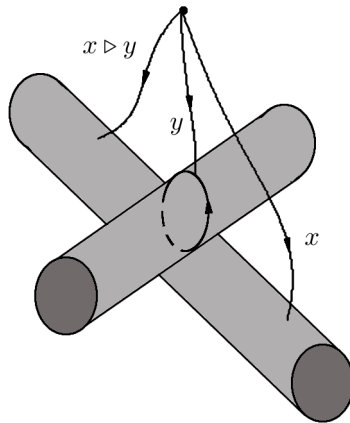
The set  $G$  with these quandle structures is called  $\text{Conj}_n(G)$  and  $\text{Core}(G)$  respectively.

**Example 3.** Any module  $M$  over the ring  $\Lambda = \mathbb{Z}[t^{\pm 1}]$  is a quandle under the operation

$$x \triangleright y = tx + (1 - t)y$$

called an *Alexander quandle*. More generally, if  $A$  is any abelian group and  $t \in \text{Aut}(A)$  is an automorphism of abelian groups, then  $A$  is an Alexander quandle under the operation above where 1 is the identity map.

**Example 4.** Let  $K$  be a link in  $S^3$  and  $N(K)$  a regular neighborhood of  $K$ . Then the *fundamental quandle* of  $K$  is the set of homotopy classes of paths in  $S^3 \setminus N(K)$  from a base point to  $N(K)$  such that the initial point stays fixed at the base point while the terminal point is free to wander on  $N(K)$ . The quandle operation is then given by setting  $x \triangleright y$  to the homotopy class of the path given by first following  $y$ , then going around a canonical meridian on  $N(K)$  linking  $K$  once, then going backward along  $y$ , then following  $x$  as illustrated. See [7] for more.

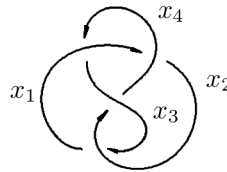


**Example 5.** The knot quandle can also be expressed combinatorially with a presentation by generators and relations as the set of equivalence classes of quandle words in a set of generators corresponding to arcs in a diagram of

$K$  under the equivalence relation generated by the quandle axioms together with the *crossing relations*

$$\begin{array}{c} \xrightarrow{x \triangleright y} \uparrow \\ | \\ \xrightarrow{x} \\ | \\ y \end{array} \quad \begin{array}{c} \xrightarrow{x} \uparrow \\ | \\ \xrightarrow{x \triangleright^{-1} y} \\ | \\ y \end{array}$$

For example, the figure 8 knot  $4_1$  below has the listed quandle presentation



$$\langle x_1, x_2, x_3, x_4 \mid \begin{array}{l} x_3 \triangleright x_2 = x_1, \quad x_2 \triangleright x_3 = x_4, \\ x_3 \triangleright x_1 = x_4, \quad x_2 \triangleright x_4 = x_1 \end{array} \rangle.$$

Say a relation in a quandle presentation is *short* if it has the form  $x_i \triangleright x_j = x_k$  for  $x_i, x_j, x_k$  generators. Then we observe that every finitely presented quandle  $Q$  has a presentation in which every relation is short, since we can add new generators  $x_k$  and short form relations abbreviating subwords of the form  $x_i \triangleright x_j$  to  $x_k$  as needed until all relations are short. If our generators are numbered  $\{x_1, \dots, x_n\}$ , then we can express a short form presentation with a matrix whose row  $i$  column  $j$  entry is  $k$  if  $x_i \triangleright x_j = x_k$  and 0 otherwise; we will call this a *presentation matrix* for  $Q$ . If a presentation matrix for  $Q$  has no zeros, then it expresses the complete operation table for  $Q$ , and  $Q$  is a finite quandle.

**Example 6.** The figure 8 knot in example 5 above has presentation matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

### 3. Quotients of the fundamental quandle

Knot quandles are generally infinite. However, it is observed in [7] and later in [12] that the *involutory* version of the fundamental quandle of a knot is often finite, and the fundamental *abelian involutory quandle* of a knot is always finite with order equal to the determinant of the knot, i.e., the absolute value of the Alexander polynomial evaluated at  $-1$ . The fundamental involutory quandle of a knot can be understood as the result of adding a fourth axiom which says

$$(iv) \quad x \triangleright y = x \triangleright^{-1} y \text{ for all } x, y \in Q,$$

or equivalently, replacing the second quandle axiom with

$$(ii') \quad (x \triangleright y) \triangleright y = x \text{ for all } x, y \in Q;$$

the fundamental abelian involutory quandle is then obtained by adding an additional axiom which says

$$(v) (x \triangleright y) \triangleright (z \triangleright w) = (x \triangleright z) \triangleright (y \triangleright w) \text{ for all } x, y, z, w \in Q.$$

We can verify that the fundamental involutory quandle of the figure eight knot is finite with cardinality 5 by observing that moves of the following types do not change the quandle presented by a presentation matrix:

- (i) filling in a zero with a value obtained as a consequence of the axioms and other relations,
- (ii) filling in a zero with a number defining a new generator and adding a row and column of zeroes corresponding to the new generator,
- (iii) deleting a row and column and replacing all instances of the larger generator with the smaller one when two generators are found to be equal, taking care to note any new equalities of generators implied.

This gives us a procedure for filling in the complete operation table of a finitely presented quandle: first, fill in all zeroes determined by consequences of the axioms and keep a list of any pairs of equal generators, reducing the presentation by eliminating redundant generators when possible. Next, if any zeroes remain, choose one to assign to a new generator and repeat the process. This procedure may or may not terminate – if the presented quandle is infinite, the process can never terminate, but even if the quandle finite then the speed of termination depends a great deal on the choice of zeroes for replacement. On the other hand, when the process does terminate, the result is a sequence of Tietze moves showing that the presented quandle is finite.

**Example 7.** Let us use the above procedure to verify that the figure eight knot has fundamental involutory quandle of cardinality 5. We start with the presentation matrix from Example 5 and fill in the zeroes as determined by the involutory quandle axioms:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 2 & 4 & 1 \\ 4 & 1 & 3 & 0 \\ 3 & 0 & 2 & 4 \end{bmatrix}.$$

For example, quandle axiom (i) says  $x_i \triangleright x_i = x_i$ , so the diagonal elements are filled in with their row numbers; the involutory condition says that since  $x_3 \triangleright x_2 = x_1$ , we have  $x_1 \triangleright x_2 = (x_3 \triangleright x_2) \triangleright x_3 = x_3$ , etc. Note that we still have some zeroes which cannot be filled in from the axioms; thus, we need to choose a zero to assign a new generator  $x_5$  – say we set  $x_5 = x_1 \triangleright x_3$ . Then we have presentation matrix below which completes via the involutory

quandle axioms to the matrix on the right

$$\begin{bmatrix} 1 & 3 & 5 & 2 & 0 \\ 0 & 2 & 4 & 1 & 0 \\ 4 & 1 & 3 & 0 & 0 \\ 3 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 5 & 2 & 4 \\ 5 & 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 5 & 2 \\ 3 & 5 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \end{bmatrix}$$

and the fundamental involutory quandle of  $4_1$  has 5 elements.

The involutory and abelian conditions have geometric motivations: the involutory condition comes from considering unoriented knots, while the abelian condition is the condition required for the set of quandle homomorphisms from the knot quandle to  $Q$  to inherit a natural quandle structure (see [3] for more). Nevertheless, we can consider these quotient quandles to be simply the result of imposing algebraic conditions on the fundamental quandle of a knot. Any such choice of conditions results in a quandle-valued knot invariant, and for each such invariant we can ask whether the resulting quandle is finite. In [7, 12] the generalizations of the involutory condition to higher numbers of operations, e.g.,

$$(\dots((x \triangleright y) \triangleright y) \dots) \triangleright y = x$$

were considered, with the notable result that the square knot and granny knot have nonisomorphic 4-quandles, i.e., quotients in which we set

$$(((x \triangleright y) \triangleright y) \triangleright y) \triangleright y = x,$$

despite having isomorphic knot groups.

We considered several examples of algebraic axioms and used our procedure outlined above, implemented in `Python`, to search for examples of knots whose fundamental quandles had finite quotients when the axioms were imposed. These included:

- *Anti-abelian axiom:*  $(x \triangleright y) \triangleright (z \triangleright w) = (w \triangleright y) \triangleright (z \triangleright x)$ .
- *Left distributive axiom:*  $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ .
- *Commutative operator axiom:*  $x \triangleright (y \triangleright z) = x \triangleright (z \triangleright y)$ .
- *Latin axiom:*  $x \triangleright y = x \triangleright z \Rightarrow y = z$ .

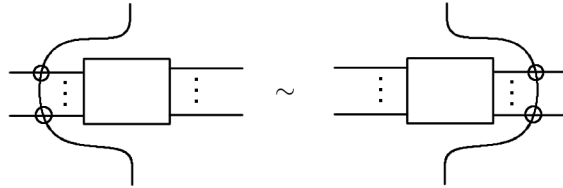
both in combination with the abelian and involutory axioms and alone. Some combinations are redundant; for instance, the abelian condition implies left distributivity. Curiously, we found that many of the above conditions yield the same results, with most knots of small crossing number having either trivial one-element quotient quandles or the three-element quandle structure  $\mathbb{Z}_3$  with  $x \triangleright y = 2x - y$ .

**Example 8.** Of the classical knots with seven or fewer crossings,  $3_1$ ,  $6_1$ ,  $7_4$ , and  $7_7$  have anti-abelian involutory quandles with three elements

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix},$$

while the rest have the trivial one-element quandle.

Recall that a *virtual knot* is an equivalence class of oriented Gauss codes under the equivalence relation determined by the Gauss code Reidemeister moves. It is standard practice to draw virtual knots with extra *virtual crossings*, circled self-intersections representing nonplanarity of Gauss codes; these virtual crossings interact with classical crossings via the *detour move*,



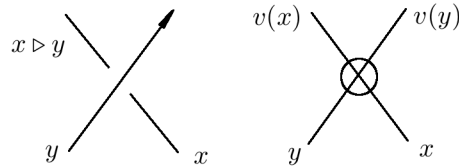
which says we can redraw any arc with only virtual crossings in its interior as any other arc with only virtual crossings in its interior. Virtual knots may be understood as equivalence classes of knots in thickened oriented surfaces  $\Sigma \times [0, 1]$  modulo stabilization. See [8, 9] for more about virtual knots.

**Definition 2.** Let  $Q$  be a quandle and  $v : Q \rightarrow Q$  a bijective map. We say that  $(Q, v)$  is a *virtual quandle* if  $v$  satisfies

$$v(x \triangleright y) = v(x) \triangleright v(y),$$

i.e., a virtual quandle is a quandle with a choice of automorphism. If  $Q$  is an involutory quandle, then  $(Q, v)$  is an *involutory virtual quandle* if  $(Q, v)$  is a virtual quandle and  $v$  is an involution, i.e., if  $v(v(x)) = x$  for all  $x \in Q$ .

Let  $K$  be a virtual knot. The *fundamental involutory virtual quandle* of  $K$  is the virtual quandle with presentation consisting of one generator for each portion of  $K$  containing only overcrossings (that is, we divide  $K$  at classical undercrossings and at virtual crossings) with relations as pictured together with the involutory quandle axioms.



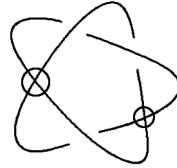
As with finite quandles, we can express virtual quandle structures on a finite set  $X = \{x_1, x_2, \dots, x_n\}$  with an  $n \times (n + 1)$  matrix whose leftmost  $n \times n$  block is a quandle operation matrix and whose last column expresses the map  $v$ , i.e., the entry in row  $j$  column  $n + 1$  is  $x_k$  where  $v(x_j) = x_k$ .

**Example 9.** The virtual knots 3.2, 3.3, 3.4, 3.5, 4.2, 4.4, 4.5, 4.9, and 4.43 all have anti-abelian involutory virtual quandle

$$\left[ \begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 3 & 2 & 1 & 1 \\ 2 & 1 & 3 & 3 \end{array} \right]$$

and the virtual knot  $4_3$  has the trivial involutory virtual quandle.

**Example 10.** However, while many knots seem to have either the trivial one-element anti-abelian involutory virtual quandle or the three-element anti-abelian involutory virtual quandle structure above, not all of them do. For example, we found via Python computations that the anti-abelian quandle of the virtual knot  $3_7$  has 27 elements and is given by the operation matrix below:



|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |   |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|---|
| 1  | 7  | 6  | 8  | 9  | 3  | 2  | 4  | 5  | 11 | 10 | 13 | 12 | 15 | 14 | 17 | 16 | 20 | 21 | 1  | 19 | 23 | 22 | 25 | 24 | 27 | 26 | 2  |   |
| 7  | 2  | 12 | 5  | 4  | 14 | 1  | 10 | 11 | 8  | 9  | 3  | 15 | 6  | 13 | 18 | 21 | 16 | 20 | 19 | 17 | 24 | 26 | 22 | 27 | 23 | 25 | 1  | 4 |
| 6  | 12 | 3  | 16 | 22 | 1  | 15 | 26 | 21 | 20 | 25 | 2  | 14 | 13 | 7  | 4  | 27 | 24 | 23 | 10 | 9  | 5  | 19 | 18 | 11 | 8  | 17 | 4  |   |
| 8  | 5  | 16 | 4  | 2  | 27 | 11 | 1  | 10 | 9  | 7  | 24 | 21 | 20 | 23 | 3  | 26 | 22 | 25 | 14 | 13 | 18 | 15 | 12 | 19 | 17 | 6  | 3  |   |
| 9  | 4  | 22 | 2  | 5  | 19 | 10 | 11 | 1  | 7  | 8  | 18 | 26 | 25 | 17 | 24 | 15 | 12 | 6  | 27 | 23 | 3  | 21 | 16 | 14 | 13 | 20 | 6  |   |
| 3  | 14 | 1  | 27 | 19 | 6  | 13 | 17 | 23 | 24 | 18 | 15 | 7  | 2  | 12 | 26 | 8  | 11 | 5  | 25 | 22 | 21 | 9  | 10 | 20 | 16 | 4  | 5  |   |
| 2  | 1  | 15 | 11 | 10 | 13 | 7  | 9  | 8  | 5  | 4  | 14 | 6  | 12 | 3  | 19 | 20 | 21 | 16 | 17 | 18 | 27 | 25 | 26 | 23 | 24 | 22 | 7  |   |
| 4  | 10 | 26 | 1  | 11 | 17 | 9  | 8  | 7  | 2  | 5  | 19 | 25 | 22 | 18 | 27 | 6  | 15 | 12 | 23 | 24 | 14 | 20 | 21 | 13 | 3  | 16 | 12 |   |
| 5  | 11 | 21 | 10 | 1  | 23 | 8  | 7  | 9  | 4  | 2  | 27 | 20 | 16 | 24 | 14 | 25 | 26 | 22 | 13 | 3  | 19 | 6  | 15 | 17 | 18 | 12 | 14 |   |
| 11 | 8  | 20 | 9  | 7  | 24 | 5  | 2  | 4  | 10 | 1  | 23 | 16 | 21 | 27 | 13 | 22 | 25 | 26 | 3  | 14 | 17 | 12 | 6  | 18 | 19 | 15 | 13 |   |
| 10 | 9  | 25 | 7  | 8  | 18 | 4  | 5  | 2  | 1  | 11 | 17 | 22 | 26 | 19 | 23 | 12 | 6  | 15 | 24 | 27 | 13 | 16 | 20 | 3  | 14 | 21 | 15 |   |
| 13 | 3  | 2  | 24 | 18 | 15 | 14 | 19 | 27 | 23 | 17 | 12 | 1  | 7  | 6  | 22 | 11 | 5  | 8  | 26 | 25 | 16 | 10 | 4  | 21 | 20 | 9  | 8  |   |
| 12 | 15 | 14 | 21 | 26 | 7  | 6  | 25 | 20 | 16 | 22 | 1  | 13 | 3  | 2  | 10 | 23 | 27 | 24 | 9  | 4  | 11 | 17 | 19 | 8  | 5  | 18 | 10 |   |
| 15 | 6  | 13 | 20 | 25 | 2  | 12 | 22 | 16 | 21 | 26 | 7  | 3  | 14 | 1  | 9  | 24 | 23 | 27 | 4  | 10 | 8  | 18 | 17 | 5  | 11 | 19 | 9  |   |
| 14 | 13 | 7  | 23 | 17 | 12 | 3  | 18 | 24 | 27 | 19 | 6  | 2  | 1  | 15 | 25 | 5  | 8  | 11 | 22 | 26 | 20 | 4  | 9  | 16 | 21 | 10 | 11 |   |
| 17 | 18 | 4  | 3  | 24 | 26 | 19 | 27 | 14 | 13 | 23 | 22 | 10 | 9  | 25 | 16 | 1  | 2  | 7  | 21 | 20 | 12 | 11 | 5  | 15 | 6  | 8  | 16 |   |
| 16 | 21 | 27 | 26 | 15 | 8  | 20 | 6  | 25 | 22 | 12 | 11 | 23 | 24 | 5  | 1  | 17 | 19 | 18 | 7  | 2  | 10 | 13 | 14 | 9  | 4  | 3  | 18 |   |
| 20 | 16 | 24 | 22 | 12 | 11 | 21 | 15 | 26 | 25 | 6  | 5  | 27 | 23 | 8  | 2  | 19 | 18 | 17 | 1  | 7  | 4  | 14 | 3  | 10 | 9  | 13 | 17 |   |
| 21 | 20 | 23 | 25 | 6  | 5  | 16 | 12 | 22 | 26 | 15 | 8  | 24 | 27 | 11 | 7  | 18 | 17 | 19 | 2  | 1  | 9  | 3  | 13 | 4  | 10 | 14 | 19 |   |
| 18 | 19 | 10 | 14 | 27 | 25 | 17 | 23 | 13 | 3  | 24 | 26 | 9  | 4  | 22 | 21 | 7  | 1  | 2  | 20 | 16 | 15 | 8  | 11 | 6  | 12 | 5  | 21 |   |
| 19 | 17 | 9  | 13 | 23 | 22 | 18 | 24 | 3  | 14 | 27 | 25 | 4  | 10 | 26 | 20 | 2  | 7  | 1  | 16 | 21 | 6  | 5  | 8  | 12 | 15 | 11 | 20 |   |
| 23 | 24 | 5  | 18 | 3  | 21 | 27 | 14 | 19 | 17 | 13 | 16 | 11 | 8  | 20 | 12 | 10 | 4  | 9  | 15 | 6  | 22 | 1  | 2  | 26 | 25 | 7  | 27 |   |
| 22 | 26 | 19 | 15 | 21 | 9  | 25 | 20 | 6  | 12 | 16 | 10 | 17 | 18 | 4  | 11 | 13 | 14 | 3  | 8  | 5  | 1  | 23 | 27 | 7  | 2  | 24 | 25 |   |
| 25 | 22 | 18 | 12 | 16 | 10 | 26 | 21 | 15 | 6  | 20 | 4  | 19 | 17 | 9  | 5  | 14 | 3  | 13 | 11 | 8  | 2  | 27 | 24 | 1  | 7  | 23 | 26 |   |
| 24 | 27 | 11 | 19 | 14 | 20 | 23 | 13 | 17 | 18 | 3  | 21 | 8  | 5  | 16 | 15 | 9  | 10 | 4  | 6  | 12 | 26 | 7  | 1  | 25 | 22 | 2  | 23 |   |
| 27 | 23 | 8  | 17 | 13 | 16 | 24 | 3  | 18 | 19 | 14 | 20 | 5  | 11 | 21 | 6  | 4  | 9  | 10 | 12 | 15 | 25 | 2  | 7  | 22 | 26 | 1  | 24 |   |
| 26 | 25 | 17 | 6  | 20 | 4  | 22 | 16 | 12 | 15 | 21 | 9  | 18 | 19 | 10 | 8  | 3  | 13 | 14 | 5  | 11 | 7  | 24 | 23 | 2  | 1  | 27 | 22 |   |

#### 4. Fundamental Latin Alexander Gröbner invariants

Let  $K$  be a knot or link. The *fundamental Alexander quandle*  $FAQ(K)$  is the  $\Lambda$ -module generated by generators corresponding to arcs in a diagram of  $K$  with Alexander quandle operations at the crossings. As a  $\Lambda$ -module, the fundamental Alexander quandle of a knot is the classical Alexander invariant.

Let  $R$  be a polynomial ring and  $M$  an  $R$ -module with presentation matrix  $P \in M_{m,n}(R)$ , i.e., the rows of  $P$  correspond to generators of  $M$  and the rows of  $P$  express relations defining  $M$ . The  $k$ th elementary ideal  $I_k$  of  $M$



is the ideal in  $R$  generated by the  $(n - k)$  (or  $m - k$  if  $m > n$ ) minors of  $P$ . It is a standard result (see [10] for instance) that changes to  $P$  reflecting Tietze moves in the presentation of  $M$  do not change the elementary ideals, and hence these ideals are invariants of  $M$ .

**Example 11.** Let  $K$  be a knot and  $P$  a  $\Lambda$ -module presentation matrix of the Alexander quandle of  $K$ . The  $k$ th Alexander Polynomial of  $K$  is any generator  $\Delta_k$  of the smallest principal ideal of  $\Lambda$  containing the  $k$ th elementary ideal of  $P$ . Note that  $\Delta_k$  is defined only up to multiplication by units in  $\Lambda$ . In particular,  $\Delta_0 = 1$  for classical knots  $K$ , and  $\Delta_1$  is often called the Alexander polynomial.

Recall that a quandle is *Latin* if in addition to the right-invertibility required by quandle axiom (ii), we also have left-invertibility. That is, a quandle  $Q$  is Latin if it satisfies the axiom:

(vi) For every  $x, y \in Q$ , there is a unique  $z \in Q$  such that  $x \triangleright z = y$ ,

or equivalently:

(vi') For every  $x \in Q$ , the map  $f_x : Q \rightarrow Q$  defined by  $f_x(y) = x \triangleright y$  is bijective.

A finite quandle is Latin if and only its operation table forms a Latin square, i.e., if every row and column is a permutation of the elements of  $Q$ .

**Example 12.** An Alexander quandle is Latin iff  $1 - t$  is invertible. For instance, the Alexander quandle structure on  $\mathbb{Z}_3$  with  $t \in \text{Aut}(\mathbb{Z}_3)$  given by multiplication by 2 is Latin, while the Alexander quandle structure on  $\mathbb{Z}_4$  with  $t \in \text{Aut}(\mathbb{Z}_4)$  given by multiplication by 3 is not Latin:

|                  |   |   |   |  |                  |   |   |   |   |
|------------------|---|---|---|--|------------------|---|---|---|---|
| $\triangleright$ | 0 | 1 | 2 |  | $\triangleright$ | 0 | 1 | 2 | 3 |
| 0                | 0 | 2 | 1 |  | 0                | 0 | 2 | 0 | 2 |
| 1                | 2 | 1 | 0 |  | 1                | 3 | 1 | 3 | 1 |
| 2                | 1 | 0 | 2 |  | 2                | 2 | 0 | 2 | 0 |
|                  |   |   |   |  | 3                | 1 | 3 | 1 | 3 |

The element  $(1 - t) \in \Lambda$  is not invertible in  $\Lambda$ ; its “natural” inverse is the Laurent series  $1 + t + t^2 + \dots$ . We prefer to stick to polynomial rings, so we define the Fundamental Latin Alexander Quandle of an oriented link  $L$  in the following way: let  $\widehat{\Lambda} = \mathbb{Z}[t, t^{-1}s, s^{-1}]$  where the variables  $t^{-1}$  and  $s^{-1}$  are new formal variables, not (yet) inverses for  $t$  and  $s$ , and then define the quotient ring  $\widehat{\Lambda}' = \widehat{\Lambda}/(ss^{-1} - 1, tt^{-1} - 1, 1 - t - s)$ . Then we define the *Fundamental Latin Alexander Quandle* of  $L$ ,  $F\text{LAQ}(L)$ , to be the  $\widehat{\Lambda}'$ -module generated by a set of generators corresponding to arcs in a diagram of  $L$  with relations of the form  $w = tx + sy$  at crossings as depicted below. Equivalently,  $F\text{LAQ}(L)$  can be regarded as the Alexander quandle of the knot with coefficients in the extension ring  $\Lambda[(1 - t)^{-1}]$  of  $\Lambda = \mathbb{Z}[t^{\pm 1}]$ .

In [2], Gröbner basis-valued invariants of knots and link were defined from the Alexander biquandle by considering the pullback ideals of the elementary

ideals to a standard (non-Laurent) polynomial ring, then taking the Gröbner basis of the resulting ideal with respect to a choice of monomial ordering. A similar idea was used in [6] to study Gröbner basis invariants of the usual (non-Latin) Alexander module of a knot. We can apply the same idea here to get a new Gröbner basis-valued invariant which we call the *Fundamental Latin Alexander Gröbner* invariant, denoted  $FLAG(L)$ .

**Definition 3.** Let  $L$  be an oriented link,  $\widehat{\Lambda} = \mathbb{Z}[t, t^{-1}, s, s^{-1}]$  a four variable polynomial ring, and  $P$  the coefficient matrix of the homogeneous system of linear equations with variables corresponding to arcs in a diagram of  $L$  and equations at crossings as depicted.

$$\begin{array}{c}
 \uparrow \\
 \hline
 w \quad \quad \quad x \\
 \hline
 y
 \end{array}
 \quad tx + sy = w$$

Then the  $k$ th FLAG ideal of  $L$  is the ideal  $I_k$  in  $\widehat{\Lambda}$  generated by the generators of the  $k$ th elementary ideal of  $P$  and the polynomials  $ss^{-1} - 1$ ,  $tt^{-1} - 1$  and  $1 - t - s$ . Given a choice of monomial ordering  $<$  of the variables  $s, s^{-1}, t, t^{-1}$ , the resulting Gröbner basis of  $I_k$  is the  $k$ th FLAG invariant of  $L$ , denoted  $FLAG_k^<(L)$ .

The FLAG invariant contains more information than the Alexander polynomial, in general; for instance, the number of elements of the FLAG basis with respect to a choice of monomial ordering,  $|FLAG_1(K)|$  is an invariant of knots, while the classical Alexander  $k = 1$  ideal is always principal for classical knots. We note that setting  $s = 1 - t$  and  $ss^{-1} = 1$  in each of the polynomials in the  $FLAG_1^<$  ideal yields either the Alexander polynomial or 0, since this is the ideal which is set to zero when defining the Alexander invariant. Since the  $FLAG_k^<$  ideals are in general not principal, these Gröbner bases in general contain more information than the usual  $k$ th Alexander polynomial.

**Example 13.** We computed the  $FLAG_1^<$  invariants with graded reverse lexicographical ordering with respect to the monomial ordering  $t < s < t^{-1} < s^{-1}$  for all prime classical knots with up to eight crossings using Python code available at the first author's website [www.esotericka.org](http://www.esotericka.org).

The results are collected in the tables below.<sup>1</sup>

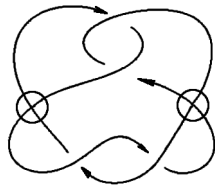
| $K$             | $ \text{FLAG}_1(K) $ | $\text{FLAG}_1(K)$  |
|-----------------|----------------------|---|
| 3 <sub>1</sub>  | 4                    | $\{t^2 - t + 1, s^{-1} - t, t^{-1} + t - 1, s + t - 1\}$  |
| 4 <sub>1</sub>  | 4                    | $\{t^2 - 3t + 1, s^{-1} + t - 2, t^{-1} + t - 3, s + t - 1\}$   |
| 5 <sub>1</sub>  | 7                    | $\{s^{-2} - s^{-1} + t^{-1} + t + 1, s^{-1}t^{-1} - s^{-1} - t^{-1},$<br>$s^{-1}t - s^{-1} + 1, s^{-1} + t^{-2} - t, t^{-1}t - 1, -s^{-1} - t^{-1} + t^2 + 1, s + t - 1\}$  |
| 5 <sub>2</sub>  | 4                    | $\{2t^2 - 3t + 2, s^{-1} - 2t + 1, 2t^{-1} + 2t - 3, s + t - 1\}$   |
| 6 <sub>1</sub>  | 4                    | $\{(t - 2)(2t - 1), s^{-1} + 2t - 3, 2t^{-1} + 2t - 5, s + t - 1\}$   |
| 6 <sub>2</sub>  | 7                    | $\{s^{-2} - s^{-1} - t^{-1} - t + 1, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1,$<br>$-s^{-1} + t^{-2} - 2t^{-1} - t + 2, t^{-1}t - 1, s^{-1} - t^{-1} + t^2 - 2t + 1, s + t - 1\}$   |
| 6 <sub>3</sub>  | 7                    | $\{s^{-2} - s^{-1} + t^{-1} + t - 1, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1,$<br>$s^{-1} + t^{-2} - 2t^{-1} - t + 2, t^{-1}t - 1, -s^{-1} - t^{-1} + t^2 - 2t + 3, s + t - 1\}$   |
| 7 <sub>1</sub>  | 7                    | $\{s^{-1} + t^{-3} + t^{-1} - t^2 - 1, -s^{-1} - t^{-2} + t^3 + t,$<br>$s^{-2} - s^{-1} + t^{-2} + t^{-1} + t^2 + t + 2, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1,$<br>$t^{-1}t - 1, s + t - 1\}$                                       |
| 7 <sub>2</sub>  | 4                    | $\{3t^2 - 5t + 3, s^{-1} - 3t + 2, 3t^{-1} + 3t - 5, s + t - 1\}$   |
| 7 <sub>3</sub>  | 7                    | $\{s^{-2} - s^{-1} + 2t^{-1} + 2t + 1, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1,$<br>$s^{-1} + 2t^{-2} - t^{-1} - 2t + 1, t^{-1}t - 1, -s^{-1} - 2t^{-1} + 2t^2 - t + 2, s + t - 1\}$   |
| 7 <sub>4</sub>  | 4                    | $\{4t^2 - 7t + 4, s^{-1} - 4t + 3, 4t^{-1} + 4t - 7, s + t - 1\}$   |
| 7 <sub>5</sub>  | 7                    | $\{s^{-2} - s^{-1} + 2t^{-1} + 2t, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1,$<br>$s^{-1} + 2t^{-2} - 2t^{-1} - 2t + 2, t^{-1}t - 1, -s^{-1} - 2t^{-1} + 2t^2 - 2t + 3,$<br>$s + t - 1\}$  |
| 7 <sub>6</sub>  | 7                    | $\{s^{-2} - s^{-1} - t^{-1} - t + 3, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1,$<br>$-s^{-1} + t^{-2} - 4t^{-1} - t + 4, t^{-1}t - 1, s^{-1} - t^{-1} + t^2 - 4t + 3, s + t - 1\}$   |
| 7 <sub>7</sub>  | 7                    | $\{s^{-2} - s^{-1} + t^{-1} + t - 3, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1,$<br>$s^{-1} + t^{-2} - 4t^{-1} - t + 4, t^{-1}t - 1, -s^{-1} - t^{-1} + t^2 - 4t + 5, s + t - 1\}$   |
| 8 <sub>1</sub>  | 4                    | $\{3t^2 - 7t + 3, s^{-1} + 3t - 4, 3t^{-1} + 3t - 7, s + t - 1\}$   |
| 8 <sub>2</sub>  | 7                    | $\{-s^{-1} + t^{-3} - 2t^{-2} + t^{-1} - t^2 + 2t - 1, s^{-1} - t^{-2} + 2t^{-1} + t^3 - 2t^2 + t - 2,$<br>$s^{-2} - s^{-1} - t^{-2} + t^{-1} - t^2 + t, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1,$<br>$t^{-1}t - 1, s + t - 1\}$       |
| 8 <sub>3</sub>  | 4                    | $\{4t^2 - 9t + 4, s^{-1} + 4t - 5, 4t^{-1} + 4t - 9, s + t - 1\}$   |
| 8 <sub>4</sub>  | 7                    | $\{s^{-2} - s^{-1} - 2t^{-1} - 2t + 1, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1,$<br>$-s^{-1} + 2t^{-2} - 3t^{-1} - 2t + 3, t^{-1}t - 1, s^{-1} - 2t^{-1} + 2t^2 - 3t + 2, s + t - 1\}$   |
| 8 <sub>5</sub>  | 7                    | $\{-s^{-1} + t^{-3} - 2t^{-2} + 2t^{-1} - t^2 + 2t - 2, s^{-1} - t^{-2} + 2t^{-1} + t^3 - 2t^2 + 2t - 3,$<br>$s^{-2} - s^{-1} - t^{-2} + t^{-1} - t^2 + t - 1, s^{-1}t^{-1} - s^{-1} - t^{-1},$<br>$s^{-1}t - s^{-1} + 1, t^{-1}t - 1, s + t - 1\}$ |
| 8 <sub>6</sub>  | 7                    | $\{s^{-2} - s^{-1} - 2t^{-1} - 2t + 2, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1,$<br>$-s^{-1} + 2t^{-2} - 4t^{-1} - 2t + 4, t^{-1}t - 1, s^{-1} - 2t^{-1} + 2t^2 - 4t + 3, s + t - 1\}$   |
| 8 <sub>7</sub>  | 7                    | $\{s^{-1} + t^{-3} - 2t^{-2} + 3t^{-1} - t^2 + 2t - 3, -s^{-1} - t^{-2} + 2t^{-1} + t^3 - 2t^2 + 3t - 2,$<br>$s^{-2} - s^{-1} + t^{-2} - t^{-1} + t^2 - t + 2, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1,$<br>$t^{-1}t - 1, s + t - 1\}$ |
| 8 <sub>8</sub>  | 7                    | $\{s^{-2} - s^{-1} + 2t^{-1} + 2t - 2, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1,$<br>$s^{-1} + 2t^{-2} - 4t^{-1} - 2t + 4, t^{-1}t - 1, -s^{-1} - 2t^{-1} + 2t^2 - 4t + 5, s + t - 1\}$   |
| 8 <sub>9</sub>  | 7                    | $\{-s^{-1} + t^{-3} - 2t^{-2} + 3t^{-1} - t^2 + 2t - 3, s^{-1} - t^{-2} + 2t^{-1} + t^3 - 2t^2 + 3t - 4,$<br>$s^{-2} - s^{-1} - t^{-2} + t^{-1} - t^2 + t - 2, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1,$<br>$t^{-1}t - 1, s + t - 1\}$ |
| 8 <sub>10</sub> | 7                    | $\{s^{-1} + t^{-3} - 2t^{-2} + 4t^{-1} - t^2 + 2t - 4, -s^{-1} - t^{-2} + 2t^{-1} + t^3 - 2t^2 + 4t - 3,$<br>$s^{-2} - s^{-1} + t^{-2} - t^{-1} + t^2 - t + 3, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1,$<br>$t^{-1}t - 1, s + t - 1\}$ |
| 8 <sub>11</sub> | 7                    | $\{s^{-2} - s^{-1} - 2t^{-1} - 2t + 3, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1,$<br>$-s^{-1} + 2t^{-2} - 5t^{-1} - 2t + 5, t^{-1}t - 1, s^{-1} - 2t^{-1} + 2t^2 - 5t + 4, s + t - 1\}$   |

<sup>1</sup>We note that each of the  $|\text{FLAG}_1(K)|$  values in the table are in the set  $\{4, 7\}$ . Whether this is related to the fact that the computation was done on the Pomona College campus is currently unknown.

| $K$           | $ \text{FLAG}_1(K) $ | $\text{FLAG}(K)$  |
|---------------|----------------------|---|
| $\delta_{12}$ | 7                    | $\{s^{-2} - s^{-1} + t^{-1} + t - 5, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1, s^{-1} + t^{-2} - 6t^{-1} - t + 6, t^{-1}t - 1, -s^{-1} - t^{-1} + t^2 - 6t + 7, s + t - 1\}$  |
| $\delta_{13}$ | 7                    | $\{s^{-2} - s^{-1} + 2t^{-1} + 2t - 3, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1, s^{-1} + 2t^{-2} - 5t^{-1} - 2t + 5, t^{-1}t - 1, -s^{-1} - 2t^{-1} + 2t^2 - 5t + 6, s + t - 1\}$  |
| $\delta_{14}$ | 7                    | $\{s^{-2} - s^{-1} - 2t^{-1} - 2t + 4, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1, -s^{-1} + 2t^{-2} - 6t^{-1} - 2t + 6, t^{-1}t - 1, s^{-1} - 2t^{-1} + 2t^2 - 6t + 5, s + t - 1\}$  |
| $\delta_{15}$ | 7                    | $\{s^{-2} - s^{-1} + 3t^{-1} + 3t - 2, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1, s^{-1} + 3t^{-2} - 5t^{-1} - 3t + 5, t^{-1}t - 1, -s^{-1} - 3t^{-1} + 3t^2 - 5t + 6, s + t - 1\}$  |
| $\delta_{16}$ | 7                    | $\{s^{-1} + t^{-3} - 3t^{-2} + 5t^{-1} - t^2 + 3t - 5, -s^{-1} - t^{-2} + 3t^{-1} + t^3 - 3t^2 + 5t - 4, s^{-2} - s^{-1} + t^{-2} - 2t^{-1} + t^2 - 2t + 3, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1, t^{-1}t - 1, s + t - 1\}$ |
| $\delta_{17}$ | 7                    | $\{-s^{-1} + t^{-3} - 3t^{-2} + 5t^{-1} - t^2 + 3t - 5, s^{-1} - t^{-2} + 3t^{-1} + t^3 - 3t^2 + 5t - 6, s^{-2} - s^{-1} - t^{-2} + 2t^{-1} - t^2 + 2t - 3, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1, t^{-1}t - 1, s + t - 1\}$ |
| $\delta_{18}$ | 7                    | $\{-s^{-1} + t^{-3} - 4t^{-2} + 6t^{-1} - t^2 + 4t - 6, s^{-1} - t^{-2} + 4t^{-1} + t^3 - 4t^2 + 6t - 7, s^{-2} - s^{-1} - t^{-2} + 3t^{-1} - t^2 + 3t - 3, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1, t^{-1}t - 1, s + t - 1\}$ |
| $\delta_{19}$ | 7                    | $\{s^{-1} + t^{-3} - t^2, -s^{-1} - t^{-2} + t^3 + 1, s^{-2} - s^{-1} + t^{-2} + t^{-1} + t^2 + t + 1, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1, t^{-1}t - 1, s + t - 1\}$  |
| $\delta_{20}$ | 7                    | $\{s^{-2} - s^{-1} + t^{-1} + t, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1, s^{-1} + t^{-2} - t^{-1} - t + 1, t^{-1}t - 1, -s^{-1} - t^{-1} + t^2 - t + 2, s + t - 1\}$  |
| $\delta_{21}$ | 7                    | $\{s^{-2} - s^{-1} - t^{-1} - t + 2, s^{-1}t^{-1} - s^{-1} - t^{-1}, s^{-1}t - s^{-1} + 1, -s^{-1} + t^{-2} - 3t^{-1} - t + 3, t^{-1}t - 1, s^{-1} - t^{-1} + t^2 - 3t + 2, s + t - 1\}$  |

The FLAG ideals are defined for virtual knots just as for classical knots since each crossing can be considered locally. Thus, we can extend the FLAG invariant to virtual knots in the usual way by simply ignoring virtual crossings.

**Example 14.** The virtual knot below, named 4.99 in the knot atlas [1], has trivial virtual Alexander polynomial, as does the trefoil  $3_1$ . However, the two are distinguished by their  $\text{FLAG}_1$  invariants; this shows that the  $\text{FLAG}_1^<$  invariant (with the same monomial ordering as above) is not determined by the virtual Alexander polynomial.



$$\text{FLAG}_1(4.99) = \{s^{-1} - 2, t^{-1} - 2, 2s - 1, 2t - 1\}$$

We remark that this virtual knot has classical Alexander polynomial  $2t - 1$ , which is not symmetric, unlike the case for all classical knots.

### 5. Questions

In this section we collect a few questions for future research.

What other quotients of the fundamental quandle yield interesting finite quandles? What is the relationship between quotients of the fundamental quandle, a variety of functorial invariant, and the homomorphism-based invariants such as the quandle counting invariant and its enhancements? What does the cardinality of the  $\text{FLAG}_k^<$  invariant tell us about a knot?

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