

On the smallest Salem series in $\mathbb{F}_q((X^{-1}))$

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ABSTRACT. The paper arose from the fact that the smallest element of the set of Salem numbers is not known. Indeed, it is not even known whether this set has a smallest element.

The aim of this paper is to prove that the minimal polynomial of the smallest Salem series of degree n in the field of formal power series over a finite field is given by $P(Y) = Y^n - XY^{n-1} - Y + X - 1$, where we suppose that 1 is the least element of the finite field \mathbb{F}_q^* (as a finite total ordered set). Consequently, we are led to deduce that $\mathbb{F}_q((X^{-1}))$ has no smallest Salem series. Moreover, we will prove that the root of $P(Y)$ of degree $n = 2^s + 1$ in $\mathbb{F}_{2^m}((X^{-1}))$ is well approximable.

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1. Introduction

A Pisot number (resp. Salem number) is a real algebraic integer $\alpha > 1$ all of whose other conjugates have modulus strictly less than 1 (resp. a real algebraic integer $\alpha > 1$, whose other conjugates have modulus at most 1, with at least one having modulus exactly 1).

The study of Pisot and Salem numbers unexpectedly or exceptionally has appeared in a number of quite different branches of mathematics. Much is known about Pisot numbers, and there are many known ways to construct them. However, little is known about Salem numbers and their difficult construction. There are still many open questions about Salem numbers, including determining the infimum of the set.

Pisot numbers have a long history, being studied as early as 1912 by Thue [13]. Salem first became interested in Pisot numbers because of their property that they can be used to generate almost integers. Indeed, higher

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powers of a Pisot number give better and better integer approximations. This property stems from the fact that for each n , the sum of n^{th} powers of an algebraic integer x and its conjugates is exactly an integer, following from an application of Newton's identities. When x is a Pisot number, the n^{th} powers of the other conjugates tend to 0 as n tends to infinity. Since the sum is an integer, the distance from x_n to the nearest integer tends to 0 at an exponential rate. For example, the root φ of $P(X) = X^2 - X - 1$, called the golden ratio ($\varphi = \frac{1+\sqrt{5}}{2}$) is a Pisot number, and $\varphi^{21} = 24476.0000409$ [10].

The set of all Pisot numbers is denoted S . Since Pisot numbers are algebraic, the set S is countable. Salem proved that this set is closed [10]. The closedness of S implies that it has a minimal element. Siegel [11] proved that this minimal element is the positive root of the equation:

$$X^3 - X - 1 = 0$$

called *the plastic constant*, which is approximately

$$1.324717957244746025960908854\dots$$

[12] and is isolated in S .

The set of Salem numbers is denoted T . It is intimately related to S . It has been proved in an older result of Salem [9] that S is contained in the set T' of the limit points of T .

The methods used in the study of Pisot numbers cannot be applied to Salem numbers, mainly, due to the fact that Salem numbers are reciprocal algebraic integers. The smallest known Salem number is $x_0 = 1.1762\dots$ of degree 10 known as Lehmer's number, its minimal polynomial is the Lehmer polynomial:

$$P(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1.$$

The smallest element of T is unknown [3]. In fact, it is not even known if the set T has a smallest element or is bounded from below by a number $1 + \delta$, where δ is a positive constant. Some algebraic relations between the conjugates of Pisot and Salem numbers have been studied in ([5],[8]).

In the field of formal power series over a finite field \mathbb{F}_q , the study of Pisot series is easier than in the real case, thanks to the theorem of Bateman and Duquette [1]. In [4], Chandoul, Jellali and Mkaouar have obtained the smallest Pisot series (SPS) of algebraic degree n in $\mathbb{F}_q((X^{-1}))$ denoted w_n . They have proved that its minimal polynomial is given by

$$P(Y) = Y^n - \alpha XY^{n-1} - \alpha^n,$$

where α is the least element of the set $\mathbb{F}_q \setminus \{0\}$. Then, they have shown that the sequence of SPS of degree n decreases and converges to αX . Finally, they have obtained the continued fraction expansion of w_n .

In the present paper, we plan to prove that there is no smallest Salem element in the field of Laurent series in characteristic p . We also give the

smallest Salem element of algebraic degree n and we prove that for $n = 2^s + 1$ the smallest Salem element is well approximable in $\mathbb{F}_{2^m}((X^{-1}))$.

The paper is organized as follows. In Section 2, we introduce the field $\mathbb{F}_q((X^{-1}))$ and we present some important results which characterize Pisot and Salem elements. In Section 3, we characterize for each $n \geq 3$ the smallest Salem element of algebraic degree n , which leads us to deduce that $\mathbb{F}_q((X^{-1}))$ does not have a smallest Salem element. In Section 4 we prove that for $n = 2^s + 1$ the smallest Salem element is well approximable in $\mathbb{F}_{2^m}((X^{-1}))$.

2. Field of power series

For p a prime and q a power of p , let \mathbb{F}_q be a field with q elements of characteristic p , $\mathbb{F}_q[X]$ the set of polynomials with coefficients in \mathbb{F}_q and $\mathbb{F}_q(X)$ its field of fractions. The set $\mathbb{F}_q((X^{-1}))$ of Laurent series over \mathbb{F}_q is defined as follows

$$\mathbb{F}_q((X^{-1})) = \left\{ f = \sum_{i \geq n_0} f_i X^{-i} : n_0 \in \mathbb{Z} \text{ and } f_i \in \mathbb{F}_q \right\}.$$

Let $f = \sum f_i X^{-i}$ be any formal power series, we denote its polynomial part by $[f]$ and by $\{f\}$ its fractional part. We remark that $f = [f] + \{f\}$. If $f \neq 0$, then the polynomial degree of f is $\gamma(f) = \sup\{-i : f_i \neq 0\}$, the degree of the highest-degree nonzero monomial in f , and $\gamma(0) = -\infty$. Note that if $[f] \neq 0$ then $\gamma(f)$ is the degree of the polynomial $[f]$. Thus, we define $|f| = q^{\gamma(f)}$. Note that $|\cdot|$ is a nonarchimedean absolute value over $\mathbb{F}_q((X^{-1}))$. It is clear that, for all $P \in \mathbb{F}_q[X]$, $|P| = q^{\deg P}$ and, for all $Q \in \mathbb{F}_q[X]$, such that $Q \neq 0$, $|\frac{P}{Q}| = q^{\deg P - \deg Q}$. We know that $\mathbb{F}_q((X^{-1}))$ is complete and locally compact with respect to the metric defined by this absolute value. We denote by $\overline{\mathbb{F}_q((X^{-1}))}$ an algebraic closure of $\mathbb{F}_q((X^{-1}))$. We note that the absolute value has a unique extension to $\overline{\mathbb{F}_q((X^{-1}))}$. Abusing the notation a little, we will use the same symbol $|\cdot|$ for the two absolute values.

Following [4], we introduce a lexicographic order \prec on $\mathbb{F}_q((X^{-1}))$. Let \mathbb{F}_q be a finite field equipped with a total order \prec , with $0 \prec 1$ and $1 \prec a$ for all $a \in \mathbb{F}_q \setminus \{0, 1\}$. Then the lexicographic order on $\mathbb{F}_q((X^{-1}))$ is defined as follows. Let

$$f = \sum_{i=n}^{+\infty} f_i X^{-i} \quad \text{and} \quad g = \sum_{i=m}^{+\infty} g_i X^{-i}.$$

If $\deg f < \deg g$ then $f \prec g$ and if we have $\deg f = \deg g$ then $f \prec g$ if $f_n \prec g_m$ or $f_n = g_m, \dots, f_{n+j} = g_{m+j}, f_{n+j+1} \prec g_{m+j+1}$ for some $j \in \mathbb{N}^*$.

Proposition 2.1 ([14]). *Let K be a complete field with respect to a non-archimedean absolute value $|\cdot|$ and L/K ($K \subset L$) be an algebraic extension of degree m . Then $|\cdot|$ has a unique extension to L defined by $|a| = \sqrt[m]{|N_{L/K}(a)|}$ and L is complete with respect to this extension.*

We apply Proposition 2.1 to algebraic extensions of $\mathbb{F}_q((X^{-1}))$. Since $\mathbb{F}_q[X] \subset \mathbb{F}_q((X^{-1}))$, every algebraic element over $\mathbb{F}_q[X]$ can be evaluated. However, since $\mathbb{F}_q((X^{-1}))$ is not algebraically closed, such an element is not necessarily expressed as a power series. For a full characterization of the algebraic closure of $\mathbb{F}_q[X]$, we refer to Kedlaya [7].

Definition 2.1. Let

$$(\dagger) \quad f(X, Y) = A_0 + A_1Y + \cdots + A_mY^m \in \mathbb{F}_q[X, Y], \quad A_i \in \mathbb{F}_q[X],$$

be irreducible of $\mathbb{F}_q[X, Y]$. To each monomial $A_iY^i \neq 0$, we assign the point $(i, \deg(A_i)) \in \mathbb{Z}^2$. For $A_i = 0$, we ignore the corresponding point $(i, -\infty)$. If we consider the upper convex hull of the set of points

$$\{(0, \deg(A_0)), \dots, (m, \deg(A_m))\},$$

we obtain the so-called *upper Newton polygon* of $f(X, Y)$ with respect to Y . The polygon is a sequence of line segments E_1, E_2, \dots, E_t with monotone decreasing slopes.

Proposition 2.2 ([14]). *Let $f(X, Y) \in \mathbb{F}_q[X, Y] \subset \mathbb{F}_q((Y^{-1})[X]$ be of the form (\dagger) . Since $\mathbb{F}_q((Y^{-1}))$ is complete with respect to $|\cdot|$, there is a unique extension of $|\cdot|$ to the splitting field L of $f(X, Y)$ over $K = \mathbb{F}_q((Y^{-1}))$.*

Let $1 \leq r < r + s \leq m$. We define E to be the line joining the points $(r, \deg(A_r))$ and $(r + s, \deg(A_{r+s}))$, which has slope

$$k = \frac{\deg(A_{r+s}) - \deg(A_r)}{s}.$$

Then $f(X, Y)$ as a polynomial in Y has s roots $\alpha_1, \dots, \alpha_s$ with $|\alpha_1| = \cdots = |\alpha_s| = q^{-k}$.

Corollary 2.3. *There are no roots in $\mathbb{F}_q((X^{-1}))$ with absolute value > 1 of the polynomial*

$$H(Y) = A_nY^n + A_{n-1}Y^{n-1} + \cdots + A_0,$$

where $|A_n| = \sup_{0 \leq i \leq n} |A_i|$.

Corollary 2.4. *Let $P(Y) = A_nY^n + A_{n-1}Y^{n-1} + \cdots + A_0$ with $A_i \in \mathbb{F}_q[X]$, $A_n = 1$, $A_0 \neq 0$ and $|A_{n-1}| > |A_i|$, for all $i \neq n - 1$. Then, P has only one root $f \in \mathbb{F}_q((X^{-1}))$ satisfying $|f| > 1$. Moreover, $[f] = -[\frac{A_{n-1}}{A_n}]$.*

Proof. The first part follow easily from Proposition 2.2. For the second part, we use the fact that $(\frac{-A_{n-1}}{A_n})$ is the sum of the roots of the polynomial P .

A Salem element $f \in \mathbb{F}_q((X^{-1}))$ is an algebraic integer over $\mathbb{F}_q[X]$ such that $|f| > 1$, whose remaining conjugates in $\overline{\mathbb{F}_q(X)}$ have an absolute value no greater than 1, and at least one has absolute value exactly 1. In 1962, Bateman and Duquette [1] introduced and characterized Pisot and Salem elements in the field of Laurent series.

Theorem 2.2. *An element f in $\mathbb{F}_q((X^{-1}))$ is a Pisot element if and only if its minimal polynomial can be written as $P(Y) = Y^s + A_{s-1}Y^{s-1} + \dots + A_0$, $A_i \in \mathbb{F}_q[X]$ for $i = 0, \dots, s - 1$ with $|A_{s-1}| > |A_i|$ for $i = 0, \dots, s - 2$.*

Theorem 2.3. *An element f in $\mathbb{F}_q((X^{-1}))$ is a Salem element if and only if its minimal polynomial can be written as $P(Y) = Y^s + A_{s-1}Y^{s-1} + \dots + A_0$, $A_i \in \mathbb{F}_q[X]$ for $i = 0, \dots, s - 1$ with $|A_{s-1}| = \sup_{i \neq s-1} |A_i|$.*

3. Smallest Salem element in $\mathbb{F}_q((X^{-1}))$

In the following results and without loss of generality, we suppose that 1 is the least element of \mathbb{F}_q^* and by the algebraic degree of an algebraic element of $\mathbb{F}_q((X^{-1}))$ we mean the degree of its minimal polynomial over $\mathbb{F}_q(X)$. Our main result is the following theorem.

Theorem 3.1. *Let $n \geq 3$. Then the equation*

$$(1) \quad Y^n - XY^{n-1} - Y + X - 1 = 0$$

has a unique root w_n in $\mathbb{F}_q((X^{-1}))$ of degree ≥ 1 . Moreover, w_n is the smallest Salem series of algebraic degree n in $\mathbb{F}_q((X^{-1}))$.

The following lemma is a direct consequence of Lemma 3 in [6].

Lemma 3.2. *Let f and $g \in \mathbb{F}_q((X^{-1}))$ be two Salem elements of algebraic degree n and d such that $f \neq g$. Then*

$$|f - g| \geq \frac{1}{H(f)^d |g|^{n-2}}$$

with $H(f)$ the height of f defined by $H(f) = \sup_{1 \leq i \leq n} |A_i|$, where

$$P(Y) = A_n Y^n + A_{n-1} Y^{n-1} + \dots + A_0 \in \mathbb{F}_q[X, Y]$$

is the minimal polynomial of f .

Lemma 3.3. *Let w be a Salem series in $\mathbb{F}_q((X^{-1}))$ of algebraic degree $n \geq 3$ such that $[w] = X$. Then we have*

$$w - X \succ \frac{1}{X^{n-1}} \quad \text{and} \quad w - X - \frac{1}{X^{n-1}} \succ \frac{1}{X^{2(n-1)}}.$$

Proof. The minimal polynomial of w is necessarily of the form

$$P_0(Y) = Y^n + (\alpha - X)Y^{n-1} + A_{n-2}Y^{n-2} + \dots + A_1Y + A_0 \in \mathbb{F}_q[X, Y]$$

with $\alpha \in \mathbb{F}_q$, $A_n = 1$, $A_{n-1} = \alpha - X$, $A_0 \neq 0$ and

$$(*) \quad \sup(|A_{n-2}|, |A_{n-3}|, \dots, |A_0|) = |X| > 1.$$

Since $[w] = X$ so $w = X + \frac{1}{g}$ with $|g| > 1$. By replacing w by $(X + \frac{1}{g})$, we get

$$\begin{aligned} \left(X + \frac{1}{g}\right)^n + (\alpha - X) \left(X + \frac{1}{g}\right)^{n-1} + A_{n-2} \left(X + \frac{1}{g}\right)^{n-2} \\ + \cdots + A_1 \left(X + \frac{1}{g}\right) + A_0 = 0. \end{aligned}$$

Thus, it follows directly from this expression that we can prove that g satisfies the following equation:

$$(*) \quad g^n B_n + g^{n-1} B_{n-1} + \cdots + g^2 B_2 + g B_1 + B_0 = 0,$$

where

$$B_k = \sum_{j=n-k}^n \binom{j}{n-k} A_j X^{j+k-n} = \sum_{j=0}^k \binom{j+n-k}{n-k} A_{j+n-k} X^j.$$

If $\alpha \neq 0$, then both B_n and B_{n-1} have degree $n-1$. Since $\deg B_k \leq k+1$ from $(*)$ we have $|B_n| = \sup_{0 \leq i \leq n-1} |B_i|$. Hence by Lemma 2.3 there is no root of $(*)$ with absolute value > 1 . So necessarily $\alpha = 0$. In this case

$$(**) \quad \begin{cases} B_n = A_{n-2} X^{n-2} + A_{n-3} X^{n-3} + \cdots + A_1 X + A_0, \\ B_{n-1} = X^{n-1} + A_{n-2} (n-2) X^{n-3} + \cdots + A_1 = X^{n-1} + H_{n-2}. \end{cases}$$

Remark that B_k has degree at most $k+1$, as the A_i are of degree at most 1. Also that B_{n-2} has degree at most $n-2$, using $A_n = 1$. These two facts give that $\deg(B_{n-1}) > \deg(B_i)$ for $i \neq n-1$. By Lemma 2.4, there is a unique root of $(*)$ denoted g with $|g| > 1$ and $[g] = -[\frac{B_{n-1}}{B_n}]$ so $\deg(g) \leq n-1$, and then $w - X \succ \frac{1}{X^{n-1}}$.

Now let us prove that $w - X - \frac{1}{X^{n-1}} \succ \frac{1}{X^{2(n-1)}}$. We separate two cases:

Case 1. $\deg(g) = n-1$. From the Newton polygon of $(*)$ $\deg(B_{n-1}) = n-1$ and $\deg(B_n) = 0$. To prove that $w - X - \frac{1}{X^{n-1}} \succ \frac{1}{X^{2(n-1)}}$, it suffices to prove that $[g] \neq X^{n-1}$.

We suppose that $[g] = X^{n-1}$. Then $B_{n-1} = X^{n-1}$ and $B_n = 1$ which gives $H_{n-2} = 0$ and $B'_n = \frac{dB_n}{dX} = 0$. However,

$$B'_n = H_{n-2} + A'_{n-2} X^{n-2} + \cdots + A'_1 X + A'_0,$$

where $(A_i)' \in \mathbb{F}_q$ for all $0 \leq i \leq n-2$, then, as $\deg A_i = 1$, $A'_i = 0$ for all $0 \leq i \leq n-2$. Hence $|A_i| \leq 1$ for $0 \leq i \leq n-2$, contradicting $(*)$.

Case 2. $\deg(g) < n-1$. Here $w - X - \frac{1}{X^{n-1}} \succ \frac{1}{X^{n-1}} \succ \frac{1}{X^{2(n-1)}}$. \square

Proof of Theorem 3.1. By Proposition 2.2, the equation

$$Y^n - XY^{n-1} - Y + X - 1 = 0$$

has a unique root w_n in $\mathbb{F}_q((X^{-1}))$ with $|w_n| > 1$ and the other roots of absolute value equal to 1. So w_n is a Salem series. Moreover, $w_n = X + \frac{1}{g}$ where $[g] = X^{n-1} - 1$ (by (**)), which implies that

$$(2) \quad w_n = X + \frac{1}{X^{n-1}} + \frac{1}{X^{2(n-1)}} + \cdots$$

We suppose that the algebraic degree of w_n is $s < n$. Then by Lemma 3.3 we have $w_n - X \succ \frac{1}{X^{s-1}}$, which is absurd because $w_n - X = \frac{1}{X^{n-1}} + \cdots \prec \frac{1}{X^{s-1}}$. Now we suppose that w_n is not the smallest Salem element of degree n . Then there exists another Salem element w'_n of algebraic degree n such that $w'_n \prec w_n$. By combining (2) and Lemma 3.3 we obtain that

$$w'_n = X + \frac{1}{X^{n-1}} + \frac{1}{X^{2(n-1)}} + \cdots,$$

so that $|w_n - w'_n| < \frac{1}{|X|^{2(n-1)}}$, which contradicts Lemma 3.2. \square

Corollary 3.4. *There is no smallest Salem series in $\mathbb{F}_q((X^{-1}))$.*

Proof. Let w_n be the unique root in $\mathbb{F}_q((X^{-1}))$ of (1). So $w_{n+1} \prec w_n$ and

$$\lim_{n \rightarrow +\infty} \{w_n\} = 0. \quad \square$$

4. Diophantine approximation of w_n in $\mathbb{F}_{2^m}((X^{-1}))$

In this section we will prove that the smallest Salem element of degree n in $\mathbb{F}_{2^m}((X^{-1}))$ is well approximable for infinitely $n \geq 2$. In order to measure the quality of rational approximation, we introduce the following notation and definition.

Let us consider the set $M_q = \{f \in \mathbb{F}_q((X^{-1})) : |f| < 1\}$ and consider the transformation

$$T : M_q \longrightarrow M_q, f \longmapsto \left\{ \frac{1}{f} \right\}.$$

For any $f \in \mathbb{F}_q((X^{-1}))$ we define a polynomial sequence $(a_n)_{n \geq 0}$ by

$$a_0 = [f] \text{ and, for } n \geq 1, a_n = \left[\frac{1}{T_{n-1}(f - [f])} \right].$$

We easily check that

$$f = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}$$

This expression is called the *continued fraction expansion* of f and it will be simply denoted by $f = [a_0, a_1, a_2, \dots]$. The sequence $(a_i)_{i \geq 0}$ is called the *sequence of partial quotients* of f .

We study the approximation of the elements of $\mathbb{F}_q((X^{-1}))$ by the elements of $\mathbb{F}_q(X)$. In particular, we consider this approximation for the elements of $\mathbb{F}_q((X^{-1}))$ that are algebraic over $\mathbb{F}_q(X)$. Let $f = [a_0, a_1, a_2, \dots]$ be an irrational element of $\mathbb{F}_q((X^{-1}))$, so that infinitely many a_i are nonzero. For all real numbers μ , we define

$$B(f, \mu) = \lim_{|Q| \rightarrow \infty} \inf |Q|^\mu |Qf - P|,$$

where P and Q run over polynomials in $\mathbb{F}_q[X]$ with $Q \neq 0$. The approximation exponent of f is defined by

$$\nu(f) = \sup\{\mu \in \mathbb{R} : B(f, \mu) < \infty\}.$$

A simple calculation proves that

$$\nu(f) = 1 + \limsup \left(\frac{\deg a_{k+1}}{\sum_{1 \leq i \leq k} \deg a_i} \right).$$

Using this quantity we can define the following classification. If $f \in \mathbb{F}_q((X^{-1}))$, we say that:

- f is *badly approximable* if $\nu(f) = 1$ and $B(f, 1) > 0$. This is equivalent to saying that the partial quotients in the continued fraction expansion for a are bounded.
- f is *normally approximable* if $\nu(f) = 1$ and $B(f, 1) = 0$.
- f is *well approximable* if $\nu(f) > 1$.

We now restrict attention to the case $q = 2^m$. Our main result in this section is given by the following theorem.

Theorem 4.1. *Let $n = 2^s + 1$ with $s \geq 1$. Then the smallest Salem element of degree n in $\mathbb{F}_{2^m}((X^{-1}))$ denoted w_s is well approximable.*

Before giving the proof of this theorem, we recall the following lemma.

Lemma 4.2 ([2]). *Let $f = [a_0, a_1, a_2, a_3, \dots]$ in $\mathbb{F}_{2^m}((X^{-1}))$. Then:*

- (1) $f^{-1} = [0, a_0, a_1, a_2, \dots]$, if $a_0 \neq 0, 1$.
- (2) $f^{2^n} = [a_0^{2^n}, a_1^{2^n}, a_2^{2^n}, \dots]$, for $n \geq 1$.

Proof of Theorem 4.1. For $n = 2^s + 1$ and $Y = w_s$, Equation (1) becomes in characteristic 2:

$$w_s^{2^n+1} + Xw_s^{2^n} + w_s + X + 1 = 0.$$

Then $(w_s^{2^n} + 1)(w_s + X) = 1$. We have $w_s \neq X$, then $w_s^{2^n} + 1 = \frac{1}{w_s + X}$.

Let $w_s = [a_0, a_1, a_2, \dots, a_n, \dots]$. By Lemma 4.2, $w_s^{2^n} = [a_0^{2^n}, a_1^{2^n}, a_2^{2^n}, \dots]$ and $\frac{1}{w_s^{2^n} + 1} = [0, a_0^{2^n} + 1, a_1^{2^n}, a_2^{2^n}, \dots]$. Then

$$\begin{aligned} w_s + X &= [a_0 + X, a_1, a_2, \dots] \\ &= [0, a_0^{2^n} + 1, a_1^{2^n}, a_2^{2^n}, \dots]. \end{aligned}$$

This implies that $a_0 = X$, $a_1 = a_0^{2^n} + 1 = X^{2^n} + 1$ and, for $i \geq 2$,

$$a_i = a_{i-1}^{2^n} = X^{2^{in}} + 1,$$

by an easy induction. So $\deg(a_i) = 2^{in}$

$$\begin{aligned} \nu(w_s) &= 1 + \limsup_{k \rightarrow \infty} \left(\frac{\deg a_{k+1}}{\sum_{1 \leq i \leq k} \deg a_i} \right) \\ &= 1 + \limsup_{k \rightarrow \infty} \left(\frac{2^{n(k+1)}}{2^n + 2^{2n} + \dots + 2^{kn}} \right) \\ &= 2^n > 1. \end{aligned} \quad \square$$

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References

- [1] BATEMAN, PAUL T.; DUQUETTE, ALFRED L. The analogue of the Pisot–Vijayaraghavan numbers in fields of formal power series. *Illinois J. Math.* **6** (1962), 594–606. MR0144883 (26 #2424), Zbl 0105.02801.
- [2] BAUM, LEONARD E.; SWEET, MELVIN M. Continued fractions of algebraic power series in characteristic 2. *Ann. of Math.* (2) **103** (1976), no. 3, 593–610. MR0409372 (53 #13127), Zbl 0312.10024.
- [3] BOYD, DAVID W. Small Salem numbers. *Duke Math. J.* **44** (1977), no. 2, 315–328. MR0453692 (56 #11952), Zbl 0353.12003, doi: 10.1215/S0012-7094-77-04413-1.
- [4] CHANDOUL, A.; JELLALI, M.; MKAOUAR, M. The smallest Pisot element in the field of formal power series over a finite field. *Canad. Math. Bull.* **56** (2013), no. 2, 258–264. MR3043053, Zbl 06166916, doi: 10.4153/CMB-2011-168-5.
- [5] DUBICKAS, ARTŪRAS; SMYTH, CHRIS. On the lines passing through two conjugates of a Salem number. *Math. Proc. Cambridge Phil. Soc.* **144** (2008), no. 1, 29–37. MR2388230 (2009f:11132), Zbl 1166.11031, doi: 10.1017/S0305004107000692.
- [6] HBAIB, M.; MKAOUAR, M.; TOUNSI, K. Un critère de transcendance de fractions continues dans $\mathbb{K}_p((X^{-1}))$. *J. Number Theory* **116** (2006), no. 1, 140–149. MR2197863 (2006i:11078), Zbl 1162.11035, doi: 10.1016/j.jnt.2005.03.008.
- [7] KEDLAYA, KIRAN S. The algebraic closure of the power series field in positive characteristic. *Proc. Amer. Math. Soc.* **129** (2001), no. 12, 3461–3470. MR1860477 (2003a:13025), Zbl 1012.12007, arXiv:math/9810142, doi: 10.1090/S0002-9939-01-06001-4.
- [8] MIGNOTTE, MAURICE. Sur les conjugués des nombres de Pisot. *C. R. Acad. Sci. Paris Sér. I Math* **298** (1984), no. 2, 21. MR0740081 (85b:11078), Zbl 0562.12001.

- [9] SALEM, R. Power series with integral coefficients. *Duke Math. J.* **12** (1945), 153–172. MR11720 (6,206b), Zbl 0060.21601.
- [10] SALEM, RAPHAËL. Algebraic numbers and Fourier analysis. *D. C. Heath and Co., Boston, Mass.* 1963. x+68 pp. MR0157941 (28 #1169), Zbl 0126.07802.
- [11] SIEGEL, CARL LUDWIG. Algebraic integers whose conjugates lie in the unit circle. *Duke Math. J.* **11** (1944), 597–602. MR0010579 (6,39b), Zbl 0063.07005, doi:10.1215/S0012-7094-44-01152-X.
- [12] SMYTH, C. J. On the product of the conjugates outside the unit circle of an algebraic number. *Bull. London Math. Soc.* **3** (1971), 169–175. MR0289451 (44 #6641), Zbl 0235.12003, doi:10.1112/blms/3.2.169.
- [13] THUE, AXEL. Über eine Eigenschaft, die keine transcendente Größe haben kann. Skrifter utg. av Videnskapsselskapet i Kristiania., 1. Mat.-naturv. Kl., vol. 20. *Kristiania: Dybwad in Komm.*, 1912. JFM 44.0480.04.
- [14] WEISS, EDWIN. Algebraic number theory. Reprint of the 1963 original. *Dover Publications, Inc., Mineola, NY*, 1998. x+275 pp. ISBN: 0-486-40189-8. MR1635455 (99c:11129), Zbl 0926.11077.

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