

Normality preserving operations for Cantor series expansions and associated fractals. II

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ABSTRACT. We investigate how nonzero rational multiplication and rational addition affect normality with respect to Q -Cantor series expansions. In particular, we show that there exists a Q such that the set of real numbers which are Q -normal but not Q -distribution normal, and which still have this property when multiplied and added by rational numbers has full Hausdorff dimension. Moreover, we give such a number that is explicit in the sense that it is computable.

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1. Introduction

We say that a real number x is *normal in base b* if for every block B consisting of k base b digits, we have that

$$(1) \quad \lim_{n \rightarrow \infty} \frac{N_n(B, x)}{n} = \frac{1}{b^k},$$

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where $N_n(B, x)$ denotes the number of times the block B occurs in the base b expansion of x within the first n digits after the decimal point. A real number x is *simply normal in base b* if (1) holds for blocks B of length 1. In essence, this says that a number x is normal if each block of digits appears with the expected frequency if the digits of x were chosen at random. We let $\mathcal{N}(b)$ denote the set of points normal in base b . While it is known that $\mathcal{N}(b)$ has full measure, it is not known if $\sqrt{2}$, π , or e or any other commonly used mathematical constant is normal in any base. We have some explicit examples of normal numbers, but generally they are numbers constructed to be normal, such as Champernowne's constant [10], which is formed by concatenating the positive integers in succession. (In base 10, this is given by 0.12345678910111213 . . .)

Given the difficulty of proving that a given number, such as π , is normal, a lot of research has focused on understanding properties of the set $\mathcal{N}(b)$ as a whole. Let f be a function from \mathbb{R} to \mathbb{R} . We say that f *preserves b -normality* if $f(\mathcal{N}(b)) \subseteq \mathcal{N}(b)$. We can make a similar definition for both normality and functions that preserve normality for many other systems, such as continued fraction expansions, β -expansions, the Lüroth series expansion, etc. (See, for example, [19].)

Some b -normality preserving functions naturally arise in H. Furstenberg's work on disjointness in ergodic theory [15]. These functions, known as selection rules, form numbers by taking subsequences of the base b digits of another number. V. N. Agafonov [1], T. Kamae [17], T. Kamae and B. Weiss [18], and W. Merkle and J. Reimann [23] studied b -normality preserving selection rules.

In this paper, we will be more interested in the simple functions of multiplication and addition. For a real number r , define real functions π_r and σ_r by $\pi_r(x) = rx$ and $\sigma_r(x) = r + x$. In 1949, D. D. Wall proved in his Ph.D. thesis [33] that for nonzero rational r the functions π_r and σ_r are b -normality preserving for all b . These results were also independently proven by K. T. Chang in 1976 [11]. D. D. Wall's method relies on the well known characterization that a real number x is normal in base b if and only if the sequence $(b^n x)$ is uniformly distributed mod 1, a result that he also proved in his Ph.D. thesis.

D. Doty, J. H. Lutz, and S. Nandakumar took a substantially different approach from D. D. Wall and strengthened his result. They proved in [12] that for every real number x and every nonzero rational number r the b -ary expansions of x , $\pi_r(x)$, and $\sigma_r(x)$ all have the same finite-state dimension and the same finite-state strong dimension. It follows that π_r and σ_r preserve b -normality. It should be noted that their proof uses different methods from those used by D. D. Wall and is unlikely to be proven using similar machinery.

C. Aistleitner generalized D. D. Wall's result on σ_r . Suppose that q is a rational number and that the digits of the b -ary expansion of z are nonzero

on a set of indices of density zero. In [4] he proved that the function σ_{qz} is b -normality preserving. It was shown in [2] that C. Aistleitner’s result does not generalize to at least one notion of normality for some of the Cantor series expansions, which we will be investigating in this paper.

There are still many open questions relating to the functions π_r and σ_r . For example, M. Mendés France asked in [22] if the function π_r preserves simple normality with respect to the regular continued fraction for every nonzero rational r . The third author proved in [32] that for any nonzero rational r , both π_r and σ_r (and indeed any nontrivial integer fractional linear transformation) preserve normality with respect to the regular continued fraction expansion, which still leaves Mendés France’s question unanswered. The authors are unaware of any theorems that state that either π_r or σ_r preserve any other form of normality.

In this paper we will be interested in the function $\tau_{r,s} = \sigma_s \circ \pi_r$ for $r \in \mathbb{Q} \setminus \{0\}$ and $s \in \mathbb{Q}$, and how this function preserves certain notions of normality of Q -Cantor series expansions, namely Q -normality and Q -distribution normality. (We will provide definitions for all these terms in Section 2.) Unlike in the other systems mentioned above, normality and distribution normality for Q -Cantor series expansions need not be equivalent. In [5], it was shown that the set of numbers that are Q -normal but not Q -distribution normal is nonempty for some basic sequences Q , but no indication was given to the size of this set.

In Theorem 2.4, we show a much stronger result: there exists a basic sequence Q and a real number x such that $\tau_{r,s}(x)$ is always Q -normal and always *not* Q -distribution normal for every $r \in \mathbb{Q} \setminus \{0\}$ and $s \in \mathbb{Q}$; in fact, the set of x with this property is big in the sense that it has full Hausdorff dimension. In other words, the set of x which not only have these peculiar normality properties, but preserve these properties under rational addition and multiplication, is a reasonably large set. Related questions for the Cantor series expansions are studied in [2].

Another question that has come into greater interest in the study of normal numbers lately is the question of how explicit a normal number construction is: it is one thing to exhibit a number and another to exhibit a number in a simple way. So we bring in definitions from recursion theory. A real number x is *computable* if there exists $b \in \mathbb{N}$ with $b \geq 2$ and a Turing machine f which given n outputs the n th digit of x in base b . A sequence of real numbers (x_n) is *computable* if there exists a Turing machine f such that on input m, n the output $f(m, n)$ satisfies $\frac{f(m,n)-1}{m} < x_n < \frac{f(m,n)+1}{m}$.

M. W. Sierpiński gave an example of an absolutely normal number¹ that is not computable in [28]. The authors feel that examples such as M. W. Sierpiński’s are not fully explicit since they are not computable real numbers, unlike Champernowne’s number. A. M. Turing gave the first example of a

¹A number is said to be absolutely normal if it is normal to every base $b \geq 2$.

computable absolutely normal number in an unpublished manuscript. This paper may be found in his collected works [30]. See [6] by V. Becher, S. Ferreira, and R. Picchi for further discussion. In Theorem 2.5 we give a basic sequence Q and real number x , with x in the set discussed in Theorem 2.4, that are fully explicit in the sense that they are computable as a sequence of integers and a real number, respectively.

Throughout this paper we will use a number of standard asymptotic notations. By $f(x) = O(g(x))$ we mean that there exists some real number $C > 0$ such that $|f(x)| \leq C|g(x)|$. By $f(x) \asymp g(x)$, we mean $f(x) = O(g(x))$ and $g(x) = O(f(x))$. By $f(x) = o(g(x))$, we mean that $f(x)/g(x) \rightarrow 0$ as $x \rightarrow \infty$.

Acknowledgements. We would like to thank Samuel Roth for posing the problem that led to Theorem 2.4 and Theorem 2.5 to the second author at the 2012 RTG conference: Logic, Dynamics and Their Interactions, with a Celebration of the Work of Dan Mauldin in Denton, Texas. He asked if it is true that $nx \in \mathcal{N}(Q)$ for all natural numbers n implies that $x \in \mathcal{DN}(Q)$.

2. Cantor series expansions

The study of normal numbers and other statistical properties of real numbers with respect to large classes of Cantor series expansions was first done by P. Erdős and A. Rényi in [13] and [14] and by A. Rényi in [24], [25], and [26] and by P. Turán in [29].

The Q -Cantor series expansions, first studied by G. Cantor in [9], are a natural generalization of the b -ary expansions.² Let $\mathbb{N}_k := \mathbb{Z} \cap [k, \infty)$. If $Q \in \mathbb{N}_2^{\mathbb{N}}$, then we say that Q is a *basic sequence*. Given a basic sequence $Q = (q_n)_{n=1}^{\infty}$, the Q -Cantor series expansion of a real number x is the (unique)³ expansion of the form

$$(2) \quad x = E_0 + \sum_{j=1}^{\infty} \frac{E_j}{q_1 q_2 \cdots q_j}$$

where $E_0 = \lfloor x \rfloor$ and E_j is in $\{0, 1, \dots, q_j - 1\}$ for $n \geq 1$ with $E_j \neq q_j - 1$ infinitely often. We abbreviate (2) with the notation $x = E_0.E_1E_2E_3 \cdots$ w.r.t. Q .

A *block* is an ordered tuple of nonnegative integers, a *block of length k* is an ordered k -tuple of nonnegative integers, and *block of length k in base b* is an ordered k -tuple of nonnegative integers in $\{0, 1, \dots, b - 1\}$.

²G. Cantor's motivation to study the Cantor series expansions was to extend the well known proof of the irrationality of the number $e = \sum 1/n!$ to a larger class of numbers. Results along these lines may be found in the monograph of J. Galambos [16].

³Uniqueness can be proven in the same way as for the b -ary expansions.

Let

$$Q_n^{(k)} := \sum_{j=1}^n \frac{1}{q_j q_{j+1} \cdots q_{j+k-1}} \text{ and } T_{Q,n}(x) := \left(\prod_{j=1}^n q_j \right) x \pmod{1}.$$

A. Rényi [25] defined a real number x to be *normal* with respect to Q if for all blocks B of length 1,

$$(3) \quad \lim_{n \rightarrow \infty} \frac{N_n^Q(B, x)}{Q_n^{(1)}} = 1,$$

where $N_n^Q(B, x)$ is the number of occurrences of the block B in the sequence $(E_i)_{i=1}^n$ of the first n digits in the Q -Cantor series expansion of x . Note that if $q_n = b$ for all n and we restrict B to consist of only digits less than b , then (3) is equivalent to *simple normality in base b* , but not equivalent to *normality in base b* . A basic sequence Q is *k -divergent* if $\lim_{n \rightarrow \infty} Q_n^{(k)} = \infty$ and *fully divergent* if Q is k -divergent for all k . A basic sequence Q is *infinite in limit* if $q_n \rightarrow \infty$.

Definition 2.1. A real number x is *Q -normal of order k* if for all blocks B of length k ,

$$\lim_{n \rightarrow \infty} \frac{N_n^Q(B, x)}{Q_n^{(k)}} = 1.$$

We let $\mathcal{N}_k(Q)$ be the set of numbers that are Q -normal of order k . The real number x is *Q -normal* if $x \in \mathcal{N}(Q) := \bigcap_{k=1}^{\infty} \mathcal{N}_k(Q)$. A real number x is *Q -distribution normal* if the sequence $(T_{Q,n}(x))_{n=0}^{\infty}$ is uniformly distributed mod 1. Let $\mathcal{DN}(Q)$ be the set of Q -distribution normal numbers.

It follows from a well known result of H. Weyl [34, 35] that $\mathcal{DN}(Q)$ is a set of full Lebesgue measure for every basic sequence Q . We will need the following results of the second author [20] later in this paper.

Theorem 2.2. ⁴ *Suppose that Q is infinite in limit. Then $\mathcal{N}_k(Q)$ (resp. $\mathcal{N}(Q)$) is of full measure if and only if Q is k -divergent (resp. fully divergent).*

We note the following simple theorem.

Theorem 2.3. *Suppose that Q is infinite in limit. Then $x = E_0.E_1E_2\dots$ is Q -distribution normal if and only if the sequence $(E_n/q_n)_{n=1}^{\infty}$ is uniformly distributed modulo 1.*

Note that in base b , where $q_n = b$ for all n , the corresponding notions of Q -normality and Q -distribution normality are equivalent. This equivalence is fundamental in the study of normality in base b .

⁴Early work in this direction has been done by A. Rényi [25], T. Šalát [31], and F. Schweiger [27].

Another definition of normality, Q -ratio normality, has also been studied. We do not introduce this notion here as this set contains the set of Q -normal numbers and all results in this paper that hold for Q -normal numbers also hold for Q -ratio normal numbers. The complete containment relation between the sets of these normal numbers and pair-wise intersections thereof is proven in [21]. The Hausdorff dimensions of difference sets such as $\mathcal{RN}(Q) \cap \mathcal{DN}(Q) \setminus \mathcal{N}(Q)$ are computed in [3]. Let $\Xi(Q)$ be the set of all $x \in [0, 1)$ such that $\tau_{r,s}(x)$ is Q -normal but not Q -distribution normal for all $r \in \mathbb{Q} \setminus \{0\}$ and $s \in \mathbb{Q}$, i.e.,

$$\Xi(Q) = \{x \in [0, 1) : \tau_{r,s}(x) \in \mathcal{N}(Q) \setminus \mathcal{DN}(Q) \ \forall r \in \mathbb{Q} \setminus \{0\}, s \in \mathbb{Q}\}.$$

Our main results of this paper will be the following:

Theorem 2.4. *There exists a basic sequence Q such that the Hausdorff dimension of $\Xi(Q)$ is 1.*

Theorem 2.5. *There exists a computable basic sequence Q such that $\Xi(Q)$ contains a computable real number.*

3. The digits of $\tau_{r,s}(x)$

In order to prove the main results of this paper, we will want to understand how the digits of $\tau_{r,s}(x)$ differ from the digits of x , when x takes a specific form. We begin with some lemmas based on elementary calculations.

Lemma 3.1. *If $x = p/q$ is a rational number with $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and $q \mid q_1 q_2 \dots q_N$ for some $N \in \mathbb{N}$, then x has a finite Q -Cantor series expansion of the form*

$$x = E_0 + \sum_{j=1}^N \frac{E_j}{q_1 q_2 \dots q_j}.$$

Alternately if x is a real number in the interval $[0, 1/q_1 q_2 \dots q_N)$, then x has a Q -Cantor series expansion of the following form,

$$x = \sum_{j=N+1}^{\infty} \frac{E_j}{q_1 q_2 \dots q_j}$$

so that $E_j = 0$ for $n \leq N$.

This allows us to prove a number of additional lemmas rather trivially.

Lemma 3.2. *Suppose that $x = E_0.E_1 E_2 \dots$ w.r.t. Q . If $s = p/q$ is rational with $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and $q \mid q_1 q_2 \dots q_N$ for some $N \in \mathbb{N}$, then $\sigma_s(x)$ has a Q -Cantor series expansion of the form*

$$\sigma_s(x) = E'_0 + \sum_{j=1}^N \frac{E'_j}{q_1 q_2 \dots q_j} + \sum_{j=N+1}^{\infty} \frac{E_j}{q_1 q_2 \dots q_j}$$

so that $\sigma_s(x)$ and x differ only in their first $N+1$ digits, including the zeroth digit.

Corollary 3.3. *Suppose that Q has the property that for every integer n there exists an integer m such that $n|q_m$. Then for any rational number s , the Q -Cantor series expansion of x and of $\sigma_s(x)$ differ on at most finitely many digits.*

Lemma 3.4. *Suppose that x has a finite Q -Cantor series expansion of the form*

$$x = \sum_{j=N+1}^M \frac{E_j}{q_1 q_2 \cdots q_j}.$$

We write

$$\begin{aligned} E &= E_{N+1}q_{N+2}q_{N+3} \cdots q_M \\ &\quad + E_{N+2}q_{N+3}q_{N+4} \cdots q_M + \cdots + E_{M-1}q_M + E_M \\ q &= q_{N+1}q_{N+2} \cdots q_M \end{aligned}$$

so that

$$x = \frac{E}{q_1 q_2 \cdots q_N q}.$$

Suppose r is a nonzero rational number. If rE is an integer and $rE < q$, then $\pi_r(x)$ has a finite Q -Cantor series expansion of the form

$$\pi_r(x) = \sum_{j=N+1}^M \frac{E'_j}{q_1 q_2 \cdots q_j}.$$

4. Results on Hausdorff dimension

Given basic sequences $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$, sequences of nonnegative integers $s = (s_i), t = (t_i), v = (v_i)$, and $F = (F_i)$, and a sequence of sets $I = (I_i)$ such that $I_i \subseteq \{0, 1, \dots, \beta_i - 1\}$, define the set $\Theta(\alpha, \beta, s, t, v, F, I)$ as follows. Let $Q = Q(\alpha, \beta, s, t, v) = (q_n)$ be the following basic sequence:

$$(4) \quad [[\alpha_1]^{s_1} [\beta_1]^{t_1}]^{v_1} [[\alpha_2]^{s_2} [\beta_2]^{t_2}]^{v_2} [[\alpha_3]^{s_3} [\beta_3]^{t_3}]^{v_3} \cdots .$$

Define the function

$$i(n) = \max \left\{ m : \sum_{j=1}^{m-1} v_j(s_j + t_j) < n \right\}.$$

Set

$$\Phi_\alpha(i, c, d) = \sum_{j=1}^{i-1} v_j s_j + c s_i + d$$

where $0 \leq c < v_i$ and $0 \leq d < s_i$ and let the functions $i_\alpha(n)$, $c_\alpha(n)$, and $d_\alpha(n)$ be such that $\Phi_\alpha^{-1}(n) = (i_\alpha(n), c_\alpha(n), d_\alpha(n))$. Note this is possible

since Φ_α is a bijection from $\mathcal{U} = \{(i, c, d) \in \mathbb{N}^3 : 0 \leq c < v_i, 0 \leq d < s_i\}$ to \mathbb{N} . Define the function

$$G(n) = \sum_{j=1}^{i_\alpha(n)-1} v_j(s_j + t_j) + c_\alpha(n) (s_{i_\alpha(n)} + t_{i_\alpha(n)}) + d_\alpha(n).$$

We consider the condition on n

$$(5) \quad \left[\left(n - \sum_{j=1}^{i(n)-1} v_j(s_j + t_j) \right) \bmod (s_{i(n)} + t_{i(n)}) \right] \geq s_{i(n)}.$$

Define the sets

$$V(n) = \begin{cases} I_{i(n)} & \text{if condition (5) holds} \\ \{F_{G(n)}\} & \text{otherwise} \end{cases}.$$

That is, we choose digits from $I_{i(n)}$ in positions corresponding to the bases obtained from the sequence β and choose a specific digit from F for the bases obtained from the sequence α . Set

$$\Theta(\alpha, \beta, s, t, v, F, I) = \{x = 0.E_1E_2 \dots \text{ w.r.t. } Q : E_n \in V(n)\}.$$

We will need the following lemma from [3].

Lemma 4.1. *Suppose that basic sequences $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$, sequences of nonzero integers $s = (s_i), t = (t_i), v = (v_i)$, and $F = (F_i)$, and a sequence of sets $I = (I_n)$ such that $I_n \subseteq \{0, 1, \dots, \beta_n - 1\}$ are given where $\lim_{n \rightarrow \infty} |I_n| = \infty$ and*

$$\lim_{n \rightarrow \infty} \frac{s_n \log \alpha_n}{\sum_{j=1}^{n-1} v_j t_j \log \beta_j} = \lim_{n \rightarrow \infty} \frac{s_n \log \alpha_n}{t_n \log \beta_n} = 0.$$

Then $\dim_H(\Theta(\alpha, \beta, s, t, v, F, I)) = \lim_{n \rightarrow \infty} \frac{\log |I_n|}{\log \beta_n}$ provided this limit exists.

5. Lemmas on (ϵ, k) -normal sequences

Given integers $b \geq 2, n \geq 1, k \geq 1$, let $p_b(n, k)$ denote the number of blocks of length n in base b containing exactly k copies of a given digit. (By symmetry it does not matter which digit we are interested in.)

Lemma 5.1 (Lemma 4.7 in [8]). *Let $b \geq 2$ and $n \geq b^{15}$ be integers. For every real number ϵ with $n^{-1/3} \leq \epsilon \leq 1$, we have*

$$\sum_{-n \leq j \leq -\lceil \epsilon n \rceil} p_b(bn, n + j) + \sum_{\lceil \epsilon n \rceil \leq j \leq (b-1)n} p_b(bn, n + j) \leq 2^{14} b^{bn} e^{-\epsilon^2 n / (10b)}.$$

Lemma 5.2. *Let $b \geq 2$ and $n \geq b^{16}$ be integers. For every real number ϵ with $\lceil n/b \rceil^{-1/3} \leq b\epsilon/2 \leq 1$, we have*

$$\left(\sum_{j > (b^{-1} + \epsilon)n} + \sum_{j < (b^{-1} - \epsilon)n} \right) p_b(n, j) \leq 2^{14} b^{n+b} e^{-\epsilon^2 n/40}.$$

Proof. Note that $p_b(n, j)$ is increasing as a function of n , therefore

$$\left(\sum_{j > (b^{-1} + \epsilon)n} + \sum_{j < (b^{-1} - \epsilon)n} \right) p_b(n, j) \leq \left(\sum_{j > (b^{-1} + \epsilon)n} + \sum_{j < (b^{-1} - \epsilon)n} \right) p_b(b \lceil n/b \rceil, j).$$

Now let $\epsilon' = b\epsilon/2$ and note that

$$\begin{aligned} \left\lceil \frac{n}{b} \right\rceil + \left\lceil \epsilon' \left\lceil \frac{n}{b} \right\rceil \right\rceil &\leq \frac{n}{b} + \epsilon' \frac{n}{b} + 3 \\ &= (b^{-1} + \epsilon)n + \left(3 - \frac{n\epsilon}{2} \right) \\ &\leq (b^{-1} + \epsilon)n. \end{aligned}$$

Likewise one can show that

$$\left\lceil \frac{n}{b} \right\rceil - \left\lceil \epsilon' \left\lceil \frac{n}{b} \right\rceil \right\rceil \geq (b^{-1} - \epsilon)n.$$

As a result, we have that

$$\begin{aligned} &\left(\sum_{j > (b^{-1} + \epsilon)n} + \sum_{j < (b^{-1} - \epsilon)n} \right) p_b(b \lceil n/b \rceil, j) \\ &\leq \sum_{j \leq -\lceil \epsilon' \lceil n/b \rceil} p_b(b \lceil n/b \rceil, \lceil n/b \rceil + j) + \sum_{\lceil \epsilon' \lceil n/b \rceil \leq j} p_b(b \lceil n/b \rceil, \lceil n/b \rceil + j). \end{aligned}$$

We now can apply Lemma 5.1 to see that

$$\begin{aligned} \left(\sum_{j > (b^{-1} + \epsilon)n} + \sum_{j < (b^{-1} - \epsilon)n} \right) p_b(n, j) &\leq 2^{14} b^{b \lceil n/b \rceil} e^{-\epsilon'^2 \lceil n/b \rceil / (10b)} \\ &\leq 2^{14} b^{n+b} e^{-\epsilon^2 n/40}, \end{aligned}$$

as desired. □

We will say a block B of length n in base b is (ϵ, k) -normal (with respect to b), if the total number of occurrences in B of any subblock of length k in base b is between $(b^{-k} - \epsilon)n$ and $(b^{-k} + \epsilon)n$. Let $B_b(n, \epsilon, k)$ denote the number of blocks of length n that are *not* (ϵ, k) -normal with respect to b . Note that Lemma 5.2 gives a bound on $B_b(n, \epsilon, 1)$, when multiplied by b to account for all the different possibilities for the “given digit.” The following lemma will give a bound on $B_b(n, \epsilon, k)$.

Lemma 5.3. *Suppose $b \geq 2$, $k \geq 1$, $n \geq k(b^{16k} + 1)$ are integers. For every real number ϵ with $(n/2kb)^{-1/3} \leq b^k\epsilon/2 \leq 1$ we have*

$$B_b(n, \epsilon, k) \leq 2^{16} kb^{n+k(1+b^k)} e^{-\epsilon^2 n/(80k)}.$$

Proof. Let us begin by considering an arbitrary block $B = [d_1, d_2, \dots, d_n]$ of n digits in base b . Suppose that $n = n'k + r$ for some $r \in \{0, 1, \dots, k - 1\}$.

Let $D_i = d_i b^{k-1} + d_{i+1} b^{k-2} + \dots + d_{i+k-1}$ for $1 \leq i \leq n - k + 1$. Note that $D_i \in \{0, 1, \dots, b^k - 1\}$, and these D_i correspond to grouping k base- b digits into a single base- b^k digit.

For $1 \leq j \leq k$, let $B_j = [D_j, D_{j+k}, D_{j+2k}, \dots, D_{j+(n'-1)k}]$ if $j \leq r + 1$ and $B_j = [D_j, D_{j+k}, D_{j+2k}, \dots, D_{j+(n'-2)k}]$ otherwise.

By the pigeon-hole principle, if B is not (ϵ, k) -normal with respect to b , then some B_j is not $(\epsilon, 1)$ -normal with respect to b^k . Thus, the total number of blocks B which are not (ϵ, k) -normal with respect to b is at most a sum over j of the number of blocks B_j which are not $(\epsilon, 1)$ -normal with respect to b^k , times either b^r or b^{k+r} to account for all possibilities of those digits of B which are not contained in B_j .

We can apply Lemma 5.2 here provided

$$\frac{b^k \epsilon}{2} \geq \left\lceil \frac{\lfloor n/k \rfloor - 1}{b} \right\rceil^{-1/3},$$

but this is clear since the right-hand side here is smaller than $(n/2kb)^{-1/3}$. Thus, by Lemma 5.2 and the fact that $n' = \lfloor n/k \rfloor \geq n/2k$, we have

$$\begin{aligned} B_b(n, \epsilon, k) &\leq (r + 1)b^r 2^{14}(b^k)^{n'+b^k+1} e^{-\epsilon^2 n'/40} \\ &\quad + (k - r - 1)b^{k+r} 2^{14}(b^k)^{(n'+b^k)} e^{-\epsilon^2 (n'-1)/40} \\ &\leq k 2^{14} b^{k(n'+1+b^k)+r} e^{-\epsilon^2 n'/40} (1 + e^{\epsilon^2/40}) \\ &\leq 2^{16} kb^{n+k(1+b^k)} e^{-\epsilon^2 n/(80k)}, \end{aligned}$$

as desired. □

6. Proof of Theorem 2.4

Given $i \geq 2$, consider the following definitions. We let $n_i = i^{\lfloor \log i \rfloor}$, $\epsilon_i = n_i^{-1/4}$. With these definitions, we have by Lemma 5.3 that the number of (ϵ_i, k) -normal blocks of n_i digits in base i is bounded by $i^{n_i} e^{-n_i^{1/5}}$, provided that i is sufficiently large compared to k . When $i = 1$, we shall let $n_i = 0$.

Given a block $B = [d_1, d_2, \dots, d_{n_i}]$ of n_i digits in base i , let

$$\overline{B} = d_1 i^{n-1} + d_2 i^{n-2} + \dots + d_n$$

be the naturally associated integer. Let \mathcal{L}_i denote the set of all such blocks B such that $i! \overline{B} < i^{n_i}$ and $i! \mid \overline{B}$. Note that \mathcal{L}_i always contains the block

$[0, 0, \dots, 0]$. We denote the size of \mathcal{L}_i by ℓ_i , and note that $\ell_i \asymp i^{n_i}/(i!)^2$ for sufficiently large i . We will let

$$L_i = i! \left\lfloor \frac{n_{i+1}\ell_{i+1}}{n_i\ell_i} \right\rfloor.$$

In the construction given in Section 4, let $\alpha_i = i$, $\beta_i = (i!)^2$, $s_i = t_i = n_i$, and $v_i = L_i\ell_i$, with Q given by (4). We shall also let

$$I_i = \left\{ 1, 2, \dots, \left\lfloor \beta_i^{1-\log(i)^{-1}} \right\rfloor \right\} \cap \left(\left\lfloor \sqrt{i} \right\rfloor! \right) \mathbb{Z}.$$

With this definition, we have that $\log |I_i|/\log \beta_i$ tends to 1 and that, as i grows, all elements of I_i become arbitrarily small compared to β_i and are eventually divisible by any fixed integer. Since $n_1 = 0$, the smallest base in Q constructed this way is 2, so that Q really is a basic sequence.

With these definitions (and any appropriate choice of sequence (F)), it is easy to check that all such points satisfy the conditions of Theorem 4.1, so that $\dim_{\mathbb{H}}(\Theta(\alpha, \beta, s, t, v, F, I)) = 1$. It therefore suffices to show that for some proper selection of F , we have $\Theta(\alpha, \beta, s, t, v, F, I) \subset \Xi(Q)$. To make this selection of F , let

$$X_i = \left[[i]^{n_i} [(i!)^2]^{n_i} \right]^{\ell_i},$$

so that we could alternately write Q as

$$(6) \quad Q = [X_2]^{L_2} [X_3]^{L_3} [X_4]^{L_4} \dots$$

We shall then choose the digits of F in such a way so that the digits corresponding to the j th occurrence of the bases $[i]^{n_i}$ in each copy of X_i are the j th string from \mathcal{L}_i (when ordered lexicographically).

With this definition of F in mind, let x be any point in $\Theta(\alpha, \beta, s, t, v, F, I)$, $r \in \mathbb{Q} \setminus \{0\}$, and $s \in \mathbb{Q}$. We will show that $\tau_{r,s}(x)$ is Q -normal but not Q -distribution normal. By the construction of Q and Corollary 3.3, we have for any rational number s that the Q -Cantor series expansions of $\tau_{r,s}(x)$ and $\pi_r(x)$ differ on at most finitely many digits. In addition, we have that \overline{B} for $B \in \mathcal{L}_i$ is small compared with i^{n_i} and is divisible by $i!$, and each digit of I_i is small compared with $(i!)^2$ and is divisible by $\lfloor \sqrt{i} \rfloor!$. Therefore, by Lemma 3.4, we have that for any nonzero rational number r , for all sufficiently large i the digits of $\tau_{r,s}(x)$ corresponding to the bases X_i satisfy the following properties:

- Each block of digits corresponding to an appearance of $[i]^{n_i}$ is unique.
- The digits corresponding to each appearance of the base $(i!)^2$ are in the interval $\{i + 1, i + 2, \dots, \beta_i/i\}$.

To see that $\tau_{r,s}(x)$ is not in $\mathcal{DN}(Q)$, we make use of Theorem 2.3. We note that asymptotically half of the bases q_n are of the form β_i for some i , and by the previous paragraph, we have that the corresponding digits E_n are $o(q_n)$. Therefore the sequence $(E_n/q_n)_{n=1}^\infty$ is clearly not uniformly distributed modulo 1.

To show that $\tau_{r,s}(x)$ is in $\mathcal{N}(Q)$, we make use of the following lemma, whose proof is elementary.

Lemma 6.1. *Let $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ be sequences of positive real numbers such that $\sum_{n=1}^\infty b_n = \infty$. Let $(N_i)_{i=0}^\infty$ be an increasing sequence of positive integers with $n_0 = 1$ and define $A_m = \sum_{n=N_{m-1}}^{N_m-1} a_n$ and $B_m = \sum_{n=N_{m-1}}^{N_m-1} b_n$. Suppose that*

$$\lim_{m \rightarrow \infty} \frac{A_m}{B_m} = 1 \quad \text{and} \quad B_m = o\left(\sum_{i=1}^{m-1} B_i\right),$$

then

$$\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + \dots + a_n}{b_1 + b_2 + \dots + b_n} = 1.$$

Let us denote the j th appearance of X_i in the bases of Q by $X_{i,j}$. In particular, this will consist of the bases q_n where n falls into the following interval

$$[N_{i,j}, M_{i,j}] := \left[\left(\sum_{k=1}^{i-1} 2n_k v_k \right) + 2(j-1)\ell_k + 1, \left(\sum_{k=1}^{i-1} 2n_k v_k \right) + 2j\ell_k \right].$$

Let us write

$$Q^{(k)}(X_{i,j}) = \sum_{n=N_{i,j}}^{M_{i,j}} \frac{1}{q_n q_{n+1} q_{n+2} \dots q_{n+k-1}}$$

and let $N(B, \tau_{r,s}(x), X_{i,j})$ denote the number of occurrences of the block of digits B in the Q -Cantor series expansion of $\tau_{r,s}(x)$ with the first digit of the block occurring at the n th place, with $n \in [N_{i,j}, M_{i,j}]$.

Comparing these two definitions with the definition of Q -normality in (3), and using Lemma 6.1, we see that it suffices to show that

$$(7) \quad N(B, \tau_{r,s}(x), X_{i,j}) = Q^{(k)}(X_{i,j})(1 + o(1))$$

as i increases (uniformly for any $j \in [1, L_i]$) and that

$$(8) \quad Q^{(k)}(X_{i,j}) = o\left(\sum_{j=1}^{L_{i-1}} Q^{(k)}(X_{i-1,j})\right)$$

as i increases.

To estimate the size of $Q^{(k)}(X_{i,j})$, we note that most of the contribution comes from the terms when $q_n = q_{n+1} = \dots = q_{n+k-1} = i$. There are precisely $\ell_i(n_i - k + 1)$ such terms. If any of the q 's in the denominator of a term equals $(i!)^2$ (or, possibly $(i + 1)!^2$), then the entire term is at most $i^{-k+1}(i!)^{-2}$. And there are precisely $\ell_i(n_i + k - 1)$ such summands. Therefore,

$$(9) \quad Q^{(k)}(X_{i,j}) = \frac{\ell_i(n_i - k + 1)}{i^k} + O\left(\frac{\ell_i(n_i + k - 1)}{i^{k-1}(i!)^2}\right) = \frac{\ell_i n_i}{i^k}(1 + o(1))$$

where the $o(1)$ is decreasing as i increases and is uniform over $j \in [1, L_i]$.

From this, we derive

$$(10) \quad \sum_{j=1}^{L_{i-1}} Q^{(k)}(X_{i-1,j}) = L_{i-1} \frac{\ell_{i-1} n_{i-1}}{(i-1)^k} (1 + o(1))$$

and therefore (8) derives from comparing (9) and (10) and using the definition of L_{i-1} .

To estimate the size of $N(B, \tau_{r,s}(x), X_{i,j})$, let us suppose that i is sufficiently large so that the digits of B are less than i and so that all the digits of $\tau_{r,s}(x)$ corresponding to the large bases $(i!)^2$ are at least i in size. Therefore B will only occur in the digit strings corresponding to the small blocks $[i]^{n_i}$. We know that there are ℓ_i such distinct digit strings and at most $i^{n_i} e^{-n_i^{1/5}}$ of them can not be (ϵ_i, k) -normal. Therefore, we have

$$(11) \quad \begin{aligned} N(B, \tau_{r,s}(x), X_{i,j}) &= \left(i^{-k} + O(\epsilon_i) \right) \ell_i n_i + O \left(n_i i^{n_i} e^{-n_i^{1/5}} \right) \\ &= \frac{\ell_i n_i}{i^k} (1 + o(1)). \end{aligned}$$

As before, the $o(1)$ here is decreasing as i increases.

Comparing (9) and (11) gives (7) and completes the proof.

7. Proof of Theorem 2.5

We shall, in fact, prove the following, more explicit theorem.

Theorem 7.1. *The basic sequence Q given in (6) is computable. Let $\eta = 0.E_1 E_2 \dots$ w.r.t. Q be the real number from the set $\Theta(\alpha, \beta, s, t, v, F, I)$ given in Section 6 such that $E_n = i_\alpha(n)!$ if (5) holds (that is, the digits corresponding to the bases $(i!)^2$ will be $i!$). Then η is computable.*

Proof. The sequence $\lfloor \log(i) \rfloor$ is computable, so $n_i = i^{\lfloor \log(i) \rfloor}$ is a computable sequence. Lexicographically enumerate all integers in $[0, i^{n_i} - 1]$. Check if each integer \bar{B} satisfies the conditions $i! \bar{B} < i^{n_i}$ and if $i! \mid \bar{B}$. This is computable because the order relation on integers and divisibility of integers are computable relations. Now lexicographically enumerate the elements of \mathcal{L}_i . The size of \mathcal{L}_i is a computable function of i since we can lexicographically enumerate the set, so we have that (l_i) is a computable sequence. Since (n_i) and (l_i) are computable sequences, the sequence (L_i) is also computable. Furthermore, $(2L_i l_i n_i)$ is also a computable sequence.

Thus the sequences (α_i) , (β_i) , (s_i) , (t_i) , and (v_i) are all computable sequences. We can compute the n th term of $Q(\alpha, \beta, s, t, v)$ as follows. First we will compute the m th base of X_j with the following method for integers m and j . Determine the residue class of m modulo $2n_j$. If this residue is less than n_j , return j , otherwise return $(j!)^2$. This procedure gives the m th base of X_j . Now determine the maximum i such that $\sum_{j=1}^i 2L_j l_j n_j < n$

and compute $N = n - \sum_{j=1}^i 2L_j l_j n_j$. To compute this maximum test $\sum_{j=1}^{\ell} 2L_j l_j n_j < n$ for integers ℓ starting at 0. Since the sequence $(2L_j l_j n_j)$ tends to infinity, there must be some finite bound, so this procedure will terminate. Finally compute the N th base of X_i and this will be the n term of $Q(\alpha, \beta, s, t, v)$. Thus this basic sequence is computable.

Now we show the sequence (E_n) is computable. Given n first compute (a, K) where $a = \min\{j : \sum_{k=1}^j 2L_k l_k n_k < n\}$ and $K = n - \sum_{j=1}^a 2L_j l_j n_j$. If the residue class of K modulo $2n_a$ is greater than or equal to n_a , output $a!$. Otherwise, compute $z = \lfloor K/(2n_a) \rfloor$ and return the $K \bmod n_a$ th digit of the z th element of \mathcal{L}_i . This procedure computes E_n . Since both (E_n) and (q_n) are computable sequences, the real number $\eta = \sum_{j=1}^{\infty} \frac{E_j}{q_1 \cdots q_j}$ is computable. \square

8. Further problems

The effect of the rational number s on the set we constructed to prove Theorem 2.4 was negligible. We specifically constructed Q so that the denominator of s had to divide some q_n , so addition by s would never change more than a finite amount of digits by Corollary 3.3, and thus had no impact on either Q -normality or Q -distribution normality (or the lack thereof). This suggests the following natural question.

Problem 8.1. *If we were to restrict Q so that, say q_n is not divisible by 3 for every n , then addition by $1/3$ would have to change an infinite number of digits. Are results similar to those given here possible for such Q ?*

We also ask

Problem 8.2. *Does a version of Theorem 2.4 hold for all Q that are infinite in limit and fully divergent?*

Problem 8.3. *There exist some basic sequences Q where the set $\mathcal{DN}(Q)$ does not contain any computable real numbers. See [7]. What assumptions on Q must we have to guarantee that there are computable real numbers in $\Xi(Q)$?*

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