New York Journal of Mathematics

New York J. Math. **21** (2015) 759–781.

On geometrical properties of noncommutative modular function spaces

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Abstract. We introduce and study the noncommutative modular function spaces of measurable operators affiliated with a semifinite von Neumann algebra and show that they are complete with respect to their modular. We prove that these spaces satisfy the uniform Opial condition with respect to $\widetilde{\rho}$-a.e.-convergence for both the Luxemburg norm and the Amemiya norm. Moreover, these spaces have the uniform Kadec–Klee property with respect to $\widetilde{\rho}$-a.e.-convergence when they are equipped with the Luxemburg norm. The above geometric properties enable us to obtain some results in noncommutative Orlicz spaces.

Contents

1. Introduction

The first attempts to generalize the classical function spaces of Lebesgue type L^p were made in the early 1930's by Orlicz and Birnbaum in connection with orthogonal expansions. The possibility of introducing the structure of a linear metric in Orlicz spaces L^φ as well as the interesting properties of these spaces and many applications to differential and integral equations with kernels of nonpower type were among the reasons for the development of the theory of Orlicz spaces.

We note two principal directions of further development. The first one is a theory of Banach function spaces initiated in 1955 by Luxemburg [L55] and then developed in a series of joint papers with Zaanen [LZ63]. The main idea consists of considering the function space $\mathcal{L}$ of all real functions

Received May 7, 2014.

2010 *Mathematics Subject Classification.* Primary 46L52. Secondary 46B20, 46A80.

Key words and phrases. Measurable operator, von Neumann algebra, generalized singular value function, uniform Opial condition, Kadec–Klee property, modular function space.

ISSN 1076-9803/2015

$f \in \mathcal{L}^0(\mathcal{X}, \Sigma, \nu)$ such that $\|f\| < \infty$, where $(\mathcal{X}, \Sigma, \nu)$ is a measure space, $\mathcal{L}^0(\mathcal{X}, \Sigma, \nu)$ denotes the space of all real measurable functions on $\mathcal{X}$, and $\|\cdot\|$ is a function norm which satisfies

$$\|f\| \leq \|g\| \quad \text{whenever} \quad |f(x)| \leq |g(x)| \quad \nu\text{-a.e.}$$

The noncommutative version of this approach, which is called the noncommutative symmetric space of τ-measurable operators affiliated with a von Neumann algebra, was considered for the first time by Ovchinnikov [Ov71]. Some properties of these spaces were examined in [DDP89, DDS89, CDSF96, P07].

The other direction, also inspired by the successful theory of Orlicz spaces, is based on replacing the integral form of a nonlinear functional with an abstract given functional with some suitable properties. This idea was the basis behind the theory of modular spaces initiated by Nakano [Na50] in connection with the theory of order spaces, which was redefined and generalized by Musielak and Orlicz [MO59].

Presently the theory of modulars and modular spaces is applied extensively, in particular in the study of various Orlicz spaces [Or88] and interpolation theory [Kr82]. Recently, the author of this paper studied noncommutative Orlicz spaces from the point of view of modulars [Sa12]. The main objective of the current paper is to investigate the theory of noncommutative modular function spaces.

The organization of the paper is as follows. In the second section we provide some necessary preliminaries related to the theory of τ–measurable operators affiliated with a von Neumann algebra and the classical theory of modular spaces. In Section 3, we introduce the definition of noncommutative modular function spaces associated to a modular on τ-measurable operators and give some relations between the $\widetilde{\rho}$-a.e.-convergence and the convergence of the modular. In the last section, we prove that these spaces satisfy the uniform Opial condition with respect to $\widetilde{\rho}$-a.e.-convergence for both the Luxemburg norm and the Amemiya norm. Moreover, these spaces have the uniform Kadec–Klee property with respect to $\widetilde{\rho}$-a.e.-convergence when they are equipped with the Luxemburg norm.

2. Preliminaries

In this section, we collect some basic facts and give some notations related to τ–measurable operators and modular function spaces. We denote by $\mathfrak{M}$ a semifinite von Neumann algebra on a Hilbert space $\mathfrak{H}$, with a fixed faithful and normal semifinite trace τ. For standard facts concerning von Neumann algebras we refer the reader to [Ta79]. The identity in $\mathfrak{M}$ is denoted by $\mathbf{1}$ and we denote by $\mathcal{P}(\mathfrak{M})$ the complete lattice of all self–adjoint projections in $\mathfrak{M}$. A linear operator $x : \mathcal{D}(x) \to \mathfrak{H}$ with domain $\mathcal{D}(x) \subseteq \mathfrak{H}$ is affiliated with $\mathfrak{M}$ if $ux = xu$ for all unitaries u in the commutant $\mathfrak{M}^{'}$ of $\mathfrak{M}$, and this is denoted by $x\eta\mathfrak{M}$. Note that the equality $ux = xu$ involves the equality of the domains of the operators ux and xu, that is, $\mathcal{D}(x) = u^{-1}(\mathcal{D}(x))$. If x is in the algebra $\mathcal{B}(\mathfrak{H})$ of all bounded linear operators on the Hilbert space $\mathfrak{H}$, then x is affiliated with $\mathfrak{M}$ if and only if $x \in \mathfrak{M}$. If x is a self–adjoint operator in $\mathfrak{H}$ affiliated with $\mathfrak{M}$, then the spectral projection $e^x(B)$ is an element of $\mathfrak{M}$ for any Borel set $B \subseteq \mathbb{R}$.

A closed and densely defined operator x affiliated with $\mathfrak{M}$ is called τ-measurable if there exists a number $\lambda \geq 0$ such that

$$\tau\left(e^{|x|}(\lambda, \infty)\right) < \infty.$$

The collection of all τ-measurable operators is denoted by $\widetilde{\mathfrak{M}}$. With the sum and product defined as the respective closure of the algebraic sum and product, it is well known that $\widetilde{\mathfrak{M}}$ is a $*$-algebra [Ne74]. Given positive real numbers ε, δ, we define $\mathcal{V}(\varepsilon, \delta)$ to be the set of all $x \in \widetilde{\mathfrak{M}}$ for which there exists $p \in \mathcal{P}(\mathfrak{M})$ such that $\|xp\|_{\mathcal{B}(\mathfrak{H})} \leq \varepsilon$ and $\tau(\mathbf{1} - p) \leq \delta$. An alternative description of this set is

$$\mathcal{V}(\varepsilon, \delta) = \left\{x \in \widetilde{\mathfrak{M}} : \tau\left(e^{|x|}(\varepsilon, \infty)\right) < \delta\right\}.$$

The collection $\{\mathcal{V}(\varepsilon, \delta)\}_{\varepsilon,\delta>0}$ is a neighborhood base at 0 for a vector space topology τ_m on $\widetilde{\mathfrak{M}}$. For $x \in \widetilde{\mathfrak{M}}$, the generalized singular value function $\mu(x) = \mu(|x|)$ is defined by

$$\mu_t(x) = \inf\left\{\lambda \geq 0 : \quad \tau\left(e^{|x|}(\lambda, \infty)\right) \leq t\right\} \quad (t \geq 0).$$

It follows directly that the generalized singular value function $\mu(x)$ is a decreasing right-continuous function on the positive half-line $[0, \infty)$. Moreover, for all $u, v \in \mathfrak{M}$ and $x \in \widetilde{\mathfrak{M}}$,

$$\mu(uxv) \leq \|u\|\|v\|\mu(x)$$

and

$$\mu(f(x)) = f(\mu(x))$$

whenever $0 \leq x \in \widetilde{\mathfrak{M}}$ and f is an increasing continuous function on $[0, \infty)$ with $f(0) = 0$. $\widetilde{\mathfrak{M}}$ is a partially ordered vector space under the ordering $x \geq 0$ defined by $\langle x\xi, \xi\rangle \geq 0$, $\xi \in \mathcal{D}(x)$. If $0 \leq x_\alpha \uparrow x$ holds in $\widetilde{\mathfrak{M}}$, then $\sup \mu_t(x_\alpha) \uparrow_\alpha \mu_t(x)$ for each $t \geq 0$. The trace τ is extended to the positive cone of $\widetilde{\mathfrak{M}}$ as a nonnegative extended real–value functional which is positively homogeneous, additive, unitary invariant and normal. Furthermore,

$$\tau(x^*x) = \tau(xx^*)$$

for all $x \in \widetilde{\mathfrak{M}}$ and

$$\tau(f(x)) = \int_0^\infty f(\mu_t(x))dt \tag{2.1}$$

whenever $0 \leq x \in \widetilde{\mathfrak{M}}$ and f is a nonnegative Borel function which is bounded on a neighborhood of 0 and satisfies $f(0) = 0$. In the following proposition, we list some properties of the rearrangement mapping $\mu_t(\cdot)$.

Proposition 2.1. *Let x, y and z be τ-measurable operators. Then the following hold true:*

(i) *The map $t \in (0, \infty) \mapsto \mu_t(x)$ is nonincreasing and continuous from the right. Moreover,*

$$\lim_{t \downarrow 0} \mu_t(x) = \|x\| \in [0, \infty].$$

(ii) $\mu_t(x) = \mu_t(|x|) = \mu_t(x^*)$.

(iii) $\mu_t(x) \leq \mu_t(y), \quad$ *for $t > 0$, if $0 \leq x \leq y$.*

(iv) $\mu_{t+s}(x + y) \leq \mu_t(x) + \mu_s(y), \quad t, s > 0.$

(v) $\mu_t(zxy) \leq \|z\|\|y\|\mu_t(x), \quad t > 0.$

(vi) $\mu_{t+s}(xy) \leq \mu_t(x)\mu_s(y), \quad t, s > 0.$

For further details and proofs we refer the reader to [FK86, DDP89].

Lemma 2.2. [Ku90] *Let x, y be τ-measurable operators. Then there exist partial isometries u, v in $\mathfrak{M}$ such that*

$$|x + y| \leq u|x|u^* + v|y|v^*.$$

The proof for measurable operators is straightforward using the fact that the square root function is operator monotone; see [AAP82].

In the sequel, we recall some basic concepts about modular function spaces. Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. In other words, the family $\mathcal{P}$ plays the role of the δ-ring of subsets of finite measure. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}$ we denote the space of all measurable functions, i.e., all functions $f : \Omega \to \mathbb{R}$ such that there exists a sequence $\{s_n\} \subset \mathcal{E}$, $|s_n| \leq |f|$ such that $s_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. By χ_A we denote the characteristic function of the set A.

Let us recall that a set function $\nu : \Sigma \to [0, +\infty]$ is called a σ-subadditive measure if $\nu(\emptyset) = 0$, $\nu(A) \leq \nu(B)$ for any $A \subseteq B$ and $\nu(\cup_n A_n) \leq \Sigma_n \nu(A_n)$ for any sequence of sets $\{A_n\} \subseteq \Sigma$.

Definition 2.3. A functional $\rho : \mathcal{E} \times \Sigma \to [0, \infty]$ is called a function modular if:

(i) $\rho(0, A) = 0$ for any $A \in \Sigma$.

(ii) $\rho(f, A) \leq \rho(g, A)$ whenever $|f(\omega) \leq |g(\omega)|$, for $\omega \in \Omega$, $f, g \in \mathcal{E}$, $A \in \Sigma$.

(iii) $\rho(f, \cdot) : \Sigma \to [0, +\infty]$ is a σ-subadditive measure for every $f \in \mathcal{E}$.

(iv) $\rho(\alpha, A) \to 0$ as α decreases to 0 for every $A \in \mathcal{P}$, where

$$\rho(\alpha, A) = \rho(\alpha \chi_A, A).$$

(v) If there exists $\alpha > 0$ such that $\rho(\alpha, A) = 0$, then $\rho(\beta, A) = 0$ for every $\beta > 0$.
(vi) For any $\alpha > 0$, $\rho(\alpha, \cdot)$ is order continuous on $\mathcal{P}$, i.e., $\rho(\alpha, A_n) \to 0$ if $\{A_n\} \subseteq \mathcal{P}$ and decreases to $\emptyset$.

When ρ satisfies

(iii′) $\rho(f, \cdot) : \Sigma \to [0, +\infty]$ is a σ-subadditive measure,

we say that ρ is additive if $\rho(f, A \cup B) = \rho(f, A) + \rho(f, B)$ whenever $A, B \in \Sigma$ such that $A \cap B = \emptyset$ and $f \in \mathcal{M}$.

The definition of ρ can be extended to all $f \in \mathcal{M}$ by

$$\rho(f, A) = \sup\{\rho(s, A); s \in \mathcal{E}, |s(x)| \leq |f(x)| \text{ for every } \omega \in \Omega\}.$$

Similarly as in the case of measure spaces, a set $A \in \Sigma$ is called ρ-null if $\rho(\alpha, A) = 0$ for every $\alpha > 0$. A property $p(\omega)$ is said to hold ρ-almost everywhere (ρ-a.e.) if the set $\{\omega \in \Omega : p(\omega) \text{ does not hold}\}$ is ρ-null. We say that $f_n \to f$ ρ-a.e. if $\{\omega \in \Omega : f(\omega) \neq \lim_{n\to\infty} f_n(\omega)\}$ is ρ-null. As usual, we identify any pair of measurable sets whose symmetric difference is ρ-null, as well as any pair of measurable functions differing only on a ρ-null set.

In the above conditions, we define the functional $\rho : \mathcal{M} \to [0, +\infty]$ by $\rho(f) = \rho(f, \Omega)$. Then it is easy to check that ρ is a modular, that is, ρ satisfies the following properties:

(i) $\rho(f) = 0$ if and only if $f = 0$.
(ii) $\rho(\alpha f) = \rho(f)$ for every scalar α with $|\alpha| = 1$.
(iii) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$.

If (iii) is replaced by

(iii′) $\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,

then we say that ρ is a convex modular.

A modular ρ defines a corresponding modular function space, i.e., the vector space $\mathcal{L}_\rho$ given by

$$\mathcal{L}_\rho = \{x \in \mathcal{M} : \quad \rho(\lambda x) \to 0 \qquad \text{as } \lambda \to 0\}.$$

When ρ is convex, the formulas

$$\|f\|_\rho = \inf\left\{\lambda > 0 : \quad \rho\left(\frac{f}{\lambda}\right) \leq 1\right\}$$

and

$$\|f\|_\rho^A = \inf\left\{\frac{1}{\lambda}\left(1 + \rho(\lambda f)\right) : \lambda > 0\right\}$$

define two complete norms on $\mathcal{L}_\rho$ which are called the Luxemburg norm and the Amemiya norm, respectively. The spaces $(\mathcal{L}_\rho, \|\cdot\|_\rho)$ and $(\mathcal{L}_\rho, \|\cdot\|_\rho^A)$ are complete. Moreover, the Luxemburg norm and the Amemiya norm are equivalent. Indeed,

$$\|f\|_\rho \leq \|f\|_\rho^A \leq 2\|f\|_\rho \tag{2.2}$$

for every $f \in \mathcal{L}_\rho$.

A function modular is said to satisfy the Δ_2-condition if

$$\sup_n \rho(2f_n, D_k) \to 0$$

as $k \to \infty$, whenever $\{f_n\} \subseteq \mathcal{M}$, $D_k \in \mathcal{M}$ decreases to $\emptyset$ and

$$\sup_n \rho(f_n, D_k) \to 0.$$

Definition 2.4. A function modular is said to satisfy the Δ_2-type condition if there exists $k > 0$ such that for any $f \in \mathcal{L}_\rho$ it holds that $\rho(2f) \leq k\rho(f)$.

It is clear that the Δ_2-type condition implies the Δ_2-condition, but in general, they are not equivalent. Note that the Δ_2-type condition ensures that $0 < \rho(\lambda f) < \infty$ for every $\lambda > 0$ provided that $0 < \rho(f) < \infty$.

Let us recall an example of a modular function space.

Example 2.5 (Orlicz–Musielak spaces). Let (Ω, Σ, ν) be a measure space, where ν is a positive σ-finite measure. Let us denote by $\mathcal{P}$ the δ-ring of all sets of finite measure. Define the modular ρ by the formula

$$\rho_\varphi(f, E) = \int_E \varphi(t, |f(t)|) d\nu(t)$$

provided φ belongs to the class Φ. For the precise definitions of the class Φ and properties of Orlicz–Musielak spaces see e.g., [Mu83]. For an Orlicz–Musielak space, the ρ-null sets coincide with the sets of measure zero in the sense of ν [Ko88, Proposition 4.1.9]. Thus the ρ-a.e.-convergence is equivalent to the convergence almost everywhere. If the modular is given by

$$\rho_\varphi(f, E) = \int_E \varphi(|f(t)|) d\nu(t),$$

where φ is an Orlicz function, we obtain the notion of Orlicz space. In particular, if $\varphi(t) = t^p$ for $1 \leq p < +\infty$, we obtain the Lebesgue space $L^p(\nu)$, where the Luxemburg norm is the classic norm $\|\cdot\|_p$.

An Orlicz function φ is said to satisfy the Δ_2-condition if there exists $k > 0$ such that $\varphi(2\alpha) \leq k\varphi(\alpha)$ for every $\alpha \geq 0$. If φ is an Orlicz function satisfying the Δ_2-condition it is clear that the modular ρ_φ satisfies the Δ_2-type condition.

In the following theorem we recall some of the properties of modular function spaces that will be used later on in this paper. For proofs and details the reader is referred to [Mu83, Ko88].

Theorem 2.6.

(i) $(\mathcal{L}_\rho, \|\cdot\|_\rho)$ *is complete and the norm* $\|\cdot\|_\rho$ *is monotone with the natural order in* $\mathcal{M}$.

(ii) $\|f\|_\rho \to 0$ *if and only if* $\rho(\alpha f) \to 0$ *for every* $\alpha > 0$.

(iii) *If* $\rho(\alpha f) \to 0$ *for an* $\alpha > 0$ *then there exists a subsequence* $\{g_n\}$ *of* $\{f_n\}$ *such that* $g_n \to 0$ ρ*-a.e.*

(iv) *If $f_n \to f$ ρ-a.e., there exists a nondecreasing sequence of sets $E_n \in \mathcal{P}$ such that $E_n \uparrow \Omega$ and $\{f_n\}$ converges uniformly to f on every E_n (Egoroff theorem).*

(v) $\rho(f) \le \liminf_{n\to\infty} \rho(f_n)$ *whenever $f_n \to f$ ρ-a.e. (This property is equivalent to the Fatou property).*

Let X be a Banach space and τ be a topology on X. We say that X satisfies the uniform Opial condition with respect to τ if $o_\tau(\alpha) > 0$ for every $\alpha > 0$, where $o_\tau(\cdot)$ is the Opial modulus defined as

$$o_\tau(\alpha) = \inf \left\{ \liminf_{n\to\infty} \|x_n + x\| - 1 \right\},$$

and the infimum is taken over all $x \in X$ with $\|x\| \ge \alpha$ and all sequences $\{x_n\}$ such that τ-$\lim_{n\to\infty} x_n = 0$ and $\liminf_{n\to\infty} \|x_n\| \ge 1$ (see [APP13]).

A space X is said to have the uniform Kadec–Klee property with respect to τ ($UKK(\tau)$) if for every $0 < \varepsilon \le 2$ there exists $\delta > 0$ such that if $\{x_n\}$ is a sequence in the unit ball of X with $\operatorname{sep}\{x_n\} = \inf\{\|x_n - x_m\| : n \ne m\} > \varepsilon$ and $\{x_n\}$ is τ-convergent to $x \in X$, then $\|x\| < 1 - \delta$.

In connection to the uniform Kadec–Klee property, we can define the following modules:

$$k_\tau(\varepsilon) = \inf \left\{ 1 - \|x\| : \{x_n\} \in B_X, \tau - \lim_{n\to\infty} x_n = x \text{ and } \operatorname{sep}\{x_n\} > \varepsilon \right\}.$$

It is clear that X has the uniform Kadec–Klee property with respect to τ iff $k_\tau(\varepsilon) > 0$ for every $\varepsilon > 0$.

In the particular case that X is a modular function space, we can replace the convergence with respect to a topology τ by the ρ-a.e.-convergence. The uniform Opial condition and the uniform Kadec–Klee property are geometric properties, which are connected to the existence of fixed points for some kinds of mappings.

3. Noncommutative modular function spaces

In this section, we assume that $\mathfrak{M}$ is a semifinite von Neumann algebra on a Hilbert space $\mathfrak{H}$ with a normal faithful trace τ. We define noncommutative modular function spaces of τ-measurable operators affiliated to $\mathfrak{M}$. To this end, we assume $\Omega = [0, \infty)$ and ν is a Lebesgue measure on Ω. By $\mathcal{L}_0(\nu)$ we denote the linear space of all (equivalence classes of) real valued Lebesgue measurable functions on Ω, Σ is the σ-algebra of Lebesgue measurable sets and $\mathcal{P}$ is the family of such sets of finite measure. Let $\rho(f) := \rho(f, \Omega)$ be a convex modular on $\mathcal{L}_0(\nu)$ such that for every $f, g \in \mathcal{L}_0(\nu)$, $f \prec\prec g$ (i.e., f is submajorized by g in the sense of Hardy, Littlewood and Polya) implies that $\rho(f) \le \rho(g)$, and let $\mathcal{L}_\rho$ denote the modular function space associated to ρ on $\mathcal{L}_0(\nu)$.

We now define the functional $\widetilde{\rho}$ on $\widetilde{\mathfrak{M}}$ as follows:

$$\widetilde{\rho} : \widetilde{\mathfrak{M}} \to [0, \infty], \quad \widetilde{\rho}(x) = \rho(\mu(|x|)).$$

Proposition 3.1. *$\widetilde{\rho}$ is a convex modular functional on $\widetilde{\mathfrak{M}}$.*

Proof. It is sufficient to show that $\widetilde{\rho}$ satisfies

$$\widetilde{\rho}(\alpha x+\beta y)\leq \alpha\widetilde{\rho}(x)+\beta\widetilde{\rho}(y) \tag{3.1}$$

for $\alpha+\beta=1$ with $\alpha,\beta\geq 0$. Let x,y be two operators in $\widetilde{\mathfrak{M}}$. It is known that $\mu(x+y)\prec\prec\mu(x)+\mu(y)$ [FK86, Theorem 4.4]. By Lemma 2.2 there exist partial isometries u,v in $\widetilde{\mathfrak{M}}$ such that

$$|x+y|\leq u|x|u^*+v|y|v^*.$$

Hence

$$\begin{aligned}\widetilde{\rho}(\alpha x+\beta y)=\rho(\mu(|\alpha x+\beta y|))&\leq \rho(\mu(\alpha u|x|u^*+\beta v|y|v^*))\\ &\leq \alpha\rho(\mu(|x|))+\beta\rho(\mu(|y|))\\ &=\alpha\widetilde{\rho}(x)+\beta\widetilde{\rho}(y).\end{aligned}$$

Thus, $\widetilde{\rho}$ is a convex modular on $\widetilde{\mathfrak{M}}$. □

The noncommutative modular function space $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)$ is defined by

$$\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)=\left\{x\in\widetilde{\mathfrak{M}}:\quad \widetilde{\rho}(\lambda x)\to 0\quad\text{as}\quad\lambda\to 0\right\},$$

or equivalently,

$$\begin{aligned}\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)&=\left\{x\in\widetilde{\mathfrak{M}}:\quad \rho(\lambda\mu(x))\to 0\quad\text{as}\quad\lambda\to 0\right\}\\ &=\left\{x\in\widetilde{\mathfrak{M}}:\quad \mu(x)\in\mathcal{L}_\rho\right\}.\end{aligned}$$

The vector space $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)$ can be equipped with the Luxemburg norm defined by

$$\begin{aligned}\|x\|_{\widetilde{\rho}}&=\inf\left\{\lambda>0:\widetilde{\rho}\left(\frac{x}{\lambda}\right)\leq 1\right\}\\ &=\inf\left\{\lambda>0:\quad\rho\left(\frac{\mu(|x|)}{\lambda}\right)\leq 1\right\}\\ &=\|\mu(|x|)\|_\rho.\end{aligned}$$

Similar to the commutative case, one can define the Amemiya norm as follows:

$$\begin{aligned}\|x\|^A_{\widetilde{\rho}}&=\inf\left\{\frac{1}{\lambda}(1+\widetilde{\rho}(\lambda x)):\quad\lambda>0\right\}\\ &=\inf\left\{\frac{1}{\lambda}(1+\rho\left(\lambda\mu(|x|)\right)):\quad\lambda>0\right\}\\ &=\|\mu(|x|)\|^A_\rho.\end{aligned}$$

Moreover, by (2.2), these norms satisfy

$$\|x\|_{\widetilde{\rho}}\leq\|x\|^A_{\widetilde{\rho}}\leq 2\|x\|_{\widetilde{\rho}}.$$

It follows from [DDP89, Theorem 4.5] that $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)$ is a Banach space since the norm $\|\cdot\|_{\widetilde{\rho}}$ has the Fatou property.

Theorem 3.2. *Let ρ be a convex function modular with the Δ_2-type condition. Then the noncommutative modular function space $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)$ is $\widetilde{\rho}$-complete.*

Proof. Suppose that $\{x_n\}$ is a $\widetilde{\rho}$–Cauchy sequence in $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)$. We first show that $\{x_n\}$ is a Cauchy sequence in the measure topology τ_m. If $\varepsilon,\delta>0$ are given, then there exist $\eta>0$, $n_0\in\mathbb{N}$ such that $\|f\|_\rho<\varepsilon$ if $\rho(f)<\eta$ for any $f\in\mathcal{L}_0(\nu)$ and

$$\widetilde{\rho}(x_n-x_m)<\eta$$

for any $n,m\geq n_0$. It is known that $\mu(x)$ is a nonincreasing function, so we have

$$\mu(x_n-x_m)\geq\mu_\delta(x_n-x_m)\chi_{[0,\delta)}+\mu(x_n-x_m)\chi_{[\delta,\infty)}.$$

It follows that

$$\rho(\mu_\delta(x_n-x_m)\chi_{[0,\delta)})\leq\widetilde{\rho}(x_n-x_m)<\eta$$

for all $n,m\geq n_0$. This yields $\|\mu_\delta(x_n-x_m)\chi_{[0,\delta)}\|_\rho<\varepsilon$. Hence

$$\mu_\delta(x_n-x_m)<\varepsilon/\|\chi_{[0,\delta)}\|_\rho$$

for any $n,m\geq n_0$. Consequently, there exists $x\in\widetilde{\mathfrak{M}}$ such that $x_n\xrightarrow{\tau_m}x$. Moreover, by [DDP89, Theorem 3.4],

$$|\mu(x_n)-\mu(x_m)|\prec\prec\mu(x_n-x_m),$$

whence

$$\rho\left(\mu(x_n)-\mu(x_m)\right))\leq\rho\left(\mu(x_n-x_m)\right)$$

for all $n,m\in\mathbb{N}$. Therefore, $\{\mu(x_n)\}$ is Cauchy sequence in $\mathcal{L}_\rho$. It follows that there exists $f\in\mathcal{L}_\rho$ such that $\|\mu(x_n)-f\|_\rho\to0$ as $n\to\infty$. By Theorem 2.6(ii), we obtain

$$\rho\left(\mu(x_n)-f\right))\to0\quad\text{as}\quad n\to\infty. \tag{3.2}$$

Using Theorem 2.6(iii), we may suppose (by passing to a subsequence if necessary) that $\mu(x_n)\to f$ ρ-a.e. We use a known version of the Egoroff theorem for modular function spaces. There exists a nondecreasing sequence of sets E_k with finite measure such that $E_k\uparrow\Omega$ and $\{\mu(x_n)\}$ converges uniformly to f on every E_k. On the other hand, it follows from [FK86, Lemma 3.4] that $\mu(x_n)\to\mu(x)$ almost everywhere on $[0,\infty)$ and so

$$\mu(x)=f\in\mathcal{L}_\rho,$$

that is $x\in\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)$. Moreover, it follows from (3.2) that

$$\rho(\mu(x_n)-\mu(x)))\to0$$

as $n\to0$, i.e., $\mu(x_n)\xrightarrow{\rho}\mu(x)$. Observe that

$$(x_n-x_m)\to(x_n-x)$$

for the measure topology τ_m in $\widetilde{\mathfrak{M}}$. Similar to the above argument one can show that $\mu(x_n - x_m) \xrightarrow{\rho} \mu(x_n - x)$. Since ρ has the Fatou property, it follows that

$$\rho\left(\mu(x_n - x)\right) \leq \liminf_{m\to\infty} \rho\left(\mu(x_n - x_m)\right).$$

Hence $\lim_{n\to\infty} \widetilde{\rho}(x_n - x) = 0$. □

Let us give an example of a noncommutative modular function space.

Example 3.3. Let $L^{\varphi}(\Omega,\nu)$ be an Orlicz space on $\Omega = [0,\infty)$ with respect to the modular functional

$$\rho_{\varphi}(f) = \int_0^{\infty} \varphi(|f(t)|)d\nu(t)$$

for every $f \in L^{\varphi}$, where ν is the Lebesgue measure on Ω. We can consider the noncommutative Orlicz spaces from the point of view of modulars as follows:

$$L^{\varphi}(\widetilde{\mathfrak{M}},\tau) = \left\{x \in \widetilde{\mathfrak{M}} : \quad \widetilde{\rho}_{\varphi}(\lambda x) \to 0 \quad \text{ as } \quad \lambda \to 0\right\},$$

where $\widetilde{\rho}_{\varphi} : \widetilde{\mathfrak{M}} \to [0,\infty]$ is defined by

$$\widetilde{\rho}_{\varphi}(x) = \tau(\varphi(|x|)) = \int_0^{\infty} \varphi(\mu_t(|x|))dt.$$

For more details, we refer the reader to [Sa12, AB14].

Proposition 3.4. *Let $x, y \in \mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)$.*

(i) *If $\widetilde{\rho}(\lambda x) \leq \widetilde{\rho}(\lambda y)$ for every $\lambda > 0$, then $\|x\|_{\widetilde{\rho}} \leq \|y\|_{\widetilde{\rho}}$.*
(ii) *The function $\alpha \mapsto \|\alpha x\|_{\widetilde{\rho}}$ is nondecreasing for $\alpha \geq 0$.*
(iii) *If $\|x\|_{\widetilde{\rho}} < 1$, then $\widetilde{\rho}(x) \leq \|x\|_{\widetilde{\rho}}$.*
(iv) *If ρ satisfies the Δ_2- type condition, then*

$$\widetilde{\rho}(x + y) \leq \frac{k}{2}(\widetilde{\rho}(x) + \widetilde{\rho}(y))$$

for all x and y in $\widetilde{\mathfrak{M}}$, with $k > 0$ satisfying $\rho(2f) \leq k\rho(f)$ for any $f \in \mathcal{L}_{\rho}$.

Proof. The properties (i) and (ii) follow immediately from the definition of $\|\cdot\|_{\widetilde{\rho}}$.

We prove now that (iii) holds. Let $\|x\|_{\widetilde{\rho}} < \alpha < 1$. Then $\rho\left(\frac{\mu(x)}{\alpha}\right) \leq 1$ and so

$$\widetilde{\rho}(x) = \rho\left(\alpha\frac{\mu(|x|)}{\alpha}\right) \leq \alpha\rho\left(\frac{\mu(|x|)}{\alpha}\right) \leq \alpha.$$

Since α is arbitrary, we obtain that $\widetilde{\rho}(x) \leq \|x\|_{\widetilde{\rho}}$.

For (iv), it follows from Lemma 2.2 that there exist partial isometries u, v in $\widetilde{\mathfrak{M}}$ such that

$$|x + y| \leq u|x|u^* + v|y|v^*.$$

By Proposition 2.1(iii),(v) and the submajorization inequality

$$\mu(x+y) \prec\prec \mu(x)+\mu(y)$$

for $x, y \in \widetilde{\mathfrak{M}}$, we have

$$\begin{aligned}\mu(|x+y|) \leq \mu(u|x|u^* + v|y|v^*) &\prec\prec \mu(u|x|u^*) + \mu(v|y|v^*)\\ &\leq \ \mu(|x|)+\mu(|y|).\end{aligned}$$

Therefore

$$\begin{aligned}\widetilde{\rho}(x+y) = \rho(\mu(|x+y|)) &\leq \rho(\mu(u|x|u^* + v|y|v^*))\\ &\leq \rho(\mu(|x|)+\mu(|y|))\\ &\leq k\rho\left(\frac{\mu(|x|)+\mu(|y|)}{2}\right)\\ &\leq \frac{k}{2}(\rho(\mu(|x|)+\rho(\mu(|y|)))\\ &= \frac{k}{2}(\widetilde{\rho}(x)+\widetilde{\rho}(y)).\end{aligned}$$

□

Definition 3.5. The growth function $\omega_{\widetilde{\rho}}$ of the modular $\widetilde{\rho}$ is defined as follows:

$$\begin{aligned}\omega_{\widetilde{\rho}}(\alpha) &:= \sup\left\{\frac{\widetilde{\rho}(\alpha x)}{\widetilde{\rho}(x)} : 0 < \widetilde{\rho}(x) < +\infty\right\}\\ &= \sup\left\{\frac{\rho(\alpha\mu(|x|))}{\rho(\mu(|x|))} : 0 < \rho(\mu(|x|)) < +\infty\right\} \quad \text{for all } \alpha \geq 0.\end{aligned}$$

The next lemma can be easily proved.

Lemma 3.6. *Let ρ be a convex function modular with the Δ_2-type condition. Then the growth function $\omega_{\widetilde{\rho}}$ has the following properties:*

(i) $\omega_{\widetilde{\rho}}(\alpha) < +\infty$ *for every* $0 \leq \alpha < +\infty$.
(ii) $\omega_{\widetilde{\rho}} : [0,+\infty) \to [0,+\infty)$ *is a convex, strictly increasing function, so it is continuous.*
(iii) $\omega_{\widetilde{\rho}}(\alpha\beta) \leq \omega_{\widetilde{\rho}}(\alpha)\omega_{\widetilde{\rho}}(\beta)$ *for all* $\alpha, \beta \geq 0$.
(iv) $\omega_{\widetilde{\rho}}^{-1}(\alpha)\omega_{\widetilde{\rho}}^{-1}(\beta) \leq \omega_{\widetilde{\rho}}^{-1}(\alpha\beta)$ *for all* $\alpha, \beta \geq 0$, *where* $\omega_{\widetilde{\rho}}^{-1}$ *is the inverse function of* $\omega_{\widetilde{\rho}}$.

The following lemma shows that the growth function can be used to give an upper bound for the norm of an operator.

Lemma 3.7. *Let ρ be a convex function modular with the Δ_2-type condition. Then*

$$\|x\|_{\widetilde{\rho}} \leq \frac{1}{\omega_{\widetilde{\rho}}^{-1}\left(\frac{1}{\widetilde{\rho}(x)}\right)}$$

for every $x \in \mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$.

Proof. Assume that $\alpha < \|x\|_{\widetilde{\rho}} = \|\mu(x)\|_{\rho}$. We have $\widetilde{\rho}\left(\frac{x}{\alpha}\right) > 1$, which implies $\frac{1}{\widetilde{\rho}(x)} < \omega_{\widetilde{\rho}}\left(\frac{1}{\alpha}\right)$. Hence

$$\omega_{\widetilde{\rho}}^{-1}\left(\frac{1}{\widetilde{\rho}(x)}\right) < \frac{1}{\alpha}.$$

Letting $\alpha \to \|x\|_{\widetilde{\rho}}$, we obtain $\|x\|_{\widetilde{\rho}} \leq \frac{1}{\omega_{\widetilde{\rho}}^{-1}\left(\frac{1}{\widetilde{\rho}(x)}\right)}$. □

Proposition 3.8. *Let ρ be a convex function modular with the Δ_2-type condition. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|x\|_{\widetilde{\rho}} < \varepsilon \quad (\|x\|_{\widetilde{\rho}}^{A} < \varepsilon) \quad \textit{if } \widetilde{\rho}(x) < \delta.$$

Proof. For $\varepsilon > 0$, we choose $\delta = \frac{1}{\omega_{\widetilde{\rho}}^{-1}\left(\frac{1}{\varepsilon}\right)}$. For the Amemiya norm we use the fact that it is equivalent to the Luxemburg norm. □

Theorem 3.9. *Let ρ be a convex function modular with the Δ_2-type condition. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in the noncommutative modular function space $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$ such that $\widetilde{\rho}(y_n) \to 0$. Then*

$$\limsup_{n\to\infty} \widetilde{\rho}(x_n + y_n) = \limsup_{n\to\infty} \widetilde{\rho}(x_n).$$

Proof. It is known that $\mu(x_n+y_n) \prec\prec \mu(x_n)+\mu(y_n)$ [FK86, Theorem 4.4]. For every $\varepsilon \in (0,1)$ we have

$$\begin{aligned}
\widetilde{\rho}(x_n + y_n) = \rho(\mu(x_n + y_n)) &\leq \rho(\mu(x_n)) + \mu(y_n)) \\
&\leq \rho\left(\frac{\mu(x_n)}{1-\varepsilon}\right) + \rho\left(\frac{\mu(y_n)}{\varepsilon}\right) \\
&\leq \omega_{\widetilde{\rho}}\left(\frac{1}{1-\varepsilon}\right)\widetilde{\rho}(x_n) + \omega_{\widetilde{\rho}}\left(\frac{1}{\varepsilon}\right)\widetilde{\rho}(y_n),
\end{aligned}$$

and so

$$\limsup_{n\to\infty} \widetilde{\rho}(x_n + y_n) \leq \omega_{\widetilde{\rho}}\left(\frac{1}{1-\varepsilon}\right) \limsup_{n\to\infty} \widetilde{\rho}(x_n).$$

Since $\varepsilon > 0$ is arbitrary and

$$\omega_{\widetilde{\rho}}\left(\frac{1}{1-\varepsilon}\right) \to 1 \text{ as } \varepsilon \to 0^{+}$$

we get

$$\limsup_{n\to\infty} \widetilde{\rho}(x_n + y_n) \leq \limsup_{n\to\infty} \widetilde{\rho}(x_n).$$

Moreover, the same argument shows that

$$\limsup_{n\to\infty} \widetilde{\rho}(x_n) = \limsup_{n\to\infty} \widetilde{\rho}(x_n + y_n - y_n) \leq \limsup_{n\to\infty} \widetilde{\rho}(x_n + y_n).$$ □

Corollary 3.10. *Let φ satisfy the Δ_2-condition. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of τ-measurable operators in the noncommutative Orlicz space $L^{\varphi}(\widetilde{\mathfrak{M}}, \tau)$ such that $\tau(\varphi(y_n)) \to 0$. Then*

$$\limsup_{n\to\infty} \tau(\varphi(|x_n + y_n|)) = \limsup_{n\to\infty} \tau(\varphi(|x_n|)).$$

Before we give the main theorem of this section, we need the following lemma.

Lemma 3.11. *Let $\varepsilon > 0$ and $\lambda > 1$ be such that $\lambda\varepsilon < 1$. Then for every $x, y \in \mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$ such that $\widetilde{\rho}(\lambda x) < \infty$ and $\widetilde{\rho}\left(\frac{1}{\varepsilon(\lambda-1)} y\right) < \infty$, we have*

$$|\widetilde{\rho}(x + y) - \widetilde{\rho}(x)| \le \varepsilon\left[\widetilde{\rho}(\lambda x) - \lambda\widetilde{\rho}(x)\right] + \widetilde{\rho}(c_\varepsilon y),$$

where $c_\varepsilon = \frac{1}{\varepsilon(\lambda-1)}$.

Proof. The proof of this lemma is essentially the same as that of [BL83, Lemma 3] if $\mathbb{C}$ is replaced by $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$ and $j(x) = \widetilde{\rho}(x)$. □

Corollary 3.12. *Let $\varepsilon > 0$ and $\lambda > 1$ be such that $\lambda\varepsilon < 1$. Then for every $x, y \in \mathcal{L}^{\varphi}(\widetilde{\mathfrak{M}}, \tau)$ such that $\tau(\varphi(\lambda|x|)) < \infty$ and $\tau\left(\varphi\left(\frac{|y|}{\varepsilon(\lambda-1)}\right)\right) < \infty$, we have*

$$|\tau(\varphi(|x+y|)) - \tau(\varphi(|x|))| \le \varepsilon\left[\tau(\varphi(\lambda|x|)) - \lambda\tau(\varphi(|x|))\right] + \tau(\varphi(c_\varepsilon y)),$$

where $c_\varepsilon = \frac{1}{\varepsilon(\lambda-1)}$.

Definition 3.13. Let $\{x_n\}$ and x be in $\widetilde{\mathfrak{M}}$. Then the sequence $\{x_n\}$ is said to be $\widetilde{\rho}$-a.e. convergent to x if $\mu(x_n - x) \to 0$ ρ-a.e.

Theorem 3.14. *Let ρ be an additive convex modular on $\mathcal{L}_0(\nu)$ and $\{x_n\}$ be a sequence in $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$ which is $\widetilde{\rho}$-a.e.-convergent to 0. Assume that there exists $\lambda > 1$ such that $\sup_n \widetilde{\rho}(\lambda x_n) < \infty$. Then*

$$\lim_{n\to\infty} (\widetilde{\rho}(x_n + y) - \widetilde{\rho}(x_n)) = \widetilde{\rho}(y)$$

for all $y \in \mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$.

Proof. By the definition of the $\widetilde{\rho}$-a.e.-convergence we have that $\{\mu(x_n)\} \to 0$ ρ-a.e. It follows from Egoroff's theorem that there exists an increasing sequence of sets $E_k \in \mathcal{P}$ such that $\Omega = \bigcup_k E_k$ and $\{\mu(x_n)\}$ converges uniformly to 0 on every E_k. On the other hand we have

$$\begin{aligned}
&|\widetilde{\rho}(x_n + y) - \widetilde{\rho}(x_n) - \widetilde{\rho}(y)| \\
&\le |\rho(\mu(x_n + y), E_m) - \rho(\mu(x_n), E_m) - \rho(\mu(y), E_m)| \\
&\quad + |\rho(\mu(x_n + y), E_m^c) - \rho(\mu(x_n), E_m^c) - \rho(\mu(y), E_m^c)|
\end{aligned}$$

where E_m^c denotes the complement of the subset E_m. Using Lemma 3.11 we get

$$(3.3)\quad |\rho(\mu(x_n+y),E_m)-\rho(\mu(y),E_m)| \le \varepsilon|\rho(\lambda\mu(y),E_m)-\lambda\rho(\mu(y),E_m)| + \rho(c_\varepsilon\mu(x_n),E_m),$$

for every $\varepsilon>0$ such that $\lambda\varepsilon<1$. Since $\{\mu(x_n)\}$ converges uniformly to 0 on every E_m, we have

$$\limsup_{n\to\infty}|\rho(\mu(x_n+y),E_m)-\rho(\mu(x_n),E_m)-\rho(\mu(y),E_m)| \le \varepsilon\rho(\lambda\mu(y)).$$

Using the same strategy we get

$$\limsup_{n\to\infty}|\rho(\mu(x_n+y),E_m^c)-\rho(\mu(x_n),E_m^c)-\rho(\mu(y),E_m^c)| \le \varepsilon\limsup_{n\to\infty}\rho(\lambda\mu(x_n)+\rho(c_\varepsilon\mu(y),E_m^c)+\rho(c_\varepsilon\mu(y),E_m^c)+\rho(\mu(y),E_m^c).$$

Hence

$$\limsup_{n\to\infty}|\rho(\mu(x_n+y)-\rho(\mu(x_n)-\rho(\mu(y) \le \varepsilon\rho(\lambda\mu(y))+\varepsilon\sup_n\rho(\lambda\mu(x_n))+\rho(c_\varepsilon\mu(y),E_m^c)+\rho(\mu(y),E_m^c).$$

Let m tend to ∞ and use the fact that $y\in\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)$ to get

$$\limsup_{n\to\infty}|\rho(\mu(x_n+y))-\rho(\mu(x_n)-\rho(\mu(y)| \le \varepsilon\rho(\lambda\mu(y))+\varepsilon\sup_n\rho(\lambda\mu(x_n)).$$

Finally, we let ε approach 0 to get

$$\limsup_{n\to\infty}|\widetilde{\rho}(x_n+y)-\widetilde{\rho}(x_n)-\widetilde{\rho}(y)|\le 0,$$

which completes the proof. □

Remark 3.15. It is known that, for an Orlicz space, the ρ-null sets coincide with the sets of measure zero. Thus the ρ-a.e.-convergence is equivalent to the convergence almost everywhere. It follows from [FK86, Lemma 3.1] that the $\widetilde{\rho}$-a.e.-convergence in noncommutative Orlicz spaces as well as noncommutative L^p–spaces is equivalent to the convergence in the measure topology τ_m.

Corollary 3.16. *Let $\{x_n\}$ be a sequence in the noncommutative Orlicz space $L^\varphi(\widetilde{\mathfrak{M}},\tau)$ such that $\{x_n\}$ converges to x in the measure topology. Assume that there exists $\lambda>1$ such that $\sup_n\tau(\varphi(\lambda|x_n|))<\infty$. Then*

$$\liminf_{n\to\infty}\tau(\varphi(|x_n|))=\liminf_{n\to\infty}\tau(\varphi(|x_n-x|))+\tau(\varphi(|x|)).$$

Brezis and Lieb [BL83] proved that if $\{f_n\}$ is a sequence of L^P–uniformly bounded functions on a measure space and if $f_n \to f$ almost everywhere then

$$\liminf_{n\to\infty} \|f_n\|^p = \liminf_{n\to\infty} \|f_n - f\|^p + \|f\|^p,$$

for all $p \in (0,\infty)$. Now we establish an extension of this equality to the noncommutative setting.

Corollary 3.17. *Let $p \geq 1$ and $\{x_n\}$ be a bounded sequence of $L^p(\widetilde{\mathfrak{M}},\tau)$. Assume that $\{x_n\}$ converges to x in the measure topology. Then*

$$\liminf_{n\to\infty} \|x_n\|^p = \liminf_{n\to\infty} \|x_n - x\|^p + \|x\|^p.$$

We will say that a Banach space X with the τ–topology on X satisfies the Opial condition if for every bounded sequence $\{x_n\}$ in X which τ–converges to $x \in X$ we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for every $y \neq x$.

Theorem 3.18. *Let $\varepsilon > 0$ and $\{x_n\} \subseteq \mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)$ be $\widetilde{\rho}$-a.e. convergent to 0. Assume there exists $\lambda > 1$ such that*

$$\sup_n \widetilde{\rho}(\lambda x_n) < \infty.$$

Let $x \in \mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)$ such that $\widetilde{\rho}(x) > \varepsilon$. Then

$$\liminf_{n\to\infty} \widetilde{\rho}(x) + \varepsilon \leq \liminf_{n\to\infty} \widetilde{\rho}(x + x_n).$$

Proof. The proof is obvious using the conclusion of Theorem 3.14. This is a kind of Opial property. □

Corollary 3.19. *Let $\varepsilon > 0$ and $\{x_n\}$ be a sequence in the noncommutative Orlicz space $L^\varphi(\widetilde{\mathfrak{M}},\tau)$ such that $\{x_n\}$ converges to 0 in the measure topology. Assume that there exists $\lambda > 1$ such that $\sup \tau(\varphi(\lambda|x_n|)) < \infty$. Let $x \in L^\varphi(\widetilde{\mathfrak{M}},\tau)$ such that $\tau(\varphi(|x|)) > \varepsilon$. Then*

$$\liminf_{n\to\infty} \tau(\varphi(|x_n|)) + \varepsilon \leq \liminf_{n\to\infty} \tau(\varphi(|x_n + x|)).$$

4. Uniform Opial and uniform Kadec–Klee properties

In this section, we assume that ρ is a convex additive modular with the Δ_2-type condition on $\mathcal{L}_0(\nu)$ such that for every $f, g \in \mathcal{L}_0(\nu)$, $f \prec\prec g$ implies that $\rho(f) \leq \rho(g)$. Note that the additive condition may seem strong, but many interesting examples lead to additive modulars (e.g., any modular generated by a functional measure). In the paper [J04], the author showed that modular function spaces $\mathcal{L}_\rho$ satisfy the uniform Opial condition with respect to the ρ-a.e.-convergence for both the Luxemburg norm and Amemiya norm, and also showed that modular function spaces $\mathcal{L}_\rho$ have the uniform Kadec–Klee property with respect to the ρ-a.e.-convergence when equipped with the Luxemburg norm. We prove these results for noncommutative modular function spaces.

Before we give the main results of this section we need the following lemma. It is a generalization of [J04, Lemma 2.5] which was given for classical modular function spaces.

Lemma 4.1. *Let $\{x_n\}$ be a sequence in the noncommutative modular function space $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)$ such that $x_n \to 0$ $\widetilde{\rho}$-a.e. Let x be a given operator in $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)$. Then for every $\varepsilon > 0$ there exist a subsequence $\{x_{n_k}\}$, a sequence $\{y_k\}$ in $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)$ and an operator $y \in \mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)$ such that:*

(i) $\lim_{k\to\infty} \|x_{n_k} - y_k\|_{\widetilde{\rho}} = 0$.
(ii) $\operatorname{supp}(\mu(y_k)) \cap \operatorname{supp}(\mu(y)) = \emptyset$ *for every* $k \in \mathbb{N}$.
(iii) $\|x - y\|_{\widetilde{\rho}} < \varepsilon$.

Proof. Since the given sequence $\{x_n\}$ converges to 0 $\widetilde{\rho}$-a.e., $\mu(x_n) \to 0$ ρ-a.e. By using the version of the Egoroff theorem for modular function spaces, we can assume that there exists a sequence of sets $\Omega_k \in \mathcal{P}$ such that $\Omega = \cup_k \Omega_k$, $\Omega_i \cap \Omega_j = \emptyset$ and $\{\mu(x_n)\}$ converges uniformly to the null function on every Ω_k. Moreover, the Δ_2–type condition of the modular implies that $\widetilde{\rho}(x) = \rho(\mu_t(x)) < +\infty$ for every measurable operator $x \in \mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)$. Also, $\widetilde{\rho}(x) = \rho(\mu_t(x)) = \Sigma_k \rho(\mu_t(x),\Omega_k)$ since ρ is additive. Let $\varepsilon > 0$ be given. We will consider the corresponding $\delta > 0$ given in Proposition 3.8. Then there exists a positive integer k_0 such that

$$\sum_{k>k_0} \rho(\mu_t(x),\Omega_k) < \delta.$$

Let $\{\varepsilon_n\}$ be a sequence in $(0,1)$ with $\lim_{n\to\infty} \varepsilon_n = 0$ and for every $n \in \mathbb{N}$, take the corresponding δ_n given by Proposition 3.8. By Definition 2.3(iv), $\rho(\alpha, \cup_{k=1}^{k_0} \Omega_k) \to 0$ as α decreases to 0. Thus there exists a number $\alpha_0 > 0$ such that

$$\rho\left(\alpha_0, \bigcup_{k=1}^{k_0} \Omega_k\right) < \frac{\delta_1}{2}.$$

Due to the fact that $\{\mu_t(x_n)\}$ converges uniformly to 0 in $\cup_{k=1}^{k_0} \Omega_k$, there exists $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$, we have $|\mu_t(x_n)| \leq \alpha_0$ for every $t \in \cup_{k=1}^{k_0} \Omega_k$. Consider the measurable operator $x_{n_1} \in \mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)$ and a positive integer $k_1 > k_0$ such that

$$\sum_{k>k_1} \rho(\mu_t(x_{n_1}),\Omega_k) < \frac{\delta_1}{2}.$$

Define the operator $y_1 = x_{n_1} e_{k_1}$ where $e_{k_1} = e^{|x|}(\cup_{k>k_0}^{k_1} \Omega_k)$. Since

$$y_1 = x_{n_1} e_{k_1} = x_{n_1} - x_{n_1} e_{k_1}^{\perp},$$

we have

$$\mu(y_1) = \mu(x_{n_1} - x_{n_1}e_{k_1}^{\perp}) \prec\prec \mu(x_{n_1}) + \mu(x_{n_1}e_{k_1}^{\perp}) \leq 2\mu(x_{n_1}).$$

The convex additive modular ρ satisfies the Δ_2-type condition. Hence

$$\rho(\mu(y_1)) \leq \rho(2\mu(x_{n_1})) \leq k\rho(\mu(x_{n_1}))$$

for some $k > 0$. This ensures that $y_1 \in \mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$ since $x_{n_1} \in \mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$. So

$$\begin{aligned}
\widetilde{\rho}(x_{n_1} - y_1) &= \rho(\mu_t(x_{n_1} - y_1)) = \rho(\mu_t(x_{n_1}e_{k_1}^{\perp})) \\
&= \rho\left(\mu_t(x_{n_1}e_{k_1}^{\perp}), \bigcup_{k=1}^{k_0}\Omega_k \cup \bigcup_{k>k_0}^{k_1}\Omega_k \cup \bigcup_{k>k_1}^{\infty}\Omega_k\right) \\
&= \rho\left(\mu_t(x_{n_1}e_{k_1}^{\perp}), \bigcup_{k=1}^{k_0}\Omega_k \cup \bigcup_{k>k_1}^{\infty}\Omega_k\right) \\
&\leq \rho\left(\mu_t(x_{n_1}), \bigcup_{k=1}^{k_0}\Omega_k \cup \bigcup_{k>k_1}^{\infty}\Omega_k\right) \\
&\leq \rho\left(\alpha_0, \bigcup_{k=1}^{k_0}\Omega_k\right) + \sum_{k>k_1}\rho\left(\mu_t(x_{n_1}), \Omega_k\right) \\
&< \delta_1.
\end{aligned}$$

Suppose that, by induction, we have two finite sequences of positive integers $k_0 < k_1 < \cdots < k_{l-1}$ and $n_1 < n_2 < \cdots < n_{l-1}$ with $\widetilde{\rho}(x_{n_i} - y_i) < \delta_i$ for $i = 1, 2, \ldots, l-1$, where the operator y_i is defined by $y_i = x_{n_i}e^{|x|}(\cup_{k>k_{i-1}}^{k_i}\Omega_k)$. By Definition 2.3(iv), we can find α_l such that $\rho(\alpha_1, \cup_{k=1}^{k_{l-1}}\Omega_k) < \frac{\delta_l}{2}$. Since $\{\mu_t(x_n)\}$ converges uniformly to the null function on $\cup_{k=1}^{k_{l-1}}\Omega_k$, we can find $n_l > n_{l-1}$ such that $|\mu_t(x_{n_l})| \leq \alpha_l$ for every $t \in \cup_{k=1}^{k_{l-1}}\Omega_k$. Moreover, we can find $k_l > k_{l-1}$ with

$$\sum_{k>k_l}\rho\left(\mu_t(x_{n_l}), \Omega_k\right) < \frac{\delta_l}{2}.$$

Using the above argument, we define the operator $y_l = x_{n_l}e_{k_l} \in \mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$, where $e_{k_l} = e^{|x|}(\cup_{k>k_{l-1}}^{k_l}\Omega_k)$, and obtain $\widetilde{\rho}(x_{n_l} - y_l) < \delta_l$. Therefore

$$\|x_{n_l} - y_l\| < \varepsilon_l.$$

Consequently, $\lim_{k\to\infty}\|x_{n_k} - y_k\| = 0$.

Now, define the operator $y = xe_0$ where $e_0 = e^{|x|}(\cup_{k=1}^{k_0}\Omega_k)$. It is clear that $\operatorname{supp}(\mu(y_k)) \cap \operatorname{supp}(\mu(y)) = \emptyset$ for every $k \in \mathbb{N}$. Moreover

$$\widetilde{\rho}(x - y) = \rho(\mu(x - xe_0)) = \rho\left(\mu(xe_0^{\perp}), \cup_{k>k_0}\Omega_k\right)$$

$$\leq \rho\left(\mu(x), \cup_{k>k_0}\Omega_k\right) = \sum_{k>k_0} \rho(\mu(x), \Omega_k) < \delta,$$

whence $\|x-y\|_{\widetilde{\rho}} < \varepsilon$. □

Corollary 4.2. *Let φ satisfy the Δ_2-condition. Let $\{x_n\}$ be a sequence in the noncommutative Orlicz space $\mathcal{L}^{\varphi}(\widetilde{\mathfrak{M}}, \tau)$ converging to 0 in the measure topology. Let x be a given operator in $\mathcal{L}^{\varphi}(\widetilde{\mathfrak{M}}, \tau)$. Then for every $\varepsilon > 0$ there exist a subsequence $\{x_{n_k}\}$, a sequence $\{y_n\}$ in $\mathcal{L}^{\varphi}(\widetilde{\mathfrak{M}}, \tau)$ and an operator $y \in \mathcal{L}^{\varphi}(\widetilde{\mathfrak{M}}, \tau)$ such that:*

(i) $\lim_{k\to\infty} \|x_{n_k} - y_k\|_{\widetilde{\rho}_\varphi} = 0$.
(ii) $\operatorname{supp}(\mu(y_k)) \cap \operatorname{supp}(\mu(y)) = \emptyset$ *for every* $k \in \mathbb{N}$.
(iii) $\|x-y\|_{\widetilde{\rho}_\varphi} < \varepsilon$.

Theorem 4.3. *The noncommutative modular function space $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$ equipped with the Luxemburg norm $\|\cdot\|_{\widetilde{\rho}}$ satisfies the uniform Opial condition with respect to the $\widetilde{\rho}$-a.e.-convergence. In particular for every $c > 0$,*

$$1 + o_{\widetilde{\rho}}(c) \geq \omega_{\widetilde{\rho}}^{-1}(1+\alpha),$$

where $\alpha = \frac{1}{\omega_{\widetilde{\rho}}(\frac{1}{c})}$.

Proof. Let $c > 0$ and $\{x_n\}$ be a sequence in $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$ such that $x_n \to 0$ $\widetilde{\rho}$-a.e. and $\lim_{n\to\infty} \|x_n\|_{\widetilde{\rho}} \geq 1$. Consider an operator $x \in \mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$ with $\|x\|_{\widetilde{\rho}} \geq c$. We apply Lemma 4.1 to the sequence $\{x_n\}$ and some $0 < \varepsilon < c$ to obtain a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, a sequence $\{y_k\} \in \mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$ and an operator $y \in \mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$ such that $\lim_{k\to\infty} \|x_{n_k} - y_k\|_{\widetilde{\rho}} = 0$, $\operatorname{supp}(\mu(y_k)) \cap \operatorname{supp}(\mu(y)) = \emptyset$ for $k \in \mathbb{N}$ and $\|x-y\|_{\widetilde{\rho}} < \varepsilon$. We can also assume that $\liminf_{n\to\infty} \|x_n + x\|_{\widetilde{\rho}} = \lim_{n\to\infty} \|x_{n_k} + x\|_{\widetilde{\rho}}$. Consequently, $\liminf_{k\to\infty} \|y_k\|_{\widetilde{\rho}} \geq 1$ and $\|y\|_{\widetilde{\rho}} \geq c - \varepsilon$.

Fix $0 < t < 1$ and define the functions $z = \frac{y}{t}$ and $z_k = \frac{y_k}{t}$ for $k \in \mathbb{N}$. We know that $\|z\|_{\widetilde{\rho}} > c - \varepsilon$ and $\|z_k\|_{\widetilde{\rho}} > 1$ for k large enough. Set

$$\gamma_\varepsilon := \omega_{\widetilde{\rho}}^{-1}\left(1 + \frac{1}{\zeta}\right) \quad \text{where } \zeta = \omega_{\widetilde{\rho}}\left(\frac{1}{c-\varepsilon}\right).$$

Using the properties of the modular $\widetilde{\rho}$, the definition of the growth function $\omega_{\widetilde{\rho}}(\cdot)$ and Lemma 3.6(iii), we have

$$\begin{aligned}
\widetilde{\rho}\left(\frac{z_k + z}{\gamma_\varepsilon}\right) &= \rho\left(\frac{\mu(z_k + z)}{\gamma_\varepsilon}\right) \geq \rho\left(\frac{\mu(z_k) - \mu(z)}{\gamma_\varepsilon}\right) \\
&= \rho\left(\frac{\mu(z_k) - \mu(z)}{\gamma_\varepsilon}, \operatorname{supp}(\mu(z_k)) \cup \operatorname{supp}(\mu(z))\right) \\
&= \rho\left(\frac{\mu(z_k)}{\gamma_\varepsilon}, \operatorname{supp}(\mu(z_k))\right) + \rho\left(\frac{\mu(z)}{\gamma_\varepsilon}, \operatorname{supp}(\mu(z))\right) \\
&= \rho\left(\frac{\mu(z_k)}{\gamma_\varepsilon}\right) + \rho\left(\frac{\mu(z)}{\gamma_\varepsilon}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \widetilde{\rho}\left(\frac{z_k}{\gamma_\varepsilon}\right) + \widetilde{\rho}\left(\frac{z}{\gamma_\varepsilon}\right) \\
&\geq \frac{1}{\omega_{\widetilde{\rho}(\gamma_\varepsilon)}}\widetilde{\rho}(z_k) + \frac{1}{\omega_{\widetilde{\rho}(\frac{\gamma_\varepsilon}{c-\varepsilon})}}\widetilde{\rho}\left(\frac{z}{c-\varepsilon}\right) \\
&> \frac{1}{\omega_{\widetilde{\rho}(\gamma_\varepsilon)}} + \frac{1}{\omega_{\widetilde{\rho}(\gamma_\varepsilon)}\omega_{\widetilde{\rho}(\frac{1}{c-\varepsilon})}} = 1.
\end{aligned}
$$

Thus $\liminf_{k\to\infty} \|y_k + y\|_{\widetilde{\rho}} \geq t\gamma_\varepsilon$. Letting t tend to 1, we deduce that $\liminf_{k\to\infty} \|y_k + y\|_{\widetilde{\rho}} \geq \gamma_\varepsilon$. On the other hand,

$$
\begin{aligned}
\liminf_{k\to\infty} \|x_k + x\|_{\widetilde{\rho}} &= \lim_{k\to\infty} \|x_{n_k} + x\|_{\widetilde{\rho}} \\
&\geq \liminf_{k\to\infty} \|y_k + y\|_{\widetilde{\rho}} - \lim_{k\to\infty} \|x_{n_k} - y_k\|_{\widetilde{\rho}} - \|x - y\|_{\widetilde{\rho}} \\
&\geq \gamma_\varepsilon - \varepsilon.
\end{aligned}
$$

Since ε is arbitrary and the functions $\omega_{\widetilde{\rho}}(\cdot)$ and $\omega_{\widetilde{\rho}}^{-1}(\cdot)$ are continuous, we infer that

$$
\liminf_{k\to\infty} \|x_k + x\|_{\widetilde{\rho}} \geq \omega_{\widetilde{\rho}}^{-1}\left(1 + \frac{1}{\omega_{\widetilde{\rho}}(\frac{1}{c})}\right),
$$

which yields the desired lower bound for $1 + o_{\widetilde{\rho}}(c)$. This implies that $(\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau), \|\cdot\|_{\widetilde{\rho}})$ has the uniform Opial condition with respect to the $\widetilde{\rho}$-a.e.-convergence □

Theorem 4.4. *The noncommutative modular function space $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$ equipped with the Amemiya norm $\|\cdot\|_{\widetilde{\rho}}^A$ satisfies the uniform Opial condition with respect to the $\widetilde{\rho}$-a.e.-convergence. In particular for every $c > 0$,*

$$
1 + o_{\widetilde{\rho}}(c) \geq \omega_{\widetilde{\rho}}^{-1}(1 + \alpha),
$$

where $\alpha = \frac{1}{\omega_{\widetilde{\rho}}(\frac{1}{c})}$.

Proof. Let $\{x_n\}$ be a sequence in $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$ such that $x_n \to 0$ $\widetilde{\rho}$-a.e. and $\liminf \|x_n\|_{\widetilde{\rho}}^A \geq 1$. Consider an operator $x \in \mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$ with $\|x\|_{\widetilde{\rho}}^A \geq c$. By Lemma 4.1 and using the same argument as in the proof Theorem 4.3, we can assume that $\operatorname{supp}(\mu(x_n)) \cap \operatorname{supp}(\mu(x)) = \emptyset$ and $\|x\|_{\widetilde{\rho}}^A \geq 1$ for every $n \in \mathbb{N}$. Recall that

$$
\|x\|_{\widetilde{\rho}}^A = \inf\left\{\frac{1 + \widetilde{\rho}(\lambda x)}{\lambda} : \lambda > 0\right\}
$$

for every $x \in \mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$. Then if x is an operator in $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$ with $\|x\|_{\widetilde{\rho}}^A \geq 1$, we have that $\widetilde{\rho}(\lambda x) \geq \lambda - 1$ for every $\lambda > 0$.

Set $\gamma := \omega_{\widetilde{\rho}}^{-1}\left(1+\frac{1}{\omega_{\widetilde{\rho}}(\frac{1}{c})}\right)$ and fix $\lambda > 0$. Using the properties of the modular $\widetilde{\rho}$ and the growth function $\omega_{\widetilde{\rho}}(\cdot)$, we get

$$\begin{aligned}
\frac{1}{\lambda}\left[1+\widetilde{\rho}\left(\frac{\lambda(x_n+x)}{\gamma}\right)\right] &= \frac{1}{\lambda}\left[1+\widetilde{\rho}\left(\frac{\lambda x_n}{\gamma}\right)+\widetilde{\rho}\left(\frac{\lambda x}{\gamma}\right)\right] \\
&\geq \frac{1}{\lambda}\left[1+\frac{\widetilde{\rho}(\lambda x_n)}{\omega_{\widetilde{\rho}}(\gamma)}+\frac{\widetilde{\rho}(\frac{\lambda x}{c})}{\omega_{\widetilde{\rho}}(\frac{\gamma}{c})}\right] \\
&\geq \frac{1}{\lambda}\left[1+\frac{\lambda-1}{\omega_{\widetilde{\rho}}(\gamma)}+\frac{\lambda-1}{\omega_{\widetilde{\rho}}(\frac{\gamma}{c})}\right] \\
&\geq \frac{1}{\lambda}\left[1+(\lambda-1)\left(\frac{1}{\omega_{\widetilde{\rho}}(\gamma)}+\frac{1}{\omega_{\widetilde{\rho}}(\gamma)\omega_{\widetilde{\rho}}(\frac{1}{c})}\right)\right] \\
&= 1.
\end{aligned}$$

Taking infimum over all $\lambda > 0$, we infer that $\liminf_{n\to\infty}\|x_n+x\|_{\widetilde{\rho}}^{A} \geq \gamma$, which completes the proof. $\square$

Corollary 4.5. *Let φ satisfy the Δ_2–condition. Then the noncommutative Orlicz space $L^{\varphi}(\widetilde{\mathfrak{M}},\tau)$ equipped with the Luxemburg and Amemiya norms satisfies the uniform Opial condition with respect to the convergence in measure.*

Theorem 4.6. *The noncommutative modular function space $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)$ equipped with the Luxemburg norm $\|\cdot\|_{\widetilde{\rho}}$ has the uniform Kadec–Klee property with respect to the $\widetilde{\rho}$-a.e.-convergence. Moreover, for every $\varepsilon > 0$,*

$$k_{\widetilde{\rho}}(\varepsilon) \geq 1-\frac{1}{\omega_{\widetilde{\rho}}(\frac{1}{1-\zeta})},$$

where $\zeta = \frac{1}{2\omega_{\widetilde{\rho}}(\frac{1}{\varepsilon})}$.

Proof. Let $\varepsilon > 0$ and $\{x_n\}$ be a sequence in the unit ball of $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)$ with $\operatorname{sep}(x_n) > \varepsilon$. Assume that $\{x_n\}$ is convergent to some $x \in \mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}},\tau)$ $\widetilde{\rho}$-a.e. Consider any real number $t > 0$ and define $y_n := \frac{x_n}{1+t}$ and $y := \frac{x}{1+t}$. In this case $\|y_n\|_{\widetilde{\rho}} < 1$ and $\widetilde{\rho}(y_n) \leq \|y_n\| < 1$ for every $n \in \mathbb{N}$. Put $\xi = \frac{\varepsilon}{1+t}$. Since $\|y_n - y_m\| \geq \xi$ for every $n \neq m$, we see that $\widetilde{\rho}\left(\frac{y_n-y_m}{\xi}\right) > 1$ for every $n \neq m$. This implies that

$$\omega_{\widetilde{\rho}}\left(\frac{1}{\xi}\right) = \sup\left\{\frac{\widetilde{\rho}(\frac{x}{\xi})}{\widetilde{\rho}(x)} : 0 < \widetilde{\rho}(x) < \infty\right\} \geq \frac{\widetilde{\rho}\left(\frac{y_n-y_m}{\xi}\right)}{\widetilde{\rho}\left(y_n-y_m\right)} > \frac{1}{\widetilde{\rho}\left(y_n-y_m\right)},$$

and consequently $\widetilde{\rho}\,(y_n - y_m) > \omega_{\widetilde{\rho}}\left(\frac{1}{\xi}\right)$ for every $n \neq m$. If we consider $m \in \mathbb{N}$ and use Theorem 3.14, we obtain

$$\begin{aligned}\frac{1}{\omega_{\widetilde{\rho}}}\left(\frac{1}{\xi}\right) &\leq \liminf_{n\to\infty} \widetilde{\rho}\,(y_n - y_m) = \liminf_{n\to\infty} \widetilde{\rho}\,((y_n - y) - (y_m - y)) \\ &= \liminf_{n\to\infty} \widetilde{\rho}\,(y_n - y) + \widetilde{\rho}(y_m - y).\end{aligned}$$

Letting m tend to infinity, we get

$$\liminf_{n\to\infty} \widetilde{\rho}\,(y_n - y) \geq \frac{1}{2\omega_{\widetilde{\rho}}(\frac{1}{\xi})}.$$

On the other hand, using Theorem 3.14, we have

$$\liminf_{n\to\infty} \widetilde{\rho}(y_n) = \liminf_{n\to\infty} \widetilde{\rho}(y_n - y + y) = \liminf_{n\to\infty} \widetilde{\rho}(y_n - y) + \widetilde{\rho}(y).$$

Thus

$$0 \leq \widetilde{\rho}(y) = \liminf_{n\to\infty} \widetilde{\rho}(y_n) - \liminf_{n\to\infty} \widetilde{\rho}(y_n - y) \leq 1 - \frac{1}{2\omega_{\widetilde{\rho}}(\frac{1}{\xi})}.$$

Employing Lemma 3.7 and the fact that the function $\omega_{\widetilde{\rho}}^{-1}$ is increasing we get

$$\|x\|_{\widetilde{\rho}} \leq (1+t)\frac{1}{\omega_{\widetilde{\rho}}(\frac{1}{1-\zeta})},$$

where $\zeta = \frac{1}{2\omega_{\widetilde{\rho}}(\frac{1}{\xi})}$. Since $t > 0$ is arbitrary and the functions $\omega_{\widetilde{\rho}}$ and $\omega_{\widetilde{\rho}}^{-1}$ are continuous, it follows that

$$\|x\|_{\widetilde{\rho}} \leq \frac{1}{\omega_{\widetilde{\rho}}(\frac{1}{1-\zeta})} := 1 - \delta(\varepsilon),$$

where $\zeta = \frac{1}{2\omega_{\widetilde{\rho}}(\frac{1}{\xi})}$. In order to finish the proof we have to check that $1-\delta(\varepsilon) < 1$. Taking into account that $\omega_{\widetilde{\rho}}$ is an increasing function and that $\omega_{\widetilde{\rho}}(1) = 1$, the above inequality holds if and only if $\omega_{\widetilde{\rho}}(\frac{1}{\varepsilon}) > 0$. This condition is satisfied due to the Δ_2–type condition. Indeed, since we are assuming that there exists some $k > 0$ such that $\widetilde{\rho}(2x) \leq k\widetilde{\rho}(x)$ for every $x \in \mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$, we obtain

$$\omega_{\widetilde{\rho}}\left(\frac{1}{2}\right) = \sup\left\{\frac{\widetilde{\rho}(\frac{x}{2})}{\widetilde{\rho}(x)} : 0 < \widetilde{\rho}(x) < \infty\right\} \geq \frac{1}{k} > 0.$$

It follows from $0 < \varepsilon \leq 2$ that $\omega_{\widetilde{\rho}}(\frac{1}{\varepsilon}) \geq \omega_{\widetilde{\rho}}(\frac{1}{2}) > 0$ as required. □

Corollary 4.7. *Let φ satisfy the Δ_2-condition. Then the noncommutative Orlicz space $L^{\varphi}(\widetilde{\mathfrak{M}}, \tau)$ equipped with the Luxemburg norm has the uniform Kadec–Klee property with respect to the convergence in measure.*

Acknowledgement. The authors are grateful to the reviewer and the area editor for their valuable comments and suggestions.

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This paper is available via http://nyjm.albany.edu/j/2015/21-33.html.