New York Journal of Mathematics

New York J. Math. **21** (2015) 351–367.

# On a generalization of the Gasca–Maeztu conjecture

**Hakop Hakopian** and **Levon Rafayelyan**

Abstract. Denote the space of all bivariate polynomials of total degree not exceeding $n$ by $\Pi_n$. The Gasca–Maeztu conjecture [Gasca M. and Maeztu J. I., On Lagrange and Hermite interpolation in $\mathbb{R}^k$, *Numer. Math.* **39** (1982), 1–14.] states that any $\Pi_n$-poised set of nodes, all fundamental polynomials of which are products of linear factors, possesses a maximal line, i.e., a line passing through $n+1$ nodes. Till now it is proved to be true for $n \leq 5$. The case $n = 5$ was proved recently in [Hakopian H., Jetter K. and Zimmermann G., The Gasca–Maeztu conjecture for $n = 5$, *Numer. Math.* **127** (2014), 685–713]. In an earlier paper the following generalized conjecture was proposed by the authors of the present paper: Any $\Pi_n$-poised set of nodes, all fundamental polynomials of which are reducible, possesses a maximal curve of some degree $k$, $1 \leq k \leq n-1$, i.e., an algebraic curve passing through $(1/2)k(2n-k+3)$ nodes. Clearly the two above conjectures coincide in the case $n \leq 2$. In this paper we prove that the generalized conjecture is true for $n = 3$.

Contents



Received February 14, 2015.

2010 *Mathematics Subject Classification.* Primary: 41A05, 41A63; Secondary 14H50.

*Key words and phrases.* Gasca–Maeztu conjecture, polynomial interpolation, algebraic curves, maximal line, maximal curve, poised, independent nodes.

ISSN 1076-9803/2015

## 1. Introduction

The bivariate (multivariate) polynomial interpolation is much more involved than the respective univariate one. A poised interpolation with a set of nodes and a polynomial space means that there is, for arbitrary data at those nodes, exactly one polynomial from the space that matches the given information. In order to have an $n$ — poised interpolation with $\Pi_n$, the space of bivariate algebraic polynomials of total degree not exceeding $n$, the number of interpolation nodes has to fit the dimension of the space:

$$N := \dim \Pi_n = \frac{(n+2)(n+1)}{2}.$$

In contrast with the univariate interpolation, even if this is the case, the poisedness of the bivariate interpolation essentially depends on the geometrical distribution of nodes. Thus, a new problem arises, which is permanently actual in the subject — the identification of simple constructions of poised node sets.

The Gasca–Maeztu conjecture concerns perhaps the simplest such construction — the so called $GC_n$ construction, based on a geometric condition. Namely, in $GC_n$ set every fundamental polynomial is a product of $n$ linear factors, as it always takes place in the univariate case. Geometrically this means that for each fixed node there are $n$ lines which pass through all the nodes of the set but the fixed one. The conjecture states that each $GC_n$ set possesses a maximal line, i.e., line passing through $n+1$ nodes. The conjecture is equivalent to the statement that each $GC_n$ set is a particular case of an extremely simple construction — the Berzolari–Radon construction (see Definition 1.0.6). So far the conjecture was proved to be true for $n \leq 5$. Let us mention that settling the conjecture for particular $n$ enables also the classification of $GC_n$ sets of the respective order (see [3]).

In this paper a generalization of the Gasca–Maeztu conjecture is considered in terms of higher order curves. We call a node set $GC_n^*$ set if every fundamental polynomial is reducible, i.e., is the product of two or more nontrivial factors. The generalized conjecture claims that (see Conjecture 2.0.11), for each $GC_n^*$ set, there is a maximal curve of some degree k, with $1 \leq k \leq n-1$, i.e., a curve containing as much as $d(n,k) := (1/2)k(2n+3-k)$ nodes. This is maximal possible number of nodes given that the node set is $n$-poised.

The generalized conjecture in its turn is equivalent to the statement that each $GC_n^*$ set is a particular case of another simple known construction, which is a generalization of the Berzolari–Radon construction (see [15]).

The present paper settles the generalized conjecture for $n = 3$, which is the first interesting case, stating that under the assumption of reducibility, there is a maximal line (containing n + 1 nodes), or a maximal conic (containing 2n + 1 nodes).

Consider a set of nodes (points):

$$\mathcal{T}_s := \{(x_1, y_1), \dots, (x_s, y_s)\}.$$

The interpolation problem consists of finding a polynomial $p \in \Pi_n$ such that

$$p(x_i, y_i) = c_i, \quad i = 1, 2, \dots, s. \tag{1.0.1}$$

**Definition 1.0.1.** A set of nodes $\mathcal{T}_s$ is called $\Pi_n$-*poised*, or briefly $n$-*poised*, if for any data $\bar{c} = \{c_1, \dots, c_s\}$ there exists a unique polynomial $p \in \Pi_n$ satisfying the conditions (1.0.1).

A necessary condition for $n$-poisedness of $\mathcal{T}_s$ is: $s = N$.

Denote by $p|_{\mathcal{T}}$ the restriction of $p$ to $\mathcal{T}$.

A polynomial $p \in \Pi_n$ is called $n$-*fundamental* for $T = (x_k, y_k) \in \mathcal{T}_s$ if

$$p|_{\mathcal{T}_s \setminus \{T\}} = 0, \quad p(T) = 1.$$

We denote by $p_T^\star := p_k^\star$ this fundamental polynomial.

Now let us consider an important concept for node sets.

**Definition 1.0.2.** A set of nodes $\mathcal{T}_s$ is called $n$-*independent*, if all its nodes have $n$-fundamental polynomials: $p_i^\star \in \Pi_n$, $i = 1, \dots, s$.

The fundamental polynomials are linearly independent. Therefore a necessary condition of $n$-independence of $\mathcal{T}_s$ is: $s \le N$. Clearly any $n$-poised set is $n$-independent. We also have that $\mathcal{T}_s$ is $n$-independent if and only if the interpolation problem (1.0.1) is solvable, meaning that for any data $\bar{c}$ there exists a polynomial $p \in \Pi_n$ (not unique, if $s < N$) satisfying the conditions (1.0.1).

In the sequel we will need the following proposition (see [6], Proposition 1, see also [9], Theorem 9, for the case of multiple nodes).

**Proposition 1.0.3.** *Any set of $k$ nodes, with $k \le 2n + 1$, in the plane, is $n$-independent if and only if no $n + 2$ of them are collinear.*

Next we present so called $GC_n$ *sets* introduced by Chung and Yao [5].

**Definition 1.0.4.** An $n$-poised set of nodes $\mathcal{T}$ is called a $GC_n$ set, if the $n$-fundamental polynomial of each its node is a product of $n$ linear factors.

We say that a node $T \in \mathcal{T}$ *uses* an algebraic curve $q$ of degree $k$ if the latter divides the fundamental polynomial of $T$, i.e., $p_T^\star = qr$ for some $r \in \Pi_{n-k}$. Thus each node of a $GC_n$ set uses $n$ lines.

The Gasca–Maeztu conjecture is the following [7]:

**Conjecture 1.0.5** (Gasca–Maeztu)**.** If $\mathcal{T}$ is a $GC_n$ set, then there is at least one line $l$ such that $\#(\mathcal{T} \cap l) = n + 1$.

So far this conjecture has been verified only for $n \le 5$ (see [1],[2],[11] for $n \le 4$ and [12] for $n = 5$). In fact, the conjecture states that every $GC_n$ set is a particular case of a very simple construction of $n$-poised sets, called *Berzolari–Radon* (see [4]):

**Definition 1.0.6.** A set of $N = 1 + \cdots + (n+1)$ nodes is called Berzolari–Radon set for degree $n$, or briefly B-R set, if there exist lines $l_1, l_2, \ldots, l_{n+1}$, such that the sets $l_1,\ l_2 \setminus l_1,\ l_3 \setminus (l_1 \cup l_2), \ldots,\ l_{n+1} \setminus (l_1 \cup \cdots \cup l_n)$ contain exactly $(n+1), n, n-1, \ldots, 1$ nodes, respectively.

## 2. Maximal curves and the generalized conjecture

Let us start with the following well-known statement.

**Proposition 2.0.7.** *Assume that $l$ is a line and $\mathcal{T}_{n+1}$ is any subset of $l$ containing $n+1$ points. Then we have that*

$$p \in \Pi_n \quad \textit{and} \quad p|_{\mathcal{T}_{n+1}} = 0 \ \Rightarrow \quad p = lr,\ \textit{where } r \in \Pi_{n-1}.$$

Denote

$$d := d(n,k) := \dim \Pi_n - \dim \Pi_{n-k} = (1/2)k(2n+3-k).$$

The following is a generalization of Proposition 2.0.7.

**Proposition 2.0.8** ([15], Proposition 3.1)**.** *Let $q$ be an algebraic curve of degree $k \le n$ without multiple components. Then the following hold:*

(i) *Any subset of $q$ containing more than $d(n,k)$ nodes is $n$-dependent.*
(ii) *Any subset $\mathcal{T}_d$ of $q$ containing exactly $d(n,k)$ nodes is $n$-independent if and only if the following condition holds:*

$$p \in \Pi_n \quad \textit{and} \quad p|_{\mathcal{T}_d} = 0 \Rightarrow p = qr,\ \textit{where } r \in \Pi_{n-k}.$$

Suppose that $\mathcal{T}$ is an $n$-poised set of nodes and $q$ is an algebraic curve of degree $k \le n$. Then of course any subset of $\mathcal{T}$ is $n$-independent too. Therefore, according to Proposition 2.0.8(i), at most $d(n,k)$ nodes of $\mathcal{T}$ can lie in the curve $q$. Let us mention that a special case of this when $q$ is a set of $k$ lines is proved in [3]. This motivates the following (see [15], Def.3.1).

**Definition 2.0.9.** Given an $n$-poised set of nodes $\mathcal{T}$. A curve of degree $k \le n$ is called *maximal* if it passes through $d(n,k)$ nodes of the set $\mathcal{T}$.

We have that $d(n,1) = n+1,\ d(n,2) = 2n+1,\ d(n,3) = 3n$. In view of Proposition 1.0.3, any set of $n+1$ nodes located in a line is $n$-independent. Note that a maximal line, as a line passing through $n+1$ nodes of $\mathcal{T}$, is defined in [2] (see also [10] for the case of general dimension). Any irreducible conic, i.e., conic which is not a pair of lines, contains at most two collinear points. Hence by Proposition 1.0.3, any set of $2n+1$ nodes located in an irreducible conic is $n$-independent. In the case of cubics (and similarly in the case of curves of higher degree) we already deal with a new phenomenon. Namely, not any set of $3n$ nodes in an irreducible cubic is $n$-independent (see [13]). Since $d(n,n) = N-1$ we have that each $n$-fundamental polynomial of any $n$-poised set $\mathcal{T}$ is a maximal curve of degree $n$.

Next we bring a characterization of maximal curves:

**Proposition 2.0.10** ([15], Prop. 3.3)**.** *Let a node set $\mathcal{T}$ be $n$-poised. Then a polynomial $q$ of degree $k,\ k \le n$, is a maximal curve if and only if it is used by any node in $\mathcal{T} \setminus q$.*

Note that one side of this statement follows from Proposition 2.0.8(ii). In the case of lines this was proved in ([2]). For other properties of maximal curves we refer reader to [15], where (in Conjecture 7.2) we propose the following generalized:

**Conjecture 2.0.11** (H.H., L.R.)**.** Suppose that $\mathcal{T}$ is an $n$-poised node set and the fundamental polynomial of each node is reducible. Then $\mathcal{T}$ possesses a maximal curve of some degree $k,\ 1 \le k \le n-1$, i.e., a curve $q$ such that $\#(\mathcal{T} \cap q) = d(n,k)$.

Note the degree of the maximal curve here does not exceed $n-1$ and the same estimate holds for the degrees of factors of fundamental polynomials. By taking into account this fact we put forward the following refined:

**Conjecture 2.0.12.** Suppose that $\mathcal{T}$ is an $n$-poised node set and the fundamental polynomial of each node is a product of factors whose degrees do not exceed $m$, where $1 \le m \le n-1$. Then $\mathcal{T}$ possesses a maximal curve of some degree $k,\ 1 \le k \le m$.

Clearly this conjecture coincides with the Gasca–Maeztu conjecture and Conjecture 2.0.11 if $m = 1$ and $m = n-1$, respectively.

**2.1. The generalized conjecture for $n = 3$.** We start this subsection with the particular case $n = 3$ of Conjecture 2.0.11 (or, which is the same, case $n = 3,\ m = 2$ of Conjecture 2.0.12).

**Theorem 2.1.1.** *Suppose that a node set $\mathcal{T}$ is $\Pi_3$-poised and the fundamental polynomial of each node is reducible. Then $\mathcal{T}$ possesses a maximal curve of degree $\le 2$, i.e., a maximal line or a maximal conic.*

This is our main result and will be proved in Sections 3–4. Note that a $\Pi_3$-poised set contains 10 nodes, while a maximal line, in case $n = 3$, passes through 4 nodes and a maximal conic passes through 7 nodes.

The following three simple lemmas will be used frequently in the sequel.

**Lemma 2.1.2.** *Assume that a node set $\mathcal{T}$ is $\Pi_3$-poised and 2 nodes in $\mathcal{T}$ use the same line. Then $\mathcal{T}$ possesses a maximal line.*

**Proof.** Suppose that two nodes $T_0$ and $T_1 \in \mathcal{T}$ use a line $l$: $p_0^\star = lq_0 \quad p_1^\star = lq_1$, where $q_0, q_1 \in \Pi_2$. Assume also that $l$ passes through $\le 3$ nodes, since otherwise it is maximal. Then both $q_0$ and $q_1$ vanish at the set $\mathcal{S} := \mathcal{T} \backslash (\{T_0, T_1\} \cup l)$ containing $\ge 5$ nodes. Now, if the nodes in $\mathcal{S}$ are 2-independent then $q_0$ and $q_1$ determine the same conic, which means that $p_0^\star$ and $p_1^\star$ are linearly dependent, leading to a contradiction. Otherwise the nodes are 2-dependent and by Proposition 1.0.3, four of them are collinear. □

**Lemma 2.1.3.** *Assume that a node set $\mathcal{T}$ is $\Pi_3$-poised. If a node $T \in \mathcal{T}$ uses a line $l$ passing through exactly 3 nodes and there exists a line $l'$ passing through exactly 3 nodes in $\mathcal{T}\backslash(\{T\} \cup l)$, then the following hold:*

(i) *The node $T$ uses the line $l'$.*
(ii) *The remaining 3 nodes in $\mathcal{T}\backslash(\{T\} \cup l \cup l')$ lie in a line $l''$ and $T$ uses the line $l''$ too.*

**Proof.** We have $p^\star_T = lq$, where $q \in \Pi_2$ vanishes at the 3 nodes in $l'$. Thus, according to Proposition 2.0.7 we have that $q = l'l''$, with $l'' \in \Pi_1$. Therefore $p^\star_T = ll'l''$ implying that the remaining 3 nodes are in $l''$, since none of them lies in $l$ or $l'$ by assumption. □

**Lemma 2.1.4.** *Assume that a node set $\mathcal{T}$ is $\Pi_3$-poised without a maximal line and a maximal conic. Then the following hold:*

(i) *Each used line passes through exactly three nodes.*
(ii) *If $l$ and $l'$ are two lines, both used by a node of $\mathcal{T}$, then $l \cap l' \cap \mathcal{T} = \emptyset$.*

**Proof.** Suppose, for (i), that a node $T \in \mathcal{T}$ uses a line $l$ passing just through 2 nodes, then $p^\star_T = lq$, where the conic $q \in \Pi_2$ passes through 7 nodes of $\mathcal{T}\backslash(\{T\} \cup l)$, and hence is maximal, which contradicts our assumption.

For (ii), suppose that $l$ and $l'$ are two lines used by $T \in \mathcal{T}$, i.e., $p^\star_T = ll'l''$, where $l'' \in \Pi_1$. Now, assume by way of contradiction that $l \cap l' \in \mathcal{T}$, hence there are only 5 nodes in $l \cup l'$. Then $l''$ passes through the 4 nodes of $\mathcal{T}\backslash(\{T\} \cup l \cup l')$, and is maximal, which contradicts our assumption. □

### 2.2. Alternatives 1 and 2.

Let us start the proof of Theorem 2.1.1. From now on we shall assume that

$$\mathcal{T} \text{ has no maximal lines or conics,} \tag{2.2.1}$$

in order to derive a contradiction.

Now let as present the following proposition which is important for the later consideration.

**Proposition 2.2.1.** *Assume that each fundamental polynomial of a $\Pi_3$-poised node set $\mathcal{T}$ with (2.2.1) is reducible. Assume also that no node of $\mathcal{T}$ is intersection point of 4 used lines. Then the following hold.*

(1) *There are exactly 10 used lines.*
(2) *On each used line there are exactly 3 nodes.*
(3) *Each node is an intersection point of exactly 3 used lines.*
(4) *Each node uses a line and an irreducible conic.*

**Proof.** Note that the reducibility of fundamental polynomials in the case of degree 3 means that each node uses either 3 lines, or a line and an irreducible conic. By taking onto account (2.2.1), we get from Lemma 2.1.2 that there are at least 10 distinct used lines. Also we get that each node uses a line and an irreducible conic, if there are exactly 10 used lines. According to Lemma 2.1.4(i) each used line passes through exactly 3 nodes. Therefore in

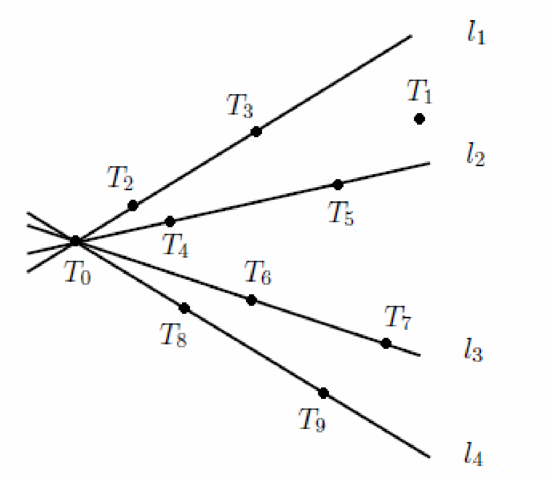


FIGURE 1. Four concurrent lines.

the case of exactly 10 distinct used lines the total number of nodes belonging to them equals 30. Thus we get that on average a node of $\mathcal{T}$ is in 3 used lines, i.e., is an intersection point of 3 used lines. Thus if there are more than 10 distinct used lines, or if a node is an intersection point of less than 3 used lines, then there is a node, which is an intersection point of four used lines, contrary to our assumption. □

In view of Proposition 2.2.1 we are to proceed in the following two alternative directions:

*Alternative* 1. Four used lines intersect at a node of $\mathcal{T}$.

*Alternative* 2. $\mathcal{T}$ satisfies the conditions (1)–(4) of Proposition 2.2.1.

We consider these two cases in the forthcoming Sections 3 and 4, respectively.

## 3. Alternative 1 — proof of the main result

Assume that we have a set of four used lines $\mathcal{L} = \{l_1, l_2, l_3, l_4\}$ that intersect at a node $T_0 \in \mathcal{T}$. Assume also, in view of Lemma 2.1.4(i) and assumption (2.2.1), that the two nodes in $l_1, l_2, l_3, l_4$, besides $T_0$, are $T_2, T_3$; $T_4, T_5$; $T_6, T_7$; $T_8, T_9$, respectively. Denote the node which does not belong to the lines of $\mathcal{L}$ by $T_1$ (see Figure 1).

Let us start with:

**Lemma 3.0.2.** *Suppose that $\mathcal{M}$ is a set of 3 or 4 lines from $\mathcal{L}$. Suppose also that a node $T$ not belonging to the lines of $\mathcal{M}$ does not use any line of $\mathcal{L}$. Then the following hold:*

(i) *The set $\mathcal{M}$ consists of 3 lines, i.e., $\#\mathcal{M} = 3$.*

(ii) *Any line used by $T$ intersects each of the three lines of $\mathcal{M}$ at a node, different from $T_0$.*

**Proof.** Suppose that the node $T$ uses a line $l$. Since $l \notin \mathcal{L}$ we have, in view of Lemma 2.1.4(i), that $l$ is not passing through $T_0$. Then, by Lemma 2.1.3(i), $l$ passes through one node from each line of $\mathcal{M}$, since otherwise $T$ uses it, contrary to our assumption. Now, in view of Lemma 2.1.4(i), we obtain that $\#\mathcal{M} = 3$. □

**Lemma 3.0.3.** *The following hold:*

(i) *No node of $\mathcal{T}$ can use 2 lines from $\mathcal{L}$.*
(ii) *The node $T_1$ uses a line from $\mathcal{L}$.*

**Proof.** Statement (i) follows immediately from Lemma 2.1.4(ii). In order to verify statement (ii), assume that $T_1$ does not use any line of $\mathcal{L}$. Then by setting $\mathcal{M} = \mathcal{L}$, we obtain a contradiction in view of Lemma 3.0.2(i). □

Now we may assume without loss of generality:

($\diamond$) The node $T_1$ uses the line $l_1$.

With that assumption, $p_1^\star = l_1 q$, with q a quadratic polynomial that must vanish at the 6 nodes $T_4, \ldots, T_9$ at which $p_1^\star$ vanishes but $l_1$ does not. In other words:

(3.0.2) The nodes $T_4, T_5, T_6, T_7, T_8, T_9$ are in a conic (possibly reducible).

Now, we turn to the nodes in the line $l_1$.

**Lemma 3.0.4.** *Each of the nodes $T_2, T_3$, lying in the line $l_1$, uses a line of $\mathcal{L}$.*

**Proof.** Suppose one of the nodes $T_2, T_3$, say $T_2$, does not use any line of $\mathcal{L}$. Assume that $l$ is a line used by $T_2$, where $l \notin \mathcal{L}$. Then from Lemma 3.0.2(ii), with $\mathcal{M} = \{l_2, l_3, l_4\}$, we get that the line $l$ intersects each of the lines $l_i,\ i = 2, 3, 4$, at a node, different from $T_0$. Hence 3 of 6 nodes mentioned in (3.0.2), which belong to a conic, are collinear. Thus the conic is reducible and the other 3 nodes, i.e., the 3 nodes in $\{T_4, T_5, T_6, T_7, T_8, T_9\} \setminus l$ are collinear too. Now, the remaining three nodes of ten: $T_0, T_1, T_3$, are not collinear, in contradiction with Lemma 2.1.3(ii). □

Thus there is no loss of generality in assuming:

($\diamond$) The nodes $T_2$ and $T_3$ use the lines $l_2$ and $l_3$, respectively.

Next we consider the nodes in the line $l_4$.

**Lemma 3.0.5.** *The nodes $T_8$ and $T_9$, lying in the line $l_4$, use certain lines $l'$ and $l''$, respectively, which intersect each of the three lines $l_i,\ i = 1, 2, 3$, at a node, different from $T_0$.*

**Proof.** We have that the line $l_i$ is used by the node $T_i, i = 1, 2, 3$. Therefore, by Lemma 2.1.2 the nodes $T_8$ and $T_9$, do not use any line from $\mathcal{L}$. Now, it remains to set $\mathcal{M} = \{l_1, l_2, l_3\}$ and use Lemma 3.0.2(ii). □

Furthermore, we have:

**Lemma 3.0.6.** *The lines $l'$ and $l''$ of Lemma 3.0.5 have a common node $T$ which belongs to $\mathcal{T}\backslash l_1$. Moreover $T$ uses the line $l_4$.*

**Proof.** Assume by way of contradiction that there is no common node. Then, in view of Lemma 2.1.3(i) and Lemma 3.0.5, the nodes $T_8$ and $T_9$ both use the lines $l'$ and $l''$, which is impossible by Lemma 2.1.2. Now suppose that $l_4$ is used by a node $T' \in \mathcal{T}\backslash l_4$, different from the common node $T$. Then $T'$ does not belong to one of the lines $l'$ and $l''$, say to $l'$. Next, by Lemma 2.1.3(i), the node $T'$ uses the line $l'$ already used by $T_8$. This contradicts Lemma 2.1.2. Finally, notice that, in view of Lemma 3.0.3(i), the common node $T$ is not in the line $l_1$, i.e., it is not coinciding with the nodes $T_2, T_3$. Indeed, $T$ uses the line $l_4 \in \mathcal{L}$, and the latter nodes use the lines $l_2, l_3 \in \mathcal{L}$, respectively. □

Therefore the common node $T$ is in the line $l_2$ or $l_3$. Hence, without loss of generality suppose that the lines $l'$ and $l''$ intersect at $T_4$.

Thus, according to Lemma 3.0.6, we have (see Figure 2)

$$(\diamond) \qquad \text{The node } T_4 = l' \cap l'' \text{ uses the line } l_4.$$

Next, completely similarly to Lemma 3.0.5 we get:

**Lemma 3.0.7.** *The nodes $T_6$ and $T_7$, lying in the line $l_3$, use certain lines $l^\star$ and $l^{\star\star}$, respectively, which intersect each of the three lines $l_i,\ i = 1, 2, 4$, at a node, different from $T_0$.*

Now we are in a position to complete:

**Proof of Theorem 2.1.1 in the case of Alternative 1.** We have that the lines $l'$ and $l''$ pass through the node $T_4$. In view of Lemma 3.0.5, we may assume without loss of generality that $l'$ passes also through the node $T_2$ and one of $T_6, T_7$, while $l''$ passes through $T_3$ and another one of $T_6, T_7$. Next, we have, in view of Lemma 3.0.7, that the lines $l^\star$ and $l^{\star\star}$ pass through the node $T_5$ of the line $l_2$. Indeed, otherwise if one of them passes through the node $T_4 \in l_2$, then it coincides with one of the lines $l'$ and $l''$. Again, in view of Lemma 3.0.7, we may assume without loss of generality that one of the lines $l^\star$ and $l^{\star\star}$, say $l^\star$, passes also through the node $T_2$ and one of $T_8, T_9$. Finally, let us turn to the node $T_3$ which uses the line $l_3$. By Lemma 2.1.3(i) it uses also $l^\star$, already used by $T_6$. This, in view of Lemma 2.1.2, is a contradiction. □

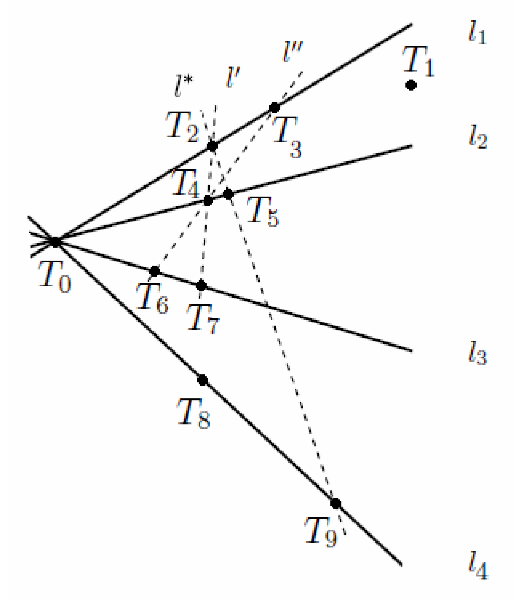


FIGURE 2. Alternative 1 — proof.

## 4. Alternative 2

**4.1. The Desargues and Pascal theorems.** Denote by $l_{AB}$ the line passing through the points $A$ and $B$.

For a set of points $A_1, A_2, A_3, B_1, B_2, B_3$ the following *cross- and v-type intersection points* (see Figure 3) will be considered in the sequel:

$$\dot{AB}_{1\vee 2} := l_{A_1A_2} \cap l_{B_1B_2} =: C_3^{\vee}, \qquad \dot{AB}_{3\vee 1} =: C_2^{\vee}, \qquad \dot{AB}_{2\vee 3} =: C_1^{\vee},$$
$$\dot{AB}_{1\times 2} := l_{A_1B_2} \cap l_{A_2B_1} =: C_3^{\times}, \qquad \dot{AB}_{3\times 1} =: C_2^{\times}, \qquad \dot{AB}_{2\times 3} =: C_1^{\times}.$$

In the brief notation $C_i^{\vee}$ and $C_i^{\times}$ we take into account the fact that $\dot{AB}_{j\vee k} = \dot{AB}_{k\vee j}$ and $\dot{AB}_{j\times k} = \dot{AB}_{k\times j}$.

**Remark 4.1.1.** Notice that the intersection points $\dot{AB}_{1\vee 2}$ and $\dot{AB}_{1\times 2}$ will be interchanged if we interchange the points $A_1, B_1$ or $A_2, B_2$ (see Figure 4).

Let us now present the well-known Desargues and Pascal theorems in terms of the above cross- and v-type intersection points (see Figure 6).

**Theorem 4.1.2** (Desargues)**.** *Suppose the lines $l_1, l_2, l_3$ are concurrent and two points $A_i, B_i$ are given on each line $l_i,\ i = 1, 2, 3$. Then the intersection points $C_1^{\vee}, C_2^{\vee}, C_3^{\vee}$ are collinear.*

**Theorem 4.1.3** (Pascal)**.** *Suppose six points: $A_1, A_2, A_3, B_1, B_2, B_3$ are given in a conic (i.e., in an algebraic curve of degree 2). Then the intersection points $C_1^{\times}, C_2^{\times}, C_3^{\times}$ are collinear.*

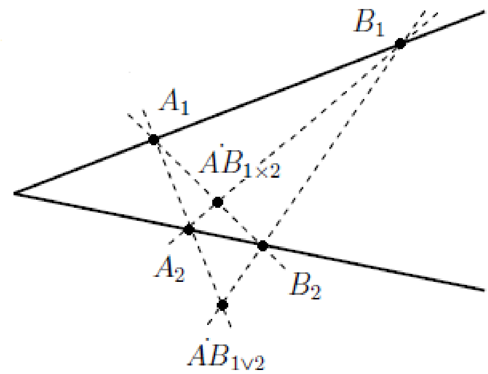


FIGURE 3. Points $\dot{AB}_{1\vee 2}$ and $\dot{AB}_{1\times 2}$.

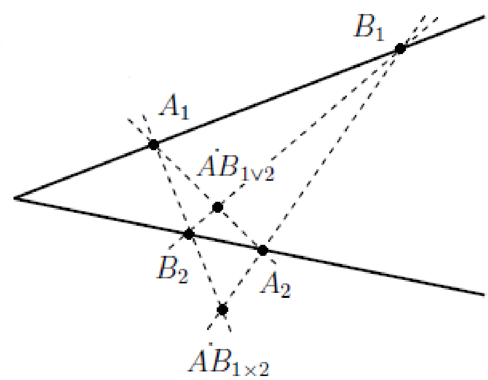


FIGURE 4. Points $A_2$ and $B_2$ interchanged.

Below, in Remark 4.1.4, we bring two other equivalent formulations of the Pascal theorem. First one is well-known and second one will be used in the proof of the forthcoming Theorem 4.1.5.

**Remark 4.1.4.**

(i) If we apply the Pascal theorem for the 6 points $T_1, \ldots, T_6$ ordered as $\{T_1, T_5, T_3, T_4, T_2, T_6\} \equiv \{A_1, A_2, A_3, B_1, B_2, B_3\}$ then we get the following equivalent formulation of the Pascal theorem:

*Suppose* 6 *points:* $T_1, \ldots, T_6$ *are given in a conic. Then the following three intersection points are collinear:*

$$l_{12} \cap l_{45}, \quad l_{23} \cap l_{56}, \quad l_{34} \cap l_{61},$$

*where* $l_{ij}$ *is the line passing through* $T_i$ *and* $T_j$.

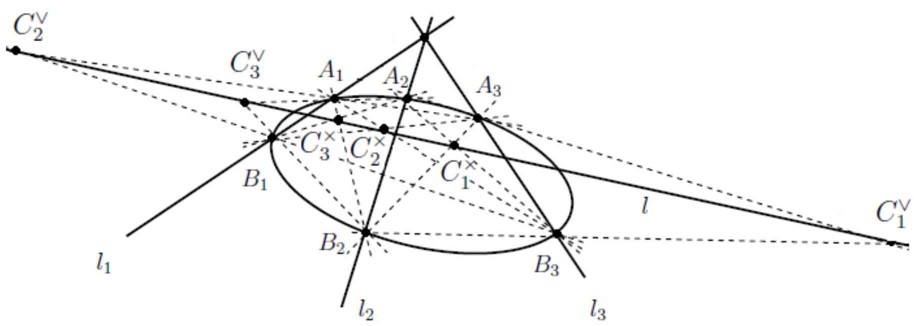


FIGURE 5. The six collinear points.

(ii) If we interchange $A_3$ and $B_3$ in Theorem 4.1.3 then, in view of Remark 4.1.1, we get the following equivalent formulation of the Pascal theorem:

*Suppose* 6 *points:* $A_1, A_2, A_3, B_1, B_2, B_3$ *are given in a conic. Then the intersection points* $C_1^\vee, C_2^\vee, C_3^\times$ *are collinear.*

On the basis of the Desargues and Pascal theorems we get the following, interesting in itself:

**Theorem 4.1.5.** *Suppose the lines* $l_1, l_2, l_3$ *are concurrent and two points* $A_i, B_i$ *are given on each line* $l_i,\ i = 1, 2, 3,$ *such that the six points* $A_1, A_2, A_3,$ $B_1, B_2, B_3$ *are in a conic. Then the following six intersection points:* $C_1^\vee$, $C_2^\vee$, $C_3^\vee$, $C_1^\times$, $C_2^\times$, $C_3^\times$ *are collinear (see Figure 5).*

**Proof.** In view of the Desargues theorem and the Pascal theorem formulated as in Remark 4.1.4(ii), we have that the points $C_1^\vee, C_2^\vee, C_3^\vee, C_3^\times$ are in a line $l$. Next we apply the Pascal theorem once more for the 6 points ordered in the following way: $A_3, A_1, A_2, B_3, B_1, B_2$, to get that the points $C_1^\vee, C_3^\vee, C_2^\times$ are collinear. Since first two of these 3 points are in $l$, also the third one: $C_2^\times$ is in $l$. To complete the proof we apply for the third time the Pascal theorem for the 6 points ordered in the following way: $A_2, A_3, A_1, B_2, B_3, B_1$, and get that the points $C_2^\vee, C_3^\vee, C_1^\times$ are collinear. Hence we obtain that $C_1^\times$ is in $l$. □

**Remark 4.1.6.** Note that the inverses of the Desargues and Pascal theorems as well as Theorem 4.1.5 also hold true.

**4.2. The construction of the node set.** Now let us turn to the case of Alternative 2 described in Subsection 2.2. Before starting the proof of Theorem 2.1.1 in this case (in Subsection 4.3), we describe the construction of $\mathcal{T}$ and make some clarifications. Now, the conditions (1)–(4) of Proposition 2.2.1 hold. It is convenient to refer to these conditions, as conditions

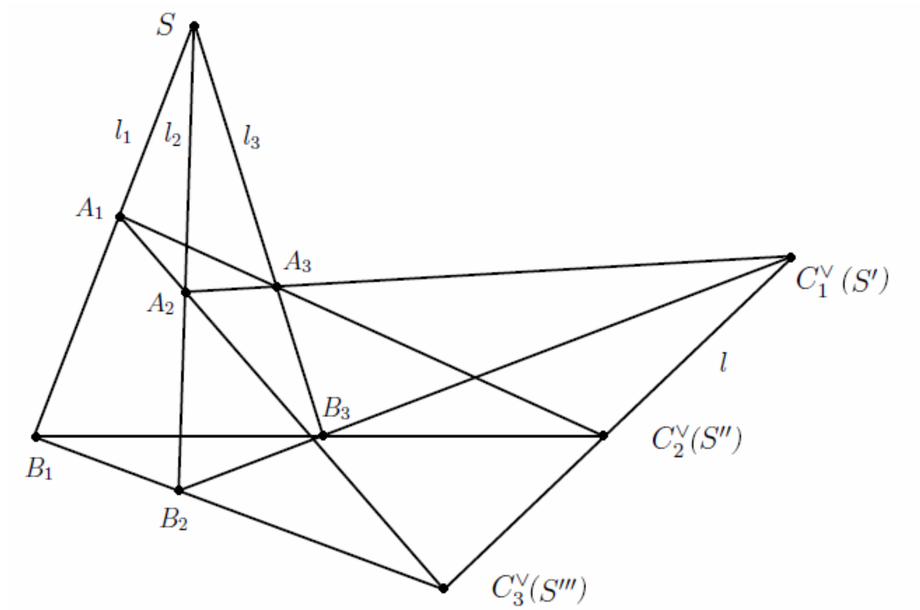


FIGURE 6. The construction of $\mathcal{T}$ with $C_3^{\vee}$.

(1)–(4) of Alternative 2, or briefly, as Alt2.1–Alt2.4. Note that conditions (1)–(3) mean that the 10 nodes of $\mathcal{T}$ and 10 used lines form a $10_3$ configuration (see [14], Chapter 3, Section 19). Recall that by condition (4) each node of $\mathcal{T}$ uses exactly one line and one irreducible conic. Below we describe the construction of $\mathcal{T}$, starting with any node (see Figure 6).

**Proposition 4.2.1.** *Let $S \in \mathcal{T}$ be any (starting) node. Assume that $l_1, l_2, l_3$ are the three used lines passing through $S$ (*Alt2.3*) Assume also that $A_i, B_i$ are the 2 nodes, besides $S$, in the line $l_i$, $i = 1, 2, 3$ (*Alt2.2*). Then the following hold.*

- (i) *The 6 nodes $A_1, A_2, A_3$, $B_1, B_2, B_3$ are in the irreducible conic used by $S$.*
- (ii) *The line $l$ used by $S$ passes through the remaining three nodes of $\mathcal{T}$, i.e., the nodes of $\mathcal{T} \setminus \{S, A_1, A_2, A_3, B_1, B_2, B_3\}$. These three nodes can be identified as $C_1^{\vee}, C_2^{\vee}$, and $C_3^{\vee}$ or $C_3^{\times}$ (for this we may interchange the nodes $A_i, B_i$ in the lines $l_i$, $i = 3, 1$, if necessary).*
- (iii) *The three nodes in $l$, i.e., $C_1^{\vee}, C_2^{\vee}, C_3^{\vee}$ or $C_3^{\times}$, use the lines $l_1, l_2, l_3$, respectively.*
- (iv) *The 6 intersection points $C_1^{\vee}, C_2^{\vee}, C_3^{\vee}, C_1^{\times}, C_2^{\times}, C_3^{\times}$ belong to the line $l$.*

**Proof.** Consider the line $l$ used by $S$. In view of conditions Alt2.2 and Alt2.3, there are 3 nodes: $S', S'', S'''$ in $l$ and through each node there pass

2 used lines, besides $l$. Thus, we have identified 7 used lines: $l$ and the other 6 ones intersecting $l$ at a node.

Now, let us verify that the remaining 3 used lines of 10, which already do not intersect $l$ at a node, pass through $S$, i.e., coincide with $l_1, l_2, l_3$. Otherwise, according to Lemma 2.1.3(i), each line not passing through $S$ and not intersecting $l$ at a node, will be used by $S$, which already uses the line $l$. But, this contradicts condition Alt2.4. Thus, the 6 nodes $A_1, A_2, A_3, B_1, B_2, B_3$ are outside the line $l$. Therefore, in view of condition Alt2.4, they are in the irreducible conic used by $S$, and (i) is proved.

Next, we observe that the line $l$ used by the node $S$ can be determined by each of the following two criteria: Take the 3 used lines $l_1, l_2, l_3$, passing through the node $S$ and then the line used by this node is

(a) the line passing through the 3 nodes of $\mathcal{T}$ not belonging to $l_1, l_2, l_3$,
(b) the line not intersecting the 3 lines $l_1, l_2, l_3$, at a node.

Then, in view of (b), we get that each of the 3 nodes in $l$ uses one of the concurrent lines $l_i,\ i = 1, 2, 3$, since all other 6 lines intersect $l$ at a node. Now let us fix one of the 3 nodes, say $S'''$, and suppose that it uses $l_3$ (see Figure 6). According to (a), all the nodes except the 3 in the line $l_3$, i.e., $S, A_3, B_3$, are in the used lines passing through $S'''$. Thus the nodes $A_1, A_2,\ B_1, B_2$ are in the two used lines passing through $S'''$, since the third line through $S'''$ is $l$, which passes through $S'$ and $S''$. Therefore we have two possibilities. The two used lines pass either through the pairs of nodes $\{A_1, A_2\},\ \{B_1, B_2\}$ or $\{A_1, B_2\},\ \{A_2, B_1\}$. In other words, we get that $S'''$ is either $C_3^{\vee}$ or $C_3^{\times}$. In the same way, by supposing that $S'$ and $S''$ use $l_1$ and $l_2$, respectively, we get that $S'$ is either $C_1^{\vee}$ or $C_1^{\times}$ and $S''$ is either $C_2^{\vee}$ or $C_2^{\times}$. Now, we interchange the nodes $A_3, B_3$ in the line $l_3$ and then the nodes $A_1, B_1$ in the line $l_1$, if necessary, to fix these nodes as they are mentioned in (ii). Thus (ii) and (iii) are proved. (Note that we cannot interchange the nodes in the line $l_2$ to fix one of the two possibilities also for the third node: $C_3^{\vee}$ or $C_3^{\times}$, since then the already fixed intersection point $C_1^{\vee}$ will turn into $C_1^{\times}$.) Finally, we get (iv) by using (i) and Theorem 4.1.5. □

The following proposition along with Proposition 4.2.1 is a main tool in proving Theorem 2.1.1 for the case of Alternative 2.

**Proposition 4.2.2.** *Suppose that two used lines $l_1$ and $l_2$ pass through a node $S \in \mathcal{T}$. Suppose also that $A_i, B_i$ are the 2 nodes, besides $S$, in the line $l_i,\ i = 1, 2$. Then one of the points $C_3^{\vee}, C_3^{\times}$ (i.e., $\dot{A}B_{1\vee 2}, \dot{A}B_{1\times 2}$) is a node in $\mathcal{T}$ and another coincides with the intersection point $l_3 \cap l$, where $l_3$ is the third used line through $S$ and $l$ is the line used by $S$.*

**Proof.** Assume, in view of Proposition 4.2.1(ii) and Remark 4.1.1, without loss of generality, that $C_3^{\vee} \in \mathcal{T}$. Then we need to prove that

$$C_3^{\times} = l_3 \cap l.$$

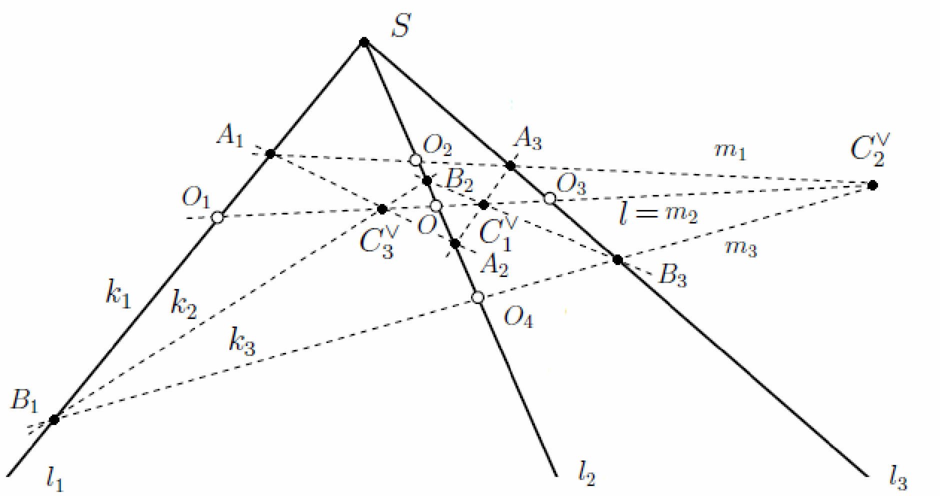


FIGURE 7. Alternative 2 — the proof.

First notice that by Proposition 4.2.1(iv) $C_3^{\times} \in l$. Next, we are going to use Proposition 4.2.1(iii) with the starting node $S = C_3^{\vee}$. We have that this node uses the line $l_3$. On the other hand the pairs of nodes $\{A_1, A_2\}, \{B_1, B_2\}$ are in the two used lines passing through the starting node $C_3^{\vee}$, respectively. Therefore, by using Proposition 4.2.1(iv), we conclude that $C_3^{\times} \in l_3$ (see Figure 3). □

**4.3. Alternative 2 — proof of the main result.** Consider a node $S \in \mathcal{T}$ satisfying the condition

$$S \notin \text{conv}\{\mathcal{T} \setminus S\}. \tag{4.3.1}$$

Suppose the 3 used lines passing through $S$ are $l_1, l_2, l_3$. Suppose also that the 2 nodes in $l_i$, besides $S$, are $A_i, B_i,\ i = 1, 2, 3$. These nodes, in view of the condition (4.3.1) are on the rays starting with $S$, which we denote by $l_i^+,\ i = 1, 2, 3$ . Assume that $A_1, B_2, A_3$ are the middle nodes in the rays, i.e., they lie in the segments $SB_1, SA_2, SB_3$, respectively. (see Figure 7). Without loss of generality assume that $l_2^+$ is between $l_1^+$ and $l_3^+$, meaning that

$$l_2^+ \text{ belongs to } \angle\alpha, \tag{4.3.2}$$

where $\angle\alpha$ is the angle ($< \pi$) with the sides $l_1^+, l_3^+$.

Then it is easily seen that

$$C_1^{\vee} := \dot{AB}_{2\vee 3} \in \text{conv}\{B_2, A_2, B_3, A_3\}, \tag{4.3.3}$$

$$C_3^{\vee} := \dot{AB}_{1\vee 2} \in \text{conv}\{A_1, B_1, A_2, B_2\}, \tag{4.3.4}$$

and

$$C_2^{\vee} := \dot{AB}_{3\vee 1} \notin \angle\alpha \cup \angle\alpha^-, \tag{4.3.5}$$

where $\angle\alpha^-$ is the opposite angle of $\angle\alpha$.

Now let us verify that the points $C_1^\vee, C_2^\vee$ and $C_3^\vee$ are nodes of $\mathcal{T}$. Indeed, otherwise, by Proposition 4.2.2, they belong to the lines $l_1, l_2$, and $l_3$, respectively, contrary to (4.3.3–4.3.5).

Next, notice that one of the nodes $B_1$ or $B_3$ has the property (4.3.1) of $S$, depending on which one is an end point in the triple $B_1, B_3, S$. Suppose it is the node $B_1$ (as in Figure 7). Note that the 3 used lines through $B_1$ are the lines $k_1 := l_{B_1S}, k_2 := l_{B_1B_2}$ and $k_3 := l_{B_1B_3}$. Then the 2 nodes in $k_i$, besides $B_1$, are in the respective rays starting with $B_1$, which we denote by $k_i^+,\ i = 1, 2, 3$, respectively.

Now, let us show that

$$k_2^+ \text{ is between } k_1^+ \text{ and } k_3^+. \tag{4.3.6}$$

For this notice that, in view of Proposition 4.2.1(ii), $S$ uses the line $l$ passing through the nodes $C_1^\vee, C_2^\vee$ and $C_3^\vee$. Hence, according to Proposition 4.2.2, $C_2^\times := \dot{A}B_{3\times 1}$, i.e., the intersection point of diagonals of the quadrangle $A_1, A_3, B_3, B_1$, coincides with the point $O := l_2 \cap l$. From here we get that

$$m_2^+ \text{ is between } m_1^+ \text{ and } m_3^+, \tag{4.3.7}$$

where $m_1^+, m_2^+, m_3^+$ are the rays starting with $C_2^\vee$ and passing through $A_1, C_1^\vee, B_1$, respectively.

Therefore

$$O_1 \in \operatorname{conv}\{A_1, B_1\}, \quad O \in \operatorname{conv}\{O_2, O_4\}, \tag{4.3.8}$$

where $O_1 := m_2 \cap l_1,\ O_2 := m_1 \cap l_2,\ O_4 = m_3 \cap l_2$.

On the other hand, by the condition (4.3.4), and the fact that $C_3^\vee \in l$, we conclude that

$$C_3^\vee \in \operatorname{conv}\{O_1, O\}.$$

This, in view of (4.3.8), establishes (4.3.6), since the ray $k_2^+$ intersects $l$ at $C_3^\vee$.

Now notice that the relations (4.3.4) and (4.3.3) proved for the starting node $S$, in the case of the starting node $B_1$ imply that $\operatorname{conv}\{A_1, S, B_2, C_3^\vee\}$ contains in its interior one of the 3 nodes in the line used by $B_1$, i.e., one of $A_2, C_1^\vee, A_3$. But it is easily seen that the later nodes belong to the angle with sides $l_2^+, l_3^+$, which is a contradiction. □

## References


[1] Busch, Jorge R. A note on Lagrange interpolation in $\mathbb{R}^2$. *Rev. Un. Mat. Argentina* **36** (1990), 33–38. MR1265693 (95d:41002), Zbl 0774.41002.

[2] Carnicer, Jesus Miguel; Gasca, Mariano. A conjecture on multivariate polynomial interpolation. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **95** (2001), no. 1, 145–153. MR1899358 (2003d:41031), Zbl 1013.41001.

[3] Carnicer, Jesus Miguel; Gasca, Mariano. Planar configurations with simple Lagrange interpolation formulae. *Mathematical methods for curves and surfaces* (Oslo, 2000), 55–62, Innov. Appl. Math., *Vanderbilt Univ. Press, Nashville, TN*, 2001. MR1857824, Zbl 1003.65008.

[4] Carnicer, Jesus Miguel; Gasca, Mariano. On Chung and Yao's geometric characterization for bivariate polynomial interpolation. *Curve and Surface Design* (Saint-Malo, 2002), 21–30, Mod. Methods Math., *Nashboro Press, Brentwood, TN*, 2003. MR2042470, Zbl 1054.41001.

[5] Chung, Kwok Chiu; Yao, Te Hai. On lattices admitting unique Lagrange interpolations. *SIAM J. Numer. Anal.* **14** (1977), no. 4, 735–743. MR0445158 (56 #3502), Zbl 0367.65003, doi: 10.1137/0714050.

[6] Eisenbud, David; Green, Mark; Harris, Joe. Cayley–Bacharach theorems and conjectures. *Bull. Amer. Math. Soc. (N.S.)* **33** (1996), no. 3, 295–324. MR1376653 (97a:14059), Zbl 0871.14024, doi: 10.1090/S0273-0979-96-00666-0.

[7] Gasca, Mariano; Maeztu, Jose Ignacio. On Lagrange and Hermite interpolation in $\mathbb{R}^k$. *Numer. Math.* **39** (1982), no. 1, 1–14. MR0664533 (83g:65012), Zbl 0457.65004, doi: 10.1007/BF01399308.

[8] Gasca, Mariano; Sauer, Thomas. Polynomial interpolation in several variables. Multivariate polynomial interpolation. *Adv. Comput. Math.* **12** (2000), no. 4, 377–410. MR1768957 (2001d:41010), Zbl 0943.41001, doi: 10.1023/A:1018981505752.

[9] Hakopian, Hakop. On a class of Hermite interpolation problems. Multivariate polynomial interpolation. *Adv. Comput. Math.* **12** (2000), no. 4, 303–309. MR1768953 (2001e:41007), Zbl 0944.41003, doi: 10.1023/A:1018933622590.

[10] Hakopian, Hakop. On recent progress in Gasca–Maeztu conjecture. *Constructive theory of functions*, 126–140, *Prof. M. Drinov Acad. Publ. House, Sofia*, 2012. MR3051104.

[11] Hakopian, Hakop; Jetter, Kurt; Zimmermann, Georg. A new proof of the Gasca–Maeztu conjecture for $n = 4$. *J. Approx. Theory* **159** (2009), no. 2, 224–242. MR2562743 (2011a:41003), Zbl 1180.41002, doi: 10.1016/j.jat.2009.04.006.

[12] Hakopian, Hakop; Jetter, Kurt; Zimmermann, Georg. The Gasca–Maeztu conjecture for $n = 5$. *Numer. Math.* **127** (2014), no. 4, 685–713. MR3229990, Zbl 1304.41002, doi: 10.1007/s00211-013-0599-4.

[13] Hakopian, Hakop; Malinyan, Argishti. Characterization of $n$-independent sets with no more than $3n$ points. *Jaen J. Approx.* **4** (2012), no. 2, 121–136. MR3203464, Zbl 1295.41035.

[14] Hilbert, David; Cohn-Vossen, Stefan. Anschauliche Geometrie. *Springer-Verlag, Berlin*, 1932. MR0430930 (55 #3935), Zbl 0005.11202, doi: 10.1007/978-3-662-36685-1.

[15] Rafayelyan, Levon. Poised nodes set constructions on algebraic curves. *East J. Approx.* **17** (2011), no. 3, 285–298. MR2953081, Zbl 06077289.

(Hakop Hakopian) Department of Informatics and Applied Mathematics, Yerevan State University, A. Manukyan Str. 1, 0025 Yerevan, Armenia
hakop@ysu.am

(Levon Rafayelyan) Department of Informatics and Applied Mathematics, Slavonic University, H. Emin Str. 123, 0051 Yerevan, Armenia
levon.rafayelyan@googlemail.com

This paper is available via http://nyjm.albany.edu/j/2015/21-17.html.