

Extending Chang’s construction to the category of m -zeroids and some category containing the category of Abelian ℓ -groups with strong unit

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ABSTRACT. In this note we prove that it is impossible to extend the natural equivalence between the category of MV-algebras and the category of Abelian ℓ -groups with strong unit described by C. C. Chang, 1958, and Cignoli & Mundici, 1997, to a natural equivalence between the category of m -zeroids and some category containing the category of Abelian ℓ -groups with strong unit.

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1. Introduction

In 1958, C. C. Chang [3] showed that there is a natural equivalence between the category of totally ordered MV-algebras and the category of totally-ordered Abelian ℓ -groups with strong unit. In 1997, Cignoli & Mundici [4] generalized Chang’s construction to show that there is a natural equivalence between the category of MV-algebras and the category of Abelian ℓ -groups with strong unit with a functor commonly called Γ . It is of interest to determine if this functor can be extended to more general classes of algebraic structures.

Received July 10, 2013.

2010 *Mathematics Subject Classification*. Primary: 06F99; Secondary: 03G20.

Key words and phrases. MV-algebras, abelian ℓ -groups, dual abelian ℓ -groups, abelian ℓ -monoids, dual abelian ℓ -monoids, m -zeroid, IMTL-algebras, abelian ℓ -groups with strong unit.

To that end, in 2001, Esteva & Godo [6] introduced a spectrum of monoidal logic systems which included involutive monoidal t-norm based logic systems, IMTL-algebras for short, and then proceeded in 2007 (see [7]) to use the IMTL-algebras to attempt a generalization of the Γ functor established by Cignoli & Mundici. They succeeded in defining a functor that is a categorical equivalence between the subcategories of *linearly-ordered* objects from the categories of lattice-ordered partially-associative Abelian groupoids with strong associative unit and IMTL-algebras. This categorical equivalence cannot be extended (yet) beyond these linearly-ordered subcategories to the corresponding non-linearly-ordered categories.

Using the fact that an m-zeroid is a generalization of an MV-algebra (see [2]), this paper attempts a generalization of Cignoli & Mundici's Γ functor to a natural equivalence between the category of m-zerooids and some category containing the category of Abelian ℓ -groups with strong unit. This is an attempted generalization of the functor Esteva & Godo [7] developed in their paper, since it is easy to see that the varieties of IMTL-algebras and m-zerooids are term equivalent (see Remark 1). The generalization being made here involves the other side of the functor: the subcategory of linearly-ordered partially-associative Abelian groupoids with strong associative unit considered by Esteva & Godo will be generalized to some category containing the category of Abelian ℓ -groups with strong unit.

In this note we prove the following result:

Main Theorem. *There does not exist an extension of the natural equivalence between the category of MV-algebras and the category of Abelian ℓ -groups with strong unit presented by Chang [3] and Cignoli & Mundici [4] to the category of m-zerooids and some category containing the category of Abelian ℓ -groups with strong unit.*

We begin in Section 2 by providing definitions for the algebraic structures, namely the m-zeroid, MV-algebra, Abelian ℓ -group, Abelian ℓ -monoid, and the dual Abelian ℓ -monoid. Section 3 summarizes the natural equivalence between the category of MV-algebras and the category of Abelian ℓ -groups with strong unit presented by Chang and Cignoli & Mundici, and then considers an extension of this natural equivalence to a natural equivalence between the category of m-zerooids and the category of dual Abelian ℓ -monoids with cancellative unit.

2. Definitions

Since C. C. Chang [3] first defined the algebraic system called an *MV-algebra*, numerous generalizations for an MV-algebra have been proposed, including the structure called an *m-zeroid*. For the purposes of this paper a combination of the definitions given by F. Paoli [8] and Cignoli & Mundici [4] for the m-zeroid and the MV-algebra will be used.

Definition 1. An *m-zeroid* is a structure $\mathbf{Z} = \langle Z, +, -, 0, \leq \rangle$ such that:

- (Z1) $x + y = y + x$;
- (Z2) $x + (y + z) = (x + y) + z$;
- (Z3) $\langle Z, \leq \rangle$ is a lattice;
- (Z4) $-(-x) = x$;
- (Z5) $x + 0 = 0$;
- (Z6) $x + -x = 0$;
- (Z7) $x \leq y$ iff $0 = -x + y$;
- (Z8) $x + (y \vee z) = (x + y) \vee (x + z)$.

Definition 2. An *MV-algebra* is an *m-zeroid* that also satisfies:

- (Z9) $-(-x + y) + y = -(-y + x) + x$.

It should be noted that both (Z3) and (Z7) can be expressed equationally strictly in terms of the algebraic operations $+$ and $-$. Thus the class of *m-zerooids* forms a variety and the class of *MV-algebras* forms a variety. As a consequence, a homomorphism between two *m-zerooids*, or a homomorphism between two *MV-algebras*, will be defined as usual—as a function that preserves the operations. However, in general, (Z3) and (Z7) will be sufficient and their equational representations will not be used.

For both the *m-zeroid* and the *MV-algebra*, the following binary operation will be defined in order to reduce the notation where possible:

$$x \cdot y = -(-x + -y).$$

The following is then an immediate consequence of property (Z4):

$$x + y = -(-x \cdot -y).$$

Note that for an *m-zeroid* $x \vee y$ can be written in terms of the other operations, including \wedge . However, in an *MV-algebra*

$$-(-x + y) + y = -(-y + x) + x,$$

so the join can be written in terms of only the operations $+$ and $-$ as follows:

$$x \vee y = -(-x + y) + y = -(-y + x) + x.$$

Similarly for the meet:

$$x \wedge y = -[-(x + -y) + -y] = -[-(y + -x) + -x].$$

In both structures, -0 plays the role of the identity element, and 0 plays the role of “collector of the opposites”, as Paoli [8] calls it. Also note that in both *m-zerooids* and *MV-algebras*, -0 represents a universal lower bound and 0 a universal upper bound.

Remark 1. We could denote an *IMTL-algebra* as $\langle Z, \star, \rightarrow, 1, \leq \rangle$, with 0 as the identity element, where the operations of the *IMTL-algebra* correspond

to those of the m-zeroid defined above as follows:

$$\begin{aligned}
1 &\iff 0 \\
0 &\iff -0 \\
x \rightarrow 0 &\iff -x \\
x \star y &\iff -(-x + -y) \\
x \rightarrow y &\iff -x + y \\
((x \rightarrow 0) \star (y \rightarrow 0)) \rightarrow 0 &\iff x + y
\end{aligned}$$

and where the lattice operations remain identical for both structures. As noted in [7], by adding the *divisibility condition* $x \vee y = (x \rightarrow y) \rightarrow y$, the IMTL-algebra will become an MV-algebra. This divisibility condition is equivalent to adding (Z9) to an m-zeroid to create an MV-algebra, since:

$$x \vee y = (x \rightarrow y) \rightarrow y \iff x \vee y = -(-x + y) + y$$

and since $x \vee y = y \vee x$, we have $-(-x + y) + y = -(-y + x) + x$.

Definition 3. An *Abelian ℓ -group* is a structure $\mathbf{G} = \langle G, +, -, 0, \leq \rangle$ such that:

- (G1) $x + y = y + x$;
- (G2) $x + (y + z) = (x + y) + z$;
- (G3) $\langle G, \leq \rangle$ is a lattice;
- (G4) $-(-x) = x$;
- (G5) $x + 0 = x$;
- (G6) $x + -x = 0$;
- (G7) $x \leq y$ iff $0 \leq -x + y$;
- (G8) $x + (y \vee z) = (x + y) \vee (x + z)$.

An Abelian ℓ -group is simply a group in which a partial ordering has been placed on the elements of the group such that the order forms a lattice (G3) and that the group operation preserves the order, an easy consequence of (G8): if $y \leq z$, then $x + z = x + (y \vee z) = (x + y) \vee (x + z)$, so $x + y \leq x + z$. As with m-zeroids and MV-algebras, (G3) and (G7) can be expressed using equations, and so the class of Abelian ℓ -groups forms a variety.

One of the fundamental properties of Abelian ℓ -groups is the following, proven in Birkhoff [1]:

Lemma 1. *Except for the trivial case $G = \{0\}$, an Abelian ℓ -group has no universal bounds.*

Definition 4. An element, $u \geq 0$, of an Abelian ℓ -group is a *strong unit* if for all $x \in \mathbf{G}$ there exists an integer $n \geq 1$ such that $nu \geq x$.

For any given Abelian ℓ -group, if a strong unit exists, it is not necessarily unique. Not every Abelian ℓ -group will have a strong unit.

Definition 5. An *Abelian ℓ -monoid* is a structure $\mathbf{M} = \langle M, +, 0, \leq \rangle$ such that:

- (M1) $x + y = y + x$;

- (M2) $x + (y + z) = (x + y) + z$;
- (M3) $\langle M, \leq \rangle$ is a lattice;
- (M4) $x + 0 = x$;
- (M5) $x + (y \vee z) = (x + y) \vee (x + z)$.

Definition 6. A *dual Abelian ℓ -monoid* is a structure $\mathbf{M}' = \langle M', +, 0, \leq \rangle$ such that:

- (M'1) $x + y = y + x$;
- (M'2) $x + (y + z) = (x + y) + z$;
- (M'3) $\langle M', \leq \rangle$ is a lattice;
- (M'4) $x + 0 = x$;
- (M'5) $x + (y \wedge z) = (x + y) \wedge (x + z)$.

Remark 2. It should be noted that these two definitions are not equivalent. In an Abelian ℓ -group,

$$x + (y \vee z) = (x + y) \vee (x + z) \iff x + (y \wedge z) = (x + y) \wedge (x + z).$$

This is *not* true in an Abelian ℓ -monoid. Isotonicity of the Abelian ℓ -monoid is a consequence of (M5). Similarly, isotonicity for the dual Abelian ℓ -monoid is a consequence of (M'5). The definition of an Abelian ℓ -monoid has been taken from Birkhoff [1]; the author has introduced the concept of a dual Abelian ℓ -monoid. And lastly, note that the dual of an Abelian ℓ -monoid is a dual Abelian ℓ -monoid.

There are significant differences between Abelian ℓ -monoids, dual Abelian ℓ -monoids, and Abelian ℓ -groups. Both Abelian ℓ -monoids and dual Abelian ℓ -monoids can have universal upper bounds and universal lower bounds, unlike Abelian ℓ -groups (see Lemma 1). In fact, it is possible for them to have both a universal upper bound and a universal lower bound.

Another difference between Abelian ℓ -groups and Abelian ℓ -monoids deals with the dual lattice. If L is a lattice, then the *dual lattice* L' is found by simply inverting the lattice, so that meets become joins and joins become meets.

Similarly, if w is a statement involving meets and joins, then the *dual statement* w' is the same statement but with the meets and joins interchanged. (This concept is described in Darnel [5].)

Lemma 2. *The dual of an Abelian ℓ -group is an Abelian ℓ -group.*

This is stated in Birkhoff [1] and is true since in an Abelian ℓ -group both statements (M5) and (M'5) hold. However, the same cannot be said for either Abelian ℓ -monoids nor dual Abelian ℓ -monoids.

3. Extension of Chang's construction

First consider the construction of the natural equivalence between the category of MV-algebras and the category of Abelian ℓ -groups with strong unit presented by Chang [3] and refined by Cignoli & Mundici [4]. Lemmas 3,

4, 5, 6, 7, and 8, and Theorems 1 and 2, have all been proven in Cignoli & Mundici [4].

Let \mathcal{MV} be the category of MV-algebras and let \mathcal{A}_1 be the category of Abelian ℓ -groups with strong unit. Let $\mathbf{G} = \langle G, +, -, 0, \leq, u \rangle$ be an Abelian ℓ -group with a specified strong unit u .

Definition 7. The *unit interval* $[0, u]$ of $\mathbf{G} = \langle G, +, -, 0, \leq, u \rangle$ is defined by:

$$[0, u] = \{x \in G \mid 0 \leq x \leq u\}.$$

Now define $\Phi(\mathbf{G}, u) = \langle [0, u], \oplus, \neg, u, \leq \rangle$ such that for each $x, y \in [0, u]$,

$$\begin{aligned} \neg x &= u - x, \\ x \oplus y &= u \wedge (x + y). \end{aligned}$$

As done with m-zerooids earlier, to shorten the notation, define:

$$x \odot y = \neg(\neg x \oplus \neg y).$$

At this point we have two different symbols for binary addition, namely $+$ and \oplus , used for a multitude of structures. \oplus will be used when the interval structure of the corresponding algebra needs to be emphasized, but it should be noted that the two symbols are interchangeable.

Lemma 3. $\Phi(\mathbf{G}, u)$ is an MV-algebra in which the natural lattice-order of $\Phi(\mathbf{G}, u)$ agrees with the order of $[0, u]$ inherited from \mathbf{G} by restriction.

A *unital ℓ -group homomorphism* will be a group homomorphism:

$$f : \langle G, +^{\mathbf{G}}, -^{\mathbf{G}}, 0^{\mathbf{G}}, \leq^{\mathbf{G}}, u \rangle \longrightarrow \langle H, +^{\mathbf{H}}, -^{\mathbf{H}}, 0^{\mathbf{H}}, \leq^{\mathbf{H}}, v \rangle$$

in \mathcal{A}_1 that also satisfies:

$$\begin{aligned} f(x \vee y) &= f(x) \vee f(y), \\ f(x \wedge y) &= f(x) \wedge f(y), \\ f(u) &= v. \end{aligned}$$

For every unital ℓ -group homomorphism f , let $\Phi_f = f|_{[0, u]}$ be the restriction of f to the interval $[0, u]$. Then:

Lemma 4. Φ is a functor from \mathcal{A}_1 into \mathcal{MV} .

The functor necessary for the other direction is a little more complicated. Every MV-algebra $\mathbf{A} = \langle A, \oplus, \neg, 0, \leq \rangle$ must give rise to an Abelian ℓ -group with strong unit. The Abelian ℓ -group with strong unit corresponding to \mathbf{A} will be denoted $\mathbf{G}_{\mathbf{A}}$. The construction from \mathbf{A} to $\mathbf{G}_{\mathbf{A}}$ is described in detail by Cignoli & Mundici [4] and begins with the following definition:

Definition 8. For every MV-algebra \mathbf{A} , a sequence $\langle a \rangle = (a_1, a_2, \dots)$ of elements of \mathbf{A} is *good* if and only if for each $i = 1, 2, \dots$, $a_i \oplus a_{i+1} = a_i$, and there exists an integer $n \geq 0$ such that $a_r = -0$ for all $r > n$.

As in Cignoli & Mundici's paper, the sequence

$$\langle a \rangle = (a_1, a_2, \dots, a_n, -0, -0, -0, \dots)$$

shall be denoted as simply $\langle a \rangle = (a_1, a_2, \dots, a_n)$. This implies that:

$$(a_1, a_2, \dots, a_n) = (a_1, a_2, \dots, a_n, -0^m)$$

for any arbitrary m -tuple of -0 s. However the sequence $(0^m, a_1, a_2, \dots, a_n)$ is distinct from the sequence (a_1, a_2, \dots, a_n) .

Definition 9. Let \mathbf{A} be an MV-algebra. Let $\langle a \rangle = (a_1, a_2, \dots, a_n)$ and $\langle b \rangle = (b_1, b_2, \dots, b_m)$ be good sequences in \mathbf{A} . Define the *sum* $\langle c \rangle = \langle a \rangle + \langle b \rangle$ to be the sequence $\langle c \rangle = (c_1, c_2, \dots)$, where for all $i = 1, 2, \dots$,

$$c_i = a_i \oplus (a_{i-1} \odot b_1) \oplus (a_{i-2} \odot b_2) \oplus \dots \oplus (a_1 \odot b_{i-1}) \oplus b_i.$$

Note that when $p > n$ and $q > m$, $a_p = b_q = -0$ and so $c_j = -0$ when $j > n + m$. In fact:

Lemma 5. *For any MV-algebra \mathbf{A} , the sum of two good sequences of \mathbf{A} is a good sequence of \mathbf{A} .*

Now consider the set of all good sequences $M_{\mathbf{A}}$ on an MV-algebra \mathbf{A} , equipped with the sum operation defined in Definition 9 and identity (-0) . This structure satisfies cancellation:

Lemma 6. *$M_{\mathbf{A}}$ is an Abelian monoid called the enveloping monoid of \mathbf{A} and satisfies cancellation: For any good sequences $\langle a \rangle$, $\langle b \rangle$, and $\langle c \rangle$, if $\langle a \rangle + \langle b \rangle = \langle a \rangle + \langle c \rangle$, then $\langle b \rangle = \langle c \rangle$.*

The Abelian group $G_{\mathbf{A}}$ is the group of quotients of this enveloping monoid $M_{\mathbf{A}}$.

Lemma 7. *$G_{\mathbf{A}}$ is an Abelian group called the enveloping group of \mathbf{A} .*

To complete the construction, a translation invariant lattice structure is defined for both $M_{\mathbf{A}}$ and $G_{\mathbf{A}}$. This lattice structure arises naturally from the underlying lattice structure of \mathbf{A} (see Cignoli & Mundici [4]). Once the translation invariant lattice structure is established, the Abelian group $G_{\mathbf{A}}$ becomes an Abelian ℓ -group. This Abelian ℓ -group $\mathbf{G}_{\mathbf{A}} = \langle G_{\mathbf{A}}, +, -, -0, \leq \rangle$ is called the *Chang ℓ -group of \mathbf{A}* . It contains a strong unit, namely $u_{\mathbf{A}} = [(0), (-0)]$, and so is an Abelian ℓ -group with strong unit.

Now, define an *mz-homomorphism* to be a function:

$$f : \langle A, +^{\mathbf{A}}, -^{\mathbf{A}}, 0^{\mathbf{A}}, \leq^{\mathbf{A}} \rangle \longrightarrow \langle B, +^{\mathbf{B}}, -^{\mathbf{B}}, 0^{\mathbf{B}}, \leq^{\mathbf{B}} \rangle$$

in \mathcal{MV} that also satisfies:

$$\begin{aligned} f(x +^{\mathbf{A}} y) &= f(x) +^{\mathbf{B}} f(y) \\ f(-^{\mathbf{A}} x) &= -^{\mathbf{B}} f(x) \\ f(x \vee y) &= f(x) \vee f(y) \\ f(x \wedge y) &= f(x) \wedge f(y) \\ f(0^{\mathbf{A}}) &= 0^{\mathbf{B}}. \end{aligned}$$

Then if $\langle a \rangle = (a_1, a_2, \dots)$ is a good sequence of \mathbf{A} , then $(f(a_1), f(a_2), \dots)$ is a good sequence of \mathbf{B} . Let $f^*(\langle a \rangle) = (f(a_1), f(a_2), \dots)$. Note that $f^* : M_{\mathbf{A}} \rightarrow M_{\mathbf{B}}$ satisfies:

$$\begin{aligned} f^*(\langle a \rangle +^{\mathbf{A}} \langle b \rangle) &= f^*(\langle a \rangle) +^{\mathbf{B}} f^*(\langle b \rangle) \\ f^*(\langle a \rangle \wedge \langle b \rangle) &= f^*(\langle a \rangle) \wedge f^*(\langle b \rangle) \\ f^*(\langle a \rangle \vee \langle b \rangle) &= f^*(\langle a \rangle) \vee f^*(\langle b \rangle). \end{aligned}$$

This is proven in Cignoli & Mundici [4]. Lastly, define

$$f^{\circledast}([\langle a \rangle, \langle b \rangle]) = [f^*(\langle a \rangle), f^*(\langle b \rangle)]$$

and let $u_{\mathbf{A}}$ and $u_{\mathbf{B}}$ be the strong units of $\mathbf{G}_{\mathbf{A}}$ and $\mathbf{G}_{\mathbf{B}}$ respectively as defined above. Then f^{\circledast} is a unital ℓ -group homomorphism from $\langle \mathbf{G}_{\mathbf{A}}, u_{\mathbf{A}} \rangle$ into $\langle \mathbf{G}_{\mathbf{B}}, u_{\mathbf{B}} \rangle$. Let $\Psi(\mathbf{A}) = \langle \mathbf{G}_{\mathbf{A}}, u_{\mathbf{A}} \rangle$ and $\Psi_f = f^{\circledast}$. Then:

Lemma 8. Ψ is a functor from \mathcal{MV} into \mathcal{A}_1 .

The two functors Φ and Ψ together will yield that:

Theorem 1. *There exists a natural equivalence between the category of MV-algebras and the category of Abelian ℓ -groups with strong unit.*

One of the more interesting properties of the construction of the Abelian ℓ -group presented by Chang and Cignoli & Mundici is the following:

Theorem 2. *Let \mathbf{A} be an MV-algebra and let $u_{\mathbf{A}}$ be the strong unit of $\mathbf{G}_{\mathbf{A}}$ defined as above. Let $\varphi_{\mathbf{A}} : \mathbf{A} \rightarrow \Phi(\mathbf{G}_{\mathbf{A}}, u_{\mathbf{A}}) \subseteq \mathbf{G}_{\mathbf{A}}$ be defined by:*

$$\varphi(a) = [(a), (-0)]$$

for all $a \in \mathbf{A}$. Then $\varphi_{\mathbf{A}}$ isomorphically maps \mathbf{A} onto $\Phi(\mathbf{G}_{\mathbf{A}}, u_{\mathbf{A}})$.

This finishes the construction of the natural equivalence between the category of MV-algebras and the category of Abelian ℓ -groups with strong unit presented in Cignoli & Mundici [4]. The question now is whether or not this natural equivalence can be extended to a natural equivalence between the category of m-zerooids and some category containing the category of Abelian ℓ -groups with strong unit. The natural instinct is to try to extend the natural equivalence to the category of m-zerooids and the category of Abelian ℓ -groups. Another possibility is to extend the natural equivalence to the category of m-zerooids and the category of Abelian ℓ -groups with some type

of unit that is a generalization of the strong unit. For our purposes, a more general category will be considered.

There are two properties that it seems reasonable, and desirable, for a natural equivalence to retain for this extension: cancellation of the specialized unit and an equivalent of Theorem 2 for the m-zerooids. The requirement of cancellation of the specialized unit is necessary for the construction of the enveloping monoid. And in order for the natural equivalence to be an *extension* of the construction above, the m-zerooid should map isomorphically onto some interval of the dual Abelian ℓ -monoid. In fact, if the extension is to work properly and u is the specialized unit in the dual Abelian ℓ -monoid, then the m-zerooid should map isomorphically onto the interval $[0, u]$. In an attempt to retain these properties, consider an extension of the natural equivalence to a natural equivalence between the category of m-zerooids and the category of dual Abelian ℓ -monoids.

To begin, consider the specialized unit:

Definition 10. Let $\mathbf{L} = \langle L, +, 0, \leq \rangle$ be an Abelian ℓ -monoid (or a dual Abelian ℓ -monoid). A non-zero element $u \geq 0$ in \mathbf{L} is a *cancellative unit* if for all $x, y \in \mathbf{L}$, $x + u = y + u$ implies $x = y$.

Now, let \mathcal{MZ} be the category of m-zerooids and let \mathcal{M}_1 be the category of dual Abelian ℓ -monoids with cancellative unit. If there is to be a natural equivalence with the required properties above, then for every m-zerooid $\mathbf{Z} = \langle Z, \oplus, \neg, 0, \leq \rangle$ there must be a corresponding dual Abelian ℓ -monoid $\mathbf{M} = \langle M, +, -, 0, \leq \rangle$ with cancellative unit u such that \mathbf{Z} gets mapped isomorphically onto $[0, u]$ in \mathbf{M} .

Theorem 3. Let $\mathbf{Z} = \langle Z, \oplus, \neg, 0, \leq \rangle$ be an m-zerooid. Let $\mathbf{M} = \langle M, +, 0, \leq \rangle$ be a dual Abelian ℓ -monoid with cancellative unit u . If $\varphi : \mathbf{Z} \rightarrow \mathbf{M}$ is an injective mapping such that for all $x, y \in \mathbf{Z}$:

- i) $\varphi(0) = u$,
- ii) $\varphi(\neg 0) = 0$,
- iii) $\varphi(x) + \varphi(\neg x) = u$, and
- iv) $\varphi(x \oplus y) = u \wedge (\varphi(x) + \varphi(y))$,

then \mathbf{Z} is an MV-algebra.

Proof. Since \mathbf{Z} is already an m-zerooid, the only property that needs to be proven is $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$. For convenience in the notation, if $x \in \mathbf{Z}$, its corresponding element $\varphi(x)$ will be denoted as simply x in \mathbf{M} . Similarly, $\varphi(\mathbf{Z})$ will simply be denoted \mathbf{Z} . Also recall that the symbols $+$ and \oplus are interchangeable, but \oplus will be used when the interval structure of \mathbf{Z} needs to be emphasized.

To prove $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$, first consider the following:

For all $a, b \in \mathbf{Z} \subset \mathbf{M}$,

$$\begin{aligned}
a + b + (\neg a \oplus \neg b) &= (a + b) + [u \wedge (\neg a + \neg b)] \\
&= [(a + b) + u] \wedge [(a + b) + (\neg a + \neg b)] \\
&= [(a + b) + u] \wedge [(a + \neg a) + (b + \neg b)] \\
&= [(a + b) + u] \wedge [u + u] \\
&= [(a + b) \wedge u] + u \\
&= (a \oplus b) + u.
\end{aligned}$$

Thus $a + b + (\neg a \oplus \neg b) = (a \oplus b) + u$.

Now let $x, y \in \mathbf{Z} \subset \mathbf{M}$. Then:

$$\begin{aligned}
x + (\neg x \oplus y) + u &= x + (\neg x) + y + (x \oplus \neg y) \\
&= u + y + (x \oplus \neg y).
\end{aligned}$$

Thus, using cancellation,

$$x + (\neg x \oplus y) = y + (x \oplus \neg y).$$

Adding $\neg(\neg x \oplus y) + \neg(x \oplus \neg y)$ to both sides yields:

$$\begin{aligned}
x + (\neg x \oplus y) + \neg(\neg x \oplus y) + \neg(x \oplus \neg y) \\
= y + (x \oplus \neg y) + \neg(\neg x \oplus y) + \neg(x \oplus \neg y).
\end{aligned}$$

This implies

$$x + u + \neg(x \oplus \neg y) = y + u + \neg(\neg x \oplus y),$$

which in turn implies

$$x + \neg(x \oplus \neg y) = y + \neg(\neg x \oplus y).$$

Now, since $x \oplus \neg y \geq x$, $\neg(x \oplus \neg y) \leq \neg x$ and so $x + \neg(x \oplus \neg y) \leq u$. Thus, $x + \neg(x \oplus \neg y) = x \oplus \neg(x \oplus \neg y)$. Similarly, $y + \neg(\neg x \oplus y) = y \oplus \neg(\neg x \oplus y)$. Therefore:

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$$

and so \mathbf{Z} is an MV-algebra. \square

Thus if there exists an isomorphic image of an m -zeroid \mathbf{Z} in a dual Abelian ℓ -monoid with a cancellative unit \mathbf{M} , the structure imposed on it by \mathbf{M} forces \mathbf{Z} to be an MV-algebra. This leads to the following corollary:

Corollary 1. *There does not exist an extension of the natural equivalence between the category of MV-algebras and the category of Abelian ℓ -groups with strong unit presented by Chang [3] and Cignoli & Mundici [4] to the category of m -zeroids and some category containing the category of Abelian ℓ -groups with strong unit.*

Proof. Suppose such an extension were possible. Let Ω be the functor from \mathcal{MZ} to \mathcal{M}_1 , and let Δ be the functor from \mathcal{M}_1 to \mathcal{MZ} . Let \mathbf{Z} be an m-zeroid which is *not* an MV-algebra. By the natural equivalence, \mathbf{Z} corresponds to a dual Abelian ℓ -monoid $\Omega(\mathbf{Z})$ with cancellative unit u . Since the natural equivalence is an extension, every m-zeroid \mathbf{Z} must be mapped isomorphically to $\Delta(\Omega(\mathbf{Z}), u) \subset \Omega(\mathbf{Z})$. But by Theorem 3, $\Delta(\Omega(\mathbf{Z}), u)$ is an MV-algebra, and thus \mathbf{Z} is an MV-algebra, which is a contradiction. Thus the natural equivalence cannot be extended. \square

4. Conclusion

Even though the result is negative, Theorem 3 is more general than the negative result stated in Remark 3 by Esteva & Godo [7]. However, their success in extending Cignoli & Mundici's Γ functor to a functor that is a categorical equivalence between the subcategories of *linearly-ordered* objects from the categories of lattice-ordered partially-associative Abelian groupoids with strong associative unit and IMTL-algebras indicates that further study needs to be made of IMTL-algebras and these lattice-ordered partially-associative Abelian groupoids.

I'd like to thank the referee for invaluable advice and suggestions during revisions of this paper.

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This paper is available via <http://nyjm.albany.edu/j/2014/20-5.html>.