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Idempotents in βS that are only products trivially

Neil Hindman and Lakeshia Legette Jones

Abstract. All results mentioned in this abstract assume Martin's Axiom. (Some of them are known to not be derivable in ZFC.) It is known that if S is the free semigroup on countably many generators, then there exists an idempotent $p \in \beta S$ such that if $q, r \in \beta S$ and $qr = p$, then $q = r = p$. We show that the same conclusion holds for the semigroups $(\mathbb{N}, \cdot)$ and $(\mathcal{F}, \cup)$ where $\mathcal{F}$ is the set of finite nonempty subsets of $\mathbb{N}$. Such a strong conclusion is not possible if S is the free group on countably many generators or is the free semigroup on finitely many (but more than one) generators, since then any idempotent can be written as a product involving elements of S. But we show that in these cases we can produce p such that if $q, r \in \beta S$ and $qr = p$, then either $q = r = p$ or q and r satisfy one of the trivial exceptions that must exist. Finally, we show that for the free semigroup on countably many generators, the conclusion can be derived from a set theoretical assumption that is at least potentially weaker than what had previously been required.

Contents

1. Introduction

Given a discrete semigroup $(S, \cdot)$, we take the points of the Stone–Čech compactification, βS, of S to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. The operation on S has a natural extension to βS making $(\beta S, \cdot)$ a right topological semigroup, meaning that for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is

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continuous. The only thing we will need to know about the operation on βS in this paper is that if $p, q \in \beta S$ and $A \subseteq S$, then $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. Much more information, including an elementary introduction, can be found in [8].

Let $\langle x_t \rangle_{t=1}^{\infty}$ be a sequence in a semigroup $(S, \cdot)$. Then

$$FP(\langle x_t \rangle_{t=1}^{\infty}) = \left\{ \prod_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N}) \right\}$$

where $\mathbb{N}$ is the set of positive integers. For any set X, $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of X and $\prod_{t \in F} x_t$ is the product in increasing order of indices. If the operation is denoted by $+$, we write

$$FS(\langle x_t \rangle_{t=1}^{\infty}) = \left\{ \sum_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N}) \right\}.$$

Given sequences $\langle x_t \rangle_{t=1}^{\infty}$ and $\langle y_t \rangle_{t=1}^{\infty}$ in S we say that $\langle y_t \rangle_{t=1}^{\infty}$ is a *product subsystem* of $\langle x_t \rangle_{t=1}^{\infty}$ if and only if there is a sequence $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $y_n = \prod_{t \in H_n} x_t$ and $\max H_n < \min H_{n+1}$. (For an additive semigroup, *sum subsystem* is defined analogously.)

An ultrafilter p on S is said to be *strongly productive* provided that, given any $A \in p$ there is a sequence $\langle x_t \rangle_{t=1}^{\infty}$ such that $FP(\langle x_t \rangle_{t=1}^{\infty}) \subseteq A$ and $FP(\langle x_t \rangle_{t=1}^{\infty}) \in p$. (The analogue in the additive situation is *strongly summable*.) See the introduction to [7] for the history behind the invention of strongly summable (or productive) ultrafilters.

It follows from [6, Theorem 2.3] that if $(S, +)$ is a countable, commutative, and cancellative semigroup, then any strongly summable ultrafilter on S is an idempotent in βS. Given any discrete semigroup S and an idempotent $p \in \beta S$, there is a largest subgroup $H(p)$ of βS with p as its identity. Often $H(p)$ is quite large. In fact, if S is an infinite cancellative semigroup with cardinality κ, then by [8, Corollary 7.39] βS contains a copy of the free group on 2^{2^κ} generators. It was shown in [5, Theorem 3.1] that if p is any strongly summable ultrafilter on $\mathbb{N}$, then any invertible element with respect to p is a member of $\mathbb{Z} + p$ and in particular, $H(p)$ is as small as possible; that is $H(p) = \mathbb{Z} + p$. And the question was asked in [5] whether a strongly summable ultrafilter p on $\mathbb{N}$ could be written as a sum of two elements, neither of which was a member of $\mathbb{Z} + p$. This question was answered in the negative in [9, Theorem 4]. (See the introduction to [7] for an explanation of why the negative answer follows.)

It was shown in [6, Theorem 4.5] that if $(G, +)$ is a countable group which can be embedded in the circle group $\mathbb{T}$, p is a *sparse* strongly summable ultrafilter on G, and $q, r \in G^* = \beta G \setminus G$ such that $q + r = p$, then p is an idempotent, $q \in G + p$, and $r \in G + p$.

Definition 1.1. Let $(S, +)$ be a semigroup and let $p \in \beta S$. Then p is a *sparse strongly summable ultrafilter* if and only if for every $A \in p$, there exist a sequence $\langle x_t\rangle_{t=1}^\infty$ and a subsequence $\langle y_t\rangle_{t=1}^\infty$ of $\langle x_t\rangle_{t=1}^\infty$ such that $FS(\langle x_t\rangle_{t=1}^\infty) \subseteq A$, $FS(\langle y_t\rangle_{t=1}^\infty) \in p$, and $\{x_n : n \in \mathbb{N}\} \setminus \{y_n : n \in \mathbb{N}\}$ is infinite.

In [7, Theorem 4.2] it was shown that if S is a countable subsemigroup of $\mathbb{T}$ and p is a nonprincipal strongly summable ultrafilter on S, then p is sparse, and thus as a consequence of [6, Theorem 4.5], if G is the group generated by S and $q, r \in G^*$ with $q + r = p$, then q and r are in $G + p$. It was recently shown in [3, Theorem 2.1] that all nonprincipal strongly summable ultrafilters on $\bigoplus_{n<\omega} \mathbb{Z}_2$ are sparse.

All of the results cited so far in this introduction deal with commutative semigroups. It was shown in [11, Theorem 3.10] that, assuming Martin's Axiom, if S is the free semigroup on countably many generators, then there is an idempotent $p \in \beta S$ such that, if $q, r \in \beta S$ and $q \cdot r = p$, then $q = r = p$. That idempotent is a strongly productive ultrafilter on S. In fact it satisfied the following stronger requirement.

Definition 1.2. Let S be the free semigroup on the generators $\langle a_t\rangle_{t=1}^\infty$ and let $p \in \beta S$. Then p is a *very strongly productive* ultrafilter on S if and only if for every $A \in p$ there is a product subsystem $\langle x_t\rangle_{t=1}^\infty$ of $\langle a_t\rangle_{t=1}^\infty$ such that $FP(\langle x_t\rangle_{t=1}^\infty) \subseteq A$ and $FP(\langle x_t\rangle_{t=1}^\infty) \in p$.

Very strongly productive ultrafilters correspond to *ordered union* ultrafilters introduced in [1]. Given a sequence $\langle A_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$,

$$FU(\langle A_n\rangle_{n=1}^\infty) = \left\{ \bigcup_{t\in F} A_t : F \in \mathcal{P}_f(\mathbb{N}) \right\}.$$

Definition 1.3. Let Θ be an ultrafilter on $\mathcal{P}_f(\mathbb{N})$.

(a) Θ is a *union ultrafilter* if and only if for each $\mathcal{A} \in \Theta$ there exists a sequence $\langle A_n\rangle_{n=1}^\infty$ of pairwise disjoint elements of $\mathcal{P}_f(\mathbb{N})$ such that $FU(\langle A_n\rangle_{n=1}^\infty) \subseteq \mathcal{A}$ and $FU(\langle A_n\rangle_{n=1}^\infty) \in \Theta$.

(b) Θ is an *ordered union ultrafilter* if and only if for each $\mathcal{A} \in \Theta$ there exists a sequence $\langle A_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $\max A_n < \min A_{n+1}$, $FU(\langle A_n\rangle_{n=1}^\infty) \subseteq \mathcal{A}$, and $FU(\langle A_n\rangle_{n=1}^\infty) \in \Theta$.

It was shown in [1, Theorem 2.4] that the Continuum Hypothesis implies the existence of ordered union ultrafilters, it was shown in [4, Theorem 4.1] that Martin's axiom implies the existence of union ultrafilters, and it was shown in [2, Theorem 3] that the existence of union ultrafilters cannot be established in ZFC.

If S is the free semigroup on a finite alphabet A with at least two members, then there is no idempotent $p \in \beta S$ such that, if $q, r \in \beta S$ and $q \cdot r = p$, then $q = r = p$. The reason is that for $p \in S^* = \beta S \setminus S$, $\bigcup_{a\in A} aS \in p$ so some $aS \in p$. Then $a^{-1}p = \{B \subseteq S : aB \in p\} \in S^*$ and thus

$$(pa) \cdot (a^{-1}p) = p \cdot p = p.$$

In Section 2 we show that the existence of ordered union ultrafilters implies the existence of an idempotent p in βS and distinct elements $b, c \in A$ such that if $q, r \in \beta S$, $q \cdot r = p$, and it is not the case that $q = r = p$, then some one of the following trivial cases must hold, and in particular $H(p) = \{p\}$.

(1) There is some $n \in \mathbb{N}$ such that $q = b^n$ and $r = b^{-n}p$;
(2) there is some $n \in \mathbb{N}$ such that $q = pb^n$ and $r = b^{-n}p$;
(3) there is some $n \in \mathbb{N}$ such that $q = pc^{-n}$ and $r = c^n$; or
(4) there is some $n \in \mathbb{N}$ such that $q = pc^{-n}$ and $r = c^n p$.

Similarly, if G is the free group on countably many generators, then there is no idempotent $p \in \beta G$ such that, if $q, r \in \beta S$ and $q \cdot r = p$, then $q = r = p$. The reason is that given any $w \in G$, one may let $q = pw$ and $r = w^{-1}p$. We show in Section 3 that Martin's axiom implies the existence of a *sparse* ordered union ultrafilter, and thus of a sparse very strongly productive ultrafilter. It is also shown that if p is a sparse very strongly productive ultrafilter, then the only way to write p nontrivially as a product in βG is as $(pw)(w^{-1}p)$, $w(w^{-1}p)$, or $(pw)w^{-1}$ for some $w \in G$.

In Section 4 we show that the existence of a union ultrafilter implies the existence of an idempotent $p \in (\beta\mathbb{N}, \cdot)$ such that if $q, r \in \beta\mathbb{N} \setminus \{1\}$ and $qr = p$, then $q = r = p$. We also show in this section that Martin's Axiom implies that there is an idempotent $p \in \beta S$, where S is the free semigroup on the generators $\langle a_t \rangle_{t=1}^{\infty}$, which is not very strongly productive, in fact not even strongly productive, but still has the property that it can only be written trivially as a product.

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2. The free semigroup on a finite alphabet

Throughout this section we shall let D be a finite alphabet with at least two members and will fix distinct elements b and c of D. We will let S be the free semigroup, with identity ι, on the alphabet D. We write $[\mathbb{N}]^{<\omega}$ for the set of finite subsets of $\mathbb{N}$. Thus $[\mathbb{N}]^{<\omega} = \mathcal{P}_f(\mathbb{N}) \cup \{\emptyset\}$. The following notions are based on the similar definitions in [11]. We agree that $\prod_{t\in\emptyset} x_t = \iota$, $\max \emptyset = 0$, and $\min \emptyset = \infty$.

We shall denote by T the subsemigroup of S generated by $\langle b^t c^t \rangle_{t=1}^{\infty}$. Then T is a copy of the free semigroup on countably many generators. Recall from [1] that the Continuum Hypothesis implies that ordered union ultrafilters exist, and by [11, Theorem 3.3] the existence of ordered union ultrafilters implies the existence of very strongly productive ultrafilters.

Lemma 2.1. *Let p be a very strongly productive ultrafilter on T. For each $A \in p$, there is a product subsystem $\langle x_t \rangle_{t=1}^{\infty}$ of $\langle b^t c^t \rangle_{t=1}^{\infty}$ such that $FP(\langle x_t \rangle_{t=1}^{\infty}) \subseteq A$ and for all $m \in \mathbb{N}$, $FP(\langle x_t \rangle_{t=m}^{\infty}) \in p$.*

Proof. For $i \in \{1,2\}$, let $C_i = \{3^n(3k+i) : n,k \in \omega\}$. (Note that C_i is the set of elements of $\mathbb{N}$ whose rightmost nonzero ternary digit is i.) For $i \in \{1,2\}$, let $D_i = \{x \in S \setminus \{\iota\} : \ell(x) \in C_i\}$, where $\ell(x)$ is the length of the word x. Pick $i \in \{1,2\}$ such that $D_i \in p$. Define $f : S \setminus \{\iota\} \to \omega$ by $f(x) = n$ where 3^n divides $\ell(x)$ and 3^{n+1} does not divide $\ell(x)$. (Thus $f(x)$ is the number of rightmost 0's in the ternary expansion of $\ell(x)$.) If $u, v \in D_i$ and $f(u) = f(v)$, then $uv \notin D_i$. Consequently, if $\{u, v, uv\} \subseteq D_i$, then $f(uv) = \min\{f(u), f(v)\}$.

Let $A \in p$ and pick a product subsystem $\langle x_t\rangle_{t=1}^\infty$ of $\langle b^t c^t\rangle_{t=1}^\infty$ such that $FP(\langle x_t\rangle_{t=1}^\infty) \subseteq A \cap D_i$ and $FP(\langle x_t\rangle_{t=1}^\infty) \in p$. Let $m \in \mathbb{N}$ and suppose that $FP(\langle x_t\rangle_{t=m}^\infty) \notin p$. Then $m > 1$. Since

$$\begin{aligned} FP(\langle x_t\rangle_{t=1}^\infty) &= FP(\langle x_t\rangle_{t=m}^\infty) \cup FP(\langle x_t\rangle_{t=1}^{m-1}) \\ &\quad \cup \bigcup\{u \cdot FP(\langle x_t\rangle_{t=m}^\infty) : u \in FP(\langle x_t\rangle_{t=1}^{m-1})\} \end{aligned}$$

and $FP(\langle x_t\rangle_{t=1}^{m-1}) \notin p$, because p is nonprincipal, there must be some $u \in FP(\langle x_t\rangle_{t=1}^{m-1})$ such that $u \cdot FP(\langle x_t\rangle_{t=m}^\infty) \in p$.

We claim that for all $x \in u \cdot FP(\langle x_t\rangle_{t=m}^\infty)$, $f(x) \le f(u)$. To see this, let $x \in u \cdot FP(\langle x_t\rangle_{t=m}^\infty)$ and pick $v \in FP(\langle x_t\rangle_{t=m}^\infty)$ such that $x = uv$. Then $\{u, v, uv\} \subseteq FP(\langle x_t\rangle_{t=1}^\infty) \subseteq D_i$ so $f(x) = \min\{f(u), f(v)\}$.

Choose a sequence $\langle y_t\rangle_{t=1}^\infty$ such that $FP(\langle y_t\rangle_{t=1}^\infty) \subseteq u \cdot FP(\langle x_t\rangle_{t=m}^\infty)$ and $FP(\langle y_t\rangle_{t=1}^\infty) \in p$. Then for all $k \in \mathbb{N}$, $f(y_k) \le f(u)$ so pick $k < t$ such that $f(y_k) = f(y_t)$. Then $y_k y_t \notin D_i$, a contradiction. □

We pause to note that every very strongly productive ultrafilter is an idempotent.

Lemma 2.2. *Let p be a very strongly productive ultrafilter on T. Then p is an idempotent.*

Proof. Let $A \in p$. We need to show that $\{y \in S : y^{-1}A \in p\} \in p$. Pick $\langle x_t\rangle_{t=1}^\infty$ as guaranteed by Lemma 2.1 for A. It suffices to show that

$$FP(\langle x_t\rangle_{t=1}^\infty) \subseteq \{y \in S : y^{-1}A \in p\},$$

so let $y \in FP(\langle x_t\rangle_{t=1}^\infty)$ and pick $F \in \mathcal{P}_f(\mathbb{N})$ such that $y = \prod_{t\in F} x_t$. Let $m = \max F + 1$. Then $FP(\langle x_t\rangle_{t=m}^\infty) \in p$ and $FP(\langle x_t\rangle_{t=m}^\infty) \subseteq y^{-1}A$. □

Definition 2.3. Let $\langle x_t\rangle_{t=1}^\infty$ be a sequence in S and let $k \in \mathbb{N}$.

(a) $$\begin{aligned} R(\langle x_t\rangle_{t=k}^\infty) = \Bigg\{ \Bigg(\prod_{t\in F} x_t\Bigg) u : {} & u \in S \setminus \{\iota\}\,,\ F \in [\mathbb{N}]^{<\omega}, \text{ and} \\ & (\exists s \in \mathbb{N})(\exists v \in S \setminus \{\iota\})(k \le \min F\,,\ \max F < s\,,\ k \le s \\ & \text{and } uv = x_s) \Bigg\}. \end{aligned}$$

(b) $$L(\langle x_t\rangle_{t=k}^\infty)=\Big\{v\Big(\prod_{t\in F}x_t\Big):v\in S\setminus\{\iota\}\,,\,F\in[\mathbb{N}]^{<\omega},\text{ and }(\exists s\in\mathbb{N})(\exists u\in S\setminus\{\iota\})(k\le s<\min F\,,\ \text{ and }uv=x_s)\Big\}.$$

Note that, with $F=\emptyset$ in the definition, we have that

$$\{u\in S\setminus\{\iota\}:(\exists s\in\mathbb{N})(\exists v\in S\setminus\{\iota\})(k\le s\text{ and }uv=x_s)\}\subseteq R(\langle x_t\rangle_{t=k}^\infty),$$
$$\{v\in S\setminus\{\iota\}:(\exists s\in\mathbb{N})(\exists u\in S\setminus\{\iota\})(k\le s\text{ and }uv=x_s)\}\subseteq L(\langle x_t\rangle_{t=k}^\infty).$$

Lemma 2.4. *Let $\langle x_t\rangle_{t=1}^\infty$ be a sequence in $S\setminus\{\iota\}$ and let $y,z\in S\setminus\{\iota\}$ such that $yz\in FP(\langle x_t\rangle_{t=1}^\infty)$. If either $y\notin FP(\langle x_t\rangle_{t=1}^\infty)$ or $z\notin FP(\langle x_t\rangle_{t=1}^\infty)$, then $y\in R(\langle x_t\rangle_{t=1}^\infty)$ and $z\in L(\langle x_t\rangle_{t=1}^\infty)$.*

Proof. Assume that either $y\notin FP(\langle x_t\rangle_{t=1}^\infty)$ or $z\notin FP(\langle x_t\rangle_{t=1}^\infty)$. Pick $H\in\mathcal{P}_f(\mathbb{N})$ such that $yz=\prod_{t\in H}x_t$ and write $H=\{n_1,n_2,\ldots,n_s\}$ where $n_1<n_2<\ldots<n_s$. Then $\ell(yz)=\sum_{i=1}^s\ell(x_{n_i})$.

Case 1. $\ell(y)\le\ell(x_{n_1})$. If $\ell(y)=\ell(x_{n_1})$, then $y=x_{n_1}$ and either $s=1$ in which case $z=\iota$ or $s>1$ in which case $z=\prod_{i=2}^s x_{n_i}$. Thus $\ell(y)<\ell(x_{n_1})$. Pick $v\in S\setminus\{\iota\}$ such that $x_{n_1}=yv$. If $s=1$, then $z=v$ and if $s>1$, then $z=v(\prod_{i=2}^s x_{n_i})$. Therefore $y\in R(\langle x_t\rangle_{t=1}^\infty)$ and $z\in L(\langle x_t\rangle_{t=1}^\infty)$.

Note that if $s=1$, then Case 1 applies.

Case 2. $s>1$ and $\ell(y)\ge\sum_{i=1}^{s-1}\ell(x_{n_i})$. If $\ell(y)=\sum_{i=1}^{s-1}\ell(x_{n_i})$, then $y\in FP(\langle x_t\rangle_{t=1}^\infty)$ and $z\in FP(\langle x_t\rangle_{t=1}^\infty)$. If $\ell(y)=\sum_{i=1}^{s}\ell(x_{n_i})$, then $z=\iota$. So we must have that $\sum_{i=1}^{s-1}\ell(x_{n_i})<\ell(y)<\sum_{i=1}^{s}\ell(x_{n_i})$. Pick $u\in S\setminus\{\iota\}$ such that $y=(\prod_{i=1}^{s-1}x_{n_i})u$. Then $x_{n_s}=uz$ so $y\in R(\langle x_t\rangle_{t=1}^\infty)$ and $z\in L(\langle x_t\rangle_{t=1}^\infty)$.

Case 3. $s>1$ and $\ell(x_{n_1})<\ell(y)<\sum_{i=1}^{s-1}\ell(x_{n_i})$. Then $s>2$. Pick $j\in\{1,2,\ldots,s-2\}$ such that

$$\sum_{i=1}^{j}\ell(x_{n_i})<\ell(y)\le\sum_{i=1}^{j+1}\ell(x_{n_i}).$$

If $\ell(y)=\sum_{i=1}^{j+1}\ell(x_{n_i})$, then $y=\prod_{i=1}^{j+1}x_{n_i}$ and $z=\prod_{i=j+2}^{s}x_{n_i}$, so

$$\sum_{i=1}^{j}\ell(x_{n_i})<\ell(y)<\sum_{i=1}^{j+1}\ell(x_{n_i}).$$

Pick $u,v\in S\setminus\{\iota\}$ such that $y=(\prod_{i=1}^{j}x_{n_i})u$ and $yv=\prod_{i=1}^{j+1}x_{n_i}$. Then $uv=x_{n_{j+1}}$ and $z=v(\prod_{i=j+2}^{s}x_{n_i})$ so $y\in R(\langle x_t\rangle_{t=1}^\infty)$ and $z\in L(\langle x_t\rangle_{t=1}^\infty)$. □

Lemma 2.5. *Let p be a very strongly productive ultrafilter on T. Assume that $q,r\in\beta S\setminus\{\iota\}$, $qr=p$, and it is not the case that $q=r=p$. Let $A\in p$. Then there is a product subsystem $\langle x_t\rangle_{t=1}^\infty$ of $\langle b^tc^t\rangle_{t=1}^\infty$ such that $FP(\langle x_t\rangle_{t=1}^\infty)\subseteq A$ and for each $k\in\mathbb{N}$, $FP(\langle x_t\rangle_{t=k}^\infty)\in p$, $R(\langle x_t\rangle_{t=k}^\infty)\in q$, and $L(\langle x_t\rangle_{t=k}^\infty)\in r$.*

Proof. Assume first that $q \neq p$ and pick $B \in q \setminus p$ such that $\iota \notin B$. By Lemma 2.1, pick a product subsystem $\langle x_t \rangle_{t=1}^{\infty}$ of $\langle b^t c^t \rangle_{t=1}^{\infty}$ such that $FP(\langle x_t \rangle_{t=1}^{\infty}) \subseteq A \setminus B$ and for all $k \in \mathbb{N}$, $FP(\langle x_t \rangle_{t=k}^{\infty}) \in p$. Let $k \in \mathbb{N}$. Then $\{y \in S : y^{-1}FP(\langle x_t \rangle_{t=k}^{\infty}) \in r\} \in q$.

Suppose that $R(\langle x_t \rangle_{t=k}^{\infty}) \notin q$ and pick $y \in B \setminus R(\langle x_t \rangle_{t=k}^{\infty})$ such that $y^{-1}FP(\langle x_t \rangle_{t=k}^{\infty}) \in r$. Pick $v \in y^{-1}FP(\langle x_t \rangle_{t=k}^{\infty})$. Then $yv = \prod_{t \in H} x_t$ for some $H \in \mathcal{P}_f(\mathbb{N})$ with $\min H \geq k$. Since $y \in B$, $y \notin FP(\langle x_t \rangle_{t=k}^{\infty})$ so by Lemma 2.4, $y \in R(\langle x_t \rangle_{t=k}^{\infty})$, a contradiction.

Now suppose that $L(\langle x_t \rangle_{t=k}^{\infty}) \notin r$. Pick $y \in B$ with $y^{-1}FP(\langle x_t \rangle_{t=k}^{\infty}) \in r$. Pick $z \in y^{-1}FP(\langle x_t \rangle_{t=k}^{\infty}) \setminus L(\langle x_t \rangle_{t=k}^{\infty})$. Then $yz \in FP(\langle x_t \rangle_{t=k}^{\infty})$ and $y \notin FP(\langle x_t \rangle_{t=k}^{\infty})$ so by Lemma 2.4, $z \in L(\langle x_t \rangle_{t=k}^{\infty})$, a contradiction.

Now assume that $r \neq p$ and pick $B \in r \setminus p$ such that $\iota \notin B$. By Lemma 2.1, pick a product subsystem $\langle x_t \rangle_{t=1}^{\infty}$ of $\langle b^t c^t \rangle_{t=1}^{\infty}$ such that $FP(\langle x_t \rangle_{t=1}^{\infty}) \subseteq A \setminus B$ and for all $k \in \mathbb{N}$, $FP(\langle x_t \rangle_{t=k}^{\infty}) \in p$. Let $k \in \mathbb{N}$. Then

$$\{y \in S \setminus \{\iota\} : y^{-1}FP(\langle x_t \rangle_{t=k}^{\infty}) \in r\} \in q.$$

Suppose that $L(\langle x_t \rangle_{t=k}^{\infty}) \notin r$. Pick $y \in S \setminus \{\iota\}$ with $y^{-1}FP(\langle x_t \rangle_{t=k}^{\infty}) \in r$ and pick $z \in B \cap y^{-1}FP(\langle x_t \rangle_{t=k}^{\infty}) \setminus L(\langle x_t \rangle_{t=k}^{\infty})$. Then $yz \in FP(\langle x_t \rangle_{t=k}^{\infty})$ and $z \notin FP(\langle x_t \rangle_{t=k}^{\infty})$ so we can again apply Lemma 2.4.

Finally suppose $R(\langle x_t \rangle_{t=k}^{\infty}) \notin q$. Pick $y \in S \setminus (R(\langle x_t \rangle_{t=k}^{\infty}) \cup \{\iota\})$ with $y^{-1}FP(\langle x_t \rangle_{t=k}^{\infty}) \in r$; pick $z \in B \cap y^{-1}FP(\langle x_t \rangle_{t=k}^{\infty})$. Then $yz \in FP(\langle x_t \rangle_{t=k}^{\infty})$ and $z \notin FP(\langle x_t \rangle_{t=k}^{\infty})$ so we can again apply Lemma 2.4. □

Lemma 2.6. *Let $p \in \beta S$ with $FP(\langle b^t c^t \rangle_{t=1}^{\infty}) \in p$. Assume that $q, r \in \beta S \setminus \{\iota\}$ and $qr = p$. Then there is some $n \in \mathbb{N}$ such that $Sb^n \notin q$.*

Proof. Suppose that for all $n \in \mathbb{N}$, $Sb^n \in q$. Let

$$A = \{x \in S : x^{-1}FP(\langle b^t c^t \rangle_{t=1}^{\infty}) \in r\}.$$

Then $A \in q$ so pick $w \in Sb \cap A$. Then there is some $n \in \mathbb{N}$ such that either $w = b^n$ or $w = uab^n$ for some $u \in S$ and some $a \in D \setminus \{b\}$. Pick $z \in Sb^{n+1} \cap A$. Then there is some $m > n$ such that either $z = b^m$ or $c = vdb^m$ for some $v \in S$ and some $d \in D \setminus \{b\}$.

Pick

$$y \in w^{-1}FP(\langle b^t c^t \rangle_{t=1}^{\infty}) \cap z^{-1}FP(\langle b^t c^t \rangle_{t=1}^{\infty}).$$

Since $wy \in FP(\langle b^t c^t \rangle_{t=1}^{\infty})$ there is some $l \geq n$ such that $y = b^{l-n}c^l$ or y begins $b^{l-n}c^l b$. Since $zy \in FP(\langle b^t c^t \rangle_{t=1}^{\infty})$ there is some $s \geq m$ such that $y = b^{s-m}c^s$ or y begins $b^{s-m}c^s b$. This is impossible, since $m > n$. □

Note that if $s \in \mathbb{N}$ and $b^s c^s$ occurs in some $z \in S$, then so does $b^t c^t$ for all $t \in \{1, 2, \ldots, s\}$. We omit the routine proof of the following lemma which allows us to conclude more from the occurrence of $bc^s b$.

Lemma 2.7. *Let $\langle x_t \rangle_{t=1}^{\infty}$ be a product subsystem of $\langle b^t c^t \rangle_{t=1}^{\infty}$ and for each $n \in \mathbb{N}$, let $H_n \in \mathcal{P}_f(\mathbb{N})$ such that $x_n = \prod_{t \in H_n} b^t c^t$. Let $s, k \in \mathbb{N}$, let $z \in L(\langle x_t \rangle_{t=k}^{\infty})$, and assume that either z ends with bc^s or $bc^s b$ occurs in z. Then $s \in H_n$ for some $n \geq k$.*

Lemma 2.8. *Let p be a very strongly productive ultrafilter on T. Assume that $q, r \in S^*$, $qr = p$, and it is not the case that $q = r = p$. If $Sb \in q$, then there is some $n \in \mathbb{N}$ such that $q = pb^n$.*

Proof. Suppose not. By Lemma 2.6 we may choose the largest $l \in \mathbb{N}$ such that $Sb^l \in q$. Then $Sb^l = \{b^l\} \cup \bigcup_{d \in D} Sdb^l$, $q \notin S$, and $Sb^{l+1} \notin q$ so there is some $d \in D \setminus \{b\}$ such that $Sdb^l \in q$. Since $Sc \in p$, we have that $p \neq q$. Pick $A \in q$ such that $\overline{A} \cap \{p, pb, pb^2, \ldots, pb^l\} = \emptyset$. Let

$$B = S \setminus (A \cup Ab^{-1} \cup Ab^{-2} \cup \ldots \cup Ab^{-l}).$$

Then $B \in p$ so pick by Lemma 2.5 a product subsystem $\langle x_t \rangle_{t=1}^{\infty}$ of $\langle b^t c^t \rangle_{t=1}^{\infty}$ such that for each $k \in \mathbb{N}$,

$$FP(\langle x_t \rangle_{t=k}^{\infty}) \in p, \quad R(\langle x_t \rangle_{t=k}^{\infty}) \in q, \quad \text{and} \quad L(\langle x_t \rangle_{t=k}^{\infty}) \in r.$$

For each $n \in \mathbb{N}$, pick $H_n \in \mathcal{P}_f(\mathbb{N})$ such that $x_n = \prod_{t \in H_n} b^t c^t$. Since $\langle x_t \rangle_{t=1}^{\infty}$ is a product subsystem of $\langle b^t c^t \rangle_{t=1}^{\infty}$ and $R(\langle x_t \rangle_{t=1}^{\infty}) \in q$, We must have $d = c$ and thus $Scb^l \in q$. Since $FP(\langle x_t \rangle_{t=l+1}^{\infty}) \in p$,

$$\{w \in S : w^{-1} FP(\langle x_t \rangle_{t=l+1}^{\infty}) \in r\} \in q.$$

Pick $w \in R(\langle x_t \rangle_{t=l+1}^{\infty}) \cap A \cap Scb^l$ such that $w^{-1} FP(\langle x_t \rangle_{t=l+1}^{\infty}) \in r$.

There are some $F \in [\mathbb{N}]^{<\omega}$ and $j \in \mathbb{N}$ with $\min F \geq l+1$, $\max F < j$ (and, if $F = \emptyset$, then $j \geq l+1$), and $v \in S$ such that $w = (\prod_{t \in F} x_t) \cdot u$ and $u \cdot v = x_j$. Since $w \in Scb^l$, we must have that u ends in cb^l. (If the length of u were at most l, then we would have $u = b^t$ for some $t \in \{1, 2, \ldots, l\}$ and thus that $\prod_{s \in F} x_s = wb^{-t} \in Ab^{-t}$, a contradiction.)

Since $uv = x_j = \prod_{i \in H_j} b^i c^i$ and u ends in cb^l, there exist $L \in \mathcal{P}_f(\mathbb{N})$, $s \in \mathbb{N}$, and (possibly empty) $M \in [\mathbb{N}]^{<\omega}$ such that $\max L < s < \min M$, $H_j = L \cup \{s\} \cup M$, $u = (\prod_{i \in L} b^i c^i) \cdot b^l$, and $v = b^{s-l} c^l \cdot \prod_{i \in M} b^i c^i$. (Note that $j > l$ so $s > l$.)

Since $L(\langle x_t \rangle_{t=j+1}^{\infty}) \in r$, pick $z \in w^{-1} FP(\langle x_t \rangle_{t=l+1}^{\infty}) \cap L(\langle x_t \rangle_{t=j+1}^{\infty})$. Then $wz \in FP(\langle x_t \rangle_{t=l+1}^{\infty})$ and $w = (\prod_{t \in F} x_t) \cdot (\prod_{i \in L} b^i c^i) \cdot b^l$. Also

$$wz = \prod_{t \in K} x_t = \prod_{t \in K} \prod_{i \in H_t} b^i c^i$$

for some $K \in \mathcal{P}_f(\mathbb{N})$ with $\min K > l$. Since $L \neq \emptyset$, pick $i \in L$. Then $b^i c^i b$ occurs in w and $i \in H_j$ so $j \in K$. Also

$$\begin{aligned} x_j &= \left(\prod_{i \in L} b^i c^i \right) \cdot b^l b^{s-l} c^s \cdot \prod_{i \in M} b^i c^i \\ &= w \cdot b^{s-l} c^s \cdot \prod_{i \in M} b^i c^i \end{aligned}$$

so z begins $b^{s-l} c^s$. So either z ends as $b^{s-l} c^s$ (if $M = \emptyset$) or $bc^s b$ occurs in z. In either case, by Lemma 2.7, $s \in H_n$ for some $n \geq j+1$. But $s \in H_j$, a contradiction. □

Theorem 2.9. *Let p be a very strongly productive ultrafilter on T. Assume that $q, r \in S^*$, $qr = p$, and it is not the case that $q = r = p$. If $Sb \in q$, then there is some $n \in \mathbb{N}$ such that $q = pb^n$ and $r = b^{-n}p$.*

Proof. By Lemma 2.8, pick $n \in \mathbb{N}$ with $q = pb^n$. Suppose $r \neq b^{-n}p$. Then $p \neq b^n r$ so pick $A \in p$ such that $A \notin b^n r$. Pick a product subsystem $\langle x_t\rangle_{t=1}^\infty$ of $\langle b^t c^t\rangle_{t=1}^\infty$ with $FP(\langle x_t\rangle_{t=1}^\infty) \in p$ and $FP(\langle x_t\rangle_{t=1}^\infty) \subseteq A$. Then

$$\{w \in S : w^{-1}FP(\langle x_t\rangle_{t=1}^\infty) \in r\} \cap FP(\langle x_t\rangle_{t=1}^\infty)b^n \in q$$

so pick $w \in FP(\langle x_t\rangle_{t=1}^\infty)b^n$ with $w^{-1}FP(\langle x_t\rangle_{t=1}^\infty) \in r$. Since $b^{-n}(S \setminus A) \in r$, pick $y \in w^{-1}FP(\langle x_t\rangle_{t=1}^\infty) \cap b^{-n}(S \setminus A)$. Pick F and H in $\mathcal{P}_f(\mathbb{N})$ such that $w = (\prod_{t\in F} x_t) \cdot b^n$ and $wy = \prod_{t\in H} x_t$. Then $\prod_{t\in H} x_t = (\prod_{t\in F} x_t) \cdot b^n \cdot y$ so F is an initial segment of H and $\prod_{t\in H\setminus F} x_t = b^n \cdot y$ and thus

$$y \in b^{-n}FP(\langle x_t\rangle_{t=1}^\infty) \subseteq b^{-n}A,$$

a contradiction. □

By a very similar sequence of lemmas, one can prove the following theorem.

Theorem 2.10. *Let p be a very strongly productive ultrafilter on T. Assume that $q, r \in S^*$, $qr = p$, and it is not the case that $q = r = p$. If $cS \in r$, then there is some $n \in \mathbb{N}$ such that $q = pc^{-n}$ and $r = c^n p$.*

Theorem 2.11. *Let p be a very strongly productive ultrafilter on T. Assume that $q, r \in \beta S$, $qr = p$, and it is not the case that $q = r = p$. Then either $Sb \in q$ or $cS \in r$. If $q \in S$ then there is some $n \in \mathbb{N}$ such that $q = b^n$. If $r \in S$, then there is some $n \in \mathbb{N}$ such that $r = c^n$.*

Proof. Suppose first that $q \in S$ and let n be the length of q. Pick by Lemma 2.1 a product subsystem $\langle x_t\rangle_{t=1}^\infty$ of $\langle b^t c^t\rangle_{t=1}^\infty$ such that $FP(\langle x_t\rangle_{t=n}^\infty) \in p$. In particular $FP(\langle b^t c^t\rangle_{t=n}^\infty) \in p = qr$ so $q^{-1}FP(\langle b^t c^t\rangle_{t=n}^\infty) \in r$. Pick $w \in q^{-1}FP(\langle b^t c^t\rangle_{t=n}^\infty)$. Then $qw \in FP(\langle b^t c^t\rangle_{t=n}^\infty)$ and thus the leftmost n letters of qw are all equal to b.

The proof for the case $r \in S$ is very similar. (At the appropriate point in the argument, pick w such that $r \in w^{-1}FP(\langle b^t c^t\rangle_{t=n}^\infty)$. Then the rightmost n letters of wr are all equal to c.)

Now assume that q and r are in S^* and suppose that $Sb \notin q$ and $cS \notin r$. Pick some $a \in D \setminus \{b\}$ and some $d \in D \setminus \{c\}$ such that $Sa \in q$ and $dS \in r$.

By Lemma 2.5 pick a product subsystem $\langle x_t\rangle_{t=1}^\infty$ of $\langle b^t c^t\rangle_{t=1}^\infty$ such that for each $k \in \mathbb{N}$, $FP(\langle x_t\rangle_{t=k}^\infty) \in p$, $R(\langle x_t\rangle_{t=k}^\infty) \in q$, and $L(\langle x_t\rangle_{t=k}^\infty) \in r$. For each $n \in \mathbb{N}$, pick $H_n \in \mathcal{P}_f(\mathbb{N})$ such that $x_n = \prod_{t\in H_n} b^t c^t$. Pick $w \in Sa \cap R(\langle x_t\rangle_{t=1}^\infty)$ such that $w^{-1}FP(\langle x_t\rangle_{t=1}^\infty) \in r$. Pick $F \in [\mathbb{N}]^{<\omega}$, $j \in \mathbb{N}$, and $u, v \in S \setminus \{\iota\}$ such that $\max F < j$, $w = (\prod_{t\in F} x_t) \cdot u$, and

$$uv = x_j = \prod_{t\in H_j} b^t c^t.$$

Since $a \neq b$ and the rightmost letter of w is the rightmost letter of u, we have $a = c$. Pick $s \in H_j$ such that

$$\sum\{2t : t \in H_j \text{ and } t < s\} < \ell(u) \leq \sum\{2t : t \in H_j \text{ and } t \leq s\},$$

where $\ell(u)$ is the length of u. Then the rightmost letter of u occurs in $b^s c^s$. We have $K_1, K_2 \in [\mathbb{N}]^{<\omega}$ and s such that $K_1 \cup \{s\} \cup K_2 = H_j$, $\max K_1 < s < \min K_2$, $u = (\prod_{t \in K_1} b^t c^t) \cdot b^s c^i$, and $v = c^{s-i} \cdot \prod_{t \in K_2} b^t c^t$ for some $i \in \{1, 2, \ldots, s\}$.

Pick $y \in w^{-1} FP(\langle x_t \rangle_{t=1}^\infty) \cap dS \cap L(\langle x_t \rangle_{t=j+1}^\infty)$. Since $wy \in FP(\langle x_t \rangle_{t=1}^\infty)$, the leftmost letter of y is b or c, and $d \neq c$, we have that $d = b$. Pick h and z in $S \setminus \{\iota\}$, $N \in [\mathbb{N}]^{<\omega}$, and $k < \min N$ with $k \geq j+1$ such that $y = z \cdot \prod_{t \in N} x_t$ and $hz = x_k = \prod_{t \in H_k} b^t c^t$. Pick $f \in H_k$ such that

$$\sum\{2t : t \in H_k \text{ and } t < f\} < \ell(z) \leq \sum\{2t : t \in H_k \text{ and } t \leq f\}.$$

Since the leftmost letter of z is the leftmost letter of y which is b, we have $M_1, M_2 \in [\mathbb{N}]^{<\omega}$ and g such that $M_1 \cup \{g\} \cup M_2 = H_k$, $\max M_1 < g < \min M_2$, $h = (\prod_{t \in M_1} b^t c^t) \cdot b^{g-\alpha}$, and $z = b^\alpha c^g \cdot \prod_{t \in M_2} b^t c^t$ for some $\alpha \in \{1, 2, \ldots, g\}$.

Pick $L \in \mathcal{P}_f(\mathbb{N})$ such that $wy = \prod_{t \in L} x_t$. Then

$$\prod_{t \in L} x_t = \left(\prod_{t \in F} x_t\right) \cdot \left(\prod_{t \in K_1} b^t c^t\right) \cdot b^s c^i b^\alpha c^g \cdot \left(\prod_{t \in M_2} b^t c^t\right) \cdot \left(\prod_{t \in N} x_t\right)$$

so

$$\prod_{t \in L \setminus (F \cup N)} x_t = \left(\prod_{t \in K_1} b^t c^t\right) \cdot b^s c^i b^\alpha c^g \cdot \left(\prod_{t \in M_2} b^t c^t\right).$$

Since $K_1 \subseteq H_j$, $s \in H_j$, $g \in H_k$, $M_2 \subseteq H_k$, and $j < k$, we must have $L \setminus (F \cup N) = \{j, k\}$, $i = s$, $\alpha = g$, $x_j = (\prod_{t \in K_1} b^t c^t) \cdot b^s c^s$, and $x_k = b^g c^g \cdot (\prod_{t \in M_2} b^t c^t)$. But then $H_j = K_1 \cup \{s\}$ so $K_2 = \emptyset$ and, since $i = s$, $v = \iota$, a contradiction. □

Corollary 2.12. *Let p be a very strongly productive ultrafilter on T. Assume that $q, r \in \beta S$, $qr = p$, and it is not the case that $q = r = p$. Then one of the following statements holds.*

1. *There is some $n \in \mathbb{N}$ such that $q = b^n$ and $r = b^{-n} p$;*
2. *there is some $n \in \mathbb{N}$ such that $q = pb^n$ and $r = b^{-n} p$;*
3. *there is some $n \in \mathbb{N}$ such that $q = pc^{-n}$ and $r = c^n$; or*
4. *there is some $n \in \mathbb{N}$ such that $q = pc^{-n}$ and $r = c^n p$.*

Proof. If q and r are in S^*, the conclusion follows from Theorems 2.9, 2.10, and 2.11. If q and r were both in S, then qr would be in S.

Assume that $q \in S$ and $r \in S^*$. Then by Theorem 2.11, pick $n \in \mathbb{N}$ such that $q = b^n$. Then $b^n r = p$ so, computing in βG, where G is the free group on the alphabet D, we have that $r = b^{-n} p$. Similarly, if $r \in S$, then there is some $n \in \mathbb{N}$ such that $r = c^n$ and $q = pc^{-n}$. □

3. The free group on a countable alphabet

Throughout this section we will let S and G be respectively the free semigroup with identity and the free group on the generators $\langle a_t\rangle_{t=1}^{\infty}$. We will let $T=\bigcap_{m=1}^{\infty}\overline{FP(\langle a_t\rangle_{t=m}^{\infty})}$. We will show that, assuming Martin's Axiom, there is an idempotent $p\in S^*$ with the property that if $q,r\in\beta G$ and $qr=p$, then there is some $w\in G$ such that (1) $q=w$ and $r=w^{-1}p$, (2) $q=pw$ and $r=w^{-1}p$, or (3) $q=pw$ and $r=w^{-1}$.

Members of G are the members of the free semigroup with identity on the alphabet $\{a_n:n\in\mathbb{N}\}\cup\{a_n^{-1}:n\in\mathbb{N}\}$ which do not have adjacent occurrences of a_n and a_n^{-1} for any n. We denote concatenation by $\frown$. Thus, for example, if $u=a_2a_3^{-1}a_2^{-1}$ and $v=a_2a_4$, then $uv=a_2a_3^{-1}{\frown}a_4$.

Definition 3.1. Let $w\in G\setminus\{\iota\}$.

(a) $A_w=\{x\in G: x \text{ begins with } w\}$.
(b) $B_w=\{x\in G: x \text{ ends with } w^{-1}\}$.

When we write "let l be a letter", we mean that

$$l\in\{a_n:n\in\mathbb{N}\}\cup\{a_n^{-1}:n\in\mathbb{N}\}.$$

Lemma 3.2. *Let $q,r\in G^*$ and assume that $qr\in T$. Let l be a letter. If $A_l\in r$, then $B_l\in q$.*

Proof. Assume first that $l=a_s^{-1}$ for some $s\in\mathbb{N}$ and suppose that $B_l\notin q$. Pick $x\in G\setminus B_l$ such that $x^{-1}FP(\langle a_t\rangle_{t=1}^{\infty})\in r$. Pick $y\in x^{-1}FP(\langle a_t\rangle_{t=1}^{\infty})\cap A_l$. Since x does not end in a_s, a_s^{-1} occurs in xy, a contradiction.

Now assume that $l=a_s$ for some $s\in\mathbb{N}$ and suppose that $B_l\notin q$. Pick $x\in G\setminus B_l$ such that $x^{-1}FP(\langle a_t\rangle_{t=s+1}^{\infty})\in r$. Pick $y\in x^{-1}FP(\langle a_t\rangle_{t=s+1}^{\infty})\cap A_l$. Then a_s occurs in xy, a contradiction. □

Lemma 3.3. *Let $q,r\in G^*$ and assume that $qr\in T$. If either $S\notin q$ or $S\notin r$, then there is a letter l such that $A_l\in r$.*

Proof. Assume first that $S\notin q$. Pick $x\in G\setminus S$ with $x^{-1}FP(\langle a_t\rangle_{t=1}^{\infty})\in r$. Pick $u\in G$, $v\in S$, and $t\in\mathbb{N}$ such that $x=u{\frown}a_t^{-1}{\frown}v$. Assume first that $v=\iota$. We claim $A_{a_t}\in r$. Suppose instead that $A_{a_t}\notin r$ and pick $y\in x^{-1}FP(\langle a_i\rangle_{i=1}^{\infty})\setminus A_{a_t}$. Then a_t^{-1} occurs in xy, a contradiction. Now assume that $v\in S$ and let a_s be the rightmost letter of v. Then as above we see that $A_{a_s^{-1}}\in r$.

The case that $S\in q$ and $S\notin r$ is handled in a similar fashion. □

Lemma 3.4. *Let $q,r\in G^*$ and assume that $qr\in T$. If either $S\notin q$ or $S\notin r$, then there is a letter l such that $A_l\in r$ and $B_l\in q$.*

Proof. Lemmas 3.2 and 3.3. □

Lemma 3.5. *Let $k\in\mathbb{N}$, let $r\in G^*$, let $w=l_1l_2\cdots l_k$ where each l_i is a letter, and assume that $A_w\in r$. Then $A_{l_k^{-1}}\notin w^{-1}r$.*

Proof. We proceed by induction on k. For $k = 1$, let l be a letter and suppose that $A_l \in r$ and $A_{l^{-1}} \in l^{-1}r$. Then $lA_{l^{-1}} \in r$. Pick $x \in A_l \cap lA_{l^{-1}}$. Since $x \in lA_{l^{-1}}$ we have $x = l(l^{-1}\frown w)$ where w does not begin with l so $x = w \notin A_l$, a contradiction.

Now assume that $k > 1$ and the lemma is valid for $k - 1$. Suppose that $A_{l_k^{-1}} \in w^{-1}r$ Let $w' = l_2 l_3 \cdots l_k$ and $r' = l_1^{-1}r$. We claim that $A_{w'} \in r'$ and $A_{l_k^{-1}} \in (w')^{-1}r'$, contradicting the induction hypothesis.

Now $A_w \in r$ so $l_1^{-1}A_w \in r'$. We claim that $l_1^{-1}A_w \subseteq A_{w'}$ so let $x \in l_1^{-1}A_w$. Then $l_1 x = l_1 l_2 \cdots l_k \frown u$ for some $u \in G$ so $l_1 x \in A_{l_1}$. If $l_1 x = l_1 \frown x$, then $x = l_2 l_3 \cdots l_k \frown u \in A_{w'}$ as desired. So suppose $l_1 x \neq l_1 \frown x$. Then $x = l_1^{-1} \frown v$ for some $v \in G \setminus A_{l_1}$ and thus $l_1 x = v \notin A_{l_1}$, a contradiction.

Finally, $(w')^{-1}r' = (w')^{-1}l_1^{-1}r = (l_1 w')^{-1}r = w^{-1}r$ so $A_{l_k^{-1}} \in (w')^{-1}r'$ as claimed. □

Lemma 3.6. *Let $q, r \in G^*$ and assume that $qr \in T$ and either $S \notin q$ or $S \notin r$. Then one of the following must hold:*

(1) *There is some $w \in G$ such that $w^{-1}r \in \beta S$ and $qw \in \beta S$.*
(2) *There exists a sequence $\langle l_t \rangle_{t=1}^{\infty}$ of letters such that $l_{t+1} \neq l_t^{-1}$ for each t and for each k, if $w_k = l_1 l_2 \cdots l_k$, then $A_{w_k} \in r$ and $B_{w_k} \in q$.*

Proof. Assume that (1) fails. By Lemma 3.4 we have some letter l_1 such that $A_{l_1} \in r$ and $B_{l_1} \in q$. Let $k \in \mathbb{N}$ and assume that $l_1, l_2, \ldots, l_k$ have been chosen. Let $w_k = l_1 l_2 \cdots l_k$. Then $A_{w_k} \in r$ and $B_{w_k} \in q$. Let $r' = w_k^{-1}r$ and $q' = qw_k$. Since (1) fails, either $S \notin r'$ or $S \notin q'$ so by Lemma 3.4, pick a letter l_{k+1} such that $A_{l_{k+1}} \in r'$ and $B_{l_{k+1}} \in q'$. By Lemma 3.5, $l_{k+1} \neq l_k^{-1}$. We claim that $A_{w_{k+1}} \in r$ and $B_{w_{k+1}} \in q$. Since $A_{l_{k+1}} \in r' = w_k^{-1}r$ and $B_{l_{k+1}} \in q' = qw_k$ we have that $w_k A_{l_{k+1}} \in r$ and $B_{l_{k+1}} w_k^{-1} \in q$. Since $l_{k+1} \neq l_k^{-1}$ we have immediately that $w_k A_{l_{k+1}} \subseteq A_{w_{k+1}}$ and $B_{l_{k+1}} w_k^{-1} \subseteq B_{w_{k+1}}$. □

We find it hard to believe that case (2) of the following theorem could hold, but we cannot prove that it does not.

Theorem 3.7. *Let p be a very strongly productive ultrafilter on S, let $q, r \in G^*$, and assume that $qr = p$ and either $S \notin q$ or $S \notin r$. Then one of the following must hold:*

(1) *There is some $w \in G$ such that $r = wp$ and $q = pw^{-1}$.*
(2) *There exists a sequence $\langle l_t \rangle_{t=1}^{\infty}$ of letters such that:*
    (a) *$l_{t+1} \neq l_t^{-1}$ for each t and for each k, if $w_k = l_1 l_2 \cdots l_k$, then $A_{w_k} \in r$ and $B_{w_k} \in q$.*
    (b) *There exists $k \in \mathbb{N}$ such that $\langle l_t \rangle_{t=k}^{\infty}$ is a subsequence of $\langle a_t \rangle_{t=1}^{\infty}$.*

Proof. We have that either conclusion (1) or conclusion (2) of Lemma 3.6 holds. Assume first that conclusion (1) of Lemma 3.6 holds. By [11, Theorem 3.10] $w^{-1}r = qw = p$.

Now assume that conclusion (2) of Lemma 3.6 holds. Let $C = FP(\langle a_t \rangle_{t=1}^{\infty})$ and pick $x \in G$ such that $x^{-1}C \in r$. Let $k = \ell(x) + 1$ and let $m > k$ be given.

Let $w_m = l_1 l_2 \cdots l_m$ and pick $y \in A_{w_m} \cap x^{-1}C$. Then $y = w_m{}^\frown v$ for some $v \in G \setminus A_{l_m^{-1}}$. In the computation of xy at most $k-1$ letters of w_m cancel so there exist $u \in G$ and $s \in \{1, 2, \ldots, k\}$ such that $xy = u^\frown l_s l_{s+1} \cdots l_m{}^\frown v$. Also $xy = \prod_{t \in F} a_t$ for some $F \in \mathcal{P}_f(\mathbb{N})$. Thus we have that for each $i \in \{0, 1, \ldots, m-s\}$, $l_{s+i} = a_{t_i}$ for some $t_0 < t_1 < \ldots < t_{m-s}$. □

Lemma 3.8. *Let p be a very strongly productive ultrafilter on S, let $q, r \in G^*$ such that $qr = p$, and assume that $\langle l_t \rangle_{t=1}^\infty$ and k are as in conclusion* (2) *of Theorem* 3.7. *Then $FP(\langle l_t \rangle_{t=k}^\infty) \in p$.*

Proof. Suppose instead $D = FP(\langle a_t \rangle_{t=1}^\infty) \setminus FP(\langle l_t \rangle_{t=k}^\infty) \in p$. Pick an increasing sequence $\langle \gamma(t) \rangle_{t=k}^\infty$ in $\mathbb{N}$ such that for each $t \geq k$, $l_t = a_{\gamma(t)}$. Pick a product subsystem $\langle x_t \rangle_{t=1}^\infty$ of $\langle a_t \rangle_{t=1}^\infty$ such that $E = FP(\langle x_t \rangle_{t=1}^\infty) \subseteq D$ and $E \in p$. For each $n \in \mathbb{N}$, pick $H_n \in \mathcal{P}_f(\mathbb{N})$ such that $x_n = \prod_{t \in H_n} a_t$. Pick $z \in B_{w_k}$ such that $z^{-1}E \in r$. Pick $\alpha \geq k$ and $u \in G$ such that $z = u^\frown l_\alpha^{-1} l_{\alpha-1}^{-1} \cdots l_1^{-1}$ and u does not end with $l_{\alpha+1}^{-1}$. (Note that $u = \iota$ is possible.)

Pick the first $\delta \in \mathbb{N}$ such that $\gamma(\alpha+1) \leq \max H_\delta$. Pick the largest $\nu \in \mathbb{N}$ such that $\gamma(\nu) \leq \max H_\delta$. Pick the first $\tau \in \mathbb{N}$ such that $\gamma(\nu+1) \leq \max H_\tau$. Pick the largest $\eta \in \mathbb{N}$ such that $\gamma(\eta) \leq \max H_\tau$. Pick $m \in \mathbb{N}$ such that $\gamma(m) > \max H_\tau$. Then $\alpha + 1 \leq \nu < \eta < m$.

Pick $y \in z^{-1}E \cap A_{w_m}$. Then $y = l_1 l_2 \cdots l_m{}^\frown v$ for some $v \in G$ which does not begin with l_m^{-1}. Then

$$(*) \qquad zy = u^\frown l_{\alpha+1} l_{\alpha+2} \cdots l_m{}^\frown v.$$

Since $zy \in E$, pick $F \in \mathcal{P}_f(\mathbb{N})$ such that $zy = \prod_{n \in F} x_n$. Pick n_1 and n_2 in F such that $\gamma(\alpha+1) \in H_{n_1}$ and $\gamma(\nu+1) \in H_{n_2}$. Then $\gamma(\alpha+1) \leq \max H_{n_1}$ and $\gamma(\alpha+1) \geq \min H_{n_1} > \max H_{n_1 - 1}$ so $n_1 = \delta$. Similarly, $n_2 = \tau$. Let $K = \{n \in F : n < \delta\}$ and $L = \{n \in F : n > \tau\}$. Then

$$(**) \qquad zy = \prod_{n \in K} x_n \cdot \prod_{t \in H_\delta} a_t \cdot \prod_{t \in H_\tau} a_t \cdot \prod_{n \in L} x_n.$$

(Recall that we take $\prod_{n \in \emptyset} x_n = \iota$.)

Comparing $(*)$ and $(**)$ we see that

$$u^\frown l_{\alpha+1} l_{\alpha+2} \cdots l_\nu = \prod_{n \in K} x_n \cdot \prod_{t \in H_\delta} a_t$$

so that $\prod_{t \in H_\tau} a_t = l_{\nu+1} l_{\nu+2} \cdots l_\eta$ and thus $x_\tau = \prod_{t \in H_\tau} a_\tau \in FP(\langle l_t \rangle_{t=k}^\infty)$, a contradiction. □

Definition 3.9. Let $p \in \beta S$. Then p is *sparse* if and only if for each $A \in p$ there exist $\langle x_t \rangle_{t=1}^\infty$ in S and an infinite set $D \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus D$ is infinite, $FP(\langle x_t \rangle_{t=1}^\infty) \subseteq A$, and $FP(\langle x_n \rangle_{n \in D}) \in p$.

We will conclude this section with a proof that Martin's Axiom implies that sparse very strongly productive ultrafilters on S exist.

Theorem 3.10. *Let p be a sparse very strongly productive ultrafilter on S and let $q, r \in G^*$ such that $qr = p$. Then there exists $w \in G$ such that $r = wp$ and $q = pw^{-1}$.*

Proof. Suppose not. Then we may pick $\langle l_t \rangle_{t=1}^{\infty}$ and k as guaranteed by conclusion (2) of Theorem 3.7. By Lemma 3.8, $FP(\langle l_t \rangle_{t=k}^{\infty}) \in p$. Pick infinite $D \subseteq \mathbb{N}$ and $\langle x_t \rangle_{t=1}^{\infty}$ in S such that $\mathbb{N} \setminus D$ is infinite, $FP(\langle x_t \rangle_{t=1}^{\infty}) \subseteq FP(\langle l_t \rangle_{t=k}^{\infty})$, and $E = FP(\langle x_n \rangle_{n \in D}) \in p$. For each $n \in \mathbb{N}$ pick $H_n \in \mathcal{P}_f(\mathbb{N})$ such that $x_n = \prod_{t \in H_n} l_t$. Note that for each n, $\max H_n < \min H_{n+1}$ because $x_n x_{n+1} = \prod_{t \in H_n} l_t \cdot \prod_{t \in H_{n+1}} l_t$ and $x_n x_{n+1} \in FP(\langle l_t \rangle_{t=k}^{\infty})$.

Pick $z \in B_{w_k}$ such that $z^{-1}E \in r$. Pick $\alpha \geq k$ and $u \in G$ such that $z = u^\frown l_\alpha^{-1} l_{\alpha-1}^{-1} \cdots l_1^{-1}$ and u does not end with $l_{\alpha+1}^{-1}$.

Pick the first $\delta \in \mathbb{N}$ such that $\alpha + 1 \leq \max H_\delta$ and let $\nu = \max H_\delta$. Pick the first $\tau > \delta$ such that $\tau \notin D$ and let $m = \max H_\tau$. Pick $y \in z^{-1}E \cap A_{w_m}$. Then $zy = u^\frown l_{\alpha+1} l_{\alpha+2} \cdots l_m{}^\frown v$ where $v \in G$ and v does not begin with l_m^{-1}. Since $zy \in E$, pick $F \in \mathcal{P}_f(D)$ such that $zy = \prod_{n \in F} x_n$. Since $l_{\alpha+1}$ occurs in zy, we may pick $n \in F$ such that $\alpha + 1 \in H_n$. Then $\alpha + 1 \leq \max H_n$ and $\alpha + 1 \geq \min H_n > \max H_{n-1}$ so $\delta = n$. Let $K = \{n \in F : n < \delta\}$. Then $\prod_{n \in K} x_n \cdot \prod_{t \in H_\delta} l_t = u^\frown l_{\alpha+1} l_{\alpha+2} \cdots l_\nu$. Now $\tau > \delta$ and $m = \max H_\tau$ so $H_\tau \subseteq \nu + 1, \nu + 2, \ldots, m$. Pick $s \in H_\tau$. Since $\tau \notin D$, l_s does not occur in $\prod_{n \in F} x_n = zy$, a contradiction. □

Corollary 3.11. *Let p be a sparse very strongly productive ultrafilter on S and let $q, r \in \beta G$ such that $qr = p$. Then there exists $w \in G$ such that:*

1. *$r = wp$ and $q = pw^{-1}$;*
2. *$r = w$ and $q = pw^{-1}$; or*
3. *$r = wp$ and $q = w^{-1}$.*

Proof. If $q, r \in G^*$, then conclusion (1) holds by Theorem 3.10. If $r \in G$, let $w = r$. Then since $wq = p$, $q = w^{-1}p$. If $q \in G$, let $w = q^{-1}$. □

Except for a question asked at the end, the rest of this section consists of a proof that Martin's Axiom implies the existence of a sparse very strongly productive ultrafilter on S (and thus that Martin's Axiom implies the existence of idempotents in βS that can only be written trivially as products of elements of βG). See [10, pages 53-61] or [8, Chapter 12] for an introduction to Martin's Axiom.

We actually produce a sparse ordered union ultrafilter on the semigroup $(\mathcal{F}, \cup)$, where $\mathcal{F} = \mathcal{P}_f(\mathbb{N})$.

Definition 3.12. Let Θ be an ultrafilter on $\mathcal{F}$. Then Θ is *sparse* if and only if for each $\mathcal{A} \in \Theta$, there exist a sequence $\langle X_n \rangle_{n=1}^{\infty}$ of members of $\mathcal{F}$ such that $\max X_n < \min X_{n+1}$ for each n and an infinite subset D of $\mathbb{N}$ such that $FU(\langle X_n \rangle_{n=1}^{\infty}) \subseteq \mathcal{A}$, $\mathbb{N} \setminus D$ is infinite, and $FU(\langle X_n \rangle_{n \in D}) \in \Theta$.

Definition 3.13.

(a) $\mathcal{I} = \{\langle X_n \rangle_{n=1}^{\infty} :$ for each $n \in \mathbb{N}$, $X_n \in \mathcal{F}$ and $\max X_n < \min X_{n+1}\}$.

(b) For $m,k\in\mathbb{N}$, $\mathcal{B}_{m,k}=FU(\langle\{2^kn\}\rangle_{n=m+1}^{\infty})$.

Note that if $(m_1,k_1),(m_2,k_2)\in\mathbb{N}\times\mathbb{N}$, $m_1\leq m_2$, and $k_1\leq k_2$, then $\mathcal{B}_{m_2,k_2}\subseteq\mathcal{B}_{m_1,k_1}$.

Definition 3.14. (Π,f) is a *sparse ordered union pair* if and only if the following hold:

(1) Π is a nonempty set of infinite subsets of $\mathcal{F}$.
(2) $f:\mathcal{P}_f(\Pi)\to\mathcal{I}$.
(3) For all $\Delta\in\mathcal{P}_f(\Pi)$, if $f(\Delta)=\langle X_n\rangle_{n=1}^{\infty}$, then:
  (a) $FU(\langle X_n\rangle_{n=1}^{\infty})\subseteq\bigcap\Delta$.
  (b) For all $m\in\mathbb{N}$, $FU(\langle X_{2n}\rangle_{n=m}^{\infty})\in\Pi$.

Lemma 3.15. *Let* $\Pi=\{\mathcal{B}_{m,k}:(m,k)\in\mathbb{N}\times\mathbb{N}\}$. *For* $F\in\mathcal{P}_f(\mathbb{N}\times\mathbb{N})$, *let*

$$\mu(F)=\max\big\{m:(\exists k)\big((m,k)\in F\big)\big\},$$
$$\kappa(F)=\max\big\{k:(\exists m)\big((m,k)\in F\big)\big\}.$$

Define $f:\mathcal{P}_f(\Pi)\to\mathcal{I}$ *as follows. Given* $\Delta\in\mathcal{P}_f(\Pi)$, *let* F *be the subset of* $\mathbb{N}\times\mathbb{N}$ *such that* $\Delta=\{\mathcal{B}_{m,k}:(m,k)\in F\}$ *and let*

$$f(\Delta)=\langle\{2^{\kappa(F)}(2\mu(F)+n)\}\rangle_{n=1}^{\infty}.$$

Then (Π,f) *is a sparse ordered union pair.*

Proof. Conditions (1) and (2) of the definition are immediate. For (3), let $\Delta\in\mathcal{P}_f(\Pi)$ be given and let F be the subset of $\mathbb{N}\times\mathbb{N}$ such that $\Delta=\{\mathcal{B}_{m,k}:(m,k)\in F\}$. For $n\in\mathbb{N}$, let $X_n=\{2^{\kappa(F)}(2\mu(F)+n)\}$. Then $FU(\langle X_n\rangle_{n=1}^{\infty})=FU(\langle\{2^{\kappa(F)}n\}\rangle_{n=2\mu(F)+1}^{\infty})=\mathcal{B}_{2\mu(F),\kappa(F)}$. For $(m,k)\in F$, $\mathcal{B}_{2\mu(F),\kappa(F)}\subseteq\mathcal{B}_{m,k}$ so $FU(\langle X_n\rangle_{n=1}^{\infty})\subseteq\mathcal{B}_{m,k}$ as required for (3a).

Also

$$\begin{aligned}FU(\langle X_{2n}\rangle_{n=1}^{\infty})&=FU(\langle\{2^{\kappa(F)}(2\mu(F)+2n)\}\rangle_{n=1}^{\infty})\\&=FU(\langle\{2^{\kappa(F)+1}(\mu(F)+n)\}\rangle_{n=1}^{\infty})\\&=\mathcal{B}_{\mu(F),\kappa(F)+1}.\end{aligned}$$ □

We now introduce the partially ordered set with which we will apply Martin's Axiom.

Given $X\in\mathcal{F}$ and $\mathcal{G}\subseteq\mathcal{F}$, by $-X+\mathcal{G}$ we mean $\{Y\in\mathcal{F}:X\cup Y\in\mathcal{G}\}$.

Definition 3.16. Let Π be a nonempty set of infinite subsets of $\mathcal{F}$. Define

$$\begin{aligned}Q(\Pi)=\{(\mathcal{G},\Delta):\ &\mathcal{G}\in\mathcal{P}_f(\mathcal{F}),\ \Delta\in\mathcal{P}_f(\Pi)\text{ and whenever }X\text{ and }Y\text{ are}\\&\text{distinct elements of }\mathcal{G},\text{ either }\max X<\min Y\\&\text{or }\max Y<\min X\}.\end{aligned}$$

We define a partial ordering on $Q(\Pi)$ as follows. for $(\mathcal{G},\Delta),(\mathcal{G}'\Delta')\in Q(\Pi)$, we set $(\mathcal{G}',\Delta')\leq(\mathcal{G},\Delta)$ if the following conditions hold:

(a) $\mathcal{G}\subseteq\mathcal{G}'$.

(b) $\Delta \subseteq \Delta'$.
(c) $(\forall Y \in \mathcal{G}' \setminus \mathcal{G})(\forall X \in \mathcal{G})(\max X < \min Y)$.
(d) $\mathcal{G}' \setminus \mathcal{G} \subseteq \bigcap \Delta$.
(e) There exists $g : \mathcal{G}' \setminus \mathcal{G} \to \Delta'$ such that:
   (i) $(\forall X \in \mathcal{G}' \setminus \mathcal{G})\big(g(X) \subseteq \bigcap \Delta \cap (-X + \bigcap \Delta)\big)$.
   (ii) $(\forall X, Y \in \mathcal{G}' \setminus \mathcal{G})\big(\max X < \min Y \Rightarrow Y \in g(X)$ and

$$g(Y) \subseteq g(X) \cap (-Y + g(X))\big).$$

Note that for applications of Martin's Axiom, partial orders need not be antisymmetric. However, the relation on $Q(\Pi)$ is trivially antisymmetric.

Lemma 3.17. *Let Π be a nonempty set of infinite subsets of $\mathcal{F}$. Then $Q(\Pi)$ is a nonempty partially ordered set.*

Proof. Pick $\mathcal{A} \in \Pi$ and pick $F \in \mathcal{F}$. Then $(\{F\}, \{\mathcal{A}\}) \in Q(\Pi)$ so $Q(\Pi) \neq \emptyset$. Trivially $\leq$ is reflexive. (For (e), $\emptyset : \emptyset \to \Delta = \Delta'$ is as required.)

To verify transitivity, let $(\mathcal{G}, \Delta), (\mathcal{G}', \Delta'), (\mathcal{G}'', \Delta'') \in Q(\Pi)$ with

$$(\mathcal{G}'', \Delta'') \leq (\mathcal{G}', \Delta') \leq (\mathcal{G}, \Delta).$$

Trivially $\mathcal{G} \subseteq \mathcal{G}''$ and $\Delta \subseteq \Delta''$. To verify (c), let $Y \in \mathcal{G}'' \setminus \mathcal{G}$ and let $X \in \mathcal{G}$. If $Y \in \mathcal{G}'$, then $\max X < \min Y$ since $Y \in \mathcal{G}' \setminus \mathcal{G}$. If $Y \notin \mathcal{G}'$, then $\max X < \min Y$ since $X \in \mathcal{G}'$.

To verify (d), let $X \in \mathcal{G}'' \setminus \mathcal{G}$. If $X \in \mathcal{G}'$, then $X \in \bigcap \Delta$ since $X \in \mathcal{G}' \setminus \mathcal{G}$. If $X \notin \mathcal{G}'$, then $X \in \mathcal{G}'' \setminus \mathcal{G}'$ so $X \in \bigcap \Delta' \subseteq \bigcap \Delta$.

To verify (e), let $g_1 : \mathcal{G}' \setminus \mathcal{G} \to \Delta'$ and $g_2 : \mathcal{G}'' \setminus \mathcal{G}' \to \Delta''$ be as guaranteed by the facts that $(\mathcal{G}', \Delta') \leq (\mathcal{G}, \Delta)$ and $(\mathcal{G}'', \Delta'') \leq (\mathcal{G}', \Delta')$. Let $g = g_1 \cup g_2$. Then $g : \mathcal{G}'' \setminus \mathcal{G} \to \Delta'$. To verifiy (ei), let $X \in \mathcal{G}'' \setminus \mathcal{G}$. If $X \in \mathcal{G}'$, then $g(X) = g_1(X) \subseteq \bigcap \Delta \cap (-X + \bigcap \Delta)$. If $X \notin \mathcal{G}'$, then

$$g(X) = g_2(X) \subseteq \bigcap \Delta' \cap \left(-X + \bigcap \Delta'\right) \subseteq \bigcap \Delta \cap \left(-X + \bigcap \Delta\right).$$

To verify (eii), let $X, Y \in \mathcal{G}'' \setminus \mathcal{G}$ with $\max X < \min Y$. If $\{X, Y\} \subseteq \mathcal{G}'' \setminus \mathcal{G}'$ or $\{X, Y\} \subseteq \mathcal{G}' \setminus \mathcal{G}$, the conclusion is immediate. By (c) the only other possibility is that $X \in \mathcal{G}' \setminus \mathcal{G}$ and $Y \in \mathcal{G}'' \setminus \mathcal{G}'$. Then $g(X) = g_1(X) \in \Delta'$ so $\bigcap \Delta' \subseteq g(X)$ and thus

$$g(Y) \subseteq \bigcap \Delta' \cap \left(-Y + \bigcap \Delta'\right) \subseteq g(X) \cap \big(-Y + g(X)\big). \qquad \square$$

Definition 3.18. Let Π be a nonempty set of infinite subsets of $\mathcal{F}$, let $\mathcal{V} \in \Pi$, and let $n \in \mathbb{N}$.

(1) $D(\mathcal{V}) = \{(\mathcal{G}, \Delta) \in Q(\Pi) : \mathcal{V} \in \Delta\}$.
(2) $E(n) = \{(\mathcal{G}, \Delta) \in Q(\Pi) : (\exists F \in \mathcal{G})(n < \min F)\}$.

Recall that in applications of Martin's Axiom, "dense" means "cofinal downward".

Lemma 3.19. *Let Π be a nonempty set of infinite subsets of $\mathcal{F}$ and let $\mathcal{V} \in \Pi$. Then $D(\mathcal{V})$ is dense in $Q(\Pi)$.*

Proof. If $(\mathcal{G},\Delta)\in Q(\Pi)$, then

$$(\mathcal{G},\Delta\cup\{\mathcal{V}\})\in Q(\Pi)\quad\text{and}\quad(\mathcal{G},\Delta\cup\{\mathcal{V}\})\le(\mathcal{G},\Delta).\qquad\square$$

Lemma 3.20. *Let Π be a nonempty set of infinite subsets of $\mathcal{F}$ and let $n\in\mathbb{N}$. If there is some f such that (Π,f) is a sparse ordered union pair, then $E(n)$ is dense in $Q(\Pi)$.*

Proof. Pick f such that (Π,f) is a sparse ordered union pair. Let $(\mathcal{G},\Delta)\in Q(\Pi)$ and let $f(\Delta)=\langle X_t\rangle_{t=1}^{\infty}$. Pick $t\in\mathbb{N}$ such that $\min X_{2t}>n$ and $\min X_{2t}>\max\bigcup\mathcal{G}$. Let $\mathcal{B}=FU(\langle X_{2m}\rangle_{m=t+1}^{\infty})$. Then $\mathcal{B}\in\Pi$ and

$$(\mathcal{G}\cup\{X_{2t}\},\Delta\cup\{\mathcal{B}\})\in Q(\Pi)\cap E(n)\,.$$

We claim that $(\mathcal{G}\cup\{X_{2t}\},\Delta\cup\{\mathcal{B}\})\le(\mathcal{G},\Delta)$. Requirements (a), (b), and (c) are immediate. Since $X_{2t}\subseteq FU(\langle X_j\rangle_{j=1}^{\infty})\subseteq\bigcap\Delta$, we have that (d) holds. To verify (e), define $g(X_{2t})=\mathcal{B}$. Then $\mathcal{B}\subseteq FU(\langle X_j\rangle_{j=1}^{\infty})\subseteq\bigcap\Delta$. To see that $\mathcal{B}\subseteq(-X_{2t}\cap\bigcap\Delta)$ let $Y\in\mathcal{B}$. Then $X_{2t}\cup Y\subseteq FU(\langle X_j\rangle_{j=1}^{\infty})\subseteq\bigcap\Delta$ so (ei) holds. And (eii) is vacuous. $\square$

Lemma 3.21. *Let Π be a nonempty set of infinite subsets of $\mathcal{F}$.*

(1) *If $(\mathcal{G},\Delta)$ and $(\mathcal{G}',\Delta')$ are incompatible, then $\mathcal{G}\neq\mathcal{G}'$. Consequently, $Q(\Pi)$ is a c.c.c. partial order.*
(2) *If $(\mathcal{G}',\Delta')\le(\mathcal{G},\Delta)$, then $FU(\mathcal{G}'\setminus\mathcal{G})\subseteq\bigcap\Delta$.*

Proof. (1) If $\mathcal{G}=\mathcal{G}'$, then $(\mathcal{G},\Delta\cup\Delta')\le(\mathcal{G},\Delta)$ and $(\mathcal{G},\Delta\cup\Delta')\le(\mathcal{G}',\Delta')$.

(2) If $\mathcal{G}'\setminus\mathcal{G}=\{X\}$, then $FU(\mathcal{G}'\setminus\mathcal{G})=\{X\}\subseteq\bigcap\Delta$ by requirement (d) of Definition 3.16. Now assume that $n>1$ and $\mathcal{G}'\setminus\mathcal{G}=\{X_1,X_2,\ldots,X_n\}$ where, for each $t\in\{1,2,\ldots,n-1\}$, $\max X_t<\min X_{t+1}$. Pick $g:\mathcal{G}'\setminus\mathcal{G}\to\Delta'$ as guaranteed by (e) of Definition 3.16. We show by induction on $|T|$ that if $\emptyset\neq T\subseteq\{2,3,\ldots,n\}$ and $\min T=t$, then $\bigcup_{i\in T}X_i\in g(X_{t-1})$. Assume first that $|T|=1$. Then $X_{t-1},X_t\in\mathcal{G}'\setminus\mathcal{G}$ so by (eii), $X_t\in g(X_{t-1})$. Now assume that $|T|>1$, let $U=T\setminus\{t\}$ and let $u=\min U$. Then $\bigcup_{i\in U}X_i\in g(X_{u-1})$. If $u-1=t$, this says that $\bigcup_{i\in U}X_i\in g(X_t)$. If $u-1>t$, then $\max X_t<\min X_{u-1}$ so by (eii), $g(X_{u-1})\subseteq X_t$. Thus in either case $\bigcup_{i\in U}X_i\in g(X_t)$. Thus by (eii), $\bigcup_{i\in U}X_i\in -X_t+g(X_{t-1})$ so $\bigcup_{i\in T}X_i\in g(X_{t-1})$ as required.

Now let $L\subseteq\{1,2,\ldots,n\}$ with $\min L=l$. Assume first that $l>1$. Then $\bigcup_{i\in L}X_i\in g(X_{l-1})\subseteq g(X_1)\subseteq\bigcap\Delta$. Now assume that $l=1$. If $L=\{1\}$ we have by (d) that $X_l\in\bigcap\Delta$, so assume that $|L|>1$. Let $T=L\setminus\{1\}$ and let $t=\min T$. Then $\bigcup_{i\in T}X_i\in g(X_{t-1})\subseteq g(X_1)\subseteq -X_1+\bigcap\Delta$ by (ei) so $\bigcup_{i\in L}X_i\subseteq\bigcap\Delta$. $\square$

Lemma 3.22. *Let $\omega\le\kappa<\mathfrak{c}$ and assume $MA(\kappa)$. Let (Π,f) be a sparse ordered union pair with $|\Pi|=\kappa$ and let $\mathcal{C}\subseteq\mathcal{F}$. There is a sparse ordered union pair (Ψ,g) such that:*

(1) $\Pi\subseteq\Psi$.
(2) $f\subseteq g$.

(3) $\mathcal{C} \in \Psi$ *or* $\mathcal{F} \setminus \mathcal{C} \in \Psi$.
(4) $|\Psi| = \kappa$.

Proof. By Lemmas 3.17 and 3.21(1), $Q(\Pi)$ is a c.c.c. partial order. By Lemmas 3.19 and 3.20, $\{D(\mathcal{V}) : \mathcal{V} \in \Pi\} \cup \{E(n) : n \in \mathbb{N}\}$ is a set of κ dense subsets of $Q(\Pi)$. Pick by $MA(\kappa)$ a filter G in $Q(\Pi)$ such that $G \cap D(\mathcal{V}) \neq \emptyset$ for each $\mathcal{V} \in \Pi$ and $G \cap E(n) \neq \emptyset$ for each $n \in \mathbb{N}$.

Since $G \cap E(n) \neq \emptyset$ for each $n \in \mathbb{N}$ we may choose a sequence $\langle F_t \rangle_{t=1}^{\infty}$ in $\mathcal{F}$ such that for each $t \in \mathbb{N}$, $\max F_t < \min F_{t+1}$ and there is some $(\mathcal{G}, \Delta) \in G$ such that $F_t \in \mathcal{G}$.

Pick by [8, Corollary 5.17] $\mathcal{D} \in \{\mathcal{C}, \mathcal{F} \setminus \mathcal{C}\}$ and a union subsystem $\langle X_t \rangle_{t=1}^{\infty}$ of $\langle F_t \rangle_{t=1}^{\infty}$ such that $FU(\langle X_t \rangle_{t=1}^{\infty}) \subseteq \mathcal{D}$. Let $\Psi = \Pi \cup \{\mathcal{D}\} \cup \{FU(\langle X_{2^k t} \rangle_{t=m}^{\infty}) : k, m \in \mathbb{N}\}$. Then conclusions (1), (3), and (4) hold. We claim that it suffices to show that

$$(*) \qquad \big(\forall \Delta \in \mathcal{P}_f(\Psi) \setminus \mathcal{P}_f(\Pi)\big)(\exists k, m \in \mathbb{N}) \left(FU(\langle X_{2^k t} \rangle_{t=m}^{\infty}) \subseteq \bigcap \Delta\right).$$

Assume we have done this. For $\Delta \in \mathcal{P}_f(\Psi)$, if $\Delta \subseteq \Pi$, let $g(\Delta) = f(\Delta)$. Otherwise, pick k and m as guaranteed by $(*)$ and let $g(\Delta) = \langle X_{2^k t} \rangle_{t=m}^{\infty}$. Then conclusion (2) holds. We need to show that (Ψ, g) is a sparse ordered union pair. Requirements (1) and (2) of Definition 3.14 hold. To verify (3), let $\Delta \in \mathcal{P}_f(\Psi)$. If $\Delta \subseteq \Pi$, then $g(\Delta) = f(\Delta)$ so (3a) and (3b) hold. So assume that $\Delta \setminus \Pi \neq \emptyset$ and pick k and m as guaranteed by $(*)$. For $t \in \mathbb{N}$, let $Y_t = X_{2^k(2m+t)}$. Then

$$FU(\langle Y_t \rangle_{t=1}^{\infty}) = FU(\langle X_{2^k(2m+t)} \rangle_{t=1}^{\infty}) \subseteq FU(\langle X_{2^k t} \rangle_{t=m}^{\infty}) \subseteq \bigcap \Delta$$

and, for $l \in \mathbb{N}$,

$$\begin{aligned} FU(\langle Y_{2t} \rangle_{t=l}^{\infty}) = FU(\langle X_{2^k(2m+2t)} \rangle_{t=l}^{\infty}) &= FU(\langle X_{2^{k+1}(m+t)} \rangle_{t=l}^{\infty}) \\ &= FU(\langle X_{2^{k+1} n} \rangle_{n=m+l}^{\infty}) \in \Psi. \end{aligned}$$

So we set out to establish $(*)$. Let $\Delta \in \mathcal{P}_f(\Psi) \setminus \mathcal{P}_f(\Pi)$. We may assume that $\Delta \cap \Pi \neq \emptyset$. We have that $\Delta \setminus \Pi \subseteq \{\mathcal{D}\} \cup \{FU(\langle X_{2^k t} \rangle_{t=m}^{\infty}) : k, m \in \mathbb{N}\}$ so pick $k, u \in \mathbb{N}$ such that $FU(\langle X_{2^k t} \rangle_{t=u}^{\infty}) \subseteq \bigcap(\Delta \setminus \Pi)$. For each $\mathcal{V} \in \Delta \cap \Pi$, pick $(\mathcal{G}_{\mathcal{V}}, \Delta_{\mathcal{V}}) \in G \cap D(\mathcal{V})$. Pick $(\mathcal{G}', \Delta') \in G$ such that $(\mathcal{G}', \Delta') \leq (\mathcal{G}_{\mathcal{V}}, \Delta_{\mathcal{V}})$ for each $\mathcal{V} \in \Delta \cap \Pi$.

Let $s = \max(\bigcup \mathcal{G}) + 1$. We claim that $FU(\langle F_t \rangle_{t=s}^{\infty}) \subseteq \bigcap(\Delta \cap \Pi)$. This will complete the proof for then we let $m = \max\{s, u\}$. Since $\langle X_t \rangle_{t=1}^{\infty}$ is a union subsystem of $\langle F_t \rangle_{t=1}^{\infty}$ we have $FU(\langle X_{2^k t} \rangle_{t=m}^{\infty}) \subseteq FU(\langle F_t \rangle_{t=s}^{\infty}) \subseteq \bigcap(\Delta \cap \Pi)$ and $FU(\langle X_{2^k t} \rangle_{t=m}^{\infty}) \subseteq FU(\langle X_{2^k t} \rangle_{t=u}^{\infty}) \subseteq \bigcap(\Delta \setminus \Pi)$.

So let $H \in \mathcal{P}_f(\mathbb{N})$ with $\min H \geq s$ be given. For $t \in H$, pick $(\mathcal{G}_t, \Delta_t) \in G$ such that $F_t \in \mathcal{G}_t$. Pick $(\mathcal{G}'', \Delta'') \in G$ such that $(\mathcal{G}'', \Delta'') \leq (\mathcal{G}', \Delta')$ and $(\mathcal{G}'', \Delta'') \leq (\mathcal{G}_t, \Delta_t)$ for each $t \in H$. Then for each $t \in H$, $F_t \in \mathcal{G}''$ and, since $\min F_t \geq t > \max \bigcup \mathcal{G}'$, we have $F_t \notin \mathcal{G}'$. By Lemma 3.21(2), we have $\bigcup_{t \in H} F_t \in FU(\mathcal{G}'' \setminus \mathcal{G}') \subseteq \bigcap \Delta'$ and $\bigcap \Delta' \subseteq \bigcap(\Delta \cap \Pi)$ since $(\mathcal{G}', \Delta') \leq (\mathcal{G}_{\mathcal{V}}, \Delta_{\mathcal{V}})$ for each $\mathcal{V} \in \Delta \cap \Pi$. □

Theorem 3.23. *Let* (Π, f) *be a sparse ordered union pair with* $\omega \leq |\Pi| < \mathfrak{c}$ *and assume Martin's Axiom. There is a sparse ordered union ultrafilter* Θ *with* $\Pi \subseteq \Theta$.

Proof. Well order $\mathcal{P}(\mathcal{F})$ as $\langle \mathcal{C}_\sigma \rangle_{\sigma<\mathfrak{c}}$ with $\mathcal{C}_0 \in \Pi$. Let $\sigma < \mathfrak{c}$ and assume that we have chosen $\langle \Psi_\delta \rangle_{\delta<\sigma}$ and $\langle g_\delta \rangle_{\delta<\sigma}$ such that for each $\delta < \sigma$:

(1) (Ψ_δ, g_δ) is a sparse ordered union pair.
(2) If $\tau < \delta$, then $\Psi_\delta \subseteq \Psi_\tau$ and $g_\delta \subseteq g_\tau$.
(3) $\mathcal{C}_\delta \in \Psi_\delta$ or $\mathcal{F} \setminus \mathcal{C}_\delta \in \Psi_\delta$.
(4) $|\Psi_\delta| \leq \max\{|\Pi|, |\delta|\}$.

These hypotheses hold at $\delta = 0$, (2) vacuously. Let $\Psi'_\sigma = \bigcup_{\delta<\sigma} \Psi_\delta$ and $g'_\sigma = \bigcup_{\delta<\sigma} g_\delta$. It is routine to verify that $(\Psi'_\sigma, g'_\sigma)$ is a sparse ordered union pair. Also $|\Psi'_\sigma| \leq \max\{|\Pi|, |\sigma|\}$. (If $\sigma \leq |\Pi|$, then $|\Psi'_\sigma| \leq \sum_{\delta<\sigma} |\Pi| = |\Pi|$. If $\sigma > |\Pi|$, then $|\Psi'_\sigma| \leq \sum_{\delta<\sigma} |\sigma| = |\sigma|$.)

Pick by Lemma 3.22 a sparse ordered union pair (Ψ_σ, g_σ) such that $\Psi'_\sigma \subseteq \Psi_\sigma$, $g'_\sigma \subseteq g_\sigma$, $|\Psi_\sigma| = |\Psi'_\sigma|$, and either $\mathcal{C}_\sigma \in \Psi_\sigma$ or $\mathcal{F} \setminus \mathcal{C}_\sigma \in \Psi_\sigma$. Hypotheses (1) through (4) all hold.

The construction being complete, let $\Theta = \bigcup_{\sigma<\mathfrak{c}} \Psi_\sigma$. If (Ξ, h) is a sparse ordered union pair, then by Definition 3.14(3a), Ξ has the finite intersection property. Therefore by induction hypotheses (1) and (3), we have that Θ is an ultrafilter on $\mathcal{F}$. To see that Θ is a sparse ordered union ultrafilter, let $\mathcal{A} \in \Theta$. Pick $\sigma < \mathfrak{c}$ such that $\mathcal{A} \in \Psi_\sigma$, let $\Delta = \{\mathcal{A}\}$, and let $\langle X_n \rangle_{n=1}^\infty = g_\sigma(\Delta)$. Then $FU(\langle X_n \rangle_{n=1}^\infty) \subseteq \mathcal{A}$ and $FU(\langle X_{2n} \rangle_{n=1}^\infty \in \Psi_\sigma \subseteq \Theta$. □

Corollary 3.24. *Assume Martin's Axiom. There exists a sparse ordered union ultrafilter on* $\mathcal{F}$.

Proof. Lemma 3.15 and Theorem 3.23. □

Recall that in this section we are taking S to be the free semigroup with identity on the generators $\langle a_t \rangle_{t=1}^\infty$.

Corollary 3.25. *Assume Martin's Axiom. There exists a sparse very strongly productive ultrafilter on* S.

Proof. By Corollary 3.24, pick a sparse ordered union ultrafilter Θ. Let $p = \{C \subseteq S : (\exists \mathcal{A} \in \Theta)(\{\prod_{n\in B} a_n : B \in \mathcal{A}\} \subseteq C)\}$. By [11, Theorem 3.3], p is a very strongly productive ultrafilter. To see that p is sparse, let $C \in p$. By Definition 3.9, we need to show that there are a sequence $\langle x_t \rangle_{t=1}^\infty$ in S and an infinite set $D \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus D$ is infinite, $FP(\langle x_t \rangle_{t=1}^\infty) \subseteq C$, and $FP(\langle x_t \rangle_{t\in D}) \in p$.

Pick $\mathcal{A} \in \Theta$ such that $\{\prod_{n\in B} a_n : B \in \mathcal{A}\} \subseteq C$. By Definition 3.12 we may pick a sequence $\langle X_n \rangle_{n=1}^\infty$ of members of $\mathcal{F}$ such that $\max X_n < \min X_{n+1}$ for each n and an infinite subset D of $\mathbb{N}$ such that $\mathbb{N} \setminus D$ is infinite, $FU(\langle X_n \rangle_{n=1}^\infty) \subseteq \mathcal{A}$, and $FU(\langle X_n \rangle_{n\in D}) \in \Theta$. For each $n \in \mathbb{N}$, let $x_n = \prod_{t\in X_n} a_t$. Since $\max X_n < \min X_{n+1}$ for each n, we have that if $H \in \mathcal{P}_f(\mathbb{N})$ and $K = \bigcup_{n\in H} X_n$, then $\prod_{n\in H} x_n = \prod_{t\in K} a_t$. Therefore $\langle x_n \rangle_{n=1}^\infty$ is as required. □

Recall from the introduction that there are many situations in which it is known that all strongly summable ultrafilters are sparse.

Question 3.26. *Let S be the free semigroup on countably many generators. Are all very strongly productive ultrafilters on S sparse?*

4. More idempotents which are products only trivially

Let S be the free semigroup on the generators $\langle a_t\rangle_{t=1}^{\infty}$ and let $\mathcal{F} = \mathcal{P}_f(\mathbb{N})$. Denote by $\uplus$ the operation on $\beta\mathcal{F}$ extending the operation $\cup$ on $\mathcal{F}$ making $(\beta\mathcal{F}, \uplus)$ a right topological semigroup with $\mathcal{F}$ contained in its topological center. (Normally we use the same symbol to denote the extended operation. But in this case, if $\Theta, \Psi \in \beta\mathcal{F}$, then $\Theta \cup \Psi$ already means something.) We show in this section that Martin's Axiom implies that there is an idempotent p in βS which is not very strongly productive, in fact is not even strongly productive, and p can only be written trivially as a product. We also show that the existence of a union ultrafilter implies that there is an idempotent p in $(\beta\mathbb{N}, \cdot)$ which can only be written trivially as a product and that there is an idempotent Θ in $(\beta\mathcal{F}, \uplus)$ so that, if Ψ and Ξ are in $\beta\mathcal{F}$ and $\Psi \uplus \Xi = \Theta$, then $\Psi = \Xi = \Theta$.

We begin by showing in Theorem 4.2 that if p is a strongly productive ultrafilter on S and $FP(\langle a_t\rangle_{t=1}^{\infty}) \in p$, then in fact p is very strongly productive.

Lemma 4.1. *Let S be the free semigroup on the generators $\langle a_t\rangle_{t=1}^{\infty}$ and let $\langle x_t\rangle_{t=1}^{\infty}$ be a sequence in S. If $FP(\langle x_t\rangle_{t=1}^{\infty}) \subseteq FP(\langle a_t\rangle_{t=1}^{\infty})$, then $\langle x_t\rangle_{t=1}^{\infty}$ is a product subsystem of $\langle a_t\rangle_{t=1}^{\infty}$.*

Proof. For each $n \in \mathbb{N}$ pick $H_n \in \mathcal{P}_f(\mathbb{N})$ such that $x_n = \prod_{t\in H_n} a_t$. We claim that for each n, $\max H_n < \min H_{n+1}$. Otherwise

$$x_n \cdot x_{n+1} = \prod_{t\in H_n} a_t \cdot \prod_{t\in H_{n+1}} a_t \notin FP(\langle a_t\rangle_{t=1}^{\infty}). \qquad \square$$

Theorem 4.2. *Let S be the free semigroup on the generators $\langle a_t\rangle_{t=1}^{\infty}$ and let p be a strongly productive ultrafilter on S such that $FP(\langle a_t\rangle_{t=1}^{\infty}) \in p$. Then p is a very strongly productive ultrafilter.*

Proof. Let $A \in p$. Pick a sequence $\langle x_t\rangle_{t=1}^{\infty}$ such that

$$FP(\langle x_t\rangle_{t=1}^{\infty}) \subseteq A \cap FP(\langle a_t\rangle_{t=1}^{\infty}) \text{ and } FP(\langle x_t\rangle_{t=1}^{\infty}) \in p\,.$$

By Lemma 4.1, $\langle x_t\rangle_{t=1}^{\infty}$ is a product subsystem of $\langle a_t\rangle_{t=1}^{\infty}$. $\square$

When we say that a sequence $\langle x_t\rangle_{t=1}^{\infty}$ satisfies uniqueness of finite products, we mean that whenever $F, H \in \mathcal{P}_f(\mathbb{N})$ and $\prod_{t\in F} x_t = \prod_{t\in H} x_t$, one must have that $F = H$.

The subsemigroup $\mathbb{H}$ of $(\beta\mathbb{N}, +)$ is defined by $\mathbb{H} = \bigcap_{n=1}^{\infty} \overline{2^n\mathbb{N}}$. This semigroup contains all of the idempotents of $(\beta\mathbb{N}, +)$ and much of the remaining known algebraic structure of $(\beta\mathbb{N}, +)$. See [8, Section 6.1]. The proof of the following lemma is only a slight variation of the proof of [8, Theorem 6.27] so we omit it.

Lemma 4.3. *Let S be any semigroup, let $\langle x_t\rangle_{t=1}^\infty$ be a sequence in S satisfying uniqueness of finite products, and let $T = \bigcap_{n=1}^\infty \overline{FP(\langle x_t\rangle_{t=n}^\infty)}$. Define $\varphi : \mathbb{N} \to S$ by, for $H \in \mathcal{P}_f(\mathbb{N})$, $\varphi(\sum_{t\in H} 2^{t-1}) = \prod_{t\in H} x_t$ and let $\widetilde{\varphi} : \beta\mathbb{N} \to \beta S$ be the continuous extension of φ. The restriction of $\widetilde{\varphi}$ to $\mathbb{H}$ is an isomorphism and a homeomorphism onto T.*

Lemma 4.4. *Define $\psi : \mathcal{F} \to \mathbb{N}$ by, for $F \in \mathcal{F}$, $\psi(F) = \sum_{t\in F} 2^{t-1}$ and let $\widetilde{\psi} : \beta\mathcal{F} \to \beta\mathbb{N}$ be its continuous extension. If Θ is a union ultrafilter on $\mathcal{F}$, then $\widetilde{\psi}(\Theta)$ is a strongly summable ultrafilter on $\mathbb{N}$.*

Proof. This is the easy half of [2, Theorem 1]. $\square$

Lemma 4.5. *Let S be the free semigroup on the generators $\langle a_t\rangle_{t=1}^\infty$, define $\varphi : \mathbb{N} \to S$ and $\psi : \mathcal{F} \to \mathbb{N}$ by, for $F \in \mathcal{F}$, $\varphi(\sum_{t\in F} 2^{t-1}) = \prod_{t\in F} a_t$ and $\psi(F) = \sum_{t\in F} 2^{t-1}$. Let $\widetilde{\varphi} : \beta\mathbb{N} \to \beta S$ and $\widetilde{\psi} : \beta\mathcal{F} \to \beta\mathbb{N}$ be the continuous extensions of φ and ψ. Let $\Theta \in \beta\mathcal{F}$ and let $p = \widetilde{\varphi}\big(\widetilde{\psi}(\Theta)\big)$. If p is a very strongly productive ultrafilter, then Θ is an ordered union ultrafilter.*

Proof. Define $\tau : \mathcal{F} \to S$ by, for $F \in \mathcal{F}$, $\tau(F) = \prod_{t\in F} a_t$. Let $\widetilde{\tau} : \beta\mathcal{F} \to \beta S$ be its continuous extension. Then $\tau = \varphi \circ \psi$ so $p = \widetilde{\tau}(\Theta)$.

To see that Θ is an ordered union ultrafilter, let $\mathcal{W} \in \Theta$. Then $\tau[\mathcal{W}] \in p$. Pick a product subsystem $\langle x_t\rangle_{t=1}^\infty$ of $\langle a_t\rangle_{t=1}^\infty$ such that $FP(\langle x_t\rangle_{t=1}^\infty) \subseteq \tau[\mathcal{W}]$ and $FP(\langle x_t\rangle_{t=1}^\infty) \in p$. Pick a sequence $\langle H_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that for each n, $x_n = \prod_{t\in H_n} a_t$ and $\max H_n < \min H_{n+1}$. For each n, pick $F_n \in \mathcal{W}$ such that $x_n = \tau(F_n)$. Then $\prod_{t\in F_n} a_t = x_n = \prod_{t\in H_n} a_t$, so $F_n = H_n$. We have $\tau^{-1}[FP(\langle x_t\rangle_{t=1}^\infty)] \in \Theta$, $\tau^{-1}[FP(\langle x_t\rangle_{t=1}^\infty)] \subseteq \tau^{-1}\big[\tau[\mathcal{W}]\big] = \mathcal{W}$, and $\tau^{-1}[FP(\langle x_t\rangle_{t=1}^\infty)] = FU(\langle F_n\rangle_{n=1}^\infty)$. $\square$

Lemma 4.6. *Let S be the free semigroup on the generators $\langle a_t\rangle_{t=1}^\infty$ and let $T = \bigcap_{n=1}^\infty \overline{FP(\langle a_t\rangle_{t=n}^\infty)}$. If $r, s \in \beta S$ and $rs \in T$, then $r \in T$ and $s \in T$.*

Proof. Let $n \in \mathbb{N}$. We will show that $FP(\langle a_t\rangle_{t=n}^\infty) \in r$ and $FP(\langle a_t\rangle_{t=n}^\infty) \in s$. Since $FP(\langle a_t\rangle_{t=n}^\infty) \in rs$, we have $B = \{x \in S : x^{-1}FP(\langle a_t\rangle_{t=n}^\infty) \in s\} \in r$. We claim $B \subseteq FP(\langle a_t\rangle_{t=n}^\infty)$, so let $x \in B$ and pick $y \in x^{-1}FP(\langle a_t\rangle_{t=n}^\infty)$. Then $xy = \prod_{t\in H} a_t$ for some $H \in \mathcal{P}_f(\mathbb{N})$ such that $\min H \geq n$. Therefore $x \in FP(\langle a_t\rangle_{t=n}^\infty)$ and $y \in FP(\langle a_t\rangle_{t=n}^\infty)$. Thus $B \subseteq FP(\langle a_t\rangle_{t=n}^\infty)$ and, since y was an arbitrary member of $x^{-1}FP(\langle a_t\rangle_{t=n}^\infty)$, $x^{-1}FP(\langle a_t\rangle_{t=n}^\infty) \subseteq FP(\langle a_t\rangle_{t=n}^\infty)$. $\square$

All previously known examples of elements of βS which could not be written nontrivially as a product were very strongly productive.

Theorem 4.7. *Let S be the free semigroup on the generators $\langle a_t\rangle_{t=1}^\infty$ and assume Martin's Axiom. There exists an idempotent $p \in \beta S$ such that:*

(1) *If $r, s \in \beta S$ and $rs = p$, then $r = s = p$.*

(2) *p is not strongly productive.*

Proof. By [2, Theorem 5] pick a union ultrafilter Θ on $\mathcal{F}$ such that Θ is not an ordered union ultrafilter. Let ψ, φ, $\widetilde{\psi}$, and $\widetilde{\varphi}$ be as in Lemma 4.5. Let $T = \bigcap_{n=1}^{\infty} \overline{FP(\langle a_t \rangle_{t=n}^{\infty})}$ and let $q = \widetilde{\psi}(\Theta)$. By Lemma 4.4, q is strongly summable.

Now $(\mathbb{N}, +)$ can be embedded in the circle group so by [7, Corollary 4.3], if $x, y \in \mathbb{N}^*$ and $x + y = q$, then $x, y \in \mathbb{Z} + q$. Consequently, if $x, y \in \mathbb{H}$ and $x + y = q$, then $x, y \in (\mathbb{Z} + q) \cap \mathbb{H} = \{q\}$.

Let $p = \widetilde{\varphi}(q)$. By Lemma 4.3, p is an idempotent and $p \in T$. Assume that $r, s \in \beta S$ and $rs = p$. By Lemma 4.6, $r \in T$ and $s \in T$ so by Lemma 4.3 pick $x, y \in \mathbb{H}$ such that $r = \widetilde{\varphi}(x)$ and $s = \widetilde{\varphi}(y)$. Then $\widetilde{\varphi}(x + y) = rs = p$ so $x + y = q$. Therefore $x = y = q$ and thus $r = s = p$.

Finally suppose that p is strongly productive. By Theorem 4.2 p is very strongly productive so by Lemma 4.5, Θ is an ordered union ultrafilter, a contradiction. □

The following corollary is an immediate consequence of the proof of Theorem 4.7.

Corollary 4.8. *Let φ, and $\widetilde{\varphi}$ be as in Lemma* 4.5 *and assume Martin's Axiom. There is a strongly summable ultrafilter q on $\mathbb{N}$ such that $\widetilde{\varphi}(q)$ is not strongly productive.*

We conclude the paper with some results which are consequences of the existence of union ultrafilters. This is certainly a weaker assumption than Martin's Axiom since it is known that the existence of union ultrafilters follows from the axiom known as $P(\mathfrak{c})$. (See the discussion in [2, Page 97].) It is not known whether this is a weaker assumption than the existence of ordered union ultrafilters.

Theorem 4.9. *Let S be any semigroup, let $\langle x_t \rangle_{t=1}^{\infty}$ be a sequence in S satisfying uniqueness of finite products, and let $T = \bigcap_{n=1}^{\infty} \overline{FP(\langle x_t \rangle_{t=n}^{\infty})}$. Define φ and $\widetilde{\varphi}$ as in Lemma* 4.3. *If whenever $r, s \in \beta S$ and $rs \in T$, one must have $r \in T$ and $s \in T$, then for any strongly summable ultrafilter q on $\mathbb{N}$, if $r, s \in \beta S$ and $\widetilde{\varphi}(q) = rs$, then $r = s = \widetilde{\varphi}(q)$.*

Proof. Pick $r, s \in \beta S$ such that $\widetilde{\varphi}(q) = rs$. Then $r, s \in T$ so by Lemma 4.3, pick $x, y \in \mathbb{H}$ such that $\widetilde{\varphi}(x) = r$ and $\widetilde{\varphi}(y) = s$. Then $x + y = q$ so by [7, Corollary 4.3], $x, y \in \mathbb{H} \cap (\mathbb{Z} + q) = \{q\}$. Thus $x = y = q$ so $r = s = \widetilde{\varphi}(q)$. □

Corollary 4.10. *Assume there exists a union ultrafilter on $\mathcal{F}$. There is an idempotent p in $(\beta\mathbb{N}, \cdot)$ such that if $r, s \in \beta\mathbb{N}$ and $rs = p$, then $r = s = p$.*

Proof. By Lemma 4.4, pick a strongly summable ultrafilter q on $\mathbb{N}$. Let $\langle x_t \rangle_{t=1}^{\infty}$ be a sequence of distinct primes and define $T = \bigcap_{n=1}^{\infty} \overline{FP(\langle x_t \rangle_{t=n}^{\infty})}$. By Theorem 4.9 it suffices to show that if $r, s \in \beta\mathbb{N}$ and $rs \in T$, then $r \in T$ and $s \in T$. This follows easily from the fact that if $n, y, z \in \mathbb{N}$ and

$yz \in FP(\langle x_t\rangle_{t=n}^{\infty})$, then all prime factors of y and of z are in $\{x_t : t \geq n\}$ and neither y nor z has a repeated prime factor. □

Corollary 4.11. *Assume there exists a union ultrafilter on $\mathcal{F}$. There is an idempotent Θ in $\beta\mathcal{F}$ such that if $\Psi, \Xi \in \beta\mathcal{F}$ and $\Psi \uplus \Xi = \Theta$, then $\Psi = \Xi = \Theta$.*

Proof. By Lemma 4.4, pick a strongly summable ultrafilter q on $\mathbb{N}$. For each $n \in \mathbb{N}$, let $X_n = \{n\}$. Then, given $n \in \mathbb{N}$,

$$FU(\langle X_t\rangle_{t=n}^{\infty}) = \{H \in \mathcal{F} : \min H \geq n\}.$$

Note that $\langle X_n\rangle_{n=1}^{\infty}$ satisfies uniqueness of finite unions. Define $\varphi : \mathbb{N} \to \mathcal{F}$ by, for $H \in \mathcal{F}$, $\varphi(\sum_{t\in H} 2^{t-1}) = H$ and let $T = \bigcap_{n=1}^{\infty} \overline{FU(\langle X_t\rangle_{t=n}^{\infty})}$.

By Theorem 4.9 it suffices to show that if $\Psi, \Xi \in \beta\mathcal{F}$ and $\Psi \uplus \Xi \in T$, then $\Psi \in T$ and $\Xi \in T$. To this end, let $n \in \mathbb{N} \setminus \{1\}$. We need to show that $\{H \in \mathcal{F} : \min H \geq n\} \in \Psi$ and $\{H \in \mathcal{F} : \min H \geq n\} \in \Xi$. Now $\{H \in \mathcal{F} : \min H < n\}$ is an ideal of $(\mathcal{F}, \cup)$ so by [8, Corollary 4.18], $\overline{\{H \in \mathcal{F} : \min H < n\}}$ is an ideal of $(\beta\mathcal{F}, \uplus)$ so if either

$$\{H \in \mathcal{F} : \min H \geq n\} \notin \Psi \text{ or } \{H \in \mathcal{F} : \min H \geq n\} \notin \Xi,$$

we would have $\{H \in \mathcal{F} : \min H \geq n\} \notin \Psi \uplus \Xi$. □

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Department of Mathematics, Howard University, Washington, DC 20059, USA
nhindman@aol.com
http://nhindman.us/

Department of Mathematics and Statistics, University of Arkansas at Little Rock, Little Rock, AR 72204, USA
lljones3@ualr.edu

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