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The Soliton-Kähler–Ricci flow over Fano manifolds

Nefton Pali

Abstract. We introduce a flow of Riemannian metrics over compact manifolds with formal limit at infinite time a shrinking Ricci soliton. We call this flow the Soliton-Ricci flow. It correspond to Perelman's modified backward Ricci type flow with some special restriction conditions. The restriction conditions are motivated by convexity results for Perelman's $\mathcal{W}$-functional over convex subsets inside adequate subspaces of Riemannian metrics. We show indeed that the Soliton-Ricci flow represents the gradient flow of the restriction of Perelman's $\mathcal{W}$-functional over such subspaces.

Over Fano manifolds we introduce a flow of Kähler structures with formal limit at infinite time a Kähler–Ricci soliton. This flow corresponds to Perelman's modified backward Kähler–Ricci type flow that we call Soliton-Kähler–Ricci flow. It can be generated by the Soliton-Ricci flow. We assume that the Soliton-Ricci flow exists for all times and the Bakry–Emery–Ricci tensor preserves a positive uniform lower bound with respect to the evolving metric. In this case we show that the corresponding Soliton-Kähler–Ricci flow converges exponentially fast to a Kähler–Ricci soliton.

Contents

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1. Introduction

The notion of Ricci soliton (in short RS) has been introduced by D.H. Friedon in [Fri]. It is a natural generalization of the notion of Einstein metric. The terminology is justified by the fact that the pull back of the RS metric via the flow of automorphisms generated by its vector field provides a Ricci flow.

In this paper we introduce the Soliton-Ricci flow, (in short SRF) which is a flow of Riemannian metrics with formal limit at infinite time a shrinking Ricci soliton.

A remarkable formula due to Perelman [Per] shows that the modified (and normalized) Ricci flow is the gradient flow of Perelman's $\mathcal{W}$ functional with respect to a fixed choice of the volume form Ω. We will denote by $\mathcal{W}_\Omega$ the corresponding Perelman's functional. However Perelman's work does not show a priori any convexity statement concerning the functional $\mathcal{W}_\Omega$.

The main attempt of this work is to fit the SRF into a gradient system picture. We mean by this the picture corresponding to the gradient flow of a convex functional.

The SRF correspond to a Perelman's modified backward Ricci type flow with 3-symmetric covariant derivative of the Ω-Bakry–Emery–Ricci (in short Ω-BER) tensor along the flow. The notion of SRF (or more precisely of Ω-SRF) is inspired from the recent work [Pal] in which we show convexity of Perelman's $\mathcal{W}_\Omega$ functional along variations with 3-symmetric covariant derivative over points with nonnegative Ω-BER tensor.

The surprising fact is that the Ω-SRF is a *forward* and strictly parabolic heat type flow with respect to such variations. They insure that the gauge modification of the backward Ricci flow via Perelman's potentials produce sufficient parabolicity which compensate the bad sign of this last flow. The difference with the Ricci flow is that the parabolicity is generated by the Hessian part of the Ω-BER tensor thanks to the particular symmetry of the variation.

However we can not expect to solve the Ω-SRF equation for arbitrary initial data. It is well-known that backward heat type equations, such as the backward Ricci flow (roughly speaking), can not be solved for arbitrary initial data.

We will call *scattering data* some special initial data which imply the existence of the Ω-SRF as a formal gradient flow for the restriction of Perelman's $\mathcal{W}$-functional over adequate subspaces of Riemannian metrics.

To be more precise, we are looking for subvarieties Σ in the space of Riemannian metrics such that at each point $g \in \Sigma$ the tangent space of Σ at g is contained in the space of variations with 3-symmetric covariant derivative and such that the gradient of the functional $\mathcal{W}_\Omega$ is tangent to Σ at each point $g \in \Sigma$.

So at first place we want that the set of initial data allow a 3-symmetric covariant derivative of the variation of the metric along the Ω-SRF. The precise definition of the set of scattering data and of the subvarieties Σ will be given in the next section.

The notion of Kähler–Ricci soliton (in short KRS) is a natural generalization of the notion of Kähler–Einstein metric. A KRS over a Fano manifold X is a Kähler metric in the class $2\pi c_1(X)$ such that the gradient of the default potential of the metric to be Kähler–Einstein is holomorphic. The terminology is justified by the fact that the pull back of the KRS metric via the flow of automorphisms generated by this gradient field provides a Kähler–Ricci flow.

We recall that the Kähler–Ricci flow (in short KRF) has been introduced by H. Cao in [Cao]. In the Fano case it exists for all positive times. Its convergence in the classic sense implies the existence of a Kähler–Einstein metric. The fact that not all Fano manifolds admit Kähler–Einstein metrics implies the nonconvergence in the classic sense of the KRF in general.

The importance of Kähler–Ricci solitons over Fano manifolds derives from the fact that if there exists one then it is a Kähler–Einstein metric provided that the Futaki invariant vanishes. It is expected that Kähler–Ricci solitons should be obtained as limits of natural geometric flows such as the Kähler–Ricci flow. Despite the substantial efforts of many well-known mathematicians, the results concerning the convergence of the Kähler–Ricci flow are still weak. This is essentially due to the fact that it is very hard to obtain a uniform lower bound on the Ricci curvature along the flow. This type of bound is necessary in order to insur compactness results which lead to Cheeger–Gromov type convergence.

Our approach for the construction of Kähler–Ricci solitons is based on the study of a flow of Kähler structures $(X, J_t, g_t)_{t\geq 0}$ associated to any normalized smooth volume form $\Omega > 0$ that we will call Ω-Soliton-Kähler–Ricci flow (in short Ω-SKRF). Using a result in [Pal] we show that the Ω-SKRF can be generated by the Ω-SRF, via an ODE flow of complex structures of Lax type.

2. Statement of the main results

Let $\Omega > 0$ be a smooth volume form over an oriented Riemannian manifold (X, g) of dimension n. We recall that the Ω-Bakry–Emery–Ricci tensor of g is defined by the formula

$$\operatorname{Ric}_g(\Omega) := \operatorname{Ric}(g) + \nabla_g \, d \log \frac{dV_g}{\Omega} \ .$$

A Riemannian metric g is called a Ω-Shrinking Ricci soliton (in short Ω-ShRS) if $g = \operatorname{Ric}_g(\Omega)$. We observe that the set of variations with 3-symmetric covariant derivative coincides with the vector space

$$\mathbb{F}_g := \Big\{ v \in C^\infty \left(X, S^2_{\mathbb{R}} T^*_X \right) \mid \ \nabla_{T_X, g} v^*_g = 0 \Big\} ,$$

where $\nabla_{T_X, g}$ denotes the covariant exterior derivative acting on T_X-valued differential forms and $v^*_g := g^{-1} v$. We define also the set of prescattering data $\mathcal{S}_\Omega$ as the subset in the space of smooth Riemannian metrics $\mathcal{M}$ over X given by

$$\mathcal{S}_\Omega := \Big\{ g \in \mathcal{M} \mid \ \nabla_{T_X, g} \operatorname{Ric}^*_g(\Omega) = 0 \Big\}.$$

Definition 1. (*The Ω-Soliton-Ricci flow*). Let $\Omega > 0$ be a smooth volume form over an oriented Riemannian manifold X. A Ω-Soliton-Ricci flow (in short Ω-SRF) is a flow of Riemannian metrics $(g_t)_{t\geqslant 0} \subset \mathcal{S}_\Omega$ solution of the evolution equation $\dot{g}_t = \operatorname{Ric}_{g_t}(\Omega) - g_t$.

We equip the set $\mathcal{M}$ with the scalar product

$$G_g(u, v) = \int_X \langle \, u, v \rangle_g \, \Omega, \tag{2.1}$$

for all $g \in \mathcal{M}$ and all $u, v \in \mathcal{H} := L^2(X, S^2_{\mathbb{R}} T^*_X)$. We denote by d_G the induced distance function. Let P^*_g be the formal adjoint of an operator P

with respect to a metric g. We observe that the operator

$$P_g^{*_\Omega} := e^f P_g^* \left(e^{-f} \bullet \right),$$

with $f := \log \frac{dV_g}{\Omega}$, is the formal adjoint of P with respect to the scalar product (2.1). We define also the Ω-Laplacian operator

$$\Delta_g^\Omega := \nabla_g^{*_\Omega} \nabla_g = \Delta_g + \nabla_g f \neg \nabla_g \, .$$

We recall (see [Pal]) that the first variation of the Ω-Bakry–Emery–Ricci tensor is given by the formula

$$2 \frac{d}{dt} \operatorname{Ric}_{g_t}(\Omega) = -\nabla_{g_t}^{*_\Omega} \mathcal{D}_{g_t} \dot{g}_t, \tag{2.2}$$

where $\mathcal{D}_g := \hat{\nabla}_g - 2 \nabla_g$, with $\hat{\nabla}_g$ being the symmetrization of ∇_g acting on symmetric 2-tensors. Explicitly

$$\hat{\nabla}_g \, \alpha \, (\xi_0, ..., \xi_p) := \sum_{j=0}^{p} \nabla_g \, \alpha \, (\xi_j, \xi_0, ..., \hat{\xi}_j, ..., \xi_p),$$

for all p-tensors α. We observe that formula (2.2) implies directly the variation formula

$$2 \frac{d}{dt} \operatorname{Ric}_{g_t}(\Omega) = -\Delta_{g_t}^\Omega \, \dot{g}_t, \tag{2.3}$$

along any smooth family $(g_t)_{t \in (0, \varepsilon)} \subset \mathcal{M}$ such that $\dot{g}_t \in \mathbb{F}_{g_t}$ for all $t \in (0, \varepsilon)$. We deduce that the Ω-SRF is a *forward* and strictly parabolic heat type flow of Riemannian metrics. In the appendix we give a direct proof of the variation formula (2.3) which shows that the Laplacian term on the right hand side is produced from the variation of the Hessian of $f_t := \log \frac{dV_{g_t}}{\Omega}$. Moreover the formula (2.3) implies directly the variation formula

$$2 \frac{d}{dt} \operatorname{Ric}^*_{g_t}(\Omega) = -\Delta_{g_t}^\Omega \, \dot{g}_t^* - 2 \, \dot{g}_t^* \operatorname{Ric}^*_{g_t}(\Omega). \tag{2.4}$$

The introduction of a "center of polarization" K of the tangent space $T_{\mathcal{M}}$ it is quite crucial and natural from the point of view of conservative differential symmetries of the Ω-SRF. We consider indeed a section $K \in C^\infty(X, \operatorname{End}(T_X))$ with n-distinct real eigenvalues almost everywhere over X and we define the vector space

$$\mathbb{F}_g^K := \Big\{ v \in \mathbb{F}_g \mid \left[\nabla_g^p T, v_g^* \right] = 0 \, , T = \mathcal{R}_g \, , K \, , \ \forall p \in \mathbb{Z}_{\geqslant 0} \Big\}.$$

(From the technical point of view it is more natural to introduce this space in a different way that we will explain in the next sections.) For any $g_0 \in \mathcal{M}$ we define the subvariety

$$\Sigma_K(g_0) := \mathbb{F}_{g_0}^K \cap \mathcal{M}.$$

It is a totally geodesic and flat subvariety of the nonpositively curved Riemannian manifold $(\mathcal{M}, G)$ which satisfies the fundamental property

$$T_{\Sigma_K(g_0),g} = \mathbb{F}_g^K, \quad \forall g \in \Sigma_K(g_0),$$

(see Lemma 7 in Section 5 below). We define the *set of scattering data* with center K as the set of metrics

$$\mathcal{S}_\Omega^K \quad := \quad \left\{ g \in \mathcal{M} \mid \operatorname{Ric}_g(\Omega) \in \mathbb{F}_g^K \right\},$$

and the subset of *positive scattering data* with center K as

$$\mathcal{S}_{\Omega,+}^K \quad := \quad \Big\{ g \in \mathcal{S}_\Omega^K \mid \operatorname{Ric}_g(\Omega) > 0 \Big\}.$$

We observe that $\mathcal{S}_{\Omega,+}^K \neq \emptyset$ if the manifold X admit a Ω-ShRS. Moreover if $g \in \mathcal{S}_{\Omega,+}^K$ and if $\dim_{\mathbb{R}} \mathbb{F}_g^K = 1$ then g solves the Ω-ShRS equation up to a constant factor $\lambda > 0$, i.e., $\lambda\, g$ is a Ω-ShRS.

In Section 18 we will explain our program for the existence of scattering data over Fano manifolds. With the notations introduced so far we can state the following result.

Theorem 1. *Let X be a n-dimensional compact and orientable manifold oriented by a smooth volume form $\Omega > 0$ and let $K \in C^\infty(X, \operatorname{End}(T_X))$ with almost everywhere n-distinct real eigenvalues over X. If $\mathcal{S}_{\Omega,+}^K \neq \emptyset$ then the following statements hold:*

(A) *For any data $g_0 \in \mathcal{S}_\Omega^K$ and any metric $g \in \Sigma_K(g_0)$ we have*

$$\nabla_G \mathcal{W}_\Omega(g) = g - \operatorname{Ric}_g(\Omega) \ \in \ T_{\Sigma_K(g_0),g},$$

$$\nabla_G^{\Sigma_K(g_0)} D\, \mathcal{W}_\Omega(g)(v, v) = \int_X \left[\left\langle v \operatorname{Ric}_g^*(\Omega), v \right\rangle_g + \frac{1}{2} \left| \nabla_g v \right|_g^2 \right] \Omega,$$

for all $v \in T_{\Sigma_K(g_0),g}$. The functional $\mathcal{W}_\Omega$ is G-convex over the G-convex set

$$\Sigma_K^-(g_0) := \left\{ g \in \Sigma_K(g_0) \mid \operatorname{Ric}_g(\Omega) \ \geqslant \ - \ \operatorname{Ric}_{g_0}(\Omega) \right\},$$

inside the totally geodesic and flat subvariety $\Sigma_K(g_0)$ of the nonpositively curved Riemannian manifold $(\mathcal{M}, G)$.

(B) *For all $g_0 \in \mathcal{S}_\Omega^K$ with $\operatorname{Ric}_{g_0}(\Omega) \geqslant \varepsilon g_0$, $\varepsilon \in \mathbb{R}_{>0}$ the functional $\mathcal{W}_\Omega$ is G-convex over the G-convex sets*

$$\Sigma_K^\delta(g_0) := \left\{ g \in \Sigma_K(g_0) \mid \operatorname{Ric}_g(\Omega) \ \geqslant \ \delta\, g \right\}, \quad \forall \delta \ \in \ [0, \varepsilon),$$

$$\Sigma_K^+(g_0) := \left\{ g \in \Sigma_K(g_0) \mid 2 \operatorname{Ric}_g(\Omega) \ + \ g_0\, \Delta_{g_0}^\Omega \log(g_0^{-1} g) \ \geqslant \ 0 \right\}.$$

In this case let $\overline{\Sigma}_K^+(g_0)$ be the closure of $\Sigma_K^+(g_0)$ with respect to the metric d_G. Then there exists a natural integral extension

$$\mathcal{W}_\Omega : \overline{\Sigma}_K^+(g_0) \longrightarrow \mathbb{R}$$

of the functional $\mathcal{W}_\Omega$ which is d_G-lower semi-continuous, uniformly bounded from below and d_G-convex over the d_G-closed and d_G-convex set $\overline{\Sigma}_K^+(g_0)$ inside the nonpositively curved length space $(\overline{\mathcal{M}}^{d_G}, d_G)$.

(C) *The formal gradient flow of the functional $\mathcal{W}_\Omega : \Sigma_K(g_0) \longrightarrow \mathbb{R}$ with initial data $g_0 \in \mathcal{S}^K_{\Omega,+}$ represents a smooth solution of the Ω-SRF equation. Assume all time existence of the Ω-SRF $(g_t)_{t\geqslant 0} \subset \Sigma_K(g_0)$ and the existence of $\delta \in \mathbb{R}_{>0}$ such that $\mathrm{Ric}_{g_t}(\Omega) \geqslant \delta g_t$ for all times $t \geqslant 0$. Then the Ω-SRF $(g_t)_{t\geqslant 0}$ converges exponentially fast with all its space derivatives to a Ω-shrinking Ricci soliton $g_{\mathrm{RS}} \in \Sigma_K(g_0)$ as $t \to +\infty$.*

We wish to point out that the G-convexity of the previous sets is part of the statement. Moreover it is possible to define a d_G-lower semi-continuous and d_G-convex extension of the functional $\mathcal{W}_\Omega$ over the closure of $\Sigma^\delta_K(g_0)$ with respect to the metric d_G.

In order to show the convexity statements we need to perform the key change of variables

$$H_t := (g_t^{-1} g_0)^{1/2} \in g_0^{-1}\Sigma_K(g_0),$$

which shows in particular that the SRF equation over $\Sigma_K(g_0)$ corresponds to the endomorphism-valued porous medium type equation.

$$2\,\dot{H}_t = -H_t^2 \Delta^\Omega_{g_0} H_t - H_t^3 \,\mathrm{Ric}^*_{g_0}(\Omega) + H_t, \tag{2.5}$$

with initial data $H_0 = 1$.

The assumption on the uniform positive lower bound of the Ω-Bakry–Emery–Ricci tensor in the statement (C) is reasonable in view of the d_G-convexity of the sets

$$\overline{\Sigma^+_K(g_0) \cap \Sigma^\delta_K(g_0)}^{d_G}.$$

Indeed we expect that this type of convexity can provide a control on $d_G(g_t, g_0)$. We will use for this purpose some known gradient flow techniques over metric spaces. Using the particular gradient flow structure of the SRF and the expression of $\mathcal{W}_\Omega$ over $\Sigma_K(g_0)$ we expect to obtain also H^1-compactness results for the evolving metrics.

Then parabolic methods applied to (2.5) can provide sufficient regularity in order to insur the required uniform lower bound of the Ω-Bakry–Emery–Ricci tensor.

The assumption on the uniform positive lower bound of the Ω-Bakry–Emery–Ricci tensor in the statement (C) allows us to obtain the exponential decay of $C^1(X)$-norms via the maximum principle. The presence of some curvature terms in the evolution equation of higher order space derivatives turns off the power of the maximum principle.

In order to show the exponentially fast convergence of higher order space derivatives we use an interpolation method introduced by Hamilton in his proof of the exponential convergence of the Ricci flow in [Ham].

The difference with the technique in [Ham] is a more involved interpolation process due to the presence of some extra curvature terms which seem to be alien to Hamilton's argument. We are able to perform our interpolation process by using some intrinsic properties of the Ω-SRF.

Let now (X, J) be a Fano manifold and let $-\operatorname{Ric}_J(\Omega)$ be the Chern curvature of the canonical bundle with respect to the hermitian metric induced by $\Omega > 0$. With this notation we give the following definition.

Definition 2. (*The Ω-Soliton-Kähler–Ricci flow*). Let (X, J_0) be a Fano manifold and let $\Omega > 0$ be a smooth volume form with $\int_X \Omega = (2\pi c_1)^n$. A flow of Kähler structures $(X, J_t, \omega_t)_{t \geqslant 0}$ which is solution of the evolution system

$$\begin{cases} \dfrac{d}{dt}\, \omega_t &= \operatorname{Ric}_{J_t}(\Omega) - \omega_t, \\ \dfrac{d}{dt}\, J_t = J_t\, \bar{\partial}_{T_{X,J_t}} \nabla_{g_t} \log \dfrac{\omega_t^n}{\Omega}, \end{cases} \tag{2.6}$$

where $g_t := -\,\omega_t J_t$, is called Ω-Soliton-Kähler–Ricci flow.

The formal limit of the Ω-SKRF at infinite time is precisely the KRS equation with corresponding volume form Ω. In this paper we denote by $\mathcal{K}_J$ the set of J-invariant Kähler metrics. We define the set of positive Kähler scattering data as the set

$$\mathcal{S}^{K,+}_{\Omega,J} := \mathcal{S}^{K}_{\Omega,+} \cap \mathcal{K}_J \,.$$

With this notation we obtain the following statement which is a consequence of the convergence result for the Ω-SRF obtained in Theorem 1(C).

Theorem 2. *Let (X, J_0) be a Fano manifold and assume there exist $g_0 \in \mathcal{S}^{K,+}_{\Omega,J_0}$, for some smooth volume form $\Omega > 0$ and some center of polarization K, such that the solution $(g_t)_t$ of the* Ω-SRF *with initial data g_0 exists for all times and satisfies $\operatorname{Ric}_{g_t}(\Omega) \geqslant \delta g_t$ for some uniform bound $\delta \in \mathbb{R}_{>0}$.*

Then the corresponding solution $(J_t, g_t)_{t \geqslant 0}$ of the Ω-SKRF *converges exponentially fast with all its space derivatives to a J_∞-invariant Kähler–Ricci soliton $g_\infty = \operatorname{Ric}_{g_\infty}(\Omega)$.*

Furtermore assume there exists a positive Kähler scattering data $g_0 \in \mathcal{S}^{K,+}_{\Omega,J_0}$ with $g_0 J_0 \in 2\pi c_1(X)$ such that the evolving complex structure J_t stays constant along a solution $(J_t, g_t)_{t \in [0,T)}$ of the Ω-SKRF *with initial data (J_0, g_0). Then g_0 is a J_0-invariant Kähler–Ricci soliton and $g_t \equiv g_0$.*

3. Conservative differential symmetries

In this section we show that some relevant differential symmetries are preserved along the geodesics induced by the scalar product (2.1).

3.1. First order conservative differential symmetries. We introduce first the cone $\mathbb{F}_g^\infty$ inside the vector space $\mathbb{F}_g$ given by;

$$\begin{aligned}\mathbb{F}_g^\infty &:= \Big\{ v \in C^\infty \left(X, S^2_{\mathbb{R}} T^*_X\right) \mid \nabla_{T_X, g} (v^*_g)^p = 0 \,, \forall p \in \mathbb{Z}_{>0} \Big\} \\ &= \Big\{ v \in C^\infty \left(X, S^2_{\mathbb{R}} T^*_X\right) \mid \nabla_{T_X, g} e^{t v^*_g} = 0 \,, \forall t \in \mathbb{R} \Big\} .\end{aligned}$$

We need also a few algebraic definitions. Let V be a real vector space. We consider the contraction operator

$$\neg \, : \, \mathrm{End}(V) \times \Lambda^2 V^* \longrightarrow \Lambda^2 V^*,$$

defined by the formula

$$H \neg (\alpha \wedge \beta) \; := \; (\alpha \cdot H) \wedge \beta + \alpha \wedge (\beta \cdot H),$$

for any $H \in \mathrm{End}(V)$, $u, v \in V$ and $\alpha, \beta \in V^*$. We can also define the contraction operator by the equivalent formula

$$(H \neg \varphi)(u, v) := \varphi(Hu, v) + \varphi(u, Hv) \,,$$

for any $\varphi \in \Lambda^2 V^*$. Moreover for any element $A \in (V^*)^{\otimes 2} \otimes V$ we define the following elementary operations over the vector space $(V^*)^{\otimes 2} \otimes V$;

$$\begin{aligned}(A\,H)(u, v) &:= A(u, Hv), \\ (HA)(u, v) &:= HA(u, v), \\ (H \bullet A)(u, v) &:= A(Hu, v), \\ (\mathrm{Alt}\, A)(u, v) &:= A(u, v) - A(v, u).\end{aligned}$$

Assume now that V is equipped with a metric g. Then we can define the g-transposed $A^T_g \in (V^*)^{\otimes 2} \otimes V$ as follows. For any $v \in V$

$$v \neg A^T_g := (v \neg A)^T_g \,.$$

We recall (see [Pal]) that the geodesics in the space of Riemannian metrics with respect to the scalar product (2.1) are given by the solutions of the equation $\dot{g}^*_t := g_t^{-1} \dot{g}_t = g_0^{-1} \dot{g}_0$. Thus the geodesic curves write explicitly as

$$g_t = g_0 \, e^{\, t g_0^{-1} \dot{g}_0} \,. \tag{3.1}$$

With this notations we can show now the following fact.

Lemma 1. *Let $(g_t)_{t \in \mathbb{R}}$ be a geodesic such that $\dot{g}_0 \in \mathbb{F}^\infty_{g_0}$. Then $\dot{g}_t \in \mathbb{F}^\infty_{g_t}$ for all $t \in \mathbb{R}$.*

Proof. Let $H \in C^\infty(X, \mathrm{End}(T_X))$ and let $(g_t)_{t\in\mathbb{R}} \subset \mathcal{M}$ be an arbitrary smooth family. We expand first the time derivative

$$\begin{aligned}\dot{\nabla}_{T_X,g_t} H(\xi,\eta) &= \dot{\nabla}_{g_t} H(\xi,\eta) - \dot{\nabla}_{g_t} H(\eta,\xi)\\ &= \dot{\nabla}_{g_t}(\xi, H\eta) - \dot{\nabla}_{g_t}(\eta, H\xi)\\ &\quad - H\left[\dot{\nabla}_{g_t}(\xi,\eta) - \dot{\nabla}_{g_t}(\eta,\xi)\right]\\ &= \dot{\nabla}_{g_t}(H\eta,\xi) - \dot{\nabla}_{g_t}(H\xi,\eta),\end{aligned}$$

since $\dot{\nabla}_{g_t} \in C^\infty(X, S^2_{\mathbb{R}} T^*_X \otimes T_X)$ thanks to the variation identity (see [Bes])

$$2\, g_t\left(\dot{\nabla}_{g_t}(\xi,\eta), \mu\right) = \nabla_{g_t} \dot{g}_t(\xi,\eta,\mu) + \nabla_{g_t} \dot{g}_t(\eta,\xi,\mu) - \nabla_{g_t} \dot{g}_t(\mu,\xi,\eta)\,. \tag{3.2}$$

We observe now that the variation formula (3.2) rewrites as

$$2\,\dot{\nabla}_{g_t}(\xi,\eta) = \nabla_{g_t} \dot{g}^*_t(\xi,\eta) + \nabla_{g_t} \dot{g}^*_t(\eta,\xi) - (\nabla_{g_t} \dot{g}^*_t\, \eta)^T_{g_t}\, \xi\,.$$

Thus

$$\begin{aligned}2\,\dot{\nabla}_{T_X,g_t} H(\xi,\eta) &= \nabla_{g_t} \dot{g}^*_t(H\eta,\xi) + \nabla_{g_t} \dot{g}^*_t(\xi,H\eta) - (\nabla_{g_t} \dot{g}^*_t\, \xi)^T_{g_t} H\eta\\ &\quad - \nabla_{g_t} \dot{g}^*_t(H\xi,\eta) - \nabla_{g_t} \dot{g}^*_t(\eta,H\xi) + (\nabla_{g_t} \dot{g}^*_t\, \eta)^T_{g_t} H\xi.\end{aligned}$$

Applying the identity

$$(\nabla_{g_t} \dot{g}^*_t\, \xi)^T_{g_t} = -\left(\xi \neg \nabla_{T_X,g_t} \dot{g}^*_t\right)^T_{g_t} + \xi \neg \nabla_{g_t} \dot{g}^*_t,$$

we obtain the equalities

$$\begin{aligned}2\,\dot{\nabla}_{T_X,g_t} H(\xi,\eta) &= \nabla_{g_t} \dot{g}^*_t(H\eta,\xi) - \nabla_{g_t} \dot{g}^*_t(H\xi,\eta)\\ &\quad + \left(\xi \neg \nabla_{T_X,g_t} \dot{g}^*_t\right)^T_{g_t} H\eta - \left(\eta \neg \nabla_{T_X,g_t} \dot{g}^*_t\right)^T_{g_t} H\xi\\ &= -\left(H \neg \nabla_{T_X,g_t} \dot{g}^*_t\right)(\xi,\eta)\\ &\quad + \nabla_{g_t} \dot{g}^*_t(\xi,H\eta) - \nabla_{g_t} \dot{g}^*_t(\eta,H\xi)\\ &\quad + \mathrm{Alt}\left[\left(\nabla_{T_X,g_t} \dot{g}^*_t\right)^T_{g_t} H\right](\xi,\eta).\end{aligned}$$

We infer the variation formula

$$\begin{aligned}2\,\dot{\nabla}_{T_X,g_t} H &= -H \neg \nabla_{T_X,g_t} \dot{g}^*_t + \nabla_{T_X,g_t}(\dot{g}^*_t H) - \dot{g}^*_t\, \nabla_{T_X,g_t} H\\ &\quad + \mathrm{Alt}\left[\left(\nabla_{T_X,g_t} \dot{g}^*_t\right)^T_{g_t} H\right].\end{aligned} \tag{3.3}$$

Thus along any geodesic we obtain the upper triangular type infinite dimensional ODE system

$$\begin{aligned}2\,\frac{d}{dt}\Big[\nabla_{T_X,g_t}(\dot g_t^*)^p\Big] &= -(\dot g_t^*)^p \neg \nabla_{T_X,g_t}\dot g_t^*\\ &\quad + \nabla_{T_X,g_t}(\dot g_t^*)^{p+1} - \dot g_t^*\,\nabla_{T_X,g_t}(\dot g_t^*)^p\\ &\quad + \mathrm{Alt}\Big[\Big(\nabla_{T_X,g_t}\dot g_t^*\Big)^T_{g_t}(\dot g_t^*)^p\Big],\end{aligned}$$

for all $p \in \mathbb{Z}_{>0}$. We recall now that $\dot g_t^* \equiv \dot g_0^*$ and we observe the formula

$$\frac{d}{dt}\Big(\nabla_{T_X,g_t}\dot g_t^*\Big)^T_{g_t} = \Big[\Big(\nabla_{T_X,g_t}\dot g_t^*\Big)^T_{g_t}, \dot g_t^*\Big] + \Big[\frac{d}{dt}\Big(\nabla_{T_X,g_t}\dot g_t^*\Big)\Big]^T_{g_t}.$$

This combined once again with the identity $\dot g_t^* \equiv \dot g_0^*$ and with the previous variation formula implies that for all $k, p \in \mathbb{Z}_{\geqslant 0}$ we have

$$\frac{d^k}{dt^k}_{|t=0}\Big[\nabla_{T_X,g_t}(\dot g_t^*)^p\Big] = 0.$$

Indeed this follows from an increasing induction in k. The conclusion follows from the fact that the curves

$$t \longmapsto \nabla_{T_X,g_t}(\dot g_t^*)^p,$$

are real analytic over the real line. □

Let now $A \in (V^*)^{\otimes p} \otimes V$, $B \in (V^*)^{\otimes q} \otimes V$ and let $k = 1, \ldots q$. We define the generalized product operation

$$(A\,B)(u_1, \ldots, u_{p-1}, v_1, \ldots, v_q) := A(u_1, \ldots, u_{p-1}, B(v_1, \ldots, v_q)).$$

With this notations we define the vector space

$$\mathbb{E}_g := \Big\{v \in C^\infty\left(X, S^2_{\mathbb{R}}T^*_X\right) \mid \left[\mathcal{R}_g, v_g^*\right] = 0\,,\ \left[\mathcal{R}_g, \nabla_{g,\xi} v_g^*\right] = 0\,,\ \forall \xi \in T_X\Big\},$$

and we show the following crucial fact.

Lemma 2. *Let $(g_t)_{t\in\mathbb{R}} \subset \mathcal{M}$ be a geodesic such that $\dot g_0 \in \mathbb{F}^\infty_{g_0} \cap \mathbb{E}_{g_0}$. Then $\dot g_t \in \mathbb{F}^\infty_{g_t} \cap \mathbb{E}_{g_t}$ for all $t \in \mathbb{R}$.*

Proof. We observe first that the variation identity (3.2) combined with the fact that $\dot g_t \in \mathbb{F}_{g_t}$ implies the variation identity

$$2\,\dot\nabla_{g_t} = \nabla_{g_t}\dot g_t^*. \tag{3.4}$$

Thus the variation formula (see [Bes])

$$\dot{\mathcal{R}}_{g_t}(\xi,\eta)\mu = \nabla_{g_t}\dot\nabla_{g_t}(\xi,\eta,\mu) - \nabla_{g_t}\dot\nabla_{g_t}(\eta,\xi,\mu), \tag{3.5}$$

rewrites as

$$\begin{aligned}2\,\dot{\mathcal{R}}_{g_t}(\xi,\eta) &= \nabla_{g_t,\xi}\nabla_{g_t,\eta}\,\dot g_t^* - \nabla_{g_t,\eta}\nabla_{g_t,\xi}\,\dot g_t^* - \nabla_{g_t,[\xi,\eta]}\,\dot g_t^*\\ &= \left[\mathcal{R}_{g_t}(\xi,\eta), \dot g_t^*\right].\end{aligned}$$

We recall in fact the general identity

$$\nabla_{g,\xi}\nabla_{g,\eta} H - \nabla_{g,\eta}\nabla_{g,\xi} H = [\mathcal{R}_g(\xi,\eta), H] + \nabla_{g,[\xi,\eta]} H, \tag{3.6}$$

for any $H \in C^\infty(X, \mathrm{End}(T_X))$. We deduce the variation identity

$$2\,\dot{\mathcal{R}}_{g_t} = [\mathcal{R}_{g_t}, \dot{g}_t^*]\,, \tag{3.7}$$

(for any smooth curve $(g_t)_t$ such that $\dot{g}_t \in \mathbb{F}_{g_t}$), and the variation formula

$$2\,\frac{d}{dt}[\mathcal{R}_{g_t}, \dot{g}_t^*] = \Big[[\mathcal{R}_{g_t}, \dot{g}_t^*]\,, \dot{g}_t^*\Big].$$

Thus the identity $[\mathcal{R}_{g_t}, \dot{g}_t^*] = 0$ holds for all times by Cauchy uniqueness. We infer in particular $\mathcal{R}_{g_t} = \mathcal{R}_{g_0}$ for all $t \in \mathbb{R}$ thanks to the identity (3.7). Using the variation formula

$$2\,\dot{\nabla}_{g_t} H = 2\,\dot{\nabla}_{g_t} \cdot H - 2\,H\,\dot{\nabla}_{g_t} = [\nabla_{g_t}\dot{g}_t^*, H]\,, \tag{3.8}$$

we deduce

$$\begin{aligned} 2\,\frac{d}{dt}[\mathcal{R}_{g_t}, \nabla_{g_t,\xi}\,\dot{g}_t^*] &= \Big[\mathcal{R}_{g_t}, [\nabla_{g_t,\xi}\,\dot{g}_t^*, \dot{g}_t^*]\Big] \\ &= -\Big[\nabla_{g_t,\xi}\,\dot{g}_t^*, [\mathcal{R}_{g_t}, \dot{g}_t^*]\Big] - \Big[\dot{g}_t^*, [\mathcal{R}_{g_t}, \nabla_{g_t,\xi}\,\dot{g}_t^*]\Big] \\ &= \Big[[\mathcal{R}_{g_t}, \nabla_{g_t,\xi}\,\dot{g}_t^*]\,, \dot{g}_t^*\Big], \end{aligned}$$

by the Jacobi identity and by the previous result. We infer the conclusion by Cauchy uniqueness. □

3.2. Conservation of the prescattering condition. This subsection is the hart of the paper. We will show the conservation of the prescattering condition along curves with variations in $\mathbb{F}_g \cap \mathbb{E}_g$. We need to introduce first a few other product notations. Let $(e_k)_k$ be a g-orthonormal basis. For any elements $A \in (T_X^*)^{\otimes 2} \otimes T_X$ and $B \in \Lambda^2 T_X^* \otimes \mathrm{End}(T_X)$ we define the generalized products

$$\begin{aligned} (B * A)(u,v) &:= B(u,e_k)A(e_k,v), \\ (B \circledast A)(u,v) &:= [B(u,e_k)\,, e_k \neg A]\,v, \\ (A * B)(u,v) &:= A(e_k, B(u,v)e_k). \end{aligned}$$

We observe that the algebraic Bianchi identity implies

$$\mathrm{Alt}(\mathcal{R}_g \circledast A) = \mathrm{Alt}(\mathcal{R}_g * A) - A * \mathcal{R}_g\,. \tag{3.9}$$

Let also $H \in C^\infty(X, \mathrm{End}(T_X))$. Then

$$\nabla_{T_X,g} H * \mathcal{R}_g = 2\,\nabla_g H * \mathcal{R}_g. \tag{3.10}$$

We observe in fact the equalities

$$\nabla_{T_X,g} H * \mathcal{R}_g = \nabla_g H * \mathcal{R}_g - \nabla_g H\,(\mathcal{R}_g e_k, e_k) = 2\,\nabla_g H * \mathcal{R}_g.$$

This follows writing with respect to the g-orthonormal basis (e_k) the identity

$$\mathcal{R}_g(\xi,\eta) = -\,(\mathcal{R}_g(\xi,\eta))_g^T\,,$$

which is a consequence of the alternating property of the $(4,0)$-Riemann curvature operator.

For any $A \in C^\infty(X, (T_X^*)^{\otimes p+1} \otimes T_X)$ we define the divergence type operations

$$\underline{\operatorname{div}}_g A(u_1, \dots, u_p) := \operatorname{Tr}_g \left[\nabla_g A(\cdot, u_1, \dots, u_p, \cdot)\right],$$
$$\underline{\operatorname{div}}_g^\Omega A(u_1, \dots, u_p) := \underline{\operatorname{div}}_g A(u_1, \dots, u_p) - A(u_1, \dots, u_p, \nabla_g f).$$

We recall that the once contracted differential Bianchi identity writes often as $\underline{\operatorname{div}}_g \mathcal{R}_g = -\nabla_{T_X,g} \operatorname{Ric}_g^*$. This combined with the identity

$$\nabla_{T_X,g} \nabla_g^2 f = \mathcal{R}_g \cdot \nabla_g f$$

implies

$$\underline{\operatorname{div}}_g^\Omega \mathcal{R}_g = -\nabla_{T_X,g} \operatorname{Ric}_g^*(\Omega). \tag{3.11}$$

With the previous notations we obtain the following lemma.

Lemma 3. *Let $(g_t)_{t\in\mathbb{R}} \subset \mathcal{M}$ be a smooth family such that $\dot{g}_t \in \mathbb{F}_{g_t}$ for all $t \in \mathbb{R}$. Then the following variation formula holds:*

$$2\,\frac{d}{dt}\left[\nabla_{T_X,g_t} \operatorname{Ric}_{g_t}^*(\Omega)\right] = \underline{\operatorname{div}}_{g_t}^\Omega \left[\mathcal{R}_{g_t}, \dot{g}_t^*\right] + \operatorname{Alt}\left(\mathcal{R}_{g_t} \circledast \nabla_{g_t}\dot{g}_t^*\right)$$
$$- 2\,\dot{g}_t^*\, \nabla_{T_X,g_t} \operatorname{Ric}_{g_t}^*(\Omega)\ .$$

Proof. We will show the above variation formula by means of the identity (3.11). Consider any $B \in C^\infty(X, \Lambda^2 T_X^* \otimes \operatorname{End}(T_X))$. Time deriving the definition of the covariant derivative $\nabla_{g_t} B$ we deduce the formula

$$\dot{\nabla}_{g_t} B(\xi, u, v)\, w = \dot{\nabla}_{g_t}(\xi, B(u,v)\,w) - B\left(\dot{\nabla}_{g_t}(\xi, u), v\right) w$$
$$- B\left(u, \dot{\nabla}_{g_t}(\xi, v)\right) w - B(u,v) \dot{\nabla}_{g_t}(\xi, w).$$

We infer the expression

$$2\,\dot{\nabla}_{g_t} B(\xi, u, v)\, w = \nabla_{g_t,\xi}\, \dot{g}_t^*\, B(u,v)\, w - B\left(\nabla_{g_t,\xi}\, \dot{g}_t^* u, v\right) w$$
$$- B\left(u, \nabla_{g_t,\xi}\, \dot{g}_t^*\, v\right) w - B\left(u,v\right) \nabla_{g_t,\xi}\, \dot{g}_t^*\, w, \tag{3.12}$$

thanks to the formula (3.4). We fix now an arbitrary space-time point (x_0, t_0) and we pick a local tangent frame $(e_k)_k$ in a neighborhood of x_0 which is $g_{t_0}(x_0)$-orthonormal at the point x_0 and satisfies $\nabla_{g_t} e_j(x_0) = 0$ at the time t_0 for all j. Then time deriving the term

$$(\underline{\operatorname{div}}_{g_t}^\Omega B)(\xi, \eta) = \nabla_{g_t, e_k} B\,(\xi, \eta)\, g_t^{-1} e_k^* - B(\xi, \eta)\, \nabla_{g_t} f_t,$$

and using the expression (3.12) we obtain the identity

$$2\,\frac{d}{dt}\,(\underline{\operatorname{div}}_{g_t}^\Omega B)(\xi, \eta) = \nabla_{g_t, e_k}\, \dot{g}_t^*\, B\,(\xi, \eta)\, e_k - B\left(\nabla_{g_t, e_k}\, \dot{g}_t^*\, \xi, \eta\right) e_k$$
$$- B\left(\xi, \nabla_{g_t, e_k}\, \dot{g}_t^*\, \eta\right) e_k - B\left(\xi, \eta\right) \nabla_{g_t, e_k}\, \dot{g}_t^*\, e_k$$
$$- 2\, \nabla_{g_t, e_k} B\,(\xi, \eta)\, \dot{g}_t^* e_k - 2\, B(\xi, \eta)\, \frac{d}{dt}\, \nabla_{g_t} f_t, \tag{3.13}$$

at the space-time (x_0, t_0). Moreover we have the elementary formula

$$2\,\frac{d}{dt}\,\nabla_{g_t} f_t = \nabla_{g_t}\,\mathrm{Tr}_{g_t}\,\dot{g}_t - 2\,\dot{g}_t^*\,\nabla_{g_t} f_t.$$

We observe also that at the space time point (x_0, t_0) we have the trivial equalities

$$\begin{aligned}
\nabla_{g_t}\,\mathrm{Tr}_{g_t}\,\dot{g}_t &= e_k\,.\,(\mathrm{Tr}_{\mathbb{R}}\,\dot{g}_t^*)\,e_k\\
&= e_k\,.\,g_t\,(\dot{g}_t^*\,e_j, e_j)\,e_k\\
&= g_t\,(\nabla_{g_t,e_k}\,\dot{g}_t^*\,e_j, e_j)\,e_k\\
&= \nabla_{g_t}\,\dot{g}_t\,(e_k, e_j, e_j)\,e_k\\
&= \nabla_{g_t}\,\dot{g}_t\,(e_j, e_j, e_k)e_k\\
&= g_t\left(\nabla_{g_t,e_j}\,\dot{g}_t^*\,e_j, e_k\right)e_k\\
&= -\nabla_{g_t}^*\,\dot{g}_t^*,
\end{aligned}$$

thanks to the assumption $\dot{g}_t \in \mathbb{F}_{g_t}$. We deduce the identity

$$2\,\frac{d}{dt}\,\nabla_{g_t} f_t = -\nabla_{g_t}^*\,\dot{g}_t^* - 2\,\dot{g}_t^*\,\nabla_{g_t} f_t. \tag{3.14}$$

Thus the identity (3.13) rewrites as follows;

$$\begin{aligned}
2\,\frac{d}{dt}\,(\underline{\mathrm{div}}_{g_t}^{\Omega} B)(\xi, \eta) = (\nabla_{g_t}\,\dot{g}_t^* * B)\,(\xi, \eta)&\\
- B\,(\nabla_{g_t,e_k}\,\dot{g}_t^*\,\xi, \eta)\,e_k - B\,(\xi, \nabla_{g_t,e_k}\,\dot{g}_t^*\,\eta)\,e_k&\\
- 2\,\nabla_{g_t,e_k} B\,(\xi, \eta)\;\dot{g}_t^* e_k + 2\,B(\xi, \eta)\,\nabla_{g_t}^{*_\Omega}\,\dot{g}_t^*.&
\end{aligned}$$

The assumption $\dot{g}_t \in \mathbb{F}_{g_t}$ implies that the endomorphism $\nabla_{g_t,\bullet}\,\dot{g}_t^*\,\xi$ is g_t-symmetric. Thus we can choose a $g_{t_0}(x_0)$-orthonormal basis $(e_k) \subset T_{X,x_0}$ which diagonalize it at the space-time point (x_0, t_0). It is easy to see that with respect to this basis

$$B\,(\nabla_{g_t,e_k}\,\dot{g}_t^*\,\xi, \eta)\,e_k = -\,(B * \nabla_{g_t}\,\dot{g}_t^*)\,(\eta, \xi),$$

by the alternating property of B. But the term on the left hand side is independent of the choice of the $g_{t_0}(x_0)$-orthonormal basis. In a similar way choosing a $g_{t_0}(x_0)$-orthonormal basis $(e_k)_k \subset T_{X,x_0}$ which diagonalizes $\nabla_{g_t,\bullet}\,\dot{g}_t^*\,\eta$ at the space-time point (x_0, t_0) we obtain the identity

$$B\,(\xi, \nabla_{g_t,e_k}\,\dot{g}_t^*\,\eta)\,e_k = (B * \nabla_{g_t}\,\dot{g}_t^*)\,(\xi, \eta).$$

We infer the equality

$$\begin{aligned}
2\left(\frac{d}{dt}\,\underline{\mathrm{div}}_{g_t}^{\Omega}\right) B = \nabla_{g_t}\,\dot{g}_t^* * B - \mathrm{Alt}\,(B * \nabla_{g_t}\,\dot{g}_t^*)&\\
- 2\,\nabla_{g_t,e_k} B\,\dot{g}_t^* e_k + 2\,B\,\nabla_{g_t}^{*_\Omega}\,\dot{g}_t^*,&
\end{aligned}$$

with respect to any $g_{t_0}(x_0)$-orthonormal basis (e_k) at the arbitrary space-time point (x_0, t_0). This combined with (3.7) and (3.9) implies the equalities

$$2\,\frac{d}{dt}\left(\underline{\operatorname{div}}^{\Omega}_{g_t}\mathcal{R}_{g_t}\right) = 2\left(\frac{d}{dt}\,\underline{\operatorname{div}}^{\Omega}_{g_t}\right)\mathcal{R}_{g_t} + \underline{\operatorname{div}}^{\Omega}_{g_t}\left[\mathcal{R}_{g_t}, \dot{g}^*_t\right]$$
$$= -\operatorname{Alt}\left(\mathcal{R}_{g_t} \circledast \nabla_{g_t}\dot{g}^*_t\right) - 2\,\nabla_{g_t,e_k}\mathcal{R}_{g_t}\,\dot{g}^*_t e_k$$
$$+ 2\,\mathcal{R}_{g_t}\nabla^{*_\Omega}_{g_t}\dot{g}^*_t + \underline{\operatorname{div}}^{\Omega}_{g_t}\left[\mathcal{R}_{g_t}, \dot{g}^*_t\right].$$

We observe now that for any smooth curve $(g_t)_{t\in\mathbb{R}} \subset \mathcal{M}$

$$\underline{\operatorname{div}}^{\Omega}_{g_t}\left[\mathcal{R}_{g_t}, \dot{g}^*_t\right] = \nabla_{g_t,e_k}\mathcal{R}_{g_t}\,\dot{g}^*_t e_k - \mathcal{R}_{g_t}\nabla^{*_\Omega}_{g_t}\dot{g}^*_t$$
$$- \nabla_{g_t}\dot{g}^*_t * \mathcal{R}_{g_t} - \dot{g}^*_t\,\underline{\operatorname{div}}^{\Omega}_{g_t}\mathcal{R}_{g_t}. \tag{3.15}$$

But our assumption $\dot{g}_t \in \mathbb{F}_{g_t}$ implies $\nabla_{g_t}\dot{g}^*_t * \mathcal{R}_{g_t} \equiv 0$ thanks to the identity (3.10). Thus we obtain the formula

$$2\,\frac{d}{dt}\left(\underline{\operatorname{div}}^{\Omega}_{g_t}\mathcal{R}_{g_t}\right) = -\operatorname{Alt}\left(\mathcal{R}_{g_t} \circledast \nabla_{g_t}\dot{g}^*_t\right) - \underline{\operatorname{div}}^{\Omega}_{g_t}\left[\mathcal{R}_{g_t}, \dot{g}^*_t\right]$$
$$- 2\,\dot{g}^*_t\,\underline{\operatorname{div}}^{\Omega}_{g_t}\mathcal{R}_{g_t},$$

which implies the required conclusion thanks to the identity (3.11). □

Corollary 1 (Conservation of the prescattering condition)**.** *Let*

$$(g_t)_{t\in\mathbb{R}} \subset \mathcal{M}$$

be a smooth family such that $\dot{g}_t \in \mathbb{F}_{g_t} \cap \mathbb{E}_{g_t}$ *for all* $t \in \mathbb{R}$. *If* $g_0 \in \mathcal{S}_{\Omega}$ *then* $g_t \in \mathcal{S}_{\Omega}$ *for all* $t \in \mathbb{R}$.

Proof. We observe that the assumption $\dot{g}_t \in \mathbb{E}_{g_t}$ implies in particular the identity $\mathcal{R}_{g_t} \circledast \nabla_{g_t}\dot{g}^*_t \equiv 0$. By Lemma 3 we infer the variation formula

$$\frac{d}{dt}\left[\nabla_{T_X,g_t}\operatorname{Ric}^*_{g_t}(\Omega)\right] = -\dot{g}^*_t\,\nabla_{T_X,g_t}\operatorname{Ric}^*_{g_t}(\Omega),$$

and thus the conclusion by Cauchy uniqueness. □

The total variation of the prescattering operator is given in Lemma 26 in the appendix. It provides in particular an alternative proof of the conservation of the prescattering condition.

3.3. Higher order conservative differential symmetries. In this subsection we will show that some higher order differential symmetries are conserved along the geodesics. This type of higher order differential symmetries is needed in order to stabilize the scattering conditions with respect to the variations produced by the SRF. We observe first that given any diagonal $n \times n$-matrix Λ,

$$[\Lambda, M] = \left((\lambda_i - \lambda_j)M_{i,j}\right),$$

for any other $n \times n$-matrix M. Thus if the values λ_j are all distinct then $[\Lambda, M] = 0$ if and only if M is also a diagonal matrix.

In this subsection and in Sections 4, 5 that will follow we will always denote by $K \in \Gamma(X, \mathrm{End}(T_X))$ an element with point wise n-distinct real eigenvalues, where $n = \dim_{\mathbb{R}} X$.

The previous remark shows that if $(e_k) \subset T_{X,p}$ is a basis diagonalizing $K(p)$ then it diagonalizes any element $M \in \mathrm{End}(T_{X,p})$ such that $[K(p), M] = 0$.

We deduce that if also $N \in \mathrm{End}(T_{X,p})$ satisfies $[K(p), N] = 0$ then $[M, N] = 0$.

We define now the vector space inside $\mathbb{F}_g^\infty$

$$\mathbb{F}_g(K) := \Big\{ v \in \mathbb{F}_g \mid \big[K, v_g^*\big] = 0 \,,\; \big[K, \nabla_g v_g^*\big] = 0 \Big\} \subset \mathbb{F}_g^\infty \,.$$

We observe in fact the definition implies $\big[\nabla_g v_g^*, v_g^*\big] = 0$, and thus the last inclusion. With this notation we obtain the following corollary analogous to Lemma 1.

Corollary 2. *Let $(g_t)_{t\in\mathbb{R}}$ be a geodesic such that $\dot g_0 \in \mathbb{F}_{g_0}(K)$. Then $\dot g_t \in \mathbb{F}_{g_t}(K)$ for all $t \in \mathbb{R}$.*

Proof. By Lemma 1 we just need to show the identity $[K, \nabla_{g_t} \dot g_t^*] \equiv 0$. In fact using the variation formula (3.8) we obtain

$$2\,\frac{d}{dt}\,[K, \nabla_{g_t} \dot g_t^*] = \Big[K, [\nabla_{g_t} \dot g_t^*, \dot g_t^*]\Big] = -\Big[\dot g_t^*, [K, \nabla_{g_t} \dot g_t^*]\Big],$$

since $[K, \dot g_t^*] \equiv 0$. Then the conclusion follows by Cauchy uniqueness. $\square$

We define now the subvector space $\mathbb{F}_g^K \subset \mathbb{F}_g(K)$,

$$\mathbb{F}_g^K := \Big\{ v \in \mathbb{F}_g \mid \Big[T, \nabla_{g,\xi}^p v_g^*\Big] = 0 \,,\; T = K \,, \mathcal{R}_g \,,\; \forall \xi \in T_X^{\otimes p},\; \forall p \in \mathbb{Z}_{\geqslant 0} \Big\} \,,$$

and we show the following elementary lemmas:

Lemma 4. *If $u, v \in \mathbb{F}_g^K$ then $u\, v_g^*, u\, e^{v_g^*} \in \mathbb{F}_g^K$.*

Proof. By assumption follows that u_g^* commutes with v_g^*. This shows that $u\, v_g^*$ is a symmetric form. Again by assumption we infer

$$[\nabla_g u_g^*, v_g^*] = [\nabla_g v_g^*, u_g^*] = 0,$$

and thus $u_g^*\, v_g^* \in \mathbb{F}_g$. We observe now that for any $A, B, C \in \mathrm{End}(V)$ such that $[C, A] = 0$,

$$[C, A\,B] = A\,[C, B] \,. \tag{3.16}$$

Thus if also $[C, B] = 0$ then

$$[C, A\,B] = 0. \tag{3.17}$$

Applying (3.17) with $C = T$ and with $A = \nabla_{g,\eta}^r u_g^*$, $\eta \in T_X^{\otimes r}$, $B = \nabla_{g,\mu}^{p-r} v_g^*$, $\mu \in T_X^{\otimes p-r}$ we infer the identity

$$[T, \nabla_{g,\xi}^p (u_g^* v_g^*)] = 0,$$

thus the conclusion $u\, v_g^* \in \mathbb{F}_g^K$. The fact $u\, e^{v_g^*} \in \mathbb{F}_g^K$ follow directly from the previous one. □

For all $\xi \equiv (\xi_1, \dots, \xi_p) \in C^\infty(X, T_X)^{\oplus p}$ we denote

$$\nabla^{(p)}_{g,\xi}\, v_g^* := \nabla_{g,\xi_1} \dots \nabla_{g,\xi_p} v_g^*,$$

and we observe that a simple induction based on the formula

$$\nabla^p_{g,\xi}\, v_g^* = \nabla^{(p)}_{g,\xi}\, v_g^* - \sum_{r=1}^{p-1} \sum_{I \in J^{p-1}_{p-r}} \varepsilon_I \left(\nabla_{g,\xi_I}\, \xi_{\complement I}\right) \neg \nabla^r_g\, v_g^*,$$

with $J^{p-1}_{p-r} := \{I \subset \{1, \dots, p-1\} : |I| = p - r\}$, $\varepsilon_I = 0, 1$, $\complement I \equiv (j_1, \dots j_r) := \{1, \dots, p\} \smallsetminus I$ and with

$$\nabla_{g,\xi_I}\, \xi_{\complement I} := \nabla_{g,\xi_{i_1}} \dots \nabla_{g,\xi_{i_{p-r}}} \left(\xi_{j_1} \otimes \dots \otimes \xi_{j_r}\right),$$

$I \equiv (i_1, \dots, i_{p-r})$, shows the identity

$$\mathbb{F}_g^K = \Big\{ v \in \mathbb{F}_g \mid \Big[T, \nabla^{(p)}_{g,\xi}\, v_g^*\Big] = 0,\ T = K, \mathcal{R}_g, \forall \xi \in C^\infty(X, T_X)^{\oplus p}, \forall p \in \mathbb{Z}_{\geqslant 0} \Big\}.$$

Lemma 5. *Let $(g_t)_{t\in\mathbb{R}}$ be a geodesic such that $\dot{g}_0 \in \mathbb{F}^K_{g_0}$. Then $\dot{g}_t \in \mathbb{F}^K_{g_t}$ for all $t \in \mathbb{R}$.*

Proof. We observe that $\dot{g}_t \in \mathbb{F}_{g_t}(K) \cap \mathbb{E}_{g_t}$ for all $t \in \mathbb{R}$, thanks to Corollary 2 and Lemma 2. This implies in particular that $\mathcal{R}_{g_t} = \mathcal{R}_{g_0}$ for all $t \in \mathbb{R}$, by the variation formula (3.7). Then the conclusion will follow from the property

$$\nabla^{(p)}_{g_0,\xi}\, \dot{g}_0^* \equiv \nabla^{(p)}_{g_t,\xi}\, \dot{g}_t^*\,, \quad \forall \xi \in C^\infty(X, T_X)^{\oplus p}, \quad \forall t \in \mathbb{R}. \tag{3.18}$$

This certainly holds true for $p = 0$ by the geodesic equation $\dot{g}_0^* \equiv \dot{g}_t^*$. We assume now the statement (3.18) true for $p - 1$ and we show it for $p > 0$. We observe first that thanks to the variation formula (3.8),

$$\dot{\nabla}_{g_t} H \equiv 0 \tag{3.19}$$

along any smooth curve $(g_t)_t \subset \mathcal{M}$ such that $\dot{g}_t \in \mathbb{F}_g$ and $[\nabla_{g_t}\, \dot{g}_t^*, H] = 0$. In our situation the identity $[K, \nabla_{g_t}\, \dot{g}_t^*] \equiv 0$ combined with the assumption on the initial data

$$\Big[K, \nabla^{(p-1)}_{g_0,\xi}\, \dot{g}_0^*\Big] = 0,$$

for all $\xi \in C^\infty(X, T_X)^{\oplus(p-1)}$, implies thanks to (3.19) the equalities

$$\nabla_{g_0} \nabla^{(p-1)}_{g_0,\xi}\, \dot{g}_0^* = \nabla_{g_t} \nabla^{(p-1)}_{g_0,\xi}\, \dot{g}_0^* \equiv \nabla_{g_t} \nabla^{(p-1)}_{g_t,\xi}\, \dot{g}_t^*,$$

by the inductive hypothesis. We infer the conclusion of the induction. □

4. The set of scattering data

We define the *set of scattering data* with center K as the set of metrics

$$\mathcal{S}_{\Omega}^{K} := \left\{g \in \mathcal{M} \mid \operatorname{Ric}_g(\Omega) \in \mathbb{F}_g^K\right\}$$
$$= \left\{g \in \mathcal{S}_{\Omega} \mid \left[T, \nabla^p_{g,\xi} \operatorname{Ric}^*_g(\Omega)\right] = 0\,,\ T = K, \mathcal{R}_g, \forall \xi \in T_X^{\otimes p}, \forall p \in \mathbb{Z}_{\geqslant 0}\right\}.$$

We observe that $\mathcal{S}_{\Omega}^{K} \neq \emptyset$ if the manifold X admit a Ω-ShRS. We introduce now a few new product notations. For any $A \in (V^*)^{\otimes p} \otimes V$, $B \in (V^*)^{\otimes q} \otimes V$ and for any $k = 1, \ldots, q$ we define the products $A \bullet_k B$ as

$$(A \bullet_k B)(u, v) := B\Big(v_1, \ldots, v_{k-1}, A(u, v_k), v_{k+1}, \ldots, v_q\Big),$$

for all $u \equiv (u_1, \ldots, u_{p-1})$ and $v \equiv (v_1, \ldots, v_q)$. We note $\bullet := \bullet_1$ for simplicity. For any $\sigma \in S_{p+k-2}$ we define $A \bullet_k^\sigma B$ as

$$(A \bullet_k^\sigma B)(u, v) := (A \bullet_k B)(\xi_\sigma, v_k, \ldots, v_q),$$

where $\xi \equiv (\xi_1, \ldots, \xi_{p+k-2}) := (u_1, \ldots, u_{p-1}, v_1, \ldots, v_{k-1})$. We should notice that $A \bullet_k^\sigma B \equiv A \bullet_k B$ if $p + k - 2 \leqslant 1$. We define

$$A \mathbin{\hat{\neg}} B := \sum_{k=1}^{q-1} A \bullet_k B.$$

For $p > 1$ and $k = 1, \ldots p - 1$ we define the trace operation

$$(\operatorname{Tr}_{g,k} A)\,(u_1, \ldots, u_{p-2}) := \operatorname{Tr}_g \Big[A\,(u_1, \ldots, u_{k-1}, \cdot, \cdot, u_k, \ldots, u_{p-2})\Big].$$

For any $v \in T_X$ and $k = 1, \ldots, p$ we define the contraction operation

$$(v \neg_k A)\,(u_1, \ldots, u_{p-1}) := A\,(u_1, \ldots, u_{k-1}, v, u_k, \ldots, u_{p-1})\,.$$

For any $B \in (V^*)^{\otimes q} \otimes V$ and $k = 1, \ldots, q-1$ we define the generalized type products

$$\begin{aligned}
&(A *_k B)\,(u_1, \ldots, u_p, v_1, \ldots, v_{q-1})\\
&\qquad := B\Big(v_1, \ldots, v_{k-1}, A(u_1, \ldots, u_p), v_k, \ldots, v_{q-1}\Big),\\
&(A *_k^\sigma B)\,(u_1, \ldots, u_p, v_1, \ldots, v_{q-1})\\
&\qquad := (A *_k B)\,(\xi_\sigma, v_k, \ldots, v_{q-1})\,, \quad \forall \sigma \in S_{p+k-1},
\end{aligned}$$

and $\xi \equiv (\xi_1, \ldots, \xi_{p+k-1}) := (u_1, \ldots, u_p, v_1, \ldots, v_{k-1})$. We observe now that if $A \in C^\infty(X, (T_X^*)^{\otimes p} \otimes T_X)$ and $(g_t)_{t \in \mathbb{R}} \subset \mathcal{M}$ is a smooth family such that $[\nabla_{g_t,\xi}\, \dot{g}_t^*, A] \equiv 0$ for all $\xi \in T_X$ then

$$2\,\dot{\nabla}_{g_t} A = -\nabla_{g_t}\, \dot{g}_t^* \mathbin{\hat{\neg}} A,$$

thanks to the variation formula (3.4). We infer by this and by the variation formula (3.19) that if $H \in C^\infty(X, \operatorname{End}(T_X))$ satisfies $[\nabla_{g_t,\xi}\, \dot{g}_t^*, \nabla^p_{g_t} H] \equiv 0$, for all $\xi \in T_X$ and $p = 0, 1$ then

$$2\,\dot{\nabla}^2_{g_t} H = -\nabla_{g_t}\, \dot{g}_t^* \mathbin{\hat{\neg}} \nabla_{g_t} H.$$

A simple induction shows that if $[\nabla_{g_t,\xi}\, \dot{g}_t^*, \nabla^r_{g_t} H] \equiv 0$, for all $\xi \in T_X$ and $r = 0, \dots p-1$ then

$$2\, \dot{\nabla}^p_{g_t} H = -\sum_{r=1}^{p-1} \sum_{k=1}^{r} \sum_{\sigma \in S_{p-r+k-1}} C^{p,r}_{k,\sigma} \nabla^{p-r}_{g_t} \dot{g}_t^* \bullet^\sigma_k \nabla^r_{g_t} H, \tag{4.1}$$

with $C^{p,r}_{k,\sigma} = 0, 1$. We show now the following fundamental result.

Corollary 3. *Let $(g_t)_{t\in\mathbb{R}} \subset \mathcal{M}$ be a smooth family such that $\dot{g}_t \in \mathbb{F}^K_{g_t}$ for all $t \in \mathbb{R}$. If $g_0 \in \mathcal{S}^K_\Omega$ then $g_t \in \mathcal{S}^K_\Omega$ for all $t \in \mathbb{R}$.*

Proof. Thanks to Corollary 1 we just need to show the condition on the brackets. We will proceed by induction on the order of covariant differentiation p. For notations simplicity we set $\rho_t := \mathrm{Ric}_{g_t}(\Omega)$. We observe now that the assumption implies $\mathcal{R}_{g_t} = \mathcal{R}_{g_0}$ for all $t \in \mathbb{R}$, thanks to the variation formula (3.7). This combined with the variation formula (2.4) gives

$$\begin{aligned} 2\,\frac{d}{dt}\,[T, \rho_t^*] &= -\left[T, \Delta^\Omega_{g_t} \dot{g}_t^* + 2\, \dot{g}_t^* \rho_t^*\right] \\ &= -2\,[T, \dot{g}_t^* \rho_t^*] \\ &= -2\, \dot{g}_t^*\, [T, \rho_t^*]\,, \end{aligned}$$

thanks to the assumption $[T, \nabla^p_{g_t,\xi}\, \dot{g}_t^*] \equiv 0$ and (3.16). By Cauchy uniqueness we infer $[T, \rho_t^*] \equiv 0$. We assume now as inductive hypothesis $\left[T, \nabla^r_{g_t,\eta}\, \rho_t^*\right] \equiv 0$, for all $r = 0, \dots, p-1$, $p > 1$ and $\eta \in T_X^{\otimes r}$. We deduce thanks to this for $T = K$ and thanks to the assumption, the identity

$$\left[\nabla_{g_t}\, \dot{g}_t^*, \nabla^r_{g_t,\eta}\, \rho_t^*\right] = 0,$$

This combined with the variation formula (4.1) with $H = \rho_t^*$, combined with the inductive hypothesis and with (2.4) provides the identity

$$2\,\frac{d}{dt}\left[T, \nabla^p_{g_t,\xi}\, \rho_t^*\right] = -\left[T, \nabla^p_{g_t,\xi}\left(\Delta^\Omega_{g_t} \dot{g}_t^* + 2\, \dot{g}_t^* \rho_t^*\right)\right].$$

We can express the p-derivative of the Ω-Laplacian as

$$\begin{aligned} \nabla^p_{g_t} \Delta^\Omega_{g_t} \dot{g}_t^* = &-\mathrm{Tr}_{g_t,p+1} \nabla^{p+2}_{g_t} \dot{g}_t^* + \nabla_{g_t} f_t \neg_{p+1} \nabla^{p+1}_{g_t} \dot{g}_t^* \\ &+ \sum_{r=1}^{p} \sum_{\sigma \in S_{p+1}} C^{p,r}_\sigma\, \nabla^{p+2-r}_{g_t} f_t *^\sigma_r \nabla^r_{g_t} \dot{g}_t^*, \end{aligned} \tag{4.2}$$

with $C^{p,r}_\sigma = 0, 1$. Moreover the assumption combined with the expression of the p-derivative of the Ω-Laplacian and with the identity

$$\left[T, \xi \neg \left(\nabla^{p+2-r}_{g_t} f_t *^\sigma_r \nabla^r_{g_t}\, \dot{g}_t^*\right)\right] \equiv 0,$$

implies the variation formula

$$2\,\frac{d}{dt}\left[T,\nabla^p_{g_t,\xi}\,\rho^*_t\right]=-2\left[T,\nabla^p_{g_t,\xi}\,(\dot g^*_t\rho^*_t)\right]$$
$$=-2\sum_{r=0}^{p}\sum_{|I|=r}\left[T,\nabla^{p-r}_{g_0,\xi_{\complement I}}\dot g^*_t\,\nabla^r_{g_0,\xi_I}\rho^*_t\right].$$

Using the assumption $[T,\nabla^{p-r}_{g_t,\mu}\,\dot g^*_t]\equiv 0$, $\mu\in T_X^{\otimes p-r}$ and the inductive hypothesis we can apply the identity (3.17) to the products of type $\nabla^{p-r}_{g_0,\xi_{\complement I}}\dot g^*_t\,\nabla^r_{g_0,\xi_I}\rho^*_t$ in order to obtain the identity

$$2\,\frac{d}{dt}\left[T,\nabla^p_{g_t,\xi}\,\rho^*_t\right]=-2\,\dot g^*_t\left[T,\nabla^p_{g_t,\xi}\,\rho^*_t\right].$$

Then the conclusion follows by Cauchy uniqueness. □

5. Integrability of the distribution $\mathbb{F}^K$

We start first with a basic calculus fact.

Lemma 6. *Let $B>0$ be a g-symmetric endomorphism smooth section of T_X such that $[B,\nabla_{g,\xi}B]=0$. Then*

$$\nabla_{g,\xi}\log B=B^{-1}\nabla_{g,\xi}B.$$

Proof. We set $A:=\log B$ and we observe that by definition $[B,A]=0$, i.e., $\left[e^A,A\right]=0$. The assumption $\left[e^A,\nabla_{g,\xi}\,e^A\right]=0$ is equivalent to the condition $\left[A,\nabla_{g,\xi}\,e^A\right]=0$ since the endomorphisms $e^A,\nabla_{g,\xi}\,e^A$ can be diagonalized simultaneously. Thus deriving the identity $\left[e^A,A\right]=0$ we infer $\left[\nabla_{g,\xi}\,A,e^A\right]=0$, which is equivalent to $[\nabla_{g,\xi}\,A,A]=0$. But this last implies

$$\nabla_{g,\xi}\,e^A=\nabla_{g,\xi}\,A\,e^A=e^A\nabla_{g,\xi}\,A\,.$$

We infer $\nabla_{g,\xi}\,A=e^{-A}\nabla_{g,\xi}\,e^A$, i.e., the required conclusion. □

We show now the following key lemma.

Lemma 7. *For any $g_0\in\mathcal{M}$,*

$$\Sigma_K(g_0):=\mathbb{F}^K_{g_0}\cap\mathcal{M}=\exp_{G,g_0}\left(\mathbb{F}^K_{g_0}\right),\tag{5.1}$$

$$T_{\Sigma_K(g_0),g}=\mathbb{F}^K_g\,,\quad\forall g\in\Sigma_K(g_0).\tag{5.2}$$

Moreover $\Sigma_K(g_0)$ is a totally geodesic and flat subvariety inside the nonpositively curved Riemannian manifold $(\mathcal{M},G)$.

Proof. Step 1 (A). We observe first the inclusion $\Sigma_K(g_0)\supseteq\exp_{G,g_0}\left(\mathbb{F}^K_{g_0}\right)$. In fact let $(g_t)_{t\in\mathbb{R}}$ be a geodesic such that $\dot g_0\in\mathbb{F}^K_{g_0}$. Then using the expression (3.1) of the geodesics we obtain

$$\nabla_{T_X,g_0}(g_0^{-1}g_t)=\nabla_{T_X,g_0}e^{t\dot g^*_0}=0\,,\quad\forall t\in\mathbb{R},$$

since the last equality is equivalent to the condition $\dot{g}_0 \in \mathbb{F}^\infty_{g_0}$. Moreover the condition

$$\left[T, \nabla^p_{g_0,\xi}\, e^{t\dot{g}_0^*}\right] = 0\,, \quad \forall t \in \mathbb{R}\,,\ \forall p \in \mathbb{Z}_{\geqslant 0},$$

is equivalent to the identity

$$\left[T, \nabla^p_{g_0,\xi}(\dot{g}_0^*)^q\right] = 0,$$

for all $p, q \in \mathbb{Z}_{\geqslant 0}$ and this last follows from a repetitive use of the identity (3.17). We conclude the inclusion $\Sigma_K(g_0) \supseteq \exp_{G,g_0}\left(\mathbb{F}^K_{g_0}\right)$.

Step 1 (B1). We observe now that for any smooth curve $(u_t)_{t\in(-\varepsilon,\varepsilon)} \in \mathbb{F}^K_{g_0}$ the identity $[K, g_0^{-1}u_t] = 0$ implies $[K, g_0^{-1}\dot{u}_t] = 0$ and thus $[g_0^{-1}u_t, g_0^{-1}\dot{u}_t] = 0$. We infer that the differential

$$D_u \exp_{G,g_0} : \mathbb{F}^K_{g_0} \longrightarrow \mathbb{F}^K_{g_0},$$

of the exponential map $\exp_{G,g_0} : \mathbb{F}^K_{g_0} \longrightarrow \Sigma_K(g_0)$ at a point $u \in \mathbb{F}^K_{g_0}$ is given by the formula

$$D_u \exp_{G,g_0}(v) = v\, e^{g_0^{-1}u},$$

for all $v \in \mathbb{F}^K_{g_0}$. We deduce by Lemma 4 that this differential map is an isomorphism. This combined with the fact that the exponential map is injective implies that $\exp_{G,g_0}(\mathbb{F}^K_{g_0})$ is an open subset of $\Sigma_K(g_0)$. But $\exp_{G,g_0}(\mathbb{F}^K_{g_0})$ is also a closed set of $\mathcal{M}$ and thus a closed set of $\Sigma_K(g_0)$. The fact that this last is connected implies the required equality (5.1).

Step I (B2). We give now an explicit proof of the inclusion

$$\Sigma_K(g_0) \subseteq \exp_{G,g_0}\left(\mathbb{F}^K_{g_0}\right).$$

(This is useful also for other considerations.) Indeed we show that if $g \in \Sigma_K(g_0)$ then $g_0 \log(g_0^{-1}g) \in \mathbb{F}^K_{g_0}$. We observe first that the assumptions $[K, g_0^{-1}g] = 0$, and $[K, \nabla_{g_0}(g_0^{-1}g)] = 0$, imply

$$\left[g_0^{-1}g, \nabla_{g_0,\xi}(g_0^{-1}g)\right] = 0,$$

which allows to apply Lemma 6 in order to obtain the formula

$$\nabla_{g_0,\xi} \log(g_0^{-1}g) = g^{-1}g_0\nabla_{g_0,\xi}(g_0^{-1}g). \tag{5.3}$$

Thus the assumption $\nabla_{T_{X,g_0}}(g_0^{-1}g) = 0$ implies the identity

$$\nabla_{T_{X,g_0}} \log(g_0^{-1}g) = 0\,.$$

Let $\varepsilon > 0$ be sufficiently small such that $\varepsilon g < 2g_0$. Then the expansion

$$\log(\varepsilon\, g_0^{-1}g) = \sum_{p=1}^{+\infty} \frac{(-1)^{p+1}}{p} \left(\varepsilon\, g_0^{-1}g - \mathbb{I}\right)^p\,,$$

implies $[T, \log(\varepsilon g_0^{-1} g)] = 0$, and thus $[T, \log(g_0^{-1} g)] = 0$. Moreover the identity (3.16) implies

$$g_0^{-1} g \left[T, g^{-1} g_0\right] = [T, \mathbb{I}] = 0\,,$$

and thus $[T, g^{-1} g_0] = 0$ which combined with the formula

$$\nabla_{g_0,\xi}(g^{-1} g_0) = -g^{-1} g_0 \,\nabla_{g_0,\xi}(g_0^{-1} g)\, g^{-1} g_0,$$

and with the identity (3.17) implies the equality $[T, \nabla_{g_0}(g_0^{-1} g)] = 0$. We infer $[T, \nabla^p_{g_0,\xi}(g_0^{-1} g)] = 0$ by a simple induction based on a repetitive use of the identity (3.17). We conclude

$$\left[T, \nabla^p_{g_0,\xi} \log(g_0^{-1} g)\right] = 0,$$

by deriving the identity (5.3) and using (3.17).

Step II. We show now the identity (5.2), i.e., the identity $\mathbb{F}^K_{g_0} = \mathbb{F}^K_g$. We can consider, thanks to the equality (5.1), a geodesic $(g_t)_{t\in\mathbb{R}} \subset \Sigma_K(g_0)$ joining $g = g_1$ with g_0. We observe also that Lemma 5 combined with the variation formula (3.7) implies the identity $\mathcal{R}_{g_t} \equiv \mathcal{R}_{g_0} = \mathcal{R}_g$. Moreover $[K, \nabla_{g_t} \dot{g}_t^*] \equiv 0$, thanks to Lemma 5. Then the variation formula (3.19) implies

$$\nabla_g H = \nabla_{g_0} H\,, \quad \forall H \in C^\infty(X, \mathrm{End}(T_X)) \;:\; [K, H] = 0. \tag{5.4}$$

On the other hand using (3.16) we obtain the equalities

$$\left[T, g_0^{-1} v\right] = \left[T, (g_0^{-1} g)(g^{-1} v)\right] = (g_0^{-1} g)\left[T, g^{-1} v\right]\ .$$

Thus $[T, g_0^{-1} v] = 0$ iff $[T, g^{-1} v] = 0$. This last for $T = K$ implies

$$\nabla_g(g^{-1} v) = \nabla_{g_0}(g^{-1} v), \tag{5.5}$$

thanks to (5.4). We consider the identities

$$\begin{aligned}
\nabla_{T_{X,g_0}}(g_0^{-1} v) &= \nabla_{T_{X,g_0}}\left[(g_0^{-1} g)(g^{-1} v)\right] \\
&= g^{-1} v \,\nabla_{T_{X,g_0}}(g_0^{-1} g) + g_0^{-1} g \,\nabla_{T_{X,g_0}}(g^{-1} v) \\
&= g_0^{-1} g \,\nabla_{T_{X,g}}(g^{-1} v),
\end{aligned}$$

since $g \in \Sigma_K(g_0)$ and thanks to (5.5). We deduce $v \in \mathbb{F}_{g_0}$ iff $v \in \mathbb{F}_g$ provided that $[K, g^{-1} v] = 0$. We show now by induction on $p \geqslant 0$ the properties

$$\begin{aligned}
\left[T, \nabla^{(r)}_{g_0,\xi}(g_0^{-1} v)\right] \;&=\; 0 \\
\Longleftrightarrow\; \left[T, \nabla^{(r)}_{g,\xi}(g^{-1} v)\right] \;&=\; 0\,,\ \forall \xi \in C^\infty(X, T_X)^{\oplus r},
\end{aligned} \tag{5.6}$$

and

$$\nabla^{(r+1)}_{g,\xi}(g^{-1} v) = \nabla^{(r+1)}_{g_0,\xi}(g^{-1} v)\,, \quad \forall \xi \in C^\infty(X, T_X)^{\oplus (r+1)}\,, \tag{5.7}$$

for all $r = 0, \dots, p$. These properties hold true for $p = 0$ as we observed previously. The assumption $g \in \Sigma_K(g_0)$ combined with (3.16) implies the identity.

$$\left[T, \nabla^{(p)}_{g_0,\xi}(g_0^{-1}v)\right] = \sum_{r=0}^{p} \sum_{|I|=r} \nabla^{(p-r)}_{g_0,\xi_{\complement I}}(g_0^{-1}g) \left[T, \nabla^{(r)}_{g_0,\xi_I}(g^{-1}v)\right].$$

We assume that the step $p-1$ of the induction holds true. We infer the equality

$$\left[T, \nabla^{(p)}_{g_0,\xi}(g_0^{-1}v)\right] = g_0^{-1}g \left[T, \nabla^{(p)}_{g,\xi}(g^{-1}v)\right],$$

which implies (5.6) for $r = p$. Moreover the identity

$$\left[K, \nabla^{(p)}_{g,\xi}(g^{-1}v)\right] = 0,$$

implies the equalities

$$\nabla_g \nabla^{(p)}_{g,\xi}(g^{-1}v) = \nabla_{g_0} \nabla^{(p)}_{g,\xi}(g^{-1}v) = \nabla_{g_0} \nabla^{(p)}_{g_0,\xi}(g^{-1}v),$$

thanks to (5.4) and to the inductive assumption. We obtain (5.7) for $r = p$ and thus the conclusion of the induction.

Step III. We show now the last statement of the lemma. We observe indeed that the identities $\Sigma_K(g_0) = \mathcal{M} \cap \mathbb{F}_g^K = \exp_{G,g}\left(\mathbb{F}_g^K\right)$, hold thanks to the equalities (5.2) and (5.1). But this implies that the second fundamental form of $\Sigma_K(g_0)$ inside $(\mathcal{M}, G)$ vanishes identically. Thus using Gauss equation, the identity (5.2) and the expression of the curvature tensor

$$\mathcal{R}_{\mathcal{M}}(g)(u,v)w = -\frac{1}{4}\, g\Big[\left[u_g^*, v_g^*\right], w_g^*\Big],$$

we infer the equalities of the curvature forms $R_{\Sigma_K(g_0)}(g) = R_{\mathcal{M}}(g)_{|\mathbb{F}_g^K} \equiv 0$ for all $g \in \Sigma_K(g_0)$. This concludes the proof of Lemma 7. □

6. Reinterpretation of the space $\mathbb{F}_g^K$

In this section we conciliate the definition of the vector space $\mathbb{F}_g^K$ given in Section 2 with the definition so far used.

We consider indeed $K \in C^\infty(X, \mathrm{End}(T_X))$ with n-distinct real eigenvalues almost everywhere over X. If $A, B \in C^\infty(X, \mathrm{End}(T_X))$ commute with K then $[A, B] = 0$ over X. We observe that this is all we need in order to make work the previous arguments. Thus all the previous results hold true if we use such K. In this case hold an equivalent definition of the vector space $\mathbb{F}_g^K$. We show in fact the following lemma.

Lemma 8. *Let $K \in C^\infty(X, \mathrm{End}(T_X))$ with n-distinct real eigenvalues almost everywhere over X. Then hold the identity*

$$\mathbb{F}_g^K = \left\{ v \in \mathbb{F}_g \mid \left[\nabla_g^p T, v_g^*\right] = 0\,,\ T = \mathcal{R}_g\,,\ K\,,\ \forall p \in \mathbb{Z}_{\geqslant 0} \right\}. \tag{6.1}$$

Proof. It is sufficient to show by induction on $p \geqslant 1$ that

$$\forall r = 0, \dots, p\,; \ \left[\nabla_g^r T, v_g^*\right] = 0 \iff \left[T, \nabla_{g,\xi}^r v_g^*\right] = 0\,, \ \forall \xi \in T_X^{\otimes r}. \tag{6.2}$$

We assume true this statement for $p-1$ and we show it for p. The inductive hypothesis implies

$$\forall r = 0, \dots, p-1\,; \ \left[\nabla_g^{r-s} T, \nabla_{g,\xi}^s v_g^*\right] = 0\,, \ \forall \xi \in T_X^{\otimes s}, \ \forall s = 0, \dots, r. \tag{6.3}$$

We show the statement (6.3) by a finite increasing induction on s. The statement (6.3) holds true obviously for $s = 0, 1$. We assume (6.3) true for s and we show it for $s+1$. Indeed by the inductive assumption on s,

$$\left[\nabla_g^{r-1-s} T, \nabla_{g,\xi}^s v_g^*\right] = 0$$

for all $\xi \in C^\infty(X, T_X^{\otimes s})$. We take in particular ξ such that $\nabla_g \xi(x) = 0$, at some arbitrary point $x \in X$. Thus

$$\begin{aligned} 0 &= \nabla_{g,\eta} \left[\nabla_g^{r-1-s} T, \nabla_{g,\xi}^s v_g^*\right] \\ &= \left[\eta \neg \nabla_g^{r-s} T, \nabla_{g,\xi}^s v_g^*\right] + \left[\nabla_g^{r-1-s} T, \nabla_{g,\eta\otimes\xi}^{s+1} v_g^*\right] \\ &= \left[\nabla_g^{r-1-s} T, \nabla_{g,\eta\otimes\xi}^{s+1} v_g^*\right], \end{aligned}$$

thanks to the inductive assumption on s. This completes the proof of (6.3). The conclusion of the induction on p for (6.2) will follow from the statement; for all $s = 0, \dots, p-1$ we have

$$\begin{aligned} &\left[\nabla_g^{p-s} T, \nabla_{g,\xi}^s v_g^*\right] = 0\,, \ \forall \xi \in T_X^{\otimes s} \iff \\ &\qquad \left[\nabla_g^{p-s-1} T, \nabla_{g,\eta}^{s+1} v_g^*\right] = 0\,, \ \forall \eta \in T_X^{\otimes s+1}. \end{aligned} \tag{6.4}$$

By (6.3) for $r = p-1$ and $s = 0, \dots, p-1$,

$$\left[\nabla_g^{p-1-s} T, \nabla_{g,\xi}^s v_g^*\right] = 0$$

for all $\xi \in C^\infty(X, T_X^{\otimes s})$. We take as before ξ such that $\nabla_g \xi(x) = 0$, at some arbitrary point $x \in X$. Thus

$$\begin{aligned} 0 &= \nabla_{g,\eta} \left[\nabla_g^{p-1-s} T, \nabla_{g,\xi}^s v_g^*\right] \\ &= \left[\eta \neg \nabla_g^{p-s} T, \nabla_{g,\xi}^s v_g^*\right] + \left[\nabla_g^{p-1-s} T, \nabla_{g,\eta\otimes\xi}^{s+1} v_g^*\right], \end{aligned}$$

which shows (6.4) and thus the conclusion of the induction on p for (6.2). □

7. Representation of the Ω-SRF as the gradient flow of the functional $\mathcal{W}_\Omega$ over $\Sigma_K(g_0)$

The properties of the subvariety $\Sigma_K(g_0)$ are elucidated by the following proposition.

Proposition 1. *For any $g_0 \in \mathcal{S}_\Omega^K$ and any $g \in \Sigma_K(g_0)$ we have the identities*

$$\nabla_G \mathcal{W}_\Omega\,(g) = g - \mathrm{Ric}_g(\Omega) \;\in\; T_{\Sigma_K(g_0),g},$$
$$\nabla_G^{\Sigma_K(g_0)} D\,\mathcal{W}_\Omega\,(g)\,(v,v) = \int_X \left[\left\langle v\,\mathrm{Ric}^*_g(\Omega), v\right\rangle_g + \frac{1}{2}\,|\nabla_g\, v|^2_g\right]\Omega,$$

for all $v \in T_{\Sigma_K(g_0),g}$. Moreover for all $g_0 \in \mathcal{S}_\Omega^K$ the Ω-SRF $(g_t)_{t\in[0,T)} \subset \Sigma_K(g_0)$ with initial data g_0 represents the formal gradient flow of the functional $\mathcal{W}_\Omega$ over the totally geodesic and flat subvariety $\Sigma_K(g_0)$ inside the nonpositively curved Riemannian manifold $(\mathcal{M}, G)$.

Proof. By the identity (5.1) in Lemma 7 there exist a geodesic $(g_t)_{t\in\mathbb{R}} \subset \Sigma_K(g_0)$, $\dot g_0 \in \mathbb{F}^K_{g_0}$, joining $g = g_1$ with g_0. Then Lemma 5 combined with Corollary 3 and with the identity (5.2) in Lemma 7 implies $\mathrm{Ric}_g(\Omega) \in T_{\Sigma_K(g_0),g}$.

Moreover $g \in T_{\Sigma_K(g_0),g}$ thanks to the identity (5.2). We conclude the fundamental property $\nabla_G \mathcal{W}_\Omega(g) \in T_{\Sigma_K(g_0),g}$ for all $g \in \Sigma_K(g_0)$.
The second variation formula in the statement follows directly from Corollary 1 in [Pal] and from the fact that $\Sigma_K(g_0)$ is a totally geodesic subvariety inside the Riemannian manifold $(\mathcal{M}, G)$. We observe however that it follows also from the tangency property $\nabla_G \mathcal{W}_\Omega(g) \in T_{\Sigma_K(g_0),g}$. Indeed we recall that for any subvariety $\Sigma \subset \mathcal{M}$ of a general Riemannian manifold $(\mathcal{M}, G)$ and for any $f \in C^2(\mathcal{M}, \mathbb{R})$ we have

$$\nabla_G^{\mathcal{M}}\, d\, f\,(\xi,\eta) = \nabla_G^{\Sigma}\, d\, f\,(\xi,\eta) - G\left(\nabla_G^{\mathcal{M}} f, \mathrm{II}^\Sigma_G(\xi,\eta)\right), \forall \xi,\eta \in T_\Sigma,$$

where $\mathrm{II}^\Sigma_G \in C^\infty\left(\Sigma, S^2 T^*_\Sigma \otimes N_{\Sigma/\mathcal{M},G}\right)$ denotes the second fundamental form of Σ inside $(\mathcal{M}, G)$. □

The result so far obtained does not allow us to see yet the Ω-SRF as the gradient flow of a convex functional inside a flat metric space. In order to see the required convexity picture we need to make a key change of variables that we explain in the next sections.

8. Explicit representations of the Ω-SRF equation

In this section we show the following fundamental expression of the Ω-BER-tensor over the variety $\Sigma_K(g_0)$.

Lemma 9. *Let $g_0 \in \mathcal{M}$. Then for any metric $g \in \Sigma_K(g_0)$,*

$$\mathrm{Ric}_g(\Omega) = \mathrm{Ric}_{g_0}(\Omega) - \frac{1}{2}\,\Delta^\Omega_{g_0}\left[g_0 \log(g_0^{-1} g)\right]$$
$$- \frac{1}{4}\, g_0\, \mathrm{Tr}_{g_0}\left[\nabla_{g_0,\bullet} \log(g_0^{-1} g) \nabla_{g_0,\bullet} \log(g_0^{-1} g)\right].$$

Proof. The fact that $\mathcal{R}_g = \mathcal{R}_{g_0}$ for all $g \in \Sigma_K(g_0)$ implies also $\operatorname{Ric}_g = \operatorname{Ric}_{g_0}$. Moreover we observe the elementary identities

$$\log \frac{dV_g}{\Omega} = \log \frac{dV_g}{dV_{g_0}} + \log \frac{dV_{g_0}}{\Omega},$$

$$\frac{dV_g}{dV_{g_0}} = \left[\frac{\det g}{\det g_0}\right]^{1/2} = \left[\det(g_0^{-1} g)\right]^{1/2},$$

$$\begin{aligned} \log \frac{dV_g}{\Omega} &= \frac{1}{2}\, \log \det(g_0^{-1} g) + \log \frac{dV_{g_0}}{\Omega} \\ &= \frac{1}{2} \operatorname{Tr}_{\mathbb{R}} \log(g_0^{-1} g) + \log \frac{dV_{g_0}}{\Omega}. \end{aligned}$$

We set $f_0 := \log \frac{dV_{g_0}}{\Omega}$ and $\nabla_g = \nabla_{g_0} + \Gamma_g^{T_X}$. We infer the equality

$$\begin{aligned} \operatorname{Ric}_g(\Omega) &= \operatorname{Ric}_{g_0}(\Omega) + \Gamma_g^{T_X^*} d\, f_0 \\ &\quad + \frac{1}{2}\, \nabla_{g_0} d \operatorname{Tr}_{\mathbb{R}} \log(g_0^{-1} g) + \frac{1}{2}\, \Gamma_g^{T_X^*} d \operatorname{Tr}_{\mathbb{R}} \log(g_0^{-1} g). \end{aligned}$$

We observe now that $\Gamma_{g,\xi}^{T_X^*} = -\left(\Gamma_{g,\xi}^{T_X}\right)^*$, with

$$\begin{aligned} 2\, \Gamma_{g,\xi}^{T_X}\, \eta &= g^{-1}\Big[\nabla_{g_0} g\,(\xi,\eta,\cdot) + \nabla_{g_0} g\,(\eta,\xi,\cdot) - \nabla_{g_0} g\,(\cdot,\xi,\eta)\Big] \\ &= \left(g^{-1} \nabla_{g_0,\xi}\, g\right) \eta, \end{aligned}$$

since $\nabla_{T_{X,g_0}}(g_0^{-1} g) = 0$. Using Lemma 6 we deduce the identity

$$\begin{aligned} 2\, \Gamma_{g,\xi}^{T_X} &= (g_0^{-1} g)^{-1} \nabla_{g_0,\xi}(g_0^{-1} g) \\ &= \nabla_{g_0,\xi} \log(g_0^{-1} g), \end{aligned}$$

since $\left[g_0^{-1} g, \nabla_{g_0,\xi}(g_0^{-1} g)\right] = 0$. Indeed we observe that this last equality follows from the fact that $[K, g_0^{-1} g] = 0$ and $[K, \nabla_{g_0}(g_0^{-1} g)] = 0$. Thus for any function u hold the identity

$$2\left(\Gamma_{g,\xi}^{T_X^*}\, d\, u\right) \eta = -g_0\left(\nabla_{g_0} u, \nabla_{g_0,\xi} \log(g_0^{-1} g)\, \eta\right).$$

Using the fact that the endomorphism $\nabla_{g_0,\xi} \log(g_0^{-1} g)$ is g_0-symmetric (since $\log(g_0^{-1} g)$ is also g_0-symmetric) we deduce

$$2\left(\Gamma_{g,\xi}^{T_X^*}\, d\, u\right) \eta = -g_0\left(\nabla_{g_0,\xi} \log(g_0^{-1} g) \nabla_{g_0} u, \eta\right). \tag{8.1}$$

We observe that $g \in \Sigma_K(g_0)$ if and only if $g_0 \log(g_0^{-1} g) \in \mathbb{F}_{g_0}^K$ by the identity (5.1). In particular

$$\nabla_{T_{X,g_0}} \log(g_0^{-1} g) = 0. \tag{8.2}$$

We deduce the equalities

$$2\left(\Gamma^{T^*_X}_{g,\xi}\, d f_0\right)\eta = -g_0\left(\nabla_{g_0,\nabla_{g_0} f_0}\log(g_0^{-1}g)\,\xi,\eta\right)$$
$$= -\nabla_{g_0,\nabla_{g_0} f_0}\left[g_0\log(g_0^{-1}g)\right](\xi,\eta).$$

Let $(e_k)_k$ be a local frame of T_X in a neighborhood of an arbitrary point $x \in X$ which is g-orthonormal at x and such that $\nabla_{g_0} e_k(x) = 0$. Deriving the identity

$$\mathrm{Tr}_{\mathbb{R}}\log(g_0^{-1}g) = g_0\big(\log(g_0^{-1}g)\, e_k, g_0^{-1}e_k^*\big),$$

we infer at the point x

$$\begin{aligned}\xi.\,\mathrm{Tr}_{\mathbb{R}}\log(g_0^{-1}g) &= g_0\left(\nabla_{g_0,\xi}\log(g_0^{-1}g)\, e_k, e_k\right)\\ &= g_0\left(\nabla_{g_0,e_k}\log(g_0^{-1}g)\,\xi, e_k\right)\\ &= g_0\left(\xi, \nabla_{g_0,e_k}\log(g_0^{-1}g)\, e_k\right),\end{aligned}$$

thanks to the identity (8.2) and thanks to the fact that the endomorphism $\nabla_{g_0,\xi}\log(g_0^{-1}g)$ is g_0-symmetric. We infer the formula

$$d\,\mathrm{Tr}_{\mathbb{R}}\log(g_0^{-1}g) = -g_0\nabla^*_{g_0}\log(g_0^{-1}g).$$

Thus using (8.2) we infer the equalities

$$\begin{aligned}\nabla_{g_0} d\,\mathrm{Tr}_{\mathbb{R}}\log(g_0^{-1}g) &= -g_0\nabla_{g_0}\nabla^*_{g_0}\log(g_0^{-1}g)\\ &= -g_0\,\Delta_{T_X,g_0}\log(g_0^{-1}g)\\ &= -g_0\,\Delta_{g_0}\log(g_0^{-1}g)\\ &= -\Delta_{g_0}\left[g_0\log(g_0^{-1}g)\right],\end{aligned}$$

by the Weitzenböck formula in Lemma 24 in the appendix and by the identity $[\mathcal{R}_g, \log(g_0^{-1}g)] = 0$. (We recall that $g_0\log(g_0^{-1}g) \in \mathbb{F}^K_{g_0}$.) We apply now the identity (8.1) to the function $u := \mathrm{Tr}_{\mathbb{R}}\log(g_0^{-1}g)$. We infer by the formula (8.3) the equality

$$\nabla_{g_0} u = \nabla_{g_0,e_k}\log(g_0^{-1}g)e_k.$$

Thus we obtain the equality

$$2\left(\Gamma^{T^*_X}_{g,\xi}\, d\,\mathrm{Tr}_{\mathbb{R}}\log(g_0^{-1}g)\right)\eta = -g_0\left(\nabla_{g_0,\xi}\log(g_0^{-1}g)\nabla_{g_0,e_k}\log(g_0^{-1}g)\, e_k,\eta\right).$$

Using the identity $[K, \nabla_{g_0}\log(g_0^{-1}g)] = 0$, we deduce the expression at the point x

$$\begin{aligned}2\left(\Gamma^{T^*_X}_{g,\xi}\, d\,\mathrm{Tr}_{\mathbb{R}}\log(g_0^{-1}g)\right)\eta &= -g_0\left(\nabla_{g_0,e_k}\log(g_0^{-1}g)\nabla_{g_0,\xi}\log(g_0^{-1}g)\, e_k,\eta\right)\\ &= -g_0\left(\nabla_{g_0,e_k}\log(g_0^{-1}g)\nabla_{g_0,e_k}\log(g_0^{-1}g)\,\xi,\eta\right),\end{aligned}$$

thanks to (8.2). Combining the expressions obtained so far we infer the required formula. □

We recall that by Proposition 1, if $g_0 \in \mathcal{S}_{\Omega}^K$ then

$$\operatorname{Ric}_g(\Omega) \;\in\; \mathbb{F}_{g_0}^K, \quad \forall g \;\in\; \Sigma_K(g_0). \tag{8.3}$$

We give an other proof (not necessarily shorter but more explicit) of this fundamental fact based on the expression of $\operatorname{Ric}_g(\Omega)$ in Lemma 9.

Proof. We recall that $g \in \Sigma_K(g_0)$ if and only if $g_0 \log(g_0^{-1} g) \in \mathbb{F}_{g_0}^K$ by the identity (5.1). We set for notation simplicity

$$U := \log(g_0^{-1} g) \;\in\; g_0^{-1} \mathbb{F}_{g_0}^K,$$

and we show first the equality

$$\nabla_{T_X, g_0} g_0^{-1} \operatorname{Ric}_g(\Omega) = 0\,. \tag{8.4}$$

The assumption $g_0 \in \mathcal{S}_{\Omega}^K$ combined with the expression of $\operatorname{Ric}_g(\Omega)$ in Lemma 9 implies the identity

$$\nabla_{T_X, g_0} g_0^{-1} \operatorname{Ric}_g(\Omega) = -\frac{1}{2} \nabla_{T_X, g_0} \Delta_{g_0}^{\Omega} U.$$

Now let $(x_1, ..., x_n)$ be g_0-geodesic coordinates centered at an arbitrary point $p \in X$ and set $e_k := \frac{\partial}{\partial x_k}$. The local tangent frame $(e_k)_k$ is $g_t(p)$-orthonormal at the point p and satisfies $\nabla_{g_0} e_j(p) = 0$ for all j.

We take now two vector fields ξ and η with constant coefficients with respect to the g_0-geodesic coordinates $(x_1, ..., x_n)$. Therefore

$$\nabla_{g_0} \xi(p) = \nabla_{g_0} \eta(p) = 0.$$

Commuting derivatives via (3.6) we infer the identities at the point p

$$\begin{aligned}
\nabla_{g_0,\xi} \operatorname{Tr}_{g_0} \left(\nabla_{g_0,\bullet} U \, \nabla_{g_0,\bullet} U\right) \eta &= \nabla_{g_0,\xi} \left(\nabla_{g_0,e_k} U \, \nabla_{g_0, g_0^{-1} e_k^*} U\right) \eta \\
&= \nabla_{g_0,e_k} \nabla_{g_0,\xi} U \, \nabla_{g_0,e_k} U \eta \\
&\quad + \nabla_{g_0,e_k} U \, \nabla_{g_0,e_k} \nabla_{g_0,\xi} U \eta \\
&= 2 \, \nabla_{g_0,e_k} U \, \nabla_{g_0,e_k} \left[\nabla_{g_0,\xi} U \eta\right],
\end{aligned}$$

since $[\mathcal{R}_{g_0}, U] = 0$, $[\xi, e_k] \equiv 0$, $[\xi, g_0^{-1} e_k^*] = 0$ at the point p and because $[\nabla_{g_0,e_k} \nabla_{g_0,\xi} U, \nabla_{g_0,e_k} U] = 0$. We infer the required identity

$$\nabla_{T_X, g_0} \operatorname{Tr}_{g_0} \left(\nabla_{g_0,\bullet} U \, \nabla_{g_0,\bullet} U\right) = 0.$$

We expand now at the point p the term

$$\begin{aligned}
&\nabla_{T_X, g_0} \Delta_{g_0} U(\xi, \eta) \\
&= -\nabla_{g_0,\xi} \left[\nabla_{g_0,e_k} \nabla_{g_0} U(e_k, \eta)\right] + \nabla_{g_0,\eta} \left[\nabla_{g_0,e_k} \nabla_{g_0} U(e_k, \xi)\right] \\
&= -\nabla_{g_0,\xi} \nabla_{g_0,e_k} \nabla_{g_0,e_k} U \eta + \nabla_{g_0} U(\nabla_{g_0,\xi} \nabla_{g_0,e_k} e_k, \eta) \\
&\quad + \nabla_{g_0,\eta} \nabla_{g_0,e_k} \nabla_{g_0,e_k} U \xi - \nabla_{g_0} U(\nabla_{g_0,\eta} \nabla_{g_0,e_k} e_k, \xi) \\
&= -\nabla_{g_0,e_k} \nabla_{g_0,\xi} \nabla_{g_0,e_k} U \eta + \nabla_{g_0} U(\nabla_{g_0,\xi} \nabla_{g_0,e_k} e_k, \eta) \\
&\quad + \nabla_{g_0,e_k} \nabla_{g_0,\eta} \nabla_{g_0,e_k} U \xi - \nabla_{g_0} U(\nabla_{g_0,\eta} \nabla_{g_0,e_k} e_k, \xi),
\end{aligned}$$

since

$$\begin{aligned}\nabla_{g_0,e_k}\nabla_{g_0}U(e_k,\eta) &= \nabla_{g_0,e_k}\left[\nabla_{g_0,e_k}U\eta\right]\\ &\quad - \nabla_{g_0}U(\nabla_{g_0,e_k}e_k,\eta) - \nabla_{g_0}U(e_k,\nabla_{g_0,e_k}\eta)\\ &= \nabla_{g_0,e_k}\nabla_{g_0,e_k}U\eta - \nabla_{g_0}U(\nabla_{g_0,e_k}e_k,\eta),\end{aligned}$$

and $[\xi,e_k]=[\eta,e_k]\equiv 0$ in a neighborhood of p and since $[\mathcal{R}_{g_0},\nabla_{g_0,e_k}U]\equiv 0$. (We use here the identity (3.6).) Moreover using the fact that $[\mathcal{R}_{g_0},U]=0$ we infer the identity at the point p

$$\begin{aligned}\nabla_{T_X,g_0}\Delta_{g_0}U(\xi,\eta) &= -\nabla_{g_0,e_k}\nabla_{g_0,e_k}\nabla_{g_0,\xi}U\eta + \nabla_{g_0}U(\nabla_{g_0,\xi}\nabla_{g_0,e_k}e_k,\eta)\\ &\quad + \nabla_{g_0,e_k}\nabla_{g_0,e_k}\nabla_{g_0,\eta}U\xi - \nabla_{g_0}U(\nabla_{g_0,\eta}\nabla_{g_0,e_k}e_k,\xi).\end{aligned}$$

We expand now at the point p the term

$$0=\Delta_{g_0}\nabla_{T_X,g_0}U(\xi,\eta) = -\nabla_{g_0,e_k}[\nabla_{g_0,e_k}\nabla_{T_X,g_0}U(\xi,\eta)].$$

Thus we expand first the term

$$\begin{aligned}\nabla_{g_0,e_k}\nabla_{T_X,g_0}U(\xi,\eta) &= \nabla_{g_0,e_k}\left[\nabla_{T_X,g_0}U(\xi,\eta)\right]\\ &\quad - \nabla_{T_X,g_0}U(\nabla_{g_0,e_k}\xi,\eta) - \nabla_{T_X,g_0}U(\xi,\nabla_{g_0,e_k}\eta)\\ &= \nabla_{g_0,e_k}\left[\nabla_{g_0,\xi}U\eta - \nabla_{g_0,\eta}U\xi\right]\\ &\quad - \nabla_{g_0}U(\nabla_{g_0,e_k}\xi,\eta) + \nabla_{g_0}U(\eta,\nabla_{g_0,e_k}\xi)\\ &\quad - \nabla_{g_0}U(\xi,\nabla_{g_0,e_k}\eta) + \nabla_{g_0}U(\nabla_{g_0,e_k}\eta,\xi)\\ &= \nabla_{g_0,e_k}\nabla_{g_0,\xi}U\eta - \nabla_{g_0,e_k}\nabla_{g_0,\eta}U\xi\\ &\quad - \nabla_{g_0}U(\nabla_{g_0,e_k}\xi,\eta) + \nabla_{g_0}U(\nabla_{g_0,e_k}\eta,\xi),\end{aligned}$$

in a neighborhood of p. We deduce the identity at the point p

$$\begin{aligned}0=\Delta_{g_0}\nabla_{T_X,g_0}U(\xi,\eta) &= -\nabla_{g_0,e_k}\nabla_{g_0,e_k}\nabla_{g_0,\xi}U\eta\\ &\quad + \nabla_{g_0,e_k}\nabla_{g_0,e_k}\nabla_{g_0,\eta}U\xi\\ &\quad + \nabla_{g_0}U(\nabla_{g_0,e_k}\nabla_{g_0,\xi}e_k,\eta)\\ &\quad - \nabla_{g_0}U(\nabla_{g_0,e_k}\nabla_{g_0,\eta}e_k,\xi),\end{aligned}$$

since $[\xi,e_k]=[\eta,e_k]\equiv 0$. Thus we obtain the identity

$$\begin{aligned}\nabla_{T_X,g_0}\Delta_{g_0}U(\xi,\eta) &= \nabla_{g_0}U(\mathcal{R}_{g_0}(\xi,e_k)e_k,\eta) - \nabla_{g_0}U(\mathcal{R}_{g_0}(\eta,e_k)e_k,\xi)\\ &= \nabla_{g_0,\eta}U\,\mathrm{Ric}^*_{g_0}\xi - \nabla_{g_0,\xi}U\,\mathrm{Ric}^*_{g_0}\eta,\end{aligned}$$

since $g_0 U \in \mathbb{F}_{g_0}$. We expand now at the point p the term

$$\begin{aligned}\nabla_{T_X, g_0}\big[\nabla_{g_0} f_0 \neg \nabla_{g_0} U\big](\xi,\eta) &= \nabla_{T_X, g_0}\big[\nabla_{g_0} U\, \nabla_{g_0} f_0\big](\xi,\eta)\\ &= \nabla_{g_0,\xi}\big[\nabla_{g_0,\eta} U\, \nabla_{g_0} f_0\big]\\ &\quad - \nabla_{g_0,\eta}\big[\nabla_{g_0,\xi} U\, \nabla_{g_0} f_0\big]\\ &= \nabla_{g_0,\xi}\nabla_{g_0,\eta} U\, \nabla_{g_0} f_0 + \nabla_{g_0,\eta} U\, \nabla^2_{g_0} f_0\, \xi\\ &\quad - \nabla_{g_0,\eta}\nabla_{g_0,\xi} U\, \nabla_{g_0} f_0 - \nabla_{g_0,\xi} U\, \nabla^2_{g_0} f_0\, \eta\\ &= \nabla_{g_0,\eta} U\, \nabla^2_{g_0} f_0\, \xi - \nabla_{g_0,\xi} U\, \nabla^2_{g_0} f_0\, \eta,\end{aligned}$$

since $[\mathcal{R}_{g_0}, U] = 0$ and $[\xi,\eta] \equiv 0$. We deduce the identity

$$\nabla_{T_X, g_0} \Delta^{\Omega}_{g_0} U = -\operatorname{Alt}\big[\nabla_{g_0} U \operatorname{Ric}^*_{g_0}(\Omega)\big].$$

But $[\nabla_{g_0} U, \operatorname{Ric}^*_{g_0}(\Omega)] = 0$ since $g_0 \in \mathcal{S}^K_{\Omega}$. We infer

$$\nabla_{T_X, g_0} \Delta^{\Omega}_{g_0} U = -\operatorname{Ric}^*_{g_0}(\Omega)\, \nabla_{T_X, g_0} U = 0\,.$$

Moreover

$$\big[T, \nabla^p_{g_0,\xi}\, \Delta^{\Omega}_{g_0} U\big] = 0,$$

thanks to the identity (4.2) applied to U. We observe now the decomposition

$$\nabla^p_{g_0,\xi} \operatorname{Tr}_{g_0}\left(\nabla_{g_0,\bullet} U\, \nabla_{g_0,\bullet} U\right) = \sum_{r=0}^{p} \sum_{|I|=r} \nabla^{r+1}_{g_0, e_k, \xi_I} U\, \nabla^{p-r+1}_{g_0, g_0^{-1} e_k^*, \xi_{\complement I}} U.$$

Then the conclusion follows from the identity (3.17) with $C = T$, $A = \nabla^{r+1}_{g_0, e_k, \xi_I} U$ and $B = \nabla^{p-r+1}_{g_0, g_0^{-1} e_k^*, \xi_{\complement I}} U$. □

In the following lemma we introduce a fundamental change of variables.

Lemma 10. *Let $g_0 \in \mathcal{M}$ and set $H \equiv H_g := (g^{-1} g_0)^{1/2}$ for any metric $g \in \Sigma_K(g_0)$. Then*

$$\operatorname{Ric}^*_g(\Omega) = H \Delta^{\Omega}_{g_0} H + H^2 \operatorname{Ric}^*_{g_0}(\Omega).$$

Proof. By Lemma 9 we obtain the formula

$$\begin{aligned}\operatorname{Ric}^*_g(\Omega) &= g^{-1} g_0 \operatorname{Ric}^*_{g_0}(\Omega) - \frac{1}{2}\, g^{-1} g_0\, \Delta^{\Omega}_{g_0} \log(g_0^{-1} g)\\ &\quad - \frac{1}{4}\, g^{-1} g_0 \operatorname{Tr}_{g_0}\left[\nabla_{g_0,\bullet} \log(g_0^{-1} g) \nabla_{g_0,\bullet} \log(g_0^{-1} g)\right].\end{aligned}$$

We set

$$A := \log H = -\frac{1}{2}\, \log(g_0^{-1} g) \in g_0^{-1} \mathbb{F}^K_{g_0}.$$

and we observe that $H = e^A \in g_0^{-1} \Sigma_K(g_0)$ thanks to Lemma 4. With this notations the previous formula rewrites as

$$\operatorname{Ric}^*_g(\Omega) = e^{2A}\left[\Delta^{\Omega}_{g_0} A - \operatorname{Tr}_{g_0}\left(\nabla_{g_0,\bullet} A \nabla_{g_0,\bullet} A\right) + \operatorname{Ric}^*_{g_0}(\Omega)\right]\,. \tag{8.5}$$

We expand now, at an arbitrary center of geodesic coordinates, the term

$$\begin{aligned}\Delta^{\Omega}_{g_0} e^A &= -\nabla_{g_0,e_k}\nabla_{g_0,e_k} e^A + \nabla_{g_0} f_0 \neg \nabla_{g_0} e^A\\ &= -\nabla_{g_0,e_k}\left(e^A \nabla_{g_0,e_k} A\right) + e^A \left(\nabla_{g_0} f_0 \neg \nabla_{g_0} A\right)\\ &= e^A \Delta^{\Omega}_{g_0} A - e^A \operatorname{Tr}_{g_0}\left(\nabla_{g_0,\bullet} A\, \nabla_{g_0,\bullet} A\right).\end{aligned}$$

We infer the expression

$$\operatorname{Ric}^*_g(\Omega) = e^A \Delta^{\Omega}_{g_0} e^A + e^{2A} \operatorname{Ric}^*_{g_0}(\Omega), \tag{8.6}$$

i.e., we have the required conclusion. □

We deduce the following corollary.

Corollary 4. *The* Ω*-SRF* $(g_t)_t \subset \Sigma_K(g_0)$ *is equivalent to the porous medium type equation*

$$2\,\dot{H}_t = -H_t^2 \Delta^{\Omega}_{g_0} H_t - H_t^3 \operatorname{Ric}^*_{g_0}(\Omega) + H_t, \tag{8.7}$$

with initial data $H_0 = \mathbb{I}$*, via the identification* $H_t = (g_t^{-1} g_0)^{1/2} \in g_0^{-1}\Sigma_K(g_0)$*.*

Proof. Let $U_t := \log(g_0^{-1} g_t) \in g_0^{-1}\mathbb{F}^K_{g_0}$ and observe that the Ω-SRF equation $(g_t)_t \subset \Sigma_K(g_0)$ is equivalent to the evolution equation

$$\dot{U}_t = \dot{g}^*_t = \operatorname{Ric}^*_{g_t}(\Omega) - \mathbb{I}.$$

Then the conclusion follows combining the identity $2\dot{H}_t = -H_t \dot{U}_t$ with the previous lemma. □

Short time existence and uniqueness of the solutions of the porous-medium type equation (8.7) with initial data $H_0 = \mathbb{I}$, follows from standard parabolic theory. We observe also that the change of variables

$$g \;\in\; \Sigma_K(g_0) \longmapsto H = (g^{-1} g_0)^{1/2} \in g_0^{-1}\Sigma_K(g_0),$$

is the one which linearizes as much as possible the expression of the SRF equation. Indeed we can rewrite it as the porous medium equation (8.7). However we will see in the next section that the change of variables $g \mapsto A = \log H$ would fit us in a gradient flow picture of a convex functional over convex sets in a Hilbert space. So from now on we will consider the change of variables

$$g \in \Sigma_K(g_0) \longmapsto A \;=\; \log H \;=\; -\,\frac{1}{2}\log(g_0^{-1} g) \in \mathbb{T}^K_{g_0} := g_0^{-1}\mathbb{F}^K_{g_0}\,. \tag{8.8}$$

We define the Ω-divergence operator of a tensor α as

$$\operatorname{div}^{\Omega}_g \alpha := e^f \operatorname{div}_g\left(e^{-f}\alpha\right) = \operatorname{div}_g \alpha - \nabla_g f \neg \alpha,$$

with $f := \log \frac{dV_g}{\Omega}$. We observe that with this notation formula (8.6) rewrites as

$$\operatorname{Ric}^*_{g_A}(\Omega) = -e^A \operatorname{div}^{\Omega}_{g_0}\left(e^A \nabla_{g_0} A\right) + e^{2A} \operatorname{Ric}^*_{g_0}(\Omega), \tag{8.9}$$

with $g_A := g_0 e^{-2A}$, for all $A \in \mathbb{T}^K_{g_0}$. With this notation we obtain the following analogue of Corollary 4.

Corollary 5. *The Ω-SRF $(g_t)_t \subset \Sigma_K(g_0)$ is equivalent to the solution $(A_t)_{t\geqslant 0} \subset \mathbb{T}^K_{g_0}$ of the forward evolution equation*

$$2\,\dot{A}_t = e^{A_t}\operatorname{div}^{\Omega}_{g_0}\left(e^{A_t}\nabla_{g_0}A_t\right) - e^{2A_t}\operatorname{Ric}^*_{g_0}(\Omega) + \mathbb{I}, \tag{8.10}$$

with initial data $A_0 = 0$, via the identification (8.8).

Proof. We observe first the identity $\dot{g}_t = -2\,g_t\,\dot{A}_t$. Then the conclusion follows combining the identity $-2\,\dot{A}_t = \dot{g}^*_t = \operatorname{Ric}^*_{g_t}(\Omega) - \mathbb{I}$, with formula (8.9). □

We observe that the assumption $g_0 \in \mathcal{S}^K_{\Omega}$ implies

$$\operatorname{Ric}^*_{g_A}(\Omega) = e^{2A}\,g_0^{-1}\operatorname{Ric}_{g_A}(\Omega) \in \mathbb{T}^K_{g_0}, \quad \forall A \in \mathbb{T}^K_{g_0},$$

thanks to the fundamental identity (8.3) combined with Lemma 4. We deduce

$$e^{A}\operatorname{div}^{\Omega}_{g_0}\left(e^{A}\nabla_{g_0}A\right) - e^{2A}\operatorname{Ric}^*_{g_0}(\Omega) \in \mathbb{T}^K_{g_0}, \quad \forall A \in \mathbb{T}^K_{g_0}, \tag{8.11}$$

thanks to the expression (8.9).

9. Convexity of $\mathcal{W}_\Omega$ over convex subsets inside $(\Sigma_K(g_0), G)$

We define the functional $\mathbf{W}_\Omega$ over $\mathbb{T}^K_{g_0}$ by the formula $\mathbf{W}_\Omega(A) := \mathcal{W}_\Omega(g_A)$, via the identification (8.8). We recall now the identity

$$\begin{aligned}\mathcal{W}_\Omega(g) &= \int_X \left[\operatorname{Tr}_g\left(\operatorname{Ric}_g(\Omega) - g\right) + 2\log\frac{dV_g}{\Omega}\right]\Omega\\ &= \int_X \operatorname{Tr}_{\mathbb{R}}\left[\operatorname{Ric}^*_g(\Omega) + \log(g_0^{-1}g)\right]\Omega + \int_X\left[2\log\frac{dV_{g_0}}{\Omega} - n\right]\Omega.\end{aligned}$$

Plugging in (8.6) and integrating by parts we infer the expression

$$\begin{aligned}\mathbf{W}_\Omega(A) = &\int_X \left[\left|\nabla_{g_0}e^{A}\right|^2_{g_0} + \operatorname{Tr}_{\mathbb{R}}\left(e^{2A}\operatorname{Ric}^*_{g_0}(\Omega) - 2\,A\right)\right]\Omega\\ &+ \int_X\left[2\log\frac{dV_{g_0}}{\Omega} - n\right]\Omega.\end{aligned}$$

We define now the vector space $\overline{\mathbb{T}}^K_{g_0}$ as the L^2-closure of $\mathbb{T}^K_{g_0}$ and we equip it with the constant L^2-product $4\int_X \langle\cdot,\cdot\rangle_{g_0}\,\Omega$. From now on all L^2-products are defined by this formula.

Lemma 11. *Let $g_0 \in \mathcal{S}^K_{\Omega}$. Then the forward equation* (8.10) *with initial data $A_0 = 0$ is equivalent to a smooth solution of the gradient flow equation $\dot{A}_t = -\nabla_{L^2}\mathbf{W}_\Omega(A_t)$.*

Proof. We compute first the L^2-gradient of the functional $\mathbf{W}_\Omega$. For this purpose we consider a line $t \mapsto A_t := A + t\,V$ with $A, V \in \mathbb{T}^K_{g_0}$ arbitrary. Then

$$\frac{d}{dt}\,\mathbf{W}_\Omega(A_t) = 2\int_X \left\langle \nabla_{g_0} e^{A_t}, \nabla_{g_0}(e^{A_t} V)\right\rangle_{g_0} \Omega + 2\int_X \mathrm{Tr}_{\mathbb{R}}\left[\left(e^{2A_t}\,\mathrm{Ric}^*_{g_0}(\Omega) - \mathbb{I}\right) V\right] \Omega. \tag{9.1}$$

Integrating by parts we obtain the first variation formula

$$\frac{d}{dt}_{|t=0}\mathbf{W}_\Omega(A_t) = -\,2\int_X \left\langle e^A\,\mathrm{div}^\Omega_{g_0}\left(e^A \nabla_{g_0} A\right) - e^{2A}\,\mathrm{Ric}^*_{g_0}(\Omega) + \mathbb{I}\,, V\right\rangle_{g_0} \Omega.$$

The assumption $g_0 \in \mathcal{S}^K_\Omega$ implies the expression of the gradient

$$2\,\nabla_{L^2}\mathbf{W}_\Omega(A) = -e^A\,\mathrm{div}^\Omega_{g_0}\left(e^A \nabla_{g_0} A\right) + e^{2A}\,\mathrm{Ric}^*_{g_0}(\Omega) - \mathbb{I} \in \mathbb{T}^K_{g_0}\,,$$

thanks to the identity (8.11). We infer the required conclusion. □

We show now the following convexity results.

Lemma 12. *For any $g_0 \in \mathcal{M}$ and for all $A, V \in \mathbb{T}^K_{g_0}$ we have the second variation formula*

$$\begin{aligned}&\nabla_{L^2} D\,\mathbf{W}_\Omega(A)(V,V)\\ &= 2\int_X \left\langle \left[-e^A\,\mathrm{div}^\Omega_{g_0}\left(e^A \nabla_{g_0} A\right) + 2\,e^{2A}\,\mathrm{Ric}^*_{g_0}(\Omega)\right] V, V\right\rangle_{g_0} \Omega\\ &\quad + 2\int_X \left|\nabla_{g_0}(e^A V)\right|^2_{g_0} \Omega.\end{aligned}$$

Moreover if $\mathrm{Ric}_{g_0}(\Omega) > 0$ then the functional $\mathbf{W}_\Omega$ is convex over the convex set

$$\mathbb{T}^{K,+}_{g_0} := \left\{ A \in \mathbb{T}^K_{g_0} \mid \int_X |U \nabla_{g_0} A|^2_{g_0} \Omega \leqslant \int_X \mathrm{Tr}_{\mathbb{R}}\left[U^2\,\mathrm{Ric}^*_{g_0}(\Omega)\right] \Omega\,, \forall U \in \mathbb{T}^K_{g_0} \right\}.$$

Proof. We observe first that the convexity of the set $\mathbb{T}^{K,+}_{g_0}$ follows directly by the convexity of the L^2-norm squared. We compute now the second variation of the functional $\mathbf{W}_\Omega$ along any line $t \mapsto A_t := A + t\,V$ with $A, V \in \mathbb{T}^K_{g_0}$. Differentiating the formula (9.1) we infer the expansion

$$\begin{aligned}\nabla_{L^2} D\,\mathbf{W}_\Omega(A)(V,V) &= \frac{d^2}{dt^2}_{|t=0}\mathbf{W}_\Omega(A_t)\\ &= 2\int_X \left[\left|\nabla_{g_0}(e^A V)\right|^2_{g_0} + \left\langle \nabla_{g_0} e^A, \nabla_{g_0}(e^A V^2)\right\rangle_{g_0}\right]\Omega\\ &\quad + 4\int_X \mathrm{Tr}_{\mathbb{R}}\left[e^{2A} V^2\,\mathrm{Ric}^*_{g_0}(\Omega)\right] \Omega.\end{aligned}$$

The required second variation formula follows by an integration by parts. We expand now the term

$$\begin{aligned}\left\langle \nabla_{g_0} e^A, \nabla_{g_0}(e^A V^2)\right\rangle_{g_0} &= \left\langle \nabla_{g_0} e^A, \nabla_{g_0}(e^A V) V + e^A V\, \nabla_{g_0} V\right\rangle_{g_0}\\ &= \left\langle V\, \nabla_{g_0} e^A, \nabla_{g_0}(e^A V) + e^A \nabla_{g_0} V\right\rangle_{g_0}\\ &= 2\left\langle V\, \nabla_{g_0} e^A, \nabla_{g_0}(e^A V)\right\rangle_{g_0} - \left|V\, \nabla_{g_0} e^A\right|^2_{g_0}.\end{aligned}$$

Using the Cauchy–Schwarz and Jensen's inequalities we obtain

$$\begin{aligned}2\left|\left\langle V\, \nabla_{g_0} e^A, \nabla_{g_0}(e^A V)\right\rangle_{g_0}\right| &\leqslant 2\left|V\, \nabla_{g_0} e^A\right|_{g_0}\left|\nabla_{g_0}(e^A V)\right|_{g_0}\\ &\leqslant \left|V\, \nabla_{g_0} e^A\right|^2_{g_0} + \left|\nabla_{g_0}(e^A V)\right|^2_{g_0},\end{aligned}$$

and thus the inequality

$$\left\langle \nabla_{g_0} e^A, \nabla_{g_0}(e^A V^2)\right\rangle_{g_0} \geqslant -2\left|V\, \nabla_{g_0} e^A\right|^2_{g_0} - \left|\nabla_{g_0}(e^A V)\right|^2_{g_0}.$$

We infer the estimate

$$\nabla_{L^2} D\, \mathbf{W}_\Omega(A)(V,V) \geqslant 4\int_X \left\{ \mathrm{Tr}_{\mathbb{R}}\left[(Ve^A)^2\, \mathrm{Ric}^*_{g_0}(\Omega)\right] - \left|Ve^A \nabla_{g_0} A\right|^2_{g_0}\right\}\Omega,$$

which implies the required convexity statement over the convex set $\mathbb{T}^{K,+}_{g_0}$. Indeed $Ve^A \in \mathbb{T}^K_{g_0}$ thanks to Lemma 4. □

Corollary 6. *Let $g_0 \in \mathcal{M}$. The functional $\mathcal{W}_\Omega$ is G-convex over the G-convex set*

$$\Sigma^-_K(g_0) := \Big\{ g \in \Sigma_K(g_0) \mid \mathrm{Ric}_g(\Omega) \;\geqslant\; -\;\mathrm{Ric}_{g_0}(\Omega)\Big\},$$

inside the totally geodesic and flat subvariety $\Sigma_K(g_0)$ of the nonpositively curved Riemannian manifold $(\mathcal{M}, G)$. Moreover if $\mathrm{Ric}_{g_0}(\Omega) \geqslant \varepsilon g_0$, for some $\varepsilon \in \mathbb{R}_{>0}$ then the functional $\mathcal{W}_\Omega$ is G-convex over the G-convex and nonempty sets

$$\begin{aligned}\Sigma^\delta_K(g_0) &:= \Big\{ g \in \Sigma_K(g_0) \mid \mathrm{Ric}_g(\Omega) \;\geqslant\; \delta\, g\Big\}, \quad \forall \delta \in [0,\varepsilon),\\ \Sigma^+_K(g_0) &:= \Big\{ g \in \Sigma_K(g_0) \mid 2\,\mathrm{Ric}_g(\Omega) \;+\; g_0\, \Delta^\Omega_{g_0} \log(g_0^{-1} g) \;\geqslant\; 0\Big\}.\end{aligned}$$

Proof. Step I (G-convexity of the sets $\Sigma^*_K(g_0)$). We observe first that the change of variables (8.8) send geodesics in to lines. Indeed the image of any geodesic $t \mapsto g_t = g\, e^{tv^*_g}$ via this map is the line $t \mapsto A_t := A - t\, v^*_g/2 \in \mathbb{T}^K_{g_0}$. We infer that the G-convexity of the set $\Sigma^-_K(g_0)$ is equivalent to the (linear) convexity of the set

$$\mathbb{T}^{K,-}_{g_0} := \Big\{ A \in \mathbb{T}^K_{g_0} \mid \mathrm{Ric}_{g_A}(\Omega) \;\geqslant\; -\;\mathrm{Ric}_{g_0}(\Omega)\Big\},$$

with $g_A := g_0 e^{-2A}$. In the same way the G-convexity of the sets $\Sigma^\delta_K(g_0)$ and $\Sigma^+_K(g_0)$ is equivalent respectively to the convexity of the sets

$$\mathbb{T}^{K,\delta}_{g_0} := \Big\{ A \in \mathbb{T}^K_{g_0} \mid \mathrm{Ric}_{g_A}(\Omega) \;\geqslant\; \delta\, g_A \Big\},$$
$$\mathbb{T}^{K,++}_{g_0} := \Big\{ A \in \mathbb{T}^K_{g_0} \mid \mathrm{Ric}_{g_A}(\Omega) \;\geqslant\; g_0\, \Delta^\Omega_{g_0} A \Big\} .$$

Given any metric $g \in \mathcal{M}$ and any sections $A, B \in C^\infty(X, \mathrm{End}_g(T_X))$ we define the bilinear product operation

$$\{A, B\}_g :=\; g\ \mathrm{Tr}_g \left(\nabla_{g,\bullet}\, A\, \nabla_{g,\bullet}\, B\right),$$

and we observe the inequality $\{A, A\}_g \geqslant 0$. This implies the convex inequality

$$\{A_t, A_t\}_g \leqslant\; (1 \;-\; t)\,\{A_0, A_0\}_g + t\,\{A_1, A_1\}_g\,,$$
$$A_t := (1 \;-\; t)\, A_0 + t\, A_1\,, \quad t \;\in\; [0,1]\,.$$

for any $A_0, A_1 \in C^\infty(X, \mathrm{End}_g(T_X))$. Indeed we observe the expansion

$$\begin{aligned}
\{A_t, A_t\}_g &= \{A_t, A_0\}_g + \{A_t, t(A_1 \;-\; A_0)\}_g\\
&= (1 \;-\; t)\,\{A_0, A_0\}_g + t\ \{A_1, A_0\}_g\\
&\quad + \{A_1 - (1 \;-\; t)(A_1 \;-\; A_0), t\,(A_1 \;-\; A_0)\}_g\\
&= (1 \;-\; t)\,\{A_0, A_0\}_g + t\,\{A_1, A_1\}_g\\
&\quad - t\,(1 \;-\; t)\,\{A_1 \;-\; A_0, A_1 \;-\; A_0\}_g\,.
\end{aligned}$$

Using this notation in formula (8.5) we infer the expression

$$\mathrm{Ric}_{g_A}(\Omega) = g_0\, \Delta^\Omega_{g_0} A - \{A, A\}_{g_0} + \mathrm{Ric}_{g_0}(\Omega). \tag{9.2}$$

We deduce the identities

$$\begin{aligned}
\mathbb{T}^{K,-}_{g_0} &= \Big\{ A \in \mathbb{T}^K_{g_0} \mid g_0\, \Delta^\Omega_{g_0} A \;\geqslant\; \{A, A\}_{g_0} \;-\; 2\,\mathrm{Ric}_{g_0}(\Omega) \Big\}\,,\\
\mathbb{T}^{K,\delta}_{g_0} &= \Big\{ A \in \mathbb{T}^K_{g_0} \mid g_0\, \Delta^\Omega_{g_0} A \;\geqslant\; \{A, A\}_{g_0} \;-\; \mathrm{Ric}_{g_0}(\Omega) \;+\; \delta\, g_0\, e^{-2A} \Big\}\,,\\
\mathbb{T}^{K,++}_{g_0} &= \Big\{ A \in \mathbb{T}^K_{g_0} \mid \{A, A\}_{g_0} \;\leqslant\; \mathrm{Ric}_{g_0}(\Omega) \Big\}\;.
\end{aligned}$$

Let now A_0, $A_1 \in \mathbb{T}^{K,\delta}_{g_0}$. The fact that $[A_0, A_1] = 0$ implies the existence of a g_0-orthonormal basis which diagonalizes simultaneously A_0 and A_1. Then the convexity of the exponential function implies the convex inequality

$$g_0\, e^{-2A_t} \leqslant (1 \;-\; t)\, g_0\, e^{-2\,A_0} + t\, g_0\, e^{-2\,A_1},$$

for all $t \in [0,1]$. Using the previous convex inequalities we obtain

$$\begin{aligned}
g_0\, \Delta^\Omega_{g_0} A_t &= (1 \;-\; t)\, g_0\, \Delta^\Omega_{g_0} A_0 + t\, g_0\, \Delta^\Omega_{g_0} A_1\\
&\geqslant (1 \;-\; t)\,\{A_0, A_0\}_{g_0} + t\,\{A_1, A_1\}_{g_0} - \mathrm{Ric}_{g_0}(\Omega)\\
&\quad + (1 \;-\; t)\,\delta\, g_0\, e^{-2\,A_0} + t\,\delta\, g_0\, e^{-2\,A_1}\\
&\geqslant \{A_t, A_t\}_{g_0} - \mathrm{Ric}_{g_0}(\Omega) + \delta\, g_0\, e^{-2\,A_t},
\end{aligned}$$

for all $t \in [0,1]$. We infer the convexity of the set $\mathbb{T}^{K,\delta}_{g_0}$. The proof of the convexity of the sets $\mathbb{T}^{K,-}_{g_0}$ and $\mathbb{T}^{K,++}_{g_0}$ is quite similar.

Step II (G-convexity of the functional $\mathcal{W}_\Omega$). Using again the fact that the change of variables (8.8) send geodesics in to lines we infer that the G-convexity of the functional $\mathcal{W}_\Omega$ over the G-convex sets $\Sigma^-_K(g_0)$, $\Sigma^\delta_K(g_0)$, $\Sigma^+_K(g_0)$ is equivalent to the convexity of the functional $\mathbf{W}_\Omega$ over the convex sets $\mathbb{T}^{K,-}_{g_0}$, $\mathbb{T}^{K,\delta}_{g_0}$, $\mathbb{T}^{K,++}_{g_0}$. Let now $g \in \Sigma^-_K(g_0)$ and observe that

$$\begin{aligned}0 &\leqslant \operatorname{Ric}_g(\Omega) + \operatorname{Ric}_{g_0}(\Omega)\\ &= -g_0\, e^{-A} \operatorname{div}^\Omega_{g_0}\left(e^A \nabla_{g_0} A\right) + 2\operatorname{Ric}_{g_0}(\Omega),\end{aligned}$$

thanks to the identity (8.9). Then the second variation formula in Lemma 12 implies the convexity of the functional $\mathbf{W}_\Omega$ over the convex set $\mathbb{T}^{K,-}_{g_0}$. The convexity of the functional $\mathbf{W}_\Omega$ over $\mathbb{T}^{K,\delta}_{g_0}$ is obvious at this point. We observe now the inclusion $\mathbb{T}^{K,++}_{g_0} \subset \mathbb{T}^{K,+}_{g_0}$. Indeed for all $A \in \mathbb{T}^{K,++}_{g_0}$ and for all $U \in \mathbb{T}^K_{g_0}$ we have the trivial identities

$$\begin{aligned}|U\,\nabla_{g_0} A|^2_{g_0} &= \sum_k |U\,\nabla_{g_0,e_k} A|^2_{g_0}\\ &= \sum_k \Big\langle \nabla_{g_0,e_k} A\,\nabla_{g_0,e_k} A\, U, U\Big\rangle_{g_0}\\ &= \Big\langle \operatorname{Tr}_{g_0}\left(\nabla_{g_0,\bullet} A\,\nabla_{g_0,\bullet} A\right) U, U\Big\rangle_{g_0}\\ &= \operatorname{Tr}_{\mathbb{R}}\Big[\operatorname{Tr}_{g_0}\left(\nabla_{g_0,\bullet} A \nabla_{g_0,\bullet} A\right) U^2\Big]\\ &\leqslant \operatorname{Tr}_{\mathbb{R}}\left[U^2 \operatorname{Ric}^*_{g_0}(\Omega)\right],\end{aligned}$$

where $(e_k)_k \subset T_{X,x}$ is a g_0-orthonormal basis at an arbitrary point $x \in X$. Then Lemma 12 implies the convexity of the functional $\mathbf{W}_\Omega$ over the convex set $\mathbb{T}^{K,++}_{g_0}$. □

10. The extension of the functional $\mathbf{W}_\Omega$ to $\overline{\mathbb{T}}^K_{g_0}$

We denote by $\operatorname{End}_{g_0}(T_X)$ the space of g_0-symmetric endomorphisms. We define the natural integral extension

$$\mathbf{W}_\Omega : \overline{\mathbb{T}}^K_{g_0} \longrightarrow (-\infty, +\infty],$$

of the functional $\mathbf{W}_\Omega$ by the integral expression in the beginning of Section 9 if $e^A \in \overline{\mathbb{T}}^K_{g_0} \cap H^1(X, \operatorname{End}_{g_0}(T_X))$ and $\mathbf{W}_\Omega(A) = +\infty$ otherwise. We show now the following elementary fact.

Lemma 13. *Let $g_0 \in \mathcal{M}$ such that $\operatorname{Ric}_{g_0}(\Omega) \geqslant \varepsilon g_0$, for some $\varepsilon \in \mathbb{R}_{>0}$. Then the natural integral extension $\mathbf{W}_\Omega : \overline{\mathbb{T}}^K_{g_0} \longrightarrow (-\infty, +\infty]$ of the functional*

$\mathbf{W}_\Omega$ *is lower semi-continuous and bounded from below. Indeed for any* $A \in \overline{\mathbb{T}}^K_{g_0}$ *we have the uniform estimate*

$$\mathbf{W}_\Omega(A) \geqslant 2\int_X \log \frac{dV_{\varepsilon g_0}}{\Omega}\,\Omega\,.$$

Proof. We observe that a function f over a metric space is l.s.c. iff

$$f(x) \leqslant \liminf_{k\to+\infty} f(x_k)$$

for any convergent sequence $x_k \to x$ such that $\sup_k f(x_k) < +\infty$. So let $(A_k)_k \subset \overline{\mathbb{T}}^K_{g_0}$ and $A \in \overline{\mathbb{T}}^K_{g_0}$ such that $A_k \to A$ in $L^2(X)$ with $\sup_k \mathbf{W}_\Omega(A_k) < +\infty$. This combined with the assumption $\mathrm{Ric}_{g_0}(\Omega) \geqslant \varepsilon g_0$, implies the estimates

$$\begin{aligned}
(10.1)\qquad & \int_X \Big[\big|\nabla_{g_0} e^{A_k}\big|^2_{g_0} + \varepsilon\big|e^{A_k}\big|^2_{g_0}\Big]\,\Omega \\
&\leqslant \int_X \Big[\big|\nabla_{g_0} e^{A_k}\big|^2_{g_0} + \big\langle \mathrm{Ric}^*_{g_0}(\Omega) e^{A_k}, e^{A_k}\big\rangle_{g_0}\Big]\,\Omega \\
&\leqslant C,
\end{aligned}$$

for some uniform constant C. The first uniform estimate combined with the Relich–Kondrachov compactness result, $H^1(X) \subset\subset L^2(X)$, implies that for every subsequence of $(e^{A_k})_k$ there exists a subsubsequence convergent to e^A in $L^2(X)$. We observe indeed that the assumption on the L^2-convergence $A_k \to A$ implies that for every subsequence of $(e^{A_k})_k$ there exists a subsubsequence convergent to e^A a.e. over X. We infer that $e^{A_k} \longrightarrow e^A$ in $L^2(X)$. Then the uniform estimates (10.1) imply that $\nabla_{g_0} e^{A_k} \longrightarrow \nabla_{g_0} e^A$ weakly in $L^2(X)$. We deduce

$$\mathbf{W}_\Omega(A) \leqslant \liminf_{k\to+\infty} \mathbf{W}_\Omega(A_k),$$

thanks to the weak lower semi-continuity of the L^2-norm. We show now the lower bound in the statement. For this purpose we observe the estimates

$$\begin{aligned}
\mathbf{W}_\Omega(A) &\geqslant \int_X \left[\mathrm{Tr}_{\mathbb{R}}\left(\varepsilon\, e^{2A} - 2\,A\right) + 2\log\frac{dV_{g_0}}{\Omega} - n\right]\Omega \\
&\geqslant \int_X \left[n(1+\log\varepsilon) + 2\log\frac{dV_{g_0}}{\Omega} - n\right]\Omega.
\end{aligned}$$

The last estimate follows from the fact that the convex function $x \mapsto \varepsilon e^{2x} - 2x$ admits a global minimum over $\mathbb{R}$ at the point $-(\log\varepsilon)/2$ in which takes the value $1+\log\varepsilon$. We infer the required lower bound. □

From now on we will assume that the polarization endomorphism K is smooth. We define the vector spaces

$$\begin{aligned}
&W^{1,\infty}(\mathbb{T}_{g_0}) := \Big\{A \in W^{1,\infty}(X, \mathrm{End}_{g_0}(T_X)) \mid \nabla_{T_{X,g_0}} A = 0\Big\}, \\
&W^{1,\infty}(\mathbb{T}^K_{g_0}) := \Big\{A \in W^{1,\infty}(\mathbb{T}_{g_0}) \mid \big[\nabla^p_{g_0} T, A\big] = 0\,,\ T = \mathcal{R}_{g_0}, K,\ \forall p \in \mathbb{Z}_{\geqslant 0}\Big\},
\end{aligned}$$

We denote by $\overline{\mathbb{T}}_{g_0}^{K,++}$ the L^2-closure of the set convex set $\mathbb{T}_{g_0}^{K,++}$. The following quite elementary lemma will be useful for convexity purposes.

Lemma 14. *Let $g_0 \in \mathcal{M}$ such that $\operatorname{Ric}_{g_0}(\Omega) > 0$. Then the L^2-closed and convex set $\overline{\mathbb{T}}_{g_0}^{K,++}$ satisfies the inclusion*

$$\overline{\mathbb{T}}_{g_0}^{K,++} \subset W^{1,\infty}(\mathbb{T}_{g_0}^{K,+}),$$

where $W^{1,\infty}(\mathbb{T}_{g_0}^{K,+})$ is the set of points $A \in W^{1,\infty}(\mathbb{T}_{g_0}^{K})$ such that

$$\int_X |U\, \nabla_{g_0} A|_{g_0}^2 \Omega \leqslant \int_X \operatorname{Tr}_{\mathbb{R}} \left[U^2 \operatorname{Ric}_{g_0}^*(\Omega) \right] \Omega\,, \quad \forall U \;\in\; \overline{\mathbb{T}}_{g_0}^{K} .$$

Proof. We recall that for all $A \in \mathbb{T}_{g_0}^{K,++}$ we have the inequality

$$|\nabla_{g_0} A|_{g_0}^2 \leqslant \operatorname{Tr}_{g_0} \operatorname{Ric}_{g_0}(\Omega),$$

which implies the uniform estimate

$$\left[\int_X |\nabla_{g_0} A|_{g_0}^{2p}\, \Omega\right]^{\frac{1}{2p}} \leqslant \left[\int_X \operatorname{Tr}_{g_0} \operatorname{Ric}_{g_0}(\Omega)\Omega\right]^{\frac{1}{2p}} , \tag{10.2}$$

for all $p \in \mathbb{N}_{>1}$. Let now $A \in \overline{\mathbb{T}}_{g_0}^{K,++}$ arbitrary and let $(A_k)_k \subset \mathbb{T}_{g_0}^{K,++}$ be a sequence L^2-convergent to A. Applying the uniform estimate (10.2) to A_k we infer that (10.2) holds also for A, by the weak L^{2p}-compactness and the weak lower semi-continuity of the L^{2p}-norm. Furtermore taking the limit as $p \to +\infty$ in (10.2) we infer the uniform estimate

$$\|\nabla_{g_0} A\|_{L^\infty(X,g_0)} \leqslant \sup_X \left[\, \operatorname{Tr}_{g_0} \operatorname{Ric}_{g_0}(\Omega)\right]^{\frac{1}{2}} ,$$

for all $A \in \overline{\mathbb{T}}_{g_0}^{K,++}$. It is clear at this point that $A \in L^\infty(X, \operatorname{End}_{g_0}(T_X))$. Indeed consider an arbitrary coordinate ball $\mathcal{B} \subset X$ with center a point $x \equiv 0$ such that $|A|_{g_0}(0) < +\infty$. Then for all $v \in \mathcal{B}$ we have the inequalities

$$\begin{aligned} |A|_{g_0}(v) &\leqslant |A|_{g_0}(0) + \int_0^1 \left| \left\langle \nabla_{g_0} |A|_{g_0}(tv), v\right\rangle_{g_0} \right| dt \\ &\leqslant |A|_{g_0}(0) + \int_0^1 |\nabla_{g_0,v} A|_{g_0}(tv)\, dt, \end{aligned}$$

which show that A is bounded. In the last inequality we used the estimate

$$\left| \left\langle \nabla_{g_0} |A|_{g_0}, \xi \right\rangle_{g_0} \right| \leqslant |\nabla_{g_0,\xi} A|_{g_0},$$

for all $\xi \in T_{X,x}$. This last follows combining the elementary identities

$$\begin{aligned} &\xi.|A|_{g_0}^2 = 2 \left\langle \nabla_{g_0,\xi} A, A \right\rangle_{g_0} , \\ &\xi.|A|_{g_0}^2 = 2\, |A|_{g_0}\, \xi.|A|_{g_0} = 2\, |A|_{g_0} \left\langle \nabla_{g_0} |A|_{g_0}, \xi \right\rangle_{g_0} . \end{aligned}$$

with the Cauchy–Schwarz inequality. It is also clear by the definition of the convex set $\overline{\mathbb{T}}_{g_0}^{K,++}$ that $A \in W^{1,\infty}(\mathbb{T}_{g_0}^K)$. Moreover the inclusion $\mathbb{T}_{g_0}^{K,++} \subset \mathbb{T}_{g_0}^{K,+}$ implies that for all $A \in \overline{\mathbb{T}}_{g_0}^{K,++}$ we have the inequality

$$\int_X |U\,\nabla_{g_0} A|_{g_0}^2 \Omega \leqslant \int_X \operatorname{Tr}_{\mathbb{R}} \left[U^2 \operatorname{Ric}_{g_0}^*(\Omega)\right] \Omega\,, \quad \forall U \;\in\; \mathbb{T}_{g_0}^K\,.$$

Indeed this follows by the weak L^2-compactness and the weak lower semi-continuity of the L^2-norm. By L^2-density we infer the inclusion in the statement of Lemma 14. □

We can show that the same result holds also for the closure of the set $\mathbb{T}_{g_0}^{K,+}$. However the proof of this case is slightly more complicated and we omit it since we will not use it.

Lemma 15. *Consider any $g_0 \in \mathcal{M}$ such that $\operatorname{Ric}_{g_0}(\Omega) > 0$. Then the natural integral extension $\mathbf{W}_\Omega : \overline{\mathbb{T}}_{g_0}^{K,++} \longrightarrow \mathbb{R}$ of the functional $\mathbf{W}_\Omega$ is lower semi-continuous, uniformly bounded from below and convex over the L^2-closed and convex set $\overline{\mathbb{T}}_{g_0}^{K,++}$.*

Proof. Thanks to Lemma 13 we just need to show the convexity of the natural integral extension $\mathbf{W}_\Omega : \overline{\mathbb{T}}_{g_0}^{K,++} \longrightarrow \mathbb{R}$. We consider for this purpose an arbitrary segment $t \in [0,1] \longmapsto A_t := A + t\,V \in \overline{\mathbb{T}}_{g_0}^{K,++} \subset W^{1,\infty}(\mathbb{T}_{g_0}^{K,+})$. In particular the fact that $A_t \in W^{1,\infty}(\mathbb{T}_{g_0}^K)$ combined with the expression

$$\begin{aligned}\mathbf{W}_\Omega(A_t) = &\int_X \Big[\left|e^{tV} e^A \nabla_{g_0} A_t\right|_{g_0}^2 + \operatorname{Tr}_{\mathbb{R}} \left(e^{2tV} e^{2A} \operatorname{Ric}_{g_0}^*(\Omega) - 2\,A_t\right) \Big] \Omega \\ &+ \int_X \left[2 \log \frac{dV_{g_0}}{\Omega} - n\right] \Omega,\end{aligned}$$

implies that the function $t \in [0,1] \longmapsto \mathbf{W}_\Omega(A_t) \in \mathbb{R}$ is of class C^∞ over the time interval $[0,1]$. Moreover $e^{A_t} \in W^{1,\infty}(\mathbb{T}_{g_0}^K)$ for all $t \in [0,1]$ thanks to the argument in the proof of Lemma 4. This is all we need in order to apply to $A_t \in W^{1,\infty}(\mathbb{T}_{g_0}^K)$ the first order computations in the proof of Lemma 12 which provide the second variation formula

$$\begin{aligned}&\frac{d^2}{dt^2} \mathbf{W}_\Omega(A_t) \\ &= 4 \int_X \Big\{ \operatorname{Tr}_{\mathbb{R}} \left[(V e^{A_t})^2 \operatorname{Ric}_{g_0}^*(\Omega)\right] - \left|V e^{A_t} \nabla_{g_0} A_t\right|_{g_0}^2 \Big\} \Omega \\ &\quad + 2 \int_X \Big[\left|\nabla_{g_0}(e^{A_t} V)\right|_{g_0}^2 + \left|V\,\nabla_{g_0} e^{A_t}\right|_{g_0}^2 + 2 \left\langle V\,\nabla_{g_0} e^{A_t}, \nabla_{g_0}(e^{A_t} V)\right\rangle_{g_0} \Big] \Omega \\ &\geqslant 4 \int_X \Big\{ \operatorname{Tr}_{\mathbb{R}} \left[(V e^{A_t})^2 \operatorname{Ric}_{g_0}^*(\Omega)\right] - \left|V e^{A_t} \nabla_{g_0} A_t\right|_{g_0}^2 \Big\} \Omega,\end{aligned}$$

thanks to the Cauchy–Schwarz and Jensen's inequalities. We show now that

$$U\, e^A \in \overline{\mathbb{T}}_{g_0}^K\,. \tag{10.3}$$

for all $U \in \overline{\mathbb{T}}^K_{g_0}$ and all $A \in \overline{\mathbb{T}}^{K,++}_{g_0}$. Indeed let $(A_j)_j \subset \mathbb{T}^{K,++}_{g_0}$ and $(U_j)_j \subset \mathbb{T}^K_{g_0}$ be two sequences convergent respectively to A and U in the L^2-topology. From a computation in the proof of Lemma 14 we know that the uniform estimate

$$|\nabla_{g_0} A_j|^2_{g_0} \leqslant \operatorname{Tr}_{\mathbb{R}} \operatorname{Ric}^*_{g_0}(\Omega),$$

combined with the convergence a.e. implies the sequence $(A_j)_j$ is bounded in norm L^∞. We infer the L^2-convergence $\mathbb{T}^K_{g_0} \ni U_k e^{A_k} \longrightarrow U e^A \in \overline{\mathbb{T}}^K_{g_0}$ thanks to the dominated convergence theorem. We observe now that the property (10.3) applied to $V e^{A_t}$ combined with the fact that $A_t \in W^{1,\infty}(\mathbb{T}^{K,+}_{g_0})$ for all $t \in [0,1]$ provides the inequality

$$\frac{d^2}{dt^2} \mathbf{W}_\Omega(A_t) \geqslant 0,$$

over the time interval $[0,1]$, which shows the required convexity statement. □

The convexity statement over the d_G-convex set $\overline{\Sigma}^+_K(g_0)$ in the main theorem 1 follows directly from Lemma 15 due to the fact that the change of variables

$$g \;\in\; \overline{\Sigma}^+_K(g_0) \;\longmapsto\; A \;=\; -\,\frac{1}{2}\,\log(g_0^{-1} g) \;\in\; \overline{\mathbb{T}}^{K,++}_{g_0},$$

represents a (d_G, L^2)-isometry map (where L^2 denotes the constant L^2-product $4\int_X \langle\cdot,\cdot\rangle_{g_0}\Omega$) which in particular send all d_G-geodesics segments

$$t \;\in\; [0,1] \;\longmapsto\; g_t \;=\; g\, e^{t v^*_g} \;\in\; \overline{\Sigma}^+_K(g_0),$$

in to linear segments $t \in [0,1] \longmapsto A_t := A - t\, v^*_g/2 \in \overline{\mathbb{T}}^{K,++}_{g_0}$.

11. On the exponentially fast convergence of the Soliton-Ricci-flow

Lemma 16. *Let $g_0 \in \mathcal{S}^K_{\Omega,+}$ and let $(g_t)_{t\geqslant 0} \subset \Sigma_K(g_0)$ be a solution of the Ω-SRF with initial data g_0. If there exist $\delta \in \mathbb{R}_{>0}$ such that $\operatorname{Ric}_{g_t}(\Omega) \geqslant \delta g_t$ for all times $t \geqslant 0$, then the Ω-SRF converges exponentially fast with all its space derivatives to a Ω-ShRS $g_{\mathrm{RS}} \in \Sigma_K(g_0)$ as $t \to +\infty$.*

Proof. Time deriving the Ω-SRF equation by means of (2.3) we infer the evolution formula

$$2\,\ddot{g}_t = -\Delta^\Omega_{g_t} \dot{g}_t - 2\,\dot{g}_t,$$

and thus the evolution equation

$$2\,\frac{d}{dt}\,\dot{g}^*_t = -\Delta^\Omega_{g_t} \dot{g}^*_t - 2\,\dot{g}^*_t - 2\,(\dot{g}^*_t)^2. \tag{11.1}$$

Using this we can compute the evolution of $|\dot{g}_t|^2_{g_t} = |\dot{g}_t^*|^2_{g_t} = \mathrm{Tr}_{\mathbb{R}}(\dot{g}_t^*)^2$. Indeed we define the heat operator

$$\Box^{\Omega}_{g_t} := \Delta^{\Omega}_{g_t} + 2\frac{d}{dt},$$

and we observe the elementary identity

$$\Delta^{\Omega}_{g_t}|\dot{g}_t|^2_{g_t} = 2\left\langle \Delta^{\Omega}_{g_t}\dot{g}_t^*, \dot{g}_t^*\right\rangle_g - 2\,|\nabla_{g_t}\dot{g}_t^*|^2_{g_t}.$$

We infer the evolution formula

$$\begin{aligned}\Box^{\Omega}_{g_t}|\dot{g}_t|^2_{g_t} &= -2\,|\nabla_{g_t}\dot{g}_t^*|^2_{g_t} - 4\,|\dot{g}_t|^2_{g_t} - 4\,\mathrm{Tr}_{\mathbb{R}}(\dot{g}_t^*)^3\\ &\leqslant -\delta|\dot{g}_t|^2_{g_t},\end{aligned}$$

thanks to the Ω-SRF equation and thanks to the assumption $\mathrm{Ric}_{g_t}(\Omega) \geqslant \delta g_t$. Applying the scalar maximum principle we infer the exponential estimate

$$|\dot{g}_t|_{g_t} \leqslant \sup_X |\dot{g}_0|_{g_0}\, e^{-\delta t/2}\,, \tag{11.2}$$

for all $t \geqslant 0$. In its turn this implies the convergence of the integral

$$\int_0^{+\infty} |\dot{g}_t|_{g_t} dt \leqslant C,$$

and thus the uniform estimate

$$e^{-C} g_0 \leqslant g_t \leqslant e^{C} g_0, \tag{11.3}$$

for all $t \geqslant 0$, (see [Ch-Kn]). Thus the convergence of the integral

$$\int_0^{+\infty} |\dot{g}_t|_{g_0} dt < +\infty,$$

implies the existence of the metric

$$g_\infty := g_0 + \int_0^{+\infty} \dot{g}_t\, dt\,,$$

thanks to Bochner's theorem. (The positivity of the metric g_∞ follows from the estimate $e^{-C}g_0 \leqslant g_t$.) Moreover the estimate

$$|g_\infty \;-\; g_t|_{g_0} \leqslant \int_t^{+\infty} |\dot{g}_s|_{g_0} ds \leqslant C' e^{-t},$$

implies the exponential convergence of the Ω-SRF to g_∞ in the uniform topology. We show now the $C^1(X)$-convergence. Indeed the fact that $(g_t)_{t\geqslant 0} \subset \Sigma_K(g_0)$ implies $\dot{g}_t \in \mathbb{F}^K_{g_t}$ for all times $t \geqslant 0$ thanks to the identity (5.2).

Thus we have the identities $[K, \dot{g}_t^*] \equiv 0$ and $[K, \nabla_{g_\tau}\dot{g}_\tau^*] \equiv 0$, which imply in their turn $[\nabla_{g_\tau}\dot{g}_\tau^*, \dot{g}_t^*] \equiv 0$. By the variation formula (3.19) we deduce the identity

$$\nabla_{g_\tau}\dot{g}_t^* \equiv \nabla_{g_0}\dot{g}_t^*, \tag{11.4}$$

for all $\tau, t \geqslant 0$. This combined with (11.1) provides the equalities

$$\begin{aligned}
2\,\frac{d}{dt}\,(\nabla_{g_t}\dot{g}_t^*) &= 2\,\nabla_{g_t}\frac{d}{dt}\,\dot{g}_t^*\\
&= -\nabla_{g_t}\Delta^{\Omega}_{g_t}\dot{g}_t^* - 2\,\nabla_{g_t}\dot{g}_t^* - 2\,\nabla_{g_t}(\dot{g}_t^*)^2\\
&= -\Delta^{\Omega}_{g_t}\nabla_{g_t}\dot{g}_t^* - \mathrm{Ric}^*_{g_t}(\Omega)\bullet\nabla_{g_t}\dot{g}_t^*\\
&\quad - 2\,\nabla_{g_t}\dot{g}_t^* - 4\,\dot{g}_t^*\nabla_{g_t}\dot{g}_t^*.
\end{aligned}$$

We justify the last equality. We pick geodesic coordinates centered at an arbitrary space time point (x_0, t_0), let $(e_k)_k$ be the coordinate local tangent frame and let ξ, η be local vector fields with constant coefficients defined in a neighborhood of x_0. We expand at the space time point (x_0, t_0) the term

$$\begin{aligned}
\nabla_{g_t,\xi}\Delta^{\Omega}_{g_t}\dot{g}_t^* &= -\nabla_{g_t,\xi}\nabla_{g_t,e_k}\nabla_{g_t,e_k}\dot{g}_t^* + \nabla_{g_t,\xi}\nabla_{g_t,e_k}e_k \neg \nabla_{g_t}\dot{g}_t^*\\
&\quad + \nabla_{g_t,\xi}\nabla_{g_t,\nabla_{g_t}f_t}\,\dot{g}_t^*\\
&= -\nabla_{g_t,e_k}\nabla_{g_t,e_k}\nabla_{g_t,\xi}\,\dot{g}_t^* + \nabla_{g_t,\xi}\nabla_{g_t,e_k}e_k \neg \nabla_{g_t}\dot{g}_t^*\\
&\quad + \nabla_{g_t,\nabla_{g_t}f_t}\nabla_{g_t,\xi}\,\dot{g}_t^* + \nabla_{g_t,\xi}\nabla_{g_t}f_t \neg \nabla_{g_t}\dot{g}_t^*,
\end{aligned}$$

thanks to the identity (3.6) and $[\mathcal{R}_{g_t}, \nabla_{g_t,e_k}\dot{g}_t^*] \equiv 0, [\mathcal{R}_{g_t}, \dot{g}_t^*] \equiv 0, [e_k, \xi] \equiv 0$. Moreover at the space time point (x_0, t_0) we have the identity

$$\begin{aligned}
\nabla_{g_t,e_k}\nabla^2_{g_t}\dot{g}_t^*(e_k,\xi,\eta) &= \nabla_{g_t,e_k}\left[\nabla_{g_t,e_k}\nabla_{g_t}\dot{g}_t^*(\xi,\eta)\right]\\
&= \nabla_{g_t,e_k}\nabla_{g_t,e_k}(\nabla_{g_t,\xi}\,\dot{g}_t^*\,\eta)\\
&\quad - \nabla_{g_t,e_k}\left[\nabla_{g_t}\dot{g}_t^*(\nabla_{g_t,e_k}\xi,\eta) + \nabla_{g_t}\dot{g}_t^*(\xi,\nabla_{g_t,e_k}\eta)\right]\\
&= \nabla_{g_t,e_k}\left[\nabla_{g_t,e_k}\nabla_{g_t,\xi}\,\dot{g}_t^*\,\eta - \nabla_{g_t}\dot{g}_t^*(\nabla_{g_t,\xi}\,e_k,\eta)\right]\\
&= \nabla_{g_t,e_k}\nabla_{g_t,e_k}\nabla_{g_t,\xi}\,\dot{g}_t^*\,\eta - \nabla_{g_t}\dot{g}_t^*(\nabla_{g_t,e_k}\nabla_{g_t,\xi}\,e_k,\eta),
\end{aligned}$$

which combined with the previous expression implies the formula

$$\nabla_{g_t}\Delta^{\Omega}_{g_t}\dot{g}_t^* = \Delta^{\Omega}_{g_t}\nabla_{g_t}\dot{g}_t^* + \mathrm{Ric}^*_{g_t}(\Omega)\bullet\nabla_{g_t}\dot{g}_t^*.$$

Thus

$$2\,\frac{d}{dt}\,(\nabla_{g_t}\dot{g}_t^*) = -\Delta^{\Omega}_{g_t}\nabla_{g_t}\dot{g}_t^* - \dot{g}_t^*\bullet\nabla_{g_t}\dot{g}_t^* - 3\,\nabla_{g_t}\dot{g}_t^* - 4\,\dot{g}_t^*\nabla_{g_t}\dot{g}_t^*,$$

by the Ω-SRF equation. We use this to compute the evolution of the norm squared

$$|\nabla_{g_t}\dot{g}_t^*|^2_{g_t} = \mathrm{Tr}_{\mathbb{R}}\left(\nabla_{g_t,e_k}\dot{g}_t^*\,\nabla_{g_t,g_t^{-1}e_k^*}\dot{g}_t^*\right).$$

Indeed at time t_0 we have the identities

$$\begin{aligned}\frac{d}{dt}\,|\nabla_{g_t}\dot{g}_t^*|^2_{g_t} &= \mathrm{Tr}_{\mathbb{R}}\left[2\,e_k \neg \frac{d}{dt}\,(\nabla_{g_t}\dot{g}_t^*)\,\nabla_{g_t,e_k}\dot{g}_t^* - \nabla_{g_t,e_k}\dot{g}_t^*\,\nabla_{g_t,\dot{g}_t^* e_k}\dot{g}_t^*\right]\\
&= -\,\mathrm{Tr}_{\mathbb{R}}\left[2\,e_k \neg \Delta^{\Omega}_{g_t}\nabla_{g_t}\dot{g}_t^*\,\nabla_{g_t,e_k}\dot{g}_t^* + 2\,\nabla_{g_t,\dot{g}_t^* e_k}\dot{g}_t^*\,\nabla_{g_t,e_k}\dot{g}_t^*\right]\\
&\quad -\,\mathrm{Tr}_{\mathbb{R}}\left[3\,\nabla_{g_t,e_k}\dot{g}_t^*\,\nabla_{g_t,e_k}\dot{g}_t^* + 4\,\dot{g}_t^*\nabla_{g_t,e_k}\dot{g}_t^*\,\nabla_{g_t,e_k}\dot{g}_t^*\right]\\
&= -\left\langle \Delta^{\Omega}_{g_t}\nabla_{g_t}\dot{g}_t^*, \nabla_{g_t}\dot{g}_t^*\right\rangle_{g_t} - 3\,|\nabla_{g_t}\dot{g}_t^*|^2_{g_t}\\
&\quad -\,2\left\langle \dot{g}_t^* \bullet \nabla_{g_t}\dot{g}_t^* + 2\,\dot{g}_t^*\,\nabla_{g_t}\dot{g}_t^*, \nabla_{g_t}\dot{g}_t^*\right\rangle_{g_t}.\end{aligned}$$

This combined with the elementary identity

$$\Delta^{\Omega}_{g_t}|\nabla_{g_t}\dot{g}_t^*|^2_{g_t} = 2\left\langle \Delta^{\Omega}_{g_t}\nabla_{g_t}\dot{g}_t^*, \nabla_{g_t}\dot{g}_t^*\right\rangle_{g_t} - 2\,|\nabla^2_{g_t}\dot{g}_t^*|^2_{g_t},$$

implies the evolution formula

$$\begin{aligned}\square^{\Omega}_{g_t}|\nabla_{g_t}\dot{g}_t^*|^2_{g_t} &= -2\,|\nabla^2_{g_t}\dot{g}_t^*|^2_{g_t} - 6\,|\nabla_{g_t}\dot{g}_t^*|^2_{g_t}\\
&\quad -\,4\left\langle \dot{g}_t^* \bullet \nabla_{g_t}\dot{g}_t^* + 2\,\dot{g}_t^*\nabla_{g_t}\dot{g}_t^*, \nabla_{g_t}\dot{g}_t^*\right\rangle_{g_t}\\
&\leqslant \left[(4 + 8\sqrt{n})|\dot{g}_t|_{g_t} - 6\right]|\nabla_{g_t}\dot{g}_t^*|^2_{g_t}\\
&\leqslant \left(Ce^{-t} - 6\right)|\nabla_{g_t}\dot{g}_t^*|^2_{g_t},\end{aligned}$$

thanks to the uniform exponential estimate (11.2). An application of the scalar maximum principle implies the estimate

$$|\nabla_{g_t}\dot{g}_t^*|_{g_t} \leqslant C_1 e^{-t}, \tag{11.5}$$

for all $t \geqslant 0$, where $C_1 > 0$ is a constant uniform in time. By abuse of notation we will allays denote by C or C_1 such type of constants. Moreover the identity (11.4) for $\tau = t$ combined with (11.3) provides the exponential estimate

$$|\nabla_{g_0}\dot{g}_t^*|_{g_0} \leqslant C_1 e^{-t}.$$

We observe now the trivial decomposition

$$\nabla_{g_0}(g_0^{-1}\dot{g}_t) = \nabla_{g_0}(g_0^{-1}g_t)\,\dot{g}_t^* + g_0^{-1}g_t\nabla_{g_0}\,\dot{g}_t^*.$$

Thus if we set $N_t := |\nabla_{g_0}(g_0^{-1}g_t)|_{g_0}$ we infer the first order differential inequality

$$\begin{aligned}\dot{N}_t &\leqslant \sqrt{n}\,N_t\,|\dot{g}_t^*|_{g_0} + \sqrt{n}\,|g_0^{-1}g_t|_{g_0}|\,\nabla_{g_0}\dot{g}_t^*|_{g_0}\\
&\leqslant C\,N_t\,e^{-t} + C\,e^{-2t},\end{aligned}$$

and thus

$$N_t \leqslant e^{C\int_0^t e^{-s}ds}\left[N_0 + C\int_0^t e^{-2s}ds\right] \leqslant e^C\,(N_0 + C),$$

by Gronwall's inequality. We deduce in conclusion the exponential estimate $\dot{N}_t \leqslant C_1 e^{-t}$, i.e.,

$$|\nabla_{g_0} \dot{g}_t|_{g_0} \leqslant C_1 e^{-t},$$

for all $t \geqslant 0$. We infer the convergence of the integral

$$\int_0^{+\infty} |\nabla_{g_0} \dot{g}_t|_{g_0} \, dt < +\infty,$$

and thus the existence of the tensor

$$A_1 := \int_0^{+\infty} \nabla_{g_0} \dot{g}_t \, dt,$$

thanks to Bochner's theorem. Moreover we have the exponential estimate

$$|A_1 \;-\; \nabla_{g_0} g_t|_{g_0} \leqslant \int_t^{+\infty} |\nabla_{g_0} \dot{g}_s|_{g_0} \, ds \leqslant C' \, e^{-t} \;.$$

A basic calculus fact implies that $A_1 = \nabla_{g_0} g_\infty$. In order to obtain the convergence of the higher order derivatives we need to combine the uniform C^1-estimate obtained so far with an interpolation method based in Hamilton's work (see [Ham]). The details will be explained in the next more technical sections. In conclusion taking the limit as $t \to +\infty$ in the Ω-SRF equation we deduce that $g_\infty = g_{\mathrm{RS}}$ is a Ω-ShRS. □

12. The commutator $\left[\nabla^p, \Delta^\Omega\right]$ along the Ω-Soliton-Ricci flow

We introduce first a few product notations. Let g be a metric over a vector space V. For any $A \in (V^*)^{\otimes p} \otimes V$, $B \in (V^*)^{\otimes q} \otimes V$ and for all integers k, l such that $1 \leqslant l \leqslant k \leqslant q-2$ we define the product $A \odot^g_{k,l} B$ as

$$\begin{aligned} &(A \odot^g_{k,l} B)(u, v) \\ &\qquad := \mathrm{Tr}_g \Big[B(v_1, \dots, v_{l-1}, \cdot, v_l, \dots, v_{k-1}, A(u, \cdot, v_k), v_{k+1}, \dots, v_{q-1}) \Big], \end{aligned}$$

for all $u \equiv (u_1, \dots, u_{p-2})$ and $v \equiv (v_1, \dots, v_{q-1})$. Moreover for any $\sigma \in S_{p+l-3}$ we define the product $A \odot^{g,\sigma}_{k,l} B$ as

$$(A \odot^{g,\sigma}_{k,l} B)(u, v) := (A \odot^g_{k,l} B)(\xi_\sigma, v_l, \dots, v_{q-1}),$$

where $\xi \equiv (\xi_1, \dots, \xi_{p+l-3}) := (u_1, \dots, u_{p-2}, v_1, \dots, v_{l-1})$. We notice also that $A \odot^{g,\sigma}_{k,l} B \equiv A \odot^g_{k,l} B$ if $p + l - 3 \leqslant 1$. Finally we define

$$A \hat{\neg}_g B := \sum_{k=1}^{q-2} A \odot^g_{k,1} B \;.$$

Let now (X, g) be a Riemannian manifold and let $A \in C^\infty(X, (T^*_X)^{\otimes p} \otimes T_X)$. We recall the classic formula

$$\left[\nabla_{g,\xi}, \nabla_{g,\eta}\right] A = \left[\mathcal{R}_g(\xi, \eta), A\right] - \mathcal{R}_g(\xi, \eta) \hat{\neg} A + \nabla_{g,[\xi,\eta]} A, \tag{12.1}$$

for all $\xi, \eta \in C^\infty(X, T_X)$. We show now the following lemma.

Lemma 17. *Let (X,g) be an oriented Riemannian manifold, let $\Omega>0$ be a smooth volume form over X and let $A\in C^\infty(X,(T_X^*)^{\otimes p}\otimes T_X)$ such that $\big[\mathcal{R}_g,\xi\neg\nabla_g^r A\big]=0$ for all $r=0,1$ and $\xi\in T_X^{\otimes p+r-1}$. Then*

$$\big[\nabla_g,\Delta_g^\Omega\big]A=\operatorname{Ric}_g^*(\Omega)\bullet\nabla_g A+2\,\mathcal{R}_g\;\hat{\neg}_g\nabla_g A+\nabla_g^{*\Omega}\mathcal{R}_g\;\hat{\neg}\;A.$$

Proof. We pick geodesic coordinates centered at an arbitrary point x_0. Let $(e_k)_k$ be the coordinate local tangent frame and let $\xi,\eta\equiv(\eta_1,\dots,\eta_p)$ be local vector fields with constant coefficients defined in a neighborhood of the point x_0. We expand at the point x_0 the term

$$\begin{aligned}
\nabla_{g,\xi}\,\Delta_g^\Omega A &= -\nabla_{g,\xi}\nabla_{g,e_k}\nabla_{g,e_k}A\\
&\quad+\nabla_{g,\xi}\nabla_{g,e_k}e_k\neg\nabla_g A+\nabla_{g,\xi}\nabla_{g,\nabla_g f}A\\
&=-\nabla_{g,e_k}\nabla_{g,\xi}\nabla_{g,e_k}A+\mathcal{R}_g(\xi,e_k)\;\hat{\neg}\;\nabla_{g,e_k}A\\
&\quad+\nabla_{g,\xi}\nabla_{g,e_k}e_k\neg\nabla_g A+\nabla_{g,\nabla_g f}\nabla_{g,\xi}A\\
&\quad-\mathcal{R}_g(\xi,\nabla_g f)\;\hat{\neg}\;A+\nabla_{g,\xi}\nabla_g f\neg\nabla_g A\\
&=-\nabla_{g,e_k}\nabla_{g,e_k}\nabla_{g,\xi}A\\
&\quad+2\,\mathcal{R}_g(\xi,e_k)\;\hat{\neg}\;\nabla_{g,e_k}A+\nabla_{g,e_k}\mathcal{R}_g(\xi,e_k)\;\hat{\neg}\;A\\
&\quad+\nabla_{g,\xi}\nabla_{g,e_k}e_k\neg\nabla_g A+\big(\nabla_g f\neg\nabla_g^2 A\big)(\xi,\cdot)\\
&\quad+\mathcal{R}_g(\nabla_g f,\xi)\;\hat{\neg}\;A+\nabla_{g,\xi}\nabla_g f\neg\nabla_g A,
\end{aligned}$$

thanks to the identity (12.1), to the assumptions on A and to the identity $[e_k,\xi]\equiv 0$. Moreover at the point x_0 we have the identity

$$\begin{aligned}
\nabla_{g,e_k}\nabla_g^2 A\,(e_k,\xi,\eta) &= \nabla_{g,e_k}\big[\nabla_{g,e_k}\nabla_g A\,(\xi,\eta)\big]\\
&=\nabla_{g,e_k}\Big\{\nabla_{g,e_k}\big[\nabla_{g,\xi}A\,(\eta)\big]-\nabla_g A\,(\nabla_{g_t,e_k}\xi,\eta)\Big\}\\
&\quad-\sum_{j=1}^p\nabla_{g,e_k}\big[\nabla_g A\,(\xi,\eta_1,\dots,\nabla_{g,e_k}\eta_j,\dots,\eta_p)\big]\\
&=\nabla_{g,e_k}\big[\nabla_{g,e_k}\nabla_{g,\xi}A\,(\eta)-\nabla_g A\,(\nabla_{g,\xi}e_k,\eta)\big]\\
&=\nabla_{g,e_k}\nabla_{g,e_k}\nabla_{g,\xi}A\,(\eta)-\nabla_g A\,(\nabla_{g,e_k}\nabla_{g,\xi}e_k,\eta),
\end{aligned}$$

which combined with the previous expression implies the required formula. □

Applying this lemma first to $A=\dot g_t^*$ and then to $A=\nabla_{g_t}\dot g_t^*$ along the Ω-SRF we infer the formula

$$\begin{aligned}
\big[\nabla_{g_t}^2,\Delta_{g_t}^\Omega\big]\dot g_t^* &= 2\,\nabla_{g_t}^2\dot g_t^*+\dot g_t^*\;\hat{\neg}\;\nabla_{g_t}^2\dot g_t^*+\nabla_{g_t}\dot g_t^*\bullet\nabla_{g_t}\dot g_t^*\\
&\quad+2\,\mathcal{R}_{g_t}\;\hat{\neg}_{g_t}\nabla_{g_t}^2\dot g_t^*+\nabla_{g_t}^{*\Omega}\mathcal{R}_{g_t}\;\hat{\neg}\;\nabla_{g_t}\dot g_t^*.
\end{aligned}$$

(We observe that the presence of the curvature factor turns off the power of the maximum principle in the exponential convergence of higher order space derivatives along the Ω-SRF.) A simple induction shows the general formula

$$\left[\nabla^p_{g_t}, \Delta^\Omega_{g_t}\right] \dot{g}^*_t = p\, \nabla^p_{g_t} \dot{g}^*_t + \sum_{r=1}^{p} \sum_{k=1}^{r} \sum_{\sigma \in S_{p-r+k-1}} C^{p,r}_{k,\sigma}\, \nabla^{p-r}_{g_t} \dot{g}^*_t \bullet^\sigma_k \nabla^r_{g_t} \dot{g}^*_t$$
$$+ \sum_{r=1}^{p-1} \sum_{k=1}^{r} \sum_{\sigma \in S_{p-r+k-1}} K^{p,r}_{k,\sigma}\, \nabla^{p-r-1}_{g_t} \nabla^{*\Omega}_{g_t} \mathcal{R}_{g_t} \bullet^\sigma_k \nabla^r_{g_t} \dot{g}^*_t$$
$$+ 2 \sum_{r=2}^{p} \sum_{k=2}^{r} \sum_{l=1}^{k-1} \sum_{\sigma \in S_{p-r+l}} Q^{p,r}_{k,l,\sigma} \nabla^{p-r}_{g_t} \mathcal{R}_{g_t} \odot^{g_t,\sigma}_{k,l} \nabla^r_{g_t} \dot{g}^*_t,$$

where $C^{p,r}_{k,\sigma}, K^{p,r}_{k,\sigma}, Q^{p,r}_{k,l,\sigma} \in \{0,1\}$.

13. Exponentially fast convergence of higher order space derivatives along the Ω-Soliton-Ricci flow

We use here an interpolation method introduced by Hamilton in his proof of the exponential convergence of the Ricci flow in [Ham]. The difference with the technique in [Ham] is a more involved interpolation process due to the presence of some extra curvature terms which seem to be alien to Hamilton's argument. We are able to perform our interpolation process by using some intrinsic properties of the Ω-SRF.

13.1. Estimate of the heat of the derivatives norm. The fact that $(g_t)_{t \geqslant 0} \subset \Sigma_K(g_0)$ implies $\dot{g}_t \in \mathbb{F}^K_{g_t}$ for all times $t \geqslant 0$ thanks to the identity (5.2). Thus we have the identities $[K, \nabla_{g_t} \dot{g}^*_t] \equiv 0$ and $[K, \nabla^p_{g_t} \dot{g}^*_t] \equiv 0$, which in their turn imply

$$\left[\nabla_{g_t,\xi}\, \dot{g}^*_t, \nabla^p_{g_t} \dot{g}^*_t\right] \equiv 0,$$

for all $p \in \mathbb{Z}_{\geqslant 0}$, $\xi \in T_X$. We deduce by using the identity (4.1)

$$2\, \dot{\nabla}^p_{g_t} \dot{g}^*_t = - \sum_{r=1}^{p-1} \sum_{k=1}^{r} \sum_{\sigma \in S_{p-r+k-1}} C^{p,r}_{k,\sigma}\, \nabla^{p-r}_{g_t} \dot{g}^*_t \bullet^\sigma_k \nabla^r_{g_t} \dot{g}^*_t, \tag{13.1}$$

for all $t \geqslant 0$ and $p \in \mathbb{N}_{>0}$. This combined with (11.1) provides the identities

$$2\, \frac{d}{dt} \left(\nabla^p_{g_t} \dot{g}^*_t\right) = - \sum_{r=1}^{p-1} \sum_{k=1}^{r} \sum_{\sigma \in S_{p-r+k-1}} C^{p,r}_{k,\sigma}\, \nabla^{p-r}_{g_t} \dot{g}^*_t \bullet^\sigma_k \nabla^r_{g_t} \dot{g}^*_t + 2\, \nabla^p_{g_t} \frac{d}{dt}\, \dot{g}^*_t$$
$$= - \sum_{r=1}^{p-1} \sum_{k=1}^{r} \sum_{\sigma \in S_{p-r+k-1}} C^{p,r}_{k,\sigma}\, \nabla^{p-r}_{g_t} \dot{g}^*_t \bullet^\sigma_k \nabla^r_{g_t} \dot{g}^*_t$$
$$- \nabla^p_{g_t} \Delta^\Omega_{g_t} \dot{g}^*_t - 2\, \nabla^p_{g_t} \dot{g}^*_t - 2\, \nabla^p_{g_t} (\dot{g}^*_t)^2.$$

We consider now a g_{t_0}-orthonormal basis $(e_k)_k \subset T_{X,x_0}$ and we define the multi-vectors $e_K := (e_{k_1}, \ldots, e_{k_p})$, $g_t^{-1} e^*_K := (g_t^{-1} e^*_{k_1}, \ldots, g_t^{-1} e^*_{k_p})$. Then at the space time point (x_0, t_0) we have

$$\begin{aligned}
&\frac{d}{dt}\,|\nabla^p_{g_t}\dot{g}^*_t|^2_{g_t}\\
&= \frac{d}{dt}\,\mathrm{Tr}_{\mathbb{R}}\left[\nabla^p_{g_t,e_K}\dot{g}^*_t\,\nabla^p_{g_t,g_t^{-1}e_K}\dot{g}^*_t\right]\\
&= \mathrm{Tr}_{\mathbb{R}}\left[e_K \neg \left(2\,\frac{d}{dt}\left(\nabla^p_{g_t}\dot{g}^*_t\right)\nabla^p_{g_t,e_K}\dot{g}^*_t - \sum_{j=1}^p \dot{g}^*_t \bullet_j \nabla^p_{g_t}\dot{g}^*_t\right)\nabla^p_{g_t,e_K}\dot{g}^*_t\right]\\
&= \left\langle 2\,\frac{d}{dt}\left(\nabla^p_{g_t}\dot{g}^*_t\right) - \sum_{j=1}^p \dot{g}^*_t \bullet_j \nabla^p_{g_t}\dot{g}^*_t, \nabla^p_{g_t}\dot{g}^*_t\right\rangle_{g_t}\\
&\leqslant -\sum_{r=1}^{p-1}\sum_{k=1}^{r}\sum_{\sigma\in S_{p-r+k-1}} C^{p,r}_{k,\sigma}\left\langle\nabla^{p-r}_{g_t}\dot{g}^*_t \bullet^\sigma_k \nabla^r_{g_t}\dot{g}^*_t, \nabla^p_{g_t}\dot{g}^*_t\right\rangle_{g_t}\\
&\quad - \left\langle\nabla^p_{g_t}\Delta^\Omega_{g_t}\dot{g}^*_t + 2\nabla^p_{g_t}\dot{g}^*_t + 2\,\nabla^p_{g_t}(\dot{g}^*_t)^2, \nabla^p_{g_t}\dot{g}^*_t\right\rangle_{g_t} + p\,|\dot{g}_t|_{g_t}|\nabla^p_{g_t}\dot{g}^*_t|^2_{g_t}\\
&\leqslant -\left\langle\Delta^\Omega_{g_t}\nabla^p_{g_t}\dot{g}^*_t, \nabla^p_{g_t}\dot{g}^*_t\right\rangle_{g_t} + C\,|\nabla^p_{g_t}\dot{g}^*_t|^2_{g_t}\\
&\quad + C\sum_{r=1}^{p-1}|\nabla^{p-r}_{g_t}\dot{g}^*_t|_{g_t}|\nabla^r_{g_t}\dot{g}^*_t|_{g_t}|\nabla^p_{g_t}\dot{g}^*_t|_{g_t}\\
&\quad + C\sum_{r=1}^{p-1}|\nabla^{p-r-1}_{g_t}\nabla^{*\Omega}_{g_t}\mathcal{R}_{g_t}|_{g_t}|\nabla^r_{g_t}\dot{g}^*_t|_{g_t}|\nabla^p_{g_t}\dot{g}^*_t|_{g_t}\\
&\quad + C\sum_{r=2}^{p}|\nabla^{p-r}_{g_t}\mathcal{R}_{g_t}|_{g_t}|\nabla^r_{g_t}\dot{g}^*_t|_{g_t}|\nabla^p_{g_t}\dot{g}^*_t|_{g_t},
\end{aligned}$$

where $C > 0$ will always denote a time independent constant. We observe now that $\mathcal{R}_{g_t} \equiv \mathcal{R}_{g_0}$ since $(g_t)_{t\geqslant 0} \subset \Sigma_K(g_0)$. Using the standard identity

$$\Delta^\Omega_{g_t}|\nabla^p_{g_t}\dot{g}^*_t|^2_{g_t} = 2\left\langle\Delta^\Omega_{g_t}\nabla^p_{g_t}\dot{g}^*_t, \nabla^p_{g_t}\dot{g}^*_t\right\rangle_{g_t} - 2\,|\nabla^{p+1}_{g_t}\dot{g}^*_t|^2_{g_t},$$

we infer the estimate of the heat of the norm squared

$$\begin{aligned}
&\Box^\Omega_{g_t}|\nabla^p_{g_t}\dot{g}^*_t|^2_{g_t}\\
&\leqslant -2\,|\nabla^{p+1}_{g_t}\dot{g}^*_t|^2_{g_t} + C\,|\nabla^p_{g_t}\dot{g}^*_t|^2_{g_t}\\
&\quad + C\sum_{r=1}^{p-1}|\nabla^{p-r}_{g_t}\dot{g}^*_t|_{g_t}|\nabla^r_{g_t}\dot{g}^*_t|_{g_t}|\nabla^p_{g_t}\dot{g}^*_t|_{g_t}\\
&\quad + C\sum_{r=2}^{p-1}\left[|\nabla^{p-r-1}_{g_t}\nabla^{*\Omega}_{g_t}\mathcal{R}_{g_t}|_{g_t} + |\nabla^{p-r}_{g_t}\mathcal{R}_{g_t}|_{g_t}\right]|\nabla^r_{g_t}\dot{g}^*_t|_{g_t}|\nabla^p_{g_t}\dot{g}^*_t|_{g_t}\\
&\quad + C\,e^{-t}|\nabla^{p-2}_{g_t}\nabla^{*\Omega}_{g_t}\mathcal{R}_{g_t}|_{g_t}|\nabla^p_{g_t}\dot{g}^*_t|_{g_t},
\end{aligned}$$

thanks to the exponential estimate (11.5). Integrating with respect to the volume form Ω we obtain the inequality

$$\begin{aligned}
&2\,\frac{d}{dt}\int_X |\nabla^p_{g_t}\dot{g}^*_t|^2_{g_t}\Omega\\
&\leqslant -2\int_X |\nabla^{p+1}_{g_t}\dot{g}^*_t|^2_{g_t}\,\Omega + C\int_X |\nabla^p_{g_t}\dot{g}^*_t|^2_{g_t}\,\Omega\\
&\quad + C\sum_{r=1}^{p-1}\int_X |\nabla^{p-r}_{g_t}\dot{g}^*_t|_{g_t}|\nabla^r_{g_t}\dot{g}^*_t|_{g_t}|\nabla^p_{g_t}\dot{g}^*_t|_{g_t}\Omega\\
&\quad + C\sum_{r=2}^{p-1}\int_X \Big[|\nabla^{p-r-1}_{g_t}\nabla^{*\Omega}_{g_t}\mathcal{R}_{g_t}|_{g_t} \;+\; |\nabla^{p-r}_{g_t}\mathcal{R}_{g_t}|_{g_t}\Big]|\nabla^r_{g_t}\dot{g}^*_t|_{g_t}|\nabla^p_{g_t}\dot{g}^*_t|_{g_t}\,\Omega\\
&\quad + C\,e^{-t}\left[\int_X |\nabla^{p-2}_{g_t}\nabla^{*\Omega}_{g_t}\mathcal{R}_{g_t}|^2_{g_t}\,\Omega\right]^{1/2}\left[\int_X |\nabla^p_{g_t}\dot{g}^*_t|^2_{g_t}\,\Omega\right]^{1/2},
\end{aligned}$$

thanks to Hölder's inequality. In order to estimate the sums we need a result of Hamilton [Ham], restated in our setting as follows.

13.2. Hamilton's interpolation inequalities.

Lemma 18. *Let $p, q, r \in \mathbb{R}$ with $r \geqslant 1$ such that $1/p + 1/q = 1/r$. There exists a time independent constant $C > 0$ such that along the Ω-SRF we have the inequality*

$$\left[\int_X |\nabla_{g_t} A|^{2r}_{g_t}\,\Omega\right]^{1/r} \leqslant C\left[\int_X |\nabla^2_{g_t} A|^p_{g_t}\,\Omega\right]^{1/p}\left[\int_X |A|^q_{g_t}\,\Omega\right]^{1/q},$$

for all tensors A.

Proof. The argument here is the same as in Hamilton [Ham]. We include it for reader's convenience. We set $u := |\nabla_{g_t} A|^{2(r-1)}_{g_t}$ and we observe the identities

$$\begin{aligned}
\int_X |\nabla_{g_t} A|^{2r}_{g_t} dV_{g_t} &= \int_X \left\langle A, \nabla^*_{g_t}(u\nabla_{g_t} A)\right\rangle_{g_t} dV_{g_t}\\
&= \int_X \left\langle A, \Delta_{g_t} A\right\rangle_{g_t} |\nabla_{g_t} A|^{2(r-1)}_{g_t} dV_{g_t}\\
&\quad - \int_X \left\langle A, \nabla_{g_t} u \neg \nabla_{g_t} A\right\rangle_{g_t} dV_{g_t},
\end{aligned}$$

and the inequalities

$$\begin{aligned}
|\left\langle A, \Delta_{g_t} A\right\rangle_{g_t}| &\leqslant \sqrt{n}\,|A|_{g_t}|\nabla^2_{g_t} A|_{g_t},\\
|\left\langle A, \nabla_{g_t} u \neg \nabla_{g_t} A\right\rangle_{g_t}| &\leqslant \sqrt{n}\,|A|_{g_t}|\nabla_{g_t} A|_{g_t}|\nabla_{g_t} u|_{g_t}.
\end{aligned}$$

Expanding the vector $\nabla_{g_t} u$ we infer the identities

$$\begin{aligned}\nabla_{g_t} u &= (r - 1)\, |\nabla_{g_t} A|_{g_t}^{2(r-2)}\, \nabla_{g_t} |\nabla_{g_t} A|_{g_t}^2 \\ &= 2\,(r - 1)\, |\nabla_{g_t} A|_{g_t}^{2(r-2)} \left\langle \nabla_{g_t, e_k} \nabla_{g_t} A, \nabla_{g_t} A \right\rangle_{g_t} e_k,\end{aligned}$$

where $(e_k)_k$ is a g_t-orthonormal basis. Thus,

$$|\nabla_{g_t} u|_{g_t} \leqslant 2\,(r - 1)\sqrt{n}\, |\nabla_{g_t} A|_{g_t}^{2r-3}\, |\nabla_{g_t}^2 A|_{g_t}.$$

Combining the previous inequalities we infer

$$\int_X |\nabla_{g_t} A|_{g_t}^{2r}\, dV_{g_t} \leqslant C_{r,n} \int_X |A|_{g_t} |\nabla_{g_t}^2 A|_{g_t} |\nabla_{g_t} A|_{g_t}^{2r-2}\, dV_{g_t},$$

with $C_{r,n} := 2\,(r - 1)\, n + \sqrt{n}$. This combined with (11.3) implies the inequality

$$\int_X |\nabla_{g_t} A|_{g_t}^{2r}\, \Omega \leqslant C \int_X |A|_{g_t} |\nabla_{g_t}^2 A|_{g_t} |\nabla_{g_t} A|_{g_t}^{2r-2}\, \Omega.$$

We can estimate the last integral using Hölder's inequality with

$$\frac{1}{p} + \frac{1}{q} + \frac{r-1}{r} = 1,$$

in order to obtain

$$\begin{aligned}&\int_X |\nabla_{g_t} A|_{g_t}^{2r}\, \Omega \\ &\qquad \leqslant C \left[\int_X |\nabla_{g_t}^2 A|_{g_t}^p\, \Omega\right]^{1/p} \left[\int_X |A|_{g_t}^q\, \Omega\right]^{1/q} \left[\int_X |\nabla_{g_t} A|_{g_t}^{2r}\, \Omega\right]^{1-1/r},\end{aligned}$$

and hence the required conclusion. □

As a corollary of this inequality Hamilton obtains in [Ham] the following two estimates. For all $r = 1, \ldots, p-1$,

$$\int_X |\nabla_{g_t}^r A|_{g_t}^{2p/r}\, \Omega \leqslant C \left[\max_X |A|_{g_t}\right]^{2(p/r-1)} \int_X |\nabla_{g_t}^p A|_{g_t}^2\, \Omega, \tag{13.2}$$

$$\int_X |\nabla_{g_t}^r A|_{g_t}^2\, \Omega \leqslant C \left[\int_X |\nabla_{g_t}^p A|_{g_t}^2\, \Omega\right]^{r/p} \left[\int_X |A|_{g_t}^2\, \Omega\right]^{1-r/p}. \tag{13.3}$$

13.3. Interpolation of the H^p-norms. We estimate now the integral

$$I_1 := \sum_{r=1}^{p-1} \int_X |\nabla_{g_t}^{p-r} \dot{g}_t^*|_{g_t} |\nabla_{g_t}^r \dot{g}_t^*|_{g_t} |\nabla_{g_t}^p \dot{g}_t^*|_{g_t}\, \Omega\, .$$

Indeed using Hölder's inequality we obtain

$$\int_X |\nabla^{p-r}_{g_t}\dot{g}^*_t|_{g_t}|\nabla^r_{g_t}\dot{g}^*_t|_{g_t}|\nabla^p_{g_t}\dot{g}^*_t|_{g_t}\,\Omega \leqslant$$
$$\left[\int_X |\nabla^{p-r}_{g_t}\dot{g}^*_t|^{2p/(p-r)}_{g_t}\,\Omega\right]^{(p-r)/2p}\left[\int_X |\nabla^r_{g_t}\dot{g}^*_t|^{2p/r}_{g_t}\,\Omega\right]^{r/2p}\left[\int_X |\nabla^p_{g_t}\dot{g}^*_t|^2_{g_t}\,\Omega\right]^{1/2}.$$

This combined with Hamilton's inequality (13.2) implies the estimate

$$I_1 \leqslant C\int_X |\nabla^p_{g_t}\dot{g}^*_t|^2_{g_t}\,\Omega.$$

In a similar way we can estimate the integral

$$I_2 := \sum_{r=2}^{p-1}\int_X |\nabla^{p-r-1}_{g_t}\nabla^{*_\Omega}_{g_t}\mathcal{R}_{g_t}|_{g_t}|\nabla^r_{g_t}\dot{g}^*_t|_{g_t}|\nabla^p_{g_t}\dot{g}^*_t|_{g_t}\,\Omega.$$

Indeed as before we obtain the inequality

$$\int_X |\nabla^{p-r-1}_{g_t}\nabla^{*_\Omega}_{g_t}\mathcal{R}_{g_t}|_{g_t}|\nabla^r_{g_t}\dot{g}^*_t|_{g_t}|\nabla^p_{g_t}\dot{g}^*_t|_{g_t}\,\Omega \leqslant$$
$$\left[\int_X |\nabla^{p-r-1}_{g_t}\nabla^{*_\Omega}_{g_t}\mathcal{R}_{g_t}|^{\frac{2p}{p-r}}_{g_t}\Omega\right]^{\frac{p-r}{2p}}\left[\int_X |\nabla^r_{g_t}\dot{g}^*_t|^{\frac{2p}{r}}_{g_t}\,\Omega\right]^{\frac{r}{2p}}\left[\int_X |\nabla^p_{g_t}\dot{g}^*_t|^2_{g_t}\,\Omega\right]^{\frac{1}{2}}.$$

Moreover the trivial inequality

$$\frac{p}{p-r} \quad\leqslant\quad \frac{p-3}{p-3-r},$$

implies the estimates

$$\begin{aligned}
&\int_X |\nabla^{p-r-1}_{g_t}\,\nabla^{*_\Omega}_{g_t}\mathcal{R}_{g_t}|^{2p/(p-r)}_{g_t}\,\Omega\\
&\leqslant \int_X \Omega + \int_X |\nabla^{p-r-1}_{g_t}\,\nabla^{*_\Omega}_{g_t}\mathcal{R}_{g_t}|^{2(p-3)/(p-3-r)}_{g_t}\,\Omega\\
&\leqslant \int_X \Omega + \int_X |\nabla^{p-3}_{g_t}\,\nabla^{*_\Omega}_{g_t}\mathcal{R}_{g_t}|^2_{g_t}\,\Omega,
\end{aligned}$$

by (13.2) since $|\nabla^{*_\Omega}_{g_t}\mathcal{R}_{g_t}|_{g_t} \leqslant C$ thanks to the uniform C^1-bound on g_t. Using again (13.2) we infer the estimate

$$I_2 \leqslant C\sum_{r=2}^{p-1}\left[1\;+\;\int_X |\nabla^{p-3}_{g_t}\nabla^{*_\Omega}_{g_t}\mathcal{R}_{g_t}|^2_{g_t}\,\Omega\right]^{(p-r)/2p}\left[\int_X |\nabla^p_{g_t}\dot{g}^*_t|^2_{g_t}\,\Omega\right]^{1/2+r/2p}.$$

We estimate finally the integral

$$I_3 \quad:=\quad \sum_{r=2}^{p-1}\int_X |\nabla^{p-r}_{g_t}\mathcal{R}_{g_t}|_{g_t}|\nabla^r_{g_t}\dot{g}^*_t|_{g_t}|\nabla^p_{g_t}\dot{g}^*_t|_{g_t}\,\Omega\,.$$

As before using the trivial inequality

$$\frac{p}{p-r} \quad\leqslant\quad \frac{p-2}{p-2-r},$$

we obtain the estimates

$$\int_X |\nabla^{p-r}_{g_t}\mathcal{R}_{g_t}|^{2p/(p-r)}_{g_t}\,\Omega \leqslant \int_X \Omega + \int_X |\nabla^{p-r}_{g_t}\mathcal{R}_{g_t}|^{2(p-2)/(p-2-r)}_{g_t}\,\Omega$$
$$\leqslant \int_X \Omega + \int_X |\nabla^{p-2}_{g_t}\mathcal{R}_{g_t}|^2_{g_t}\,\Omega,$$

by (13.2). Thus

$$I_3 \leqslant C\sum_{r=2}^{p-1}\left[1 \;+\; \int_X |\nabla^{p-2}_{g_t}\mathcal{R}_{g_t}|^2_{g_t}\,\Omega\right]^{(p-r)/2p}\left[\int_X |\nabla^{p}_{g_t}\dot{g}^*_t|^2_{g_t}\,\Omega\right]^{1/2+r/2p}.$$

In conclusion for all integers $p>1$ we have the estimate

$$\begin{aligned}
&2\,\frac{d}{dt}\int_X |\nabla^{p}_{g_t}\dot{g}^*_t|^2_{g_t}\,\Omega\\
&\leqslant -2\int_X |\nabla^{p+1}_{g_t}\dot{g}^*_t|^2_{g_t}\,\Omega + C\int_X |\nabla^{p}_{g_t}\dot{g}^*_t|^2_{g_t}\,\Omega\\
&\quad + C\sum_{r=2}^{p-1}\left[1 \;+\; \int_X |\nabla^{p-3}_{g_t}\,\nabla^{*_\Omega}_{g_t}\mathcal{R}_{g_t}|^2_{g_t}\,\Omega\right]^{(p-r)/2p}\left[\int_X |\nabla^{p}_{g_t}\dot{g}^*_t|^2_{g_t}\,\Omega\right]^{1/2+r/2p}\\
&\quad + C\sum_{r=2}^{p-1}\left[1 \;+\; \int_X |\nabla^{p-2}_{g_t}\mathcal{R}_{g_t}|^2_{g_t}\,\Omega\right]^{(p-r)/2p}\left[\int_X |\nabla^{p}_{g_t}\dot{g}^*_t|^2_{g_t}\,\Omega\right]^{1/2+r/2p}\\
&\quad + C\,e^{-t}\left[\int_X |\nabla^{p-2}_{g_t}\,\nabla^{*_\Omega}_{g_t}\mathcal{R}_{g_t}|^2_{g_t}\,\Omega\right]^{1/2}\left[\int_X |\nabla^{p}_{g_t}\dot{g}^*_t|^2_{g_t}\Omega\right]^{1/2}.
\end{aligned}$$

We set $\Gamma_t := \nabla_{g_t} - \nabla_{g_0}$, and we observe the inequality

$$|\nabla^{p}_{g_t}\mathcal{R}_{g_t}|_{g_t} \leqslant C + C\sum_{h=1}^{p-1}\sum_{q=1}^{h}\sum_{r_1+\ldots+r_{h-q+1}=q}\;\prod_{j=1}^{h-q+1}|\nabla^{r_j}_{g_0}\Gamma_t|_{g_0},$$

with $r_j = 0,\ldots,q$. Thus by using Jensen's inequality we obtain

$$|\nabla^{p}_{g_t}\mathcal{R}_{g_t}|^2_{g_t} \leqslant C + C\sum_{h=1}^{p-1}\sum_{q=1}^{h}\sum_{r_1+\ldots+r_{h-q+1}=q}\;\prod_{j=1}^{h-q+1}|\nabla^{r_j}_{g_0}\Gamma_t|^2_{g_0},$$

with $r_j = 0,\ldots,q$. Moreover using Hölder's inequality, the C^1-uniform estimate on g_t and the inequality (13.2) we infer the estimates

$$\int_X \prod_{j=1}^{h-q+1}|\nabla^{r_j}_{g_0}\Gamma_t|^2_{g_0}\,\Omega \leqslant \prod_{j=1}^{h-q+1}\left[\int_X |\nabla^{r_j}_{g_0}\Gamma_t|^{2q/r_j}_{g_0}\,\Omega\right]^{r_j/q}$$
$$\leqslant C\int_X |\nabla^{q}_{g_0}\Gamma_t|^2_{g_0}\,\Omega\,.$$

In its turn for all $q > 1$ we have

$$\begin{aligned}
&\int_X |\nabla_{g_0}^{q-1}\Gamma_t|^2_{g_0}\,\Omega \\
&\leqslant \sum_{r,h=1}^{q} \int_X |\nabla_{g_0}^{q-r} g_t|_{g_0} |\nabla_{g_0}^{r} g_t|_{g_0} |\nabla_{g_0}^{q-h} g_t|_{g_0} |\nabla_{g_0}^{h} g_t|_{g_0}\,\Omega \\
&\leqslant \sum_{r,h=1}^{q} \left[\int_X |\nabla_{g_0}^{q-r} g_t|_{g_0}^{2q/(q-r)}\,\Omega\right]^{(q-r)/2p} \left[\int_X |\nabla_{g_0}^{r} g_t|_{g_0}^{2q/r}\,\Omega\right]^{r/2q} \\
&\quad \times \left[\int_X |\nabla_{g_0}^{q-h} g_t|_{g_0}^{2q/(q-h)}\,\Omega\right]^{(q-h)/2q} \left[\int_X |\nabla_{g_0}^{h} g_t|_{g_0}^{2q/h}\,\Omega\right]^{h/2q} \\
&\leqslant C \int_X |\nabla_{g_0}^{q} g_t|^2_{g_0}\,\Omega\,.
\end{aligned}$$

We deduce the estimate

$$\int_X |\nabla_{g_t}^{p}\mathcal{R}_{g_t}|^2_{g_t}\,\Omega \quad \leqslant C + C\sum_{r=1}^{p}\int_X |\nabla_{g_0}^{r} g_t|^2_{g_0}\,\Omega.$$

In a similar way we obtain the estimate

$$\int_X |\nabla_{g_t}^{p-1}\,\nabla_{g_t}^{*_\Omega}\mathcal{R}_{g_t}|^2_{g_t}\,\Omega \leqslant C + C\sum_{r=1}^{p}\int_X |\nabla_{g_0}^{r} g_t|^2_{g_0}\,\Omega.$$

13.4. Exponential decay of the H^p-norms. We assume by induction the uniform exponential estimates

$$\int_X |\nabla_{g_0}^{r}\dot{g}_t|^2_{g_0}\,\Omega \quad\leqslant\quad C_r\, e^{-\theta_p t}\,,$$

for all $r = 0,\dots p-1$, with $C_r, \theta_r > 0$. We deduce from the previous subsection that for all $q = p, p+1$,

$$\begin{aligned}
2\,\frac{d}{dt}\int_X |\nabla_{g_t}^{q}\dot{g}_t^*|^2_{g_t}\,\Omega \leqslant& -2\int_X |\nabla_{g_t}^{q+1}\dot{g}_t^*|^2_{g_t}\,\Omega + C\int_X |\nabla_{g_t}^{q}\dot{g}_t^*|^2_{g_t}\,\Omega \\
&+ C\sum_{r=2}^{q-1}\left[\int_X |\nabla_{g_t}^{q}\dot{g}_t^*|^2_{g_t}\,\Omega\right]^{1/2+r/2q} \\
&+ Ce^{-t}\left[1+\int_X |\nabla_{g_0}^{q-1} g_t|^2_{g_0}\Omega\right]^{1/2}\left[\int_X |\nabla_{g_t}^{q}\dot{g}_t^*|^2_{g_t}\,\Omega\right]^{1/2}.
\end{aligned}$$

Thus for $q = p$,

$$\begin{aligned}
2\,\frac{d}{dt}\int_X |\nabla_{g_t}^{p}\dot{g}_t^*|^2_{g_t}\,\Omega \leqslant& -2\int_X |\nabla_{g_t}^{p+1}\dot{g}_t^*|^2_{g_t}\,\Omega + C\int_X |\nabla_{g_t}^{p}\dot{g}_t^*|^2_{g_t}\,\Omega \\
&+ C\sum_{r=0}^{p-1}\left[\int_X |\nabla_{g_t}^{p}\dot{g}_t^*|^2_{g_t}\,\Omega\right]^{1/2+r/2p}.
\end{aligned}$$

Moreover Hamilton's inequality (13.3) implies

$$\int_X |\nabla^p_{g_t} \dot{g}_t^*|^2_{g_t}\, \Omega \leqslant C \left[\int_X |\nabla^{p+1}_{g_t} \dot{g}_t^*|^2_{g_t}\, \Omega\right]^{p/(p+1)} \left[\int_X |\dot{g}_t^*|^2_{g_t}\, \Omega\right]^{1/(p+1)}. \tag{13.4}$$

Now for any $\varepsilon > 0$ and all $x, y > 0$,

$$x^p\, y \leqslant C\, \varepsilon\, x^{p+1} + C\, \varepsilon^{-p}\, y^{p+1},$$

and applying this above gives

$$\int_X |\nabla^p_{g_t} \dot{g}_t^*|^2_{g_t}\, \Omega \leqslant C\, \varepsilon \int_X |\nabla^{p+1}_{g_t} \dot{g}_t^*|^2_{g_t}\, \Omega + C\, \varepsilon^{-p} \int_X |\dot{g}_t^*|^2_{g_t}\, \Omega,$$

and also

$$\begin{aligned} &\left[\int_X |\nabla^p_{g_t} \dot{g}_t^*|^2_{g_t}\, \Omega\right]^{\alpha} \\ &\qquad \leqslant C\, \varepsilon \left[\int_X |\nabla^{p+1}_{g_t} \dot{g}_t^*|^2_{g_t}\, \Omega\right]^{\alpha} + C\, \varepsilon^{-p} \left[\int_X |\dot{g}_t^*|^2_{g_t}\, \Omega\right]^{\alpha}, \end{aligned} \tag{13.5}$$

with $\alpha := 1/2 + r/2p$. If we choose ε sufficiently small we deduce

$$\begin{aligned} 2\,\frac{d}{dt} \int_X |\nabla^p_{g_t} \dot{g}_t^*|^2_{g_t} \Omega \leqslant &- \int_X |\nabla^{p+1}_{g_t} \dot{g}_t^*|^2_{g_t}\, \Omega + C\, e^{-\,\delta t} \\ &+ C \sum_{r=0}^{p-1} \left[\int_X |\nabla^p_{g_t} \dot{g}_t^*|^2_{g_t}\, \Omega\right]^{1/2\,+\,r/2p}. \end{aligned}$$

thanks to (11.2). In order to estimate the last integral term we distinguish two cases. If for some $t > 0$

$$\int_X |\nabla^{p+1}_{g_t} \dot{g}_t^*|^2_{g_t}\, \Omega \leqslant 1,$$

then

$$C \sum_{r=0}^{p-1} \left[\int_X |\nabla^p_{g_t} \dot{g}_t^*|^2_{g_t}\, \Omega\right]^{1/2+r/2p} \quad \leqslant \quad C'\, e^{-\,\delta t/(p+1)}.$$

thanks to (13.4) and (11.2). Thus at this time we have

$$2\,\frac{d}{dt} \int_X |\nabla^p_{g_t} \dot{g}_t^*|^2_{g_t}\, \Omega \leqslant C\, e^{-\,\delta t/(p+1)}. \tag{13.6}$$

In the case $t > 0$ satisfies

$$\int_X |\nabla^{p+1}_{g_t} \dot{g}_t^*|^2_{g_t}\, \Omega > 1,$$

then

$$C \sum_{r=0}^{p-1} \left[\int_X |\nabla^p_{g_t} \dot{g}_t^*|^2_{g_t}\, \Omega\right]^{1/2+r/2p} \leqslant C'\, \varepsilon \int_X |\nabla^{p+1}_{g_t} \dot{g}_t^*|^2_{g_t}\, \Omega + C'\, \varepsilon^{-p} e^{-\,\delta t/2},$$

thanks to (13.5) and (11.2). Thus choosing $\varepsilon > 0$ sufficiently small we infer at this time the estimate

$$2\,\frac{d}{dt}\int_X |\nabla^p_{g_t}\dot{g}^*_t|^2_{g_t}\,\Omega \leqslant C\,e^{-\,\delta t/2}.$$

We deduce that the estimate (13.6) holds for all times $t > 0$. This implies the uniform estimate

$$\int_X |\nabla^p_{g_t}\dot{g}_t|^2_{g_t}\,\Omega = \int_X |\nabla^p_{g_t}\dot{g}^*_t|^2_{g_t}\,\Omega \leqslant C_p. \tag{13.7}$$

We observe now the inequality

$$\begin{aligned}|\nabla^p_{g_0}\dot{g}_t|_{g_0} \leqslant{}& |\nabla^p_{g_t}\dot{g}_t|_{g_0}\\ &+ C\sum_{h=0}^{p-1}\sum_{q=0}^{h}\sum_{r_1+\ldots+r_{h-q+1}=q}\prod_{j=1}^{h-q+1}|\nabla^{r_j}_{g_0}\Gamma_t|_{g_0}|\nabla^{p-1-h}_{g_0}\dot{g}_t|_{g_0},\end{aligned}$$

with $r_j = 0,\ldots,q$. Thus using Jensen's inequality we obtain

$$\begin{aligned}|\nabla^p_{g_0}\dot{g}_t|^2_{g_0} \leqslant{}& |\nabla^p_{g_t}\dot{g}_t|^2_{g_0}\\ &+ C\sum_{h=0}^{p-1}\sum_{q=0}^{h}\sum_{r_1+\ldots+r_{h-q+1}=q}\prod_{j=1}^{h-q+1}|\nabla^{r_j}_{g_0}\Gamma_t|^2_{g_0}|\nabla^{p-1-h}_{g_0}\dot{g}_t|^2_{g_0}.\end{aligned}$$

Let $k := p-1-h$ and $m := k+q \leqslant p-1$. Then Hölder's inequality combined with the L^2 estimate of $|\nabla^{p-1}_{g_0}\Gamma_t|_{g_0}$ given in the previous subsection and combined with Hamilton's interpolation inequality (13.2) provides the estimate

$$\begin{aligned}&\int_X\prod_{j=1}^{h-q+1}|\nabla^{r_j}_{g_0}\Gamma_t|^2_{g_0}|\nabla^{p-1-h}_{g_t}\dot{g}_t|^2_{g_0}\,\Omega\\ &\leqslant \prod_{j=1}^{h-q+1}\left[\int_X|\nabla^{r_j}_{g_0}\Gamma_t|^{2m/r_j}_{g_0}\,\Omega\right]^{r_j/m}\left[\int_X|\nabla^k_{g_0}\dot{g}_t|^{2m/k}_{g_0}\,\Omega\right]^{k/m}\\ &\leqslant C\left[\int_X|\nabla^m_{g_0}\Gamma_t|^2_{g_0}\,\Omega\right]^{q/m}\left[\int_X|\nabla^m_{g_0}\dot{g}_t|^2_{g_0}\,\Omega\right]^{k/m}\\ &\leqslant C\left[\int_X|\nabla^{m+1}_{g_0}g_t|^2_{g_0}\,\Omega\right]^{q/m}e^{-\theta_{k,m}t},\end{aligned}$$

$\theta_{k,m} > 0$, thanks to the inductive assumption. (When $r_j = 0$ or $k = 0$ in the previous estimate it means just that we are taking the corresponding L^∞-norms, which are bounded by the uniform C^1-estimate.) Thus for some $\tau_p, \rho_p > 0$, we have

$$\int_X|\nabla^p_{g_0}\dot{g}_t|^2_{g_0}\,\Omega \leqslant C\,e^{-\tau_p t}\int_X|\nabla^p_{g_0}g_t|^2_{g_0}\,\Omega + C\left(1\;+\;e^{-\rho_p t}\right),$$

thanks to the uniform estimate (13.7). We set

$$N_{p,t} := \int_X |\nabla^p_{g_0} g_t|^2_{g_0} \, \Omega \, .$$

We infer the first order differential inequality

$$\dot{N}_{p,t} \leqslant C \, N_{p,t} \, e^{-\tau_p t} + C \left(1 \; + \; e^{-\rho_p t}\right) ,$$

We deduce by Gronwall's inequality

$$N_{p,t} \leqslant e^{C \int_0^t e^{-\tau_p s} ds} \left[N_{p,0} + C \int_0^t \left(1 \; + \; e^{-\rho_p s}\right) ds \right] \leqslant C'(1 \; + \; t).$$

The fact that the function $t^{1/2} e^{-t}$ is bounded for $t \geqslant 0$ implies that for $q = p+1$,

$$2 \, \frac{d}{dt} \int_X |\nabla^q_{g_t} \dot{g}^*_t|^2_{g_t} \, \Omega \leqslant -2 \int_X |\nabla^{q+1}_{g_t} \dot{g}^*_t|^2_{g_t} \, \Omega + C \int_X |\nabla^q_{g_t} \dot{g}^*_t|^2_{g_t} \, \Omega$$
$$+ \, C \sum_{r=0}^{q-1} \left[\int_X |\nabla^q_{g_t} \dot{g}^*_t|^2_{g_t} \, \Omega \right]^{1/2 + r/2q} .$$

We deduce from the same argument showing (13.7), the uniform estimate

$$\int_X |\nabla^{p+1}_{g_t} \dot{g}^*_t|^2_{g_t} \, \Omega \leqslant C_{p+1},$$

and thus

$$\int_X |\nabla^p_{g_t} \dot{g}_t|^2_{g_t} \, \Omega = \int_X |\nabla^p_{g_t} \dot{g}^*_t|^2_{g_t} \, \Omega \leqslant C_p \, e^{-\, \delta t/(p+1)},$$

thanks to (13.4) and (11.2). Applying the previous argument to this improved estimate we infer

$$\dot{N}_{p,t} \leqslant C \, N_{p,t} \, e^{-\, \tau_p t} + C \, e^{-\, \rho_p t},$$

and thus $N_{p,t} \leqslant C$ thanks to Gronwall's inequality. We deduce the conclusion of the induction.

$$\dot{N}_{p,t} = \int_X |\nabla^p_{g_0} \dot{g}_t|^2_{g_0} \, \Omega \leqslant C_p \, e^{-\theta_p t}.$$

Thus using the Sobolev estimate we infer for all times $t > 0$ the inequality

$$|\nabla^p_{g_0} \dot{g}_t|_{g_0} \leqslant C_p \, e^{-\varepsilon_p t},$$

which implies the convergence of the integral

$$\int_0^{+\infty} |\nabla^p_{g_0} \dot{g}_t|_{g_0} \, dt < + \, \infty,$$

and thus the existence of the tensor

$$A_p := \int_0^{+\infty} \nabla^p_{g_0} \dot{g}_t \, dt,$$

thanks to Bochner's theorem. Moreover we have the exponential estimate

$$|A_p \;-\; \nabla^p_{g_0} g_t|_{g_0} \leqslant \int_t^{+\infty} |\nabla^p_{g_0} \dot{g}_s|_{g_0}\, ds \leqslant C_p\, e^{-\,\varepsilon_p t}.$$

A basic calculus fact combined with an induction on p implies that $A_p = \nabla^p_{g_0} g_\infty$. This concludes the proof of the exponential convergence of the Ω-SRF.

14. The Soliton-Kähler–Ricci Flow

Let (X, J) be a complex manifold. A J-invariant Kähler metric g is called a J-Kähler–Ricci soliton (in short J-KRS) if there exist a smooth volume form $\Omega > 0$ such that $g = \operatorname{Ric}_g(\Omega)$.

The discussion below will show that if a compact Kähler manifold admit a Kähler–Ricci soliton g then this manifold is Fano and the choice of Ω corresponding to g is unique up to a normalizing constant.

We recall first that any smooth volume form $\Omega > 0$ over a complex manifold (X, J) of complex dimension n induces a hermitian metric h_Ω over the canonical bundle $K_{X,J} := \Lambda^{n,0}_J T^*_X$ given by the formula

$$h_\Omega(\alpha, \beta) := \frac{n!\, i^{n^2} \alpha \wedge \bar{\beta}}{\Omega}.$$

By abuse of notations we will denote by Ω^{-1} the metric h_Ω. The dual metric h^*_Ω on the anti-canonical bundle $K^{-1}_{X,J} = \Lambda^{n,0}_J T_X$ is given by the formula

$$h^*_\Omega(\xi, \eta) = (-i)^{n^2} \Omega\,(\xi, \bar{\eta})\,/n!.$$

Abusing notations again, we denote by Ω the dual metric h^*_Ω. We define the Ω-Ricci form

$$\operatorname{Ric}_J(\Omega) := i\, \mathcal{C}_\Omega\big(K^{-1}_{X,J}\big) = -i\, \mathcal{C}_{\Omega^{-1}}\big(K_{X,J}\big),$$

where $\mathcal{C}_h(L)$ denotes the Chern curvature of a hermitian line bundle. In particular we observe the identity $\operatorname{Ric}_J(\omega) = \operatorname{Ric}_J(\omega^n)$. We recall also that for any J-invariant Kähler metric g the associated symplectic form $\omega := gJ$ satisfies the elementary identity

$$\operatorname{Ric}(g) = -\operatorname{Ric}_J(\omega) J. \tag{14.1}$$

Moreover for all twice differentiable function f,

$$\nabla_g\, d\, f = -\big(i\, \partial_J \bar{\partial}_J\, f\big) J + g\, \bar{\partial}_{T_{X,J}} \nabla_g\, f.$$

(See the decomposition formula (19.7) in the appendix.) We infer the decomposition identity

$$\operatorname{Ric}_g(\Omega) = -\operatorname{Ric}_J(\Omega) J + g\, \bar{\partial}_{T_{X,J}} \nabla_g \log \frac{dV_g}{\Omega}. \tag{14.2}$$

Thus a J-invariant Kähler metric g is a J-KRS iff there exist a smooth volume form $\Omega > 0$ such that

$$\begin{cases} g = -\operatorname{Ric}_{_J}(\Omega)J, \\ \bar{\partial}_{_{T_{X,J}}} \nabla_g \log \dfrac{dV_g}{\Omega} = 0\ . \end{cases}$$

The first equation of this system implies that (X, J) must be a Fano variety. We can translate the notion of Kähler–Ricci soliton in symplectic terms. In fact let (X, J_0) be a Fano manifold of complex dimension n, let $c_1 := c_1(X, [J_0])$, where $[J_0]$ is the co-boundary class of the complex structure J_0 and set

$$\mathcal{J}^+_{X,J_0} := \Big\{ J \in [J_0] \ \mid \ N_J = 0\,,\ \exists\, \omega \ \in\ \mathcal{K}^{2\pi c_1}_{_J} \Big\}\,,$$

where N_J denotes the Nijenhuis tensor and

$$\mathcal{K}^{2\pi c_1}_{_J} := \Big\{ \omega \in 2\pi c_1 \mid \omega \ = \ J^* \omega J\,, -\omega J \ > \ 0 \Big\},$$

is the set of J-invariant Kähler forms $\omega \in 2\pi c_1$. It is clear that for any complex structure $J \in \mathcal{J}^+_{X,J_0}$ and any form $\omega \in \mathcal{K}^{2\pi c_1}_{_J}$ there exist a unique smooth volume form $\Omega > 0$ with $\int_X \Omega = (2\pi c_1)^n$ such that $\omega = \operatorname{Ric}_{_J}(\Omega)$.

This induces an inverse functional $\operatorname{Ric}^{-1}_{_J}$ such that $\Omega = \operatorname{Ric}^{-1}_{_J}(\omega)$. With this notation we infer that a J-invariant form $\omega \in 2\pi c_1$ is the symplectic form associated to a J-KRS if and only if $0 < g := -\omega J$ and

$$\bar{\partial}_{_{T_{X,J}}} \nabla_g \log \frac{\omega^n}{\operatorname{Ric}^{-1}_{_J}(\omega)} = 0.$$

In equivalent volume terms we say that a smooth volume form $\Omega > 0$ with $\int_X \Omega = (2\pi c_1)^n$ is a J-Soliton-Volume-Form (in short J-SVF) if

$$\begin{cases} 0 < g := -\ \operatorname{Ric}_{_J}(\Omega)J, \\ \bar{\partial}_{_{T_{X,J}}} \nabla_g \log \dfrac{\operatorname{Ric}_{_J}(\Omega)^n}{\Omega} = 0. \end{cases}$$

We deduce a natural bijection between the sets $\{g \mid J\text{- KRS}\}$ and $\{\Omega \mid J\text{- SVF}\}$. We define also the set of Soliton-Volume-Forms over (X, J_0) as

$$\mathcal{SV}_{X,J_0} := \Big\{ \Omega \ > \ 0 \mid \int_X \Omega \ = \ (2\pi c_1)^n, \exists\, J \in \mathcal{J}^+_{X,J_0} \ :\ \Omega \text{ is a } J\text{- SVF} \Big\}.$$

We would like to investigate under which conditions $\mathcal{SV}_{X,J_0} \neq \emptyset$. For this purpose it seem natural to consider the SKRF. The Ω-SKRF equation/system (2.6) can be written in an equivalent way as

$$\begin{cases} \dfrac{d}{dt}\, \omega_t - i\, \partial_{_{J_t}} \bar{\partial}_{_{J_t}} f_t = \operatorname{Ric}_{_{J_t}}(\omega_t) \ - \ \omega_t, \\ \dfrac{d}{dt}\, J_t = J_t\, \bar{\partial}_{_{T_{X,J_t}}} \nabla_{g_t} f_t, \\ e^{-f_t} \omega_t^n = \Omega. \end{cases} \tag{14.3}$$

We observe also that (2.6) or (14.3) are equivalent to the system

$$\left\{\begin{array}{l} \dfrac{d}{dt}\,\omega_t = \operatorname{Ric}_{J_t}(\Omega) - \omega_t,\\ J_t := (\Phi_t^{-1})^* J_0 := \Big[(d\Phi_t\cdot J_0)\circ\Phi_t^{-1}\Big]\cdot d\Phi_t^{-1},\\ \dfrac{d}{dt}\,\Phi_t = -\left(\dfrac{1}{2}\,\nabla_{g_t}\log\dfrac{\omega_t^n}{\Omega}\right)\circ\Phi_t,\\ \Phi_0 = \operatorname{Id}_X\,. \end{array}\right. \tag{14.4}$$

In fact Lemma 27 combined with Lemma 28 in the appendix imply

$$\begin{aligned}\frac{d}{dt}\,(\Phi_t^* J_t) &= \Phi_t^*\left(\frac{d}{dt}\,J_t - \frac{1}{2}\,L_{\nabla_{g_t} f_t} J_t\right)\\ &= \Phi_t^*\left(\frac{d}{dt}\,J_t - J_t\,\bar{\partial}_{T_{X,J_t}}\nabla_{g_t} f_t\right) = 0.\end{aligned}$$

We define now $\hat{\omega}_t := \Phi_t^*\omega_t$, $\hat{g}_t := \Phi_t^* g_t = -\hat{\omega}_t J_0$ and we observe that the evolving family

$$(J_0, \hat{\omega}_t)_t = \Phi_t^*(J_t, \omega_t)_t,$$

represents a backward Kähler–Ricci flow over X. In fact the Kähler condition

$$\nabla_{\hat{g}_t} J_0 = \Phi_t^*\big(\nabla_{g_t} J_t\big) = 0,$$

holds and

$$\begin{aligned}\frac{d}{dt}\,\hat{\omega}_t &= \Phi_t^*\left(\frac{d}{dt}\,\omega_t - \frac{1}{2}\,L_{\nabla_{g_t} f_t}\,\omega_t\right)\\ &= \Phi_t^*\Big(\operatorname{Ric}_{J_t}(\omega_t) - \omega_t\Big)\\ &= \operatorname{Ric}_{J_0}(\hat{\omega}_t) - \hat{\omega}_t,\end{aligned}$$

by the formula (19.6) in the appendix. We observe that the volume form preserving condition $e^{-f_t}\omega_t^n = \Omega$ in the equation (14.3) is equivalent to the heat equation

$$2\,\frac{d}{dt}\,f_t = \operatorname{Tr}_{\omega_t}\frac{d}{dt}\,\omega_t = -\Delta_{g_t} f_t + \operatorname{Scal}(g_t) - 2\,n, \tag{14.5}$$

with initial data $f_0 := \log\frac{\omega_0^n}{\Omega}$. In its turn this is equivalent to the heat equation

$$2\,\frac{d}{dt}\,\hat{f}_t = -\Delta_{\hat{g}_t}\hat{f}_t - |\nabla_{\hat{g}_t}\hat{f}_t|^2_{\hat{g}_t} + \operatorname{Scal}(\hat{g}_t) - 2\,n, \tag{14.6}$$

with same initial data $\hat{f}_0 := \log \frac{\omega_0^n}{\Omega}$. In fact let $\hat{f}_t := f_t \circ \Phi_t$ and observe that the evolution equation of Φ_t in (14.4) implies

$$\begin{aligned}\frac{d}{dt}\,\hat{f}_t &= \left(\frac{d}{dt}\,f_t\right)\circ\Phi_t + \left\langle(\nabla_{g_t} f_t)\circ\Phi_t\,,\frac{d}{dt}\,\Phi_t\right\rangle_{g_t\circ\Phi_t}\\ &= \left(\frac{d}{dt}\,f_t - \frac{1}{2}\,|\nabla_{g_t} f_t|^2_{g_t}\right)\circ\Phi_t\,.\end{aligned}$$

We observe also that the derivation identity

$$0 = \frac{d}{dt}\left(\Phi_t^{-1}\circ\Phi_t\right) = \left(\frac{d}{dt}\,\Phi_t^{-1}\right)\circ\Phi_t + d\,\Phi_t^{-1}\cdot\frac{d}{dt}\,\Phi_t,$$

combined with the evolution equation of Φ_t in the system (14.4) implies

$$2\,d\,\Phi_t\cdot\left(\frac{d}{dt}\,\Phi_t^{-1}\right)\circ\Phi_t = \nabla_{g_t} f_t\circ\Phi_t = d\,\Phi_t\cdot\nabla_{\hat{g}_t}\hat{f}_t.$$

We infer the evolution formula

$$\frac{d}{dt}\,\Phi_t^{-1} = \frac{1}{2}\left(\nabla_{\hat{g}_t}\hat{f}_t\right)\circ\Phi_t^{-1}. \tag{14.7}$$

In conclusion we deduce that the Ω-SKRF $(J_t\,,\,\omega_t)_{t\geqslant 0}$ is equivalent to the system of independent equations

$$\begin{cases}\dfrac{d}{dt}\,\hat{\omega}_t = \operatorname{Ric}_{J_0}(\hat{\omega}_t) - \hat{\omega}_t,\\ 2\,\dfrac{d}{dt}\,\hat{f}_t = -\Delta_{\hat{g}_t}\hat{f}_t - |\nabla_{\hat{g}_t}\hat{f}_t|^2_{\hat{g}_t} + \operatorname{Scal}(\hat{g}_t) - 2\,n,\\ e^{-\hat{f}_0}\hat{\omega}_0^n = \Omega,\end{cases}$$

by means of the gradient flow of diffeomorphisms (14.7).

Notation. Let (X, g, J) be a Kähler manifold with symplectic form $\omega := gJ$ and consider $\alpha \in \Lambda^2_{\mathbb{R}} T^*_X$. We define the endomorphism $\alpha^*_g := \omega^{-1}\alpha$. For example we will define the endomorphism

$$\operatorname{Ric}^*_J(\Omega)_g := \omega^{-1}\operatorname{Ric}_J(\Omega).$$

With this notation formula (14.2) implies the decomposition identity

$$\operatorname{Ric}^*_g(\Omega) = \operatorname{Ric}^*_J(\Omega)_g + \bar{\partial}_{T_{X,J}}\nabla_g \log\frac{dV_g}{\Omega}\,. \tag{14.8}$$

15. The Riemannian nature of the Soliton-Kähler–Ricci Flow

The goal of this section is to show that the Kähler structure along the SKRF comes for free from the SRF by means of a Lax type ODE for the evolving complex structures. For this purpose let $(J_t, g_t)_{t\geqslant 0}$ be a Ω-SKRF. Time deriving the identity $g_t = -\,\omega_t J_t$ we obtain

$$\begin{aligned}\frac{d}{dt}\, g_t &= -\frac{d}{dt}\,\omega_t\, J_t - \omega_t\, \frac{d}{dt}\, J_t\\ &= -\operatorname{Ric}_{J_t}(\Omega) J_t + \omega_t\, J_t - \omega_t\, J_t\, \bar{\partial}_{T_{X,J_t}} \nabla_{g_t} f_t\\ &= -\operatorname{Ric}_{J_t}(\Omega) J_t + g_t\, \bar{\partial}_{T_{X,J_t}} \nabla_{g_t} f_t - g_t\\ &= \operatorname{Ric}_{g_t}(\Omega) - g_t,\end{aligned}$$

thanks to the complex decomposition (14.2). We have obtained the evolving system of Kähler structures $(J_t, g_t)_{t\geqslant 0}$,

$$\begin{cases}\dfrac{d}{dt}\, g_t = \operatorname{Ric}_{g_t}(\Omega) - g_t,\\ 2\,\dfrac{d}{dt}\, J_t = J_t \nabla^2_{g_t} \log \dfrac{dV_{g_t}}{\Omega} - \nabla^2_{g_t} \log \dfrac{dV_{g_t}}{\Omega}\, J_t,\end{cases} \tag{15.1}$$

which is equivalent to (2.6). (The second equation in the system follows from the fact that in the Kähler case the Chern connection coincides with the Levi-Civita connection.) We observe that the identity (14.1) implies that the Ricci endomorphism

$$\operatorname{Ric}^*(g) = \operatorname{Ric}^*_J(\omega)_g,$$

is J-linear. Thus the system (15.1) is equivalent to the evolution of the couple (J_t, g_t) under the system

$$\begin{cases}\dot{g}_t = \operatorname{Ric}_{g_t}(\Omega) - g_t,\\ 2\,\dot{J}_t = J_t\, \dot{g}^*_t - \dot{g}^*_t J_t,\\ J_t^2 = -\mathbb{I}_{T_X}, \quad (J_t)^T_{g_t} = -J_t, \quad \nabla_{g_t} J_t = 0,\end{cases} \tag{15.2}$$

where for notation simplicity we set $\dot{g}_t := \frac{d}{dt} g_t$ and $\dot{J}_t := \frac{d}{dt} J_t$. Moreover $(J_t)^T_{g_t}$ denotes the transpose of J_t with respect to g_t. We recall now an elementary fact (see Lemma 4 in [Pal]).

Lemma 19. *Let $(g_t)_{t\geqslant 0}$ be a smooth family of Riemannian metrics and let $(J_t)_{t\geqslant 0}$ be a family of endomorphisms of T_X solution of the* ODE

$$2\,\dot{J}_t = J_t\, \dot{g}^*_t - \dot{g}^*_t J_t,$$

with initial conditions $J_0^2 = -\mathbb{I}_{T_X}$ and $(J_0)^T_{g_0} = -J_0$. Then this conditions are preserved in time i.e., $J_t^2 = -\mathbb{I}_{T_X}$ and $(J_t)^T_{g_t} = -J_t$ for all $t \geqslant 0$.

We deduce that the system (15.2) is equivalent to the system

$$\begin{cases}\dot{g}_t = \operatorname{Ric}_{g_t}(\Omega) - g_t,\\ 2\,\dot{J}_t = \ J_t\, \dot{g}^*_t - \dot{g}^*_t J_t,\\ \nabla_{g_t} J_t = 0,\end{cases} \tag{15.3}$$

with Kähler initial data (J_0, g_0). We show now how we can get rid of the last equation. We recall the following key result obtained in [Pal].

Proposition 2. *Let $(g_t)_{t\geqslant 0}$ be a smooth family of Riemannian metrics such that $\dot{g}_t \in \mathbb{F}_{g_t}$ and let $(J_t)_{t\geqslant 0}$ be a family of endomorphisms of T_X solution of the* ODE

$$\dot{J}_t = J_t\,\dot{g}_t^* - \dot{g}_t^* J_t,$$

with Kähler initial data (J_0, g_0). Then $(J_t, g_t)_{t\geqslant 0}$ is a smooth family of Kähler structures.

In particular using Lemma 1 we infer the following corollary which provides a simple way to generate Kähler structures.

Corollary 7. *Let $(J_t, g_t)_{t\geqslant 0} \subset C^\infty(X, \mathrm{End}_{\mathbb{R}}(T_X)) \times \mathcal{M}$ be the solution of the ODE*

$$\begin{cases} \dfrac{d}{dt}\,\dot{g}_t^* = 0, \\ 2\,\dfrac{d}{dt}\,J_t = \ J_t\,\dot{g}_t^* - \dot{g}_t^* J_t, \end{cases} \tag{15.4}$$

with (J_0, g_0) Kähler data and with $\nabla_{T_X, g_0}(\dot{g}_0^*)^p = 0$ for all $p \in \mathbb{Z}_{>0}$. Then $(J_t, g_t)_{t\geqslant 0}$ is a smooth family of Kähler structures.

From Proposition 2 we deduce the following fact which shows the Riemannian nature of the Ω-SKRF. Namely that the Ω-SKRF can be generated by the Ω-SRF.

Corollary 8. *Let $\Omega > 0$ be a smooth volume form over a Kähler manifold (X, J_0) and let $(g_t)_{t\geqslant 0}$ be a solution of the* Ω-SRF *with Kähler initial data (J_0, g_0). Then the family $(J_t, g_t)_{t\geqslant 0}$ with $(J_t)_{t\geqslant 0}$ the solution of the* ODE

$$2\,\dot{J}_t = J_t\,\dot{g}_t^* - \dot{g}_t^* J_t,$$

is a solution of the Ω-SKRF *equation.*

16. The set of Kähler prescattering data

We define the set of Kähler prescattering data as $\mathcal{S}_{\Omega,J} := \mathcal{S}_\Omega \cap \mathcal{K}_J$. Using the complex decomposition formula (14.8) we infer the equality

$$\mathcal{S}_{\Omega,J} = \left\{ g \in \mathcal{K}_J \ \middle|\ \partial^g_{T_{X,J}} \bar{\partial}_{T_{X,J}} \nabla_g \log \frac{dV_g}{\Omega} = -\bar{\partial}_{T_{X,J}} \mathrm{Ric}^*_J(\Omega)_g \right\} .$$

In fact the identity $d\,\mathrm{Ric}_J(\Omega) = 0$ is equivalent to the identity $\partial_J \mathrm{Ric}_J(\Omega) = 0$, which in its turn is equivalent to the identity

$$\partial^g_{T_{X,J}} \mathrm{Ric}^*_J(\Omega)_g = 0 \, .$$

We observe now the following quite elementary facts.

Lemma 20. *Let (X, J) be a Fano manifold and let $g \in \mathcal{S}_{\Omega,J}$ such that $\omega := gJ \in 2\pi c_1(X)$. Then g is a J-invariant KRS iff $\omega = \mathrm{Ric}_J(\Omega)$, iff $\mathrm{Ric}_g(\Omega)$ is J-invariant.*

Proof. In the case $\operatorname{Ric}^*_J(\Omega)_g = \mathbb{I}_{T_X}$ the condition $g \in \mathcal{S}_{\Omega,J}$ is equivalent to the condition

$$\bar{\partial}_{T_{X,J}} \nabla_g \log \frac{dV_g}{\Omega} = 0. \tag{16.1}$$

(i.e., the J-invariance of $\operatorname{Ric}_g(\Omega)$ and thus that g is a J-invariant KRS.) In fact in this case

$$\partial^g_{T_{X,J}} \bar{\partial}_{T_{X,J}} \nabla_g \log \frac{dV_g}{\Omega} = 0,$$

which by a standard Kähler identity implies

$$\bar{\partial}^{*_g}_{T_{X,J}} \bar{\partial}_{T_{X,J}} \nabla_g \log \frac{dV_g}{\Omega} = 0\ .$$

Thus an integration by parts yields the required identity (16.1).

On the other hand if we assume that $\operatorname{Ric}_g(\Omega)$ is J-invariant, i.e., we assume (16.1) then the condition $g \in \mathcal{S}_{\Omega,J}$ is equivalent to the condition

$$\bar{\partial}_{T_{X,J}} \operatorname{Ric}^*_J(\Omega)_g = 0.$$

For cohomology reasons,

$$\omega = \operatorname{Ric}_J(\Omega) + i\, \partial_J \bar{\partial}_J u$$

for some $u \in C^\infty(X, \mathbb{R})$. We deduce the equalities

$$0 = \bar{\partial}_{T_{X,J}} \left(i \partial_J \bar{\partial}_J u\right)^*_g = \bar{\partial}_{T_{X,J}} \partial^g_{T_{X,J}} \nabla_g\, u.$$

Using again a standard Kähler identity we infer

$$\partial^{*_g}_{T_{X,J}} \partial^g_{T_{X,J}} \nabla_g\, u = 0.$$

An integration by parts yields the conclusion $i\, \partial_J \bar{\partial}_J u \equiv 0$, i.e., $u \equiv 0$, which implies the required KRS equation. □

Lemma 21. *Let (X, J) be a Kähler manifold and let $(g_t)_{t \in [0,T)}$ be a smooth family of J-invariant Kähler metrics solution of the equation*

$$\dot{g}_t = \operatorname{Ric}_{g_t}(\Omega) - g_t\ .$$

Then this family is given by the formula

$$g_t = -\operatorname{Ric}_J(\Omega) J + \big(g_0 + \operatorname{Ric}_J(\Omega) J\big)\, e^{-t}\,,$$

with J-invariant Kähler initial data g_0 solution of the equation

$$\bar{\partial}_{T_{X,J}} \nabla_{g_0} \log \frac{dV_{g_0}}{\Omega} = 0\ .$$

Proof. The fact that g_t is J-invariant implies that $\dot{g}_t$ is also J-invariant. Then the decomposition formula (14.2) combined with the evolution equation of g_t provides

$$\dot{g}_t = -\operatorname{Ric}_J(\Omega) J - g_t,$$

which implies the required conclusion. □

From the previous lemmas we deduce directly the following corollary.

Corollary 9. *Let $g_0 \in \mathcal{S}_{\Omega,J}$ with $gJ \in 2\pi c_1(X)$ be an initial data for the Ω-SKRF such that the complex structure stays constant along the flow. Then g_0 is a J-invariant KRS and $g_t \equiv g_0 = -\operatorname{Ric}_J(\Omega)J$.*

The last statement in Theorem 2 follows directly from this corollary.

17. On the smooth convergence of the Soliton-Kähler–Ricci flow

We now prove the convergence statement in Theorem 2. According to the convergence result for the Ω-SRF obtained in Theorem 1 we just need to show the smooth convergence of the complex structures. We consider the differential system

$$\begin{cases} 2\,\dot{J}_t = [J_t\,, \dot{g}_t^*]\,, \\ 2\,\ddot{g}_t = -\Delta^{\Omega}_{g_t}\dot{g}_t - 2\,\dot{g}_t, \end{cases}$$

along the Ω-SRF and we recall the uniform estimates

$$|\dot{g}_t|_{g_t} \leqslant |\dot{g}_0|_{C^0(X),g_0}\, e^{-\delta\, t/2},$$
$$e^{-C} g_0 \leqslant g_t \leqslant e^{C} g_0.$$

We consider also the estimate of the norm

$$|\dot{J}_t|_{g_t} \leqslant \sqrt{2\,n}\,|J_t|_{g_t}\,|\dot{g}_t^*|_{g_t},$$

where the constant $\sqrt{2\,n}$ comes from the equivalence between the Riemannian norm and the operator norm on the space of endomorphisms of $T_{X,x}$. We observe now the trivial identities

$$|J_t|^2_{g_t} = \operatorname{Tr}_{\mathbb{R}}\left[J_t\,(J_t)^T_{g_t}\right] = -\operatorname{Tr}_{\mathbb{R}} J_t^2 = \operatorname{Tr}_{\mathbb{R}} \mathbb{I}_{T_X} = 2\,n.$$

We deduce the exponential estimate of the variation of the complex structure

$$|\dot{J}_t|_{g_t} \leqslant 2\,n\,|\dot{g}_0|_{C^0(X),g_0}\, e^{-\delta\, t/2},$$

and thus the convergence of the integral

$$\int_0^{+\infty} |\dot{J}_t|_{g_0} dt < +\infty\ .$$

In its turn this shows the existence of the integral

$$J_\infty := J_0 + \int_0^{+\infty} \dot{J}_t\, dt,$$

thanks to Bochner's theorem. Moreover we have the exponential estimate

$$|J_\infty\ -\ J_t|_{g_0} \leqslant \int_t^{+\infty} |\dot{J}_s|_{g_0}\, ds \leqslant C'\, e^{-\delta\, t/2}\ .$$

On the other hand the Kähler identity $\nabla_{g_t} J_t \equiv 0$ implies the equality

$$2\,\nabla^p_{g_t}\dot{J}_t = \left[J_t, \nabla^p_{g_t}\dot{g}^*_t\right],$$

for all $p \in \mathbb{N}$. We deduce the estimates

$$|\nabla^p_{g_t} \dot{J}_t|_{g_t} \leqslant \sqrt{2\,n}\,|J_t|_{g_t}\,|\nabla^p_{g_t} \dot{g}^*_t|_{g_t} \leqslant 2\,n\,C_p\,e^{-\,\varepsilon_p t}\,,$$

thanks to the exponential decay of the evolving Riemannian metrics for the SRF given by Theorem 1. The fact that the flow of Riemannian metrics $(g_t)_{t\geqslant 0}$ is uniformly bounded in time for any $C^p(X)$-norm implies the uniform estimate

$$|\nabla^p_{g_0} \dot{J}_t|_{g_0} \leqslant C'_p\,e^{-\,\varepsilon_p t}\,.$$

We infer the convergence of the integral

$$\int_0^{+\infty} |\nabla^p_{g_0} \dot{J}_t|_{g_0} dt < +\infty,$$

and thus the existence of the integral

$$I_p := \nabla^p_{g_0} J_0 + \int_0^{+\infty} \nabla^p_{g_0} \dot{J}_t\,dt.$$

We deduce the exponential estimate

$$|I_p \ - \ \nabla^p_{g_0} J_t|_{g_0} \leqslant \int_t^{+\infty} |\nabla^p_{g_0} \dot{J}_s|_{g_0} ds \leqslant C'_p\,e^{-\,\varepsilon_p t}.$$

A basic calculus fact combined with an induction on p implies $I_p = \nabla^p_{g_0} J_\infty$. We deduce that (J_∞, g_∞) is a Kähler structure. Then the convergence result in Theorem 1 implies that g_∞ is a J_∞-invariant KRS.

18. On the existence of scattering data over Fano manifolds

In this section we explain our program for the existence of scattering data over Fano manifolds. We observe first that Yau's solution of the Calabi's conjecture (see [Yau]) implies that a compact Kähler manifold (X, J) is Fano if and only if there exists a J-invariant Kähler metric with strictly positive Ricci curvature. We will denote by C^{ra} the class of real analytic sections. The once contracted differential Bianchi type identity (3.11) implies that the existence of positive prescatering data can be provided by a positive answer to the following conjecture.

Conjecture 1 (Existence of prescattering data)**.** *Let* (X, J, g_0) *be a compact Kähler manifold such that* $\mathrm{Ric}(g_0) > 0$*. Then there exists a* J*-invariant Kähler metric* $g \in C^{\mathrm{ra}}(X, S^2_{\mathbb{R}} T^*_X)$ *and* $f \in C^{\mathrm{ra}}(X, \mathbb{R})$ *such that*

$$\underline{\mathrm{div}}_g \mathcal{R}_g(\xi, \eta) = \mathcal{R}_g(\xi, \eta)\,\nabla_g\,f, \quad \forall\,\xi\,,\,\eta\,\in\,T_X, \tag{18.1}$$

$$\mathrm{Ric}(g) + \nabla_g\,d\,f > 0. \tag{18.2}$$

It was kindly pointed out to us by Robert Bryant that the positivity of the Ricci curvature implies the generic inversibility of the endomorphism $\mathcal{R}_g(\xi, \eta)$ and the equation (18.1) is equivalent to the equation

$$d\left[g\left(\mathcal{R}_g^{-1}\underline{\mathrm{div}}_g \mathcal{R}_g\right)\right] = 0\,. \tag{18.3}$$

Notice indeed that the assumption $\mathrm{Ric}(g_0) > 0$ implies the vanishing of the first Betti number. The study of the involutivity of the equation (18.3) is part of a joint program with Robert Bryant. We expect the existence of a metric g solution of the equation (18.3) sufficiently close to g_0 in the C^2-topology such that the potential f is with sufficiently small C^2-norm in order to insur the positivity of the coresponding Bakry–Emery–Ricci tensor. The analytic regularity is expected for any solution of (18.3). We assume a positive answer to the conjecture 1 and we proceed to the construction of the center of polarisation K. We start with a simple application of the spectral theorem.

Lemma 22. *Let (X, g) be a compact real analytic Riemannian manifold of dimension n and let $v \in C^{\mathrm{ra}}(X, S^2_{\mathbb{R}} T^*_X)$. Then there exists a finite family of disjoint open sets $U_j \subset X$, $j = 1, \ldots, N$ with $X \smallsetminus \cup_j U_j$ of zero measure, g-orthonormal frames $(\xi_{j,k})_{k=1}^n \subset C^{\mathrm{ra}}(U_j, T_X)$ and families of functions $(\lambda_{j,k})_{k=1}^n \subset C^{\mathrm{ra}}(U_j, \mathbb{R})$ such that $v^*_g \, \xi_{j,k} = \lambda_{j,k} \, \xi_{j,k}$ over U_j for all j and k.*

Proof. Let $U^0_j \subset X$, $j = 1, \ldots, N_0$ be a finite family of disjoint and connected real analytic coordinate open sets trivialising the tangent bundle such that the set $X \smallsetminus \cup_j U^0_j$ is of zero measure. It is sufficent to show our statement over any connected open set $U := U^0_j$. We denote for notation simplicity $A_x := v^*_g(x)$. We pick a nowhere vanishing vector field η with constant coefficients over U and we consider the nondecreasing family of real analytic sets

$$Z_k := \Big\{ x \in U \mid \eta \wedge A_x \eta \wedge \ldots \wedge A^k_x \eta \;=\; 0 \Big\}, \quad k \;=\; 1, \ldots, n-1.$$

We define also $m := \max \{k \mid Z_k \subsetneq U\} \geqslant 1$ and $U' := U \smallsetminus Z_m$. The fact that the map $x \mapsto A_x$ is real analytic implies the existence of functions $c_p \in C^\omega(U', \mathbb{R})$, $p = 1, \ldots, m$ such that

$$A^{m+1}_x \eta = \sum_{p=0}^m c_p(x) \, A^p_x \, \eta,$$

for all points $x \in U'$. Moreover there exists a real analytic subset $Y \subsetneq U'$ such that the roots of the polynomial

$$z \;\; \longmapsto \;\; P(x,z) := z^{m+1} - \sum_{p=0}^m c_p(x) \, z^p,$$

are given by complex valued real analytic functions over $U'' := U' \smallsetminus Y$. We infer the existence of a unique factorisation

$$P(x,z) = \prod_{j=1}^M \left[z^2 + a_j(x) \, z + b_j(x)\right] \times \prod_{k=1}^N \left[z - \lambda_k(x)\right],$$

with a_j, b_j, $\lambda_k \in C^{\rm ra}(U'',\mathbb{R})$ such that $a_j^2 < 4\,b_j$ over U'' for all $j = 1,\dots,M$. Replacing the variable z with A_x in the previous identity we infer

$$\begin{aligned} 0 &= \left[A_x^{m+1} - \sum_{p=0}^{m} c_p(x) A_x^p\right]\eta \\ &= \prod_{j=1}^{M}\left[A_x^2 + a_j(x)\,A_x + b_j(x)\mathbb{I}\right] \times \prod_{k=1}^{N}\left[A_x - \lambda_k(x)\mathbb{I}\right]\eta, \end{aligned}$$

for all $x \in U''$. The fact that A is g-symmetric combined with the inequality $a_j^2 < 4\,b_j$ over U'' implies that the endomorphisms $A_x^2 + a_j(x)A_x + b_j(x)\mathbb{I}$ are invertible for all $x \in U''$ and all $j = 1,\dots,M$. We deduce the identity

$$0 = \prod_{k=1}^{N}\left[A_x - \lambda_k(x)\mathbb{I}\right]\eta, \tag{18.4}$$

for all $x \in U''$. For any conected component V of U'' we define the nondecreasing family of real analytic sets

$$Y_{V,r} := \left\{ x \in V \mid \prod_{k=1}^{r}\left[A_x \;-\; \lambda_k(x)\,\mathbb{I}\right]\eta \;=\; 0 \right\}, \quad r = 1,\dots,N-1,$$

and the integer $R := \max\{r \mid Y_{V,r} \subsetneq V\}$ if the set $\{r \mid Y_{V,r} \subsetneq V\}$ is nonempty and $R := 0$ otherwise. The definition of R combined with the identity (18.4) implies

$$0 = \prod_{k=1}^{R+1}\left[A_x - \lambda_k(x)\,\mathbb{I}\right]\eta,$$

for all $x \in V' := V \smallsetminus Y_{V,R}$, i.e., the nowhere vanishing real analytic vector field

$$\xi := \prod_{k=1}^{R}(A - \lambda_k\,\mathbb{I})\,\eta,$$

over V' satisfies the eigenvector equation $A_x\,\xi_x = \lambda_{R+1}(x)\,\xi_x$ for all points $x \in V'$. We deduce that $\xi_1 := \xi/|\xi|_g$ is a g-unitary vector field which represents an eigenvector field of A over all the connected components of V'. We apply the previous argument to the g-symmetric and real analytic endomorphism section $A : \langle\xi\rangle_g^{\perp} \longrightarrow \langle\xi\rangle_g^{\perp}$ over all such components. Then the conclusion follows from a finite induction argument. □

Corollary 10. *Let (X, g) be a compact real analytic Riemannian manifold of dimension n and let $v \in C^{\rm ra}(X, S^2_{\mathbb{R}}T^*_X)$. Then there exists $K \in C^\infty(X, {\rm End}_{\mathbb{R}}(T_X))$ and an open set $U \subset X$ with zero measure complementary set $X \smallsetminus U$ such that $K(x)$ has n distinct real eigenvalues for all $x \in U$ and $[K, v_g^*] = 0$ over X.*

Proof. In the set up of the lemma, we pick functions $\varphi_j \in C^\infty(U_j, \mathbb{R}_{>0})$ with infinite vanishing order along the boundary of U_j. We infer the existence of $K \in C^\infty(X, \mathrm{End}(T_X))$ which is given by the expressions

$$K = \sum_{r=1}^{n} r\, \varphi_j\, \xi_{j,r}^* \otimes \xi_{j,r},$$

over U_j and $K \equiv 0$ over $X \smallsetminus U$ with $U := \cup_j U_j$. It is clear that K and U satisfy the required properties. □

For our construction of the scattering data, the choice of the center of polarisation K will correspond to $v = \mathrm{Ric}(g) + \nabla_g\, d f$, with g and f given by the solution of Conjecture 1. Once this center of polarisation K is choosen the analysis of the PDE's for the existence of a scattering data $g_{\mathrm{scat}} \in \mathcal{S}^K_{\Omega,+}$ will be performed over the infinite dimensional vector spaces

$$\mathbb{F}^\infty_g(K) := \left\{v \in C^\infty(X, S^2_{\mathbb{R}} T^*_X) \mid \left[K, \nabla^p_g v^*_g\right] \;=\; 0\,,\; \forall\, p \;\in\; \mathbb{Z}_{\geqslant 0}\right\}.$$

It is clear by the definition that any geometric differential operator defined via g preserve this spaces. Variations with values in $\mathbb{F}^K_g$ produce the most simple variation formulas for geometric objects such as the Levi-Civita connection, the curvature operator as well as the prescattering operator. We will initiate a Nash–Moser iteration method for the solution of the scattering problem. One can not expect to obtain convergence of this method using directly variations with values in $\mathbb{F}^K_g$ exept the case when the initial data is already scattering. We will use instead variations $v_k \in \mathbb{F}^\infty_{g_k}(K)$ with

$$|\nabla_{T_X, g_k} (v_k)^*_{g_k}|_{g_k} \leqslant \varepsilon_k,$$

where $g_{k+1} := g_k + v_k$ is the iterated metric. We will keep track of the norm of the iterated error terms produced in the scattering conditions with respect to ε_k. We expect to obtain a quadratic decay of the norms which insures the convergence of the iteration scheme.

19. Appendix

19.1. Weitzenböck type formulas.

Lemma 23. *For any $g \in \mathcal{M}$ and $u \in C^\infty(X, S^2 T^*_X)$ we have*

$$-\left(\nabla^{*_\Omega}_g \mathcal{D}_g\, u\right)^*_g = \nabla^{*_\Omega}_g \nabla_{T_X, g} u^*_g + \left(\nabla^{*_\Omega}_g \nabla_{T_X, g} u^*_g\right)^T_g - \Delta^\Omega_g\, u^*_g.$$

Proof. We need to show first that for any $h \in C^\infty\left(X, (T^*_X)^{\otimes 3}\right)$,

$$\left(\nabla^*_g\, h\right)^*_g = \nabla^*_g \left(\bullet \neg h\right)^*_g, \tag{19.1}$$

where the section $(\bullet \neg h)^*_g \in C^\infty\left(X, (T^*_X)^{\otimes 2} \otimes T_X\right)$ is given by the formula

$$(\bullet \neg h)^*_g\, (\xi, \eta) := (\xi \neg h)^*_g\, \eta.$$

In fact let (e_l) be a smooth local tangent frame and let ξ, η be arbitrary smooth vector fields defined in a neighborhood of an arbitrary point x_0 such that $\nabla_g e_l(x_0) = \nabla_g \xi(x_0) = \nabla_g \eta(x_0) = 0$. Then at the point x_0,

$$\begin{aligned}
\nabla_g^* h(\xi,\eta) &= -\nabla_g h\,(e_l, e_l, \xi, \eta)\\
&= -e_l \,.\, h\,(e_l, \xi, \eta)\\
&= -e_l \,.\, g\,\Big((e_l \neg h)_g^* \xi, \eta\Big)\\
&= -g\Big(\nabla_{g,e_l}\Big[(e_l \neg h)_g^* \xi\Big], \eta\Big)\\
&= -g\Big(\nabla_{g,e_l} (\bullet \neg h)_g^* (e_l, \xi), \eta\Big)\\
&= g\Big(\nabla_g^* (\bullet \neg h)_g^* \xi, \eta\Big)\,.
\end{aligned}$$

Applying the identity (19.1) to $h = \mathcal{D}_g u$ and using the expression

$$\nabla_g^{*_\Omega} \mathcal{D}_g\, u = \nabla_g^* \mathcal{D}_g\, u + \nabla_g f \neg \mathcal{D}_g\, u,$$

we deduce the formula

$$\Big(\nabla_g^{*_\Omega} \mathcal{D}_g\, u\Big)_g^* = \nabla_g^* (\bullet \neg \mathcal{D}_g\, u)_g^* + (\nabla_g f \neg \mathcal{D}_g\, u)_g^*\,.$$

In order to explicit this expression, for any $\mu \in C^\infty(X, T_X)$, we expand the term

$$\begin{aligned}
\mathcal{D}_g\, u\,(\mu, \xi, \eta) &= \nabla_g\, u\,(\xi, \mu, \eta) + \nabla_g\, u\,(\eta, \mu, \xi) - \nabla_g\, u\,(\mu, \xi, \eta)\\
&= g\left(\nabla_g\, u_g^*\,(\xi, \mu), \eta\right) + g\left(\nabla_g\, u_g^*\,(\eta, \mu), \xi\right)\\
&\quad - g\left(\nabla_g\, u_g^*\,(\mu, \xi), \eta\right)\\
&= g\Big(\nabla_{T_X, g} u_g^*\,(\xi, \mu), \eta\Big) + g\Big(\nabla_{T_X, g} u_g^*\,(\eta, \mu), \xi\Big)\\
&\quad + g\left(\nabla_g\, u_g^*\,(\mu, \eta), \xi\right)\\
&= -g\Big(\nabla_{T_X, g} u_g^*\,(\mu, \xi), \eta\Big) - g\Big(\nabla_{T_X, g} u_g^*\,(\mu, \eta), \xi\Big)\\
&\quad + g\left(\nabla_g\, u_g^*\,(\mu, \eta), \xi\right)\\
&= -g\Big(\nabla_{T_X, g} u_g^*\,(\mu, \xi), \eta\Big) - g\Big(\Big(\mu \neg \nabla_{T_X, g} u_g^*\Big)_g^T \xi, \eta\Big)\\
&\quad + g\left(\nabla_g\, u_g^*\,(\mu, \xi), \eta\right).
\end{aligned}$$

We deduce the identity

$$(\bullet \neg \mathcal{D}_g\, u)_g^* = -\nabla_{T_X, g} u_g^* - \Big(\nabla_{T_X, g} u_g^*\Big)_g^T + \nabla_g\, u_g^*,$$

and thus the expression

$$-\left(\nabla_g^{*\Omega}\mathcal{D}_g\, u\right)_g^* = \nabla_g^{*\Omega}\nabla_{T_X,g} u_g^* + \nabla_g^*\left(\nabla_{T_X,g} u_g^*\right)_g^T + \left(\nabla_g f \neg \nabla_{T_X,g} u_g^*\right)_g^T - \nabla_g^{*\Omega}\nabla_g\, u_g^*.$$

We observe now that at the point x_0 the following equalities hold:

$$\begin{aligned}\nabla_g^*\left(\bullet \neg \nabla_{T_X,g} u_g^*\right)_g^T \xi &= -\nabla_{g,e_l}\left(\nabla_{T_X,g} u_g^*\right)_g^T (e_l,\xi)\\ &= -\nabla_{g,e_l}\left[\left(e_l \neg \nabla_{T_X,g} u_g^*\right)_g^T \xi\right]\\ &= -\left[\nabla_{g,e_l}\left(e_l \neg \nabla_{T_X,g} u_g^*\right)_g^T\right]\xi\\ &= \left(\nabla_g^*\nabla_{T_X,g} u_g^*\right)_g^T \xi,\end{aligned}$$

since at this point the identities hold:

$$\begin{aligned}\nabla_{g,e_l}\left(e_l \neg \nabla_{T_X,g} u_g^*\right)\eta &= \nabla_{g,e_l}\left[\nabla_{T_X,g} u_g^*(e_l,\eta)\right]\\ &= \nabla_{g,e_l}\nabla_{T_X,g} u_g^*(e_l,\eta).\end{aligned}$$

We infer the required formula. □

We define the Hodge Laplacian (resp. the Ω-Hodge Laplacian) operators acting on T_X-valued q-forms by the formulas

$$\begin{aligned}\Delta_{T_{X,g}} &:= \nabla_{T_{X,g}}\nabla_g^* + \nabla_g^*\nabla_{T_{X,g}},\\ \Delta_{T_{X,g}}^{\Omega} &:= \nabla_{T_{X,g}}\nabla_g^{*\Omega} + \nabla_g^{*\Omega}\nabla_{T_{X,g}}.\end{aligned}$$

Lemma 24. *Let (X,g) be a Riemannian manifold and let*

$$H \in C^\infty(X, \mathrm{End}(T_X)).$$

Then the identity

$$\Delta_{T_{X,g}} H = \Delta_g\, H - \mathcal{R}_g * H + H\,\mathrm{Ric}_g^*$$

holds, where $(\mathcal{R}_g * H)\,\xi := \mathrm{Tr}_g\left[(\xi \neg \mathcal{R}_g)\,H\right]$ for all $\xi \in T_X$.

Proof. Let $(e_k)_k$ and ξ as in the proof of Lemma 23. Then at the point x_0 we have the equalities

$$\begin{aligned}\Delta_{T_{X,g}} H\,\xi &= \nabla_{g,\xi}\nabla_g^* H - \nabla_{g,e_k}\left(\nabla_{T_{X,g}} H\right)(e_k,\xi)\\ &= -\nabla_{g,\xi}\nabla_{g,e_k} H\, e_k - \nabla_{g,e_k}\left(\nabla_{g,e_k} H\,\xi - \nabla_{g,\xi}\, H\, e_k\right)\\ &= \Delta_g H\,\xi + \nabla_{g,e_k}\nabla_{g,\xi}\, H\, e_k - \nabla_{g,\xi}\nabla_{g,e_k} H\, e_k\\ &= \Delta_g H\,\xi + \mathcal{R}_g(e_k,\xi)\, H\, e_k - H\,\mathcal{R}_g(e_k,\xi)\, e_k,\end{aligned}$$

which shows the required formula. □

Lemma 25. *Let (X,g) be a orientable Riemannian manifold, let $\Omega>0$ be a smooth volume form and let $H\in C^\infty(X,\mathrm{End}(T_X))$. Then*

$$\Delta^{\Omega}_{T_{X,g}} H = \Delta^{\Omega}_{g} H - \mathcal{R}_g * H + H\,\mathrm{Ric}^*_g(\Omega).$$

Proof. We expand the Laplacian

$$\begin{aligned}
\Delta^{\Omega}_{T_{X,g}} H &= \nabla_{T_{X,g}} \nabla^*_g H + \nabla_{T_{X,g}} (H\nabla_g f)\\
&\quad + \nabla^*_g \nabla_{T_{X,g}} H + \nabla_g f \neg \nabla_{T_{X,g}} H\\
&= \Delta_{T_{X,g}} H + \nabla_g H\, \nabla_g f + H\, \nabla^2_g f\\
&\quad + \nabla_g f \neg \nabla_g H - \nabla_g H\, \nabla_g f\\
&= \Delta^{\Omega}_g H - \mathcal{R}_g * H + H\,\mathrm{Ric}^*_g(\Omega),
\end{aligned}$$

thanks to Lemma 24. □

19.2. The first variation of the prescattering operator. We show the following:

Lemma 26. *For any smooth family of metrics $(g_t)_{t\in\mathbb{R}}\subset\mathcal{M}$ the total variation formula*

$$\begin{aligned}
2\,\frac{d}{dt}\Big[\nabla_{T_X,g_t}\mathrm{Ric}^*_{g_t}(\Omega)\Big] &= \nabla_{T_X,g_t}\Big(\nabla^{*\Omega}_{g_t}\nabla_{T_X,g_t}\dot g^*_t\Big)^T_{g_t} + \underline{\mathrm{div}}^{\Omega}_{g_t}\left[\mathcal{R}_{g_t},\dot g^*_t\right]\\
&\quad + \mathrm{Alt}\Big[\mathcal{R}_{g_t}\circledast\Big(\nabla_{g_t}-\nabla_{T_X,g_t}\Big)\dot g^*_t\Big]\\
&\quad + \mathrm{Alt}\Big[\Big(\nabla_{T_X,g_t}\dot g^*_t\Big)^T_{g_t}\mathrm{Ric}^*_{g_t}(\Omega)\Big]\\
&\quad - \mathrm{Ric}^*_{g_t}(\Omega)\neg\nabla_{T_X,g_t}\dot g^*_t - 2\,\dot g^*_t\,\nabla_{T_X,g_t}\mathrm{Ric}^*_{g_t}(\Omega)
\end{aligned}$$

holds.

Proof. Lemma 23 combined with the variation formulas (2.2) and (3.3) implies the equality

$$\begin{aligned}
2\,\frac{d}{dt}\Big[\nabla_{T_X,g_t}\mathrm{Ric}^*_{g_t}(\Omega)\Big] &= \nabla_{T_X,g_t}\Big[\nabla^{*\Omega}_{g_t}\nabla_{T_X,g_t}\dot g^*_t + \Big(\nabla^{*\Omega}_{g_t}\nabla_{T_X,g_t}\dot g^*_t\Big)^T_{g_t}\Big]\\
&\quad - \nabla_{T_X,g_t}\Big[\Delta^{\Omega}_{g_t}\dot g^*_t + \dot g^*_t\,\mathrm{Ric}^*_{g_t}(\Omega)\Big]\\
&\quad - \dot g^*_t\,\nabla_{T_X,g_t}\mathrm{Ric}^*_{g_t}(\Omega) - \mathrm{Ric}^*_{g_t}(\Omega)\neg\nabla_{T_X,g_t}\dot g^*_t\\
&\quad + \mathrm{Alt}\Big[\Big(\nabla_{T_X,g_t}\dot g^*_t\Big)^T_{g_t}\mathrm{Ric}^*_{g_t}(\Omega)\Big].
\end{aligned}$$

Moreover using Lemma 25 we deduce the identities

$$\begin{aligned}
\nabla_{T_X,g_t}\nabla^{*\Omega}_{g_t}\nabla_{T_X,g_t}\dot g^*_t &= \nabla_{T_X,g_t}\Delta^{\Omega}_{T_X,g_t}\dot g^*_t - \nabla^2_{T_X,g_t}\nabla^{*\Omega}_{g_t}\dot g^*_t\\
&= \nabla_{T_X,g_t}\Big[\Delta^{\Omega}_{g_t}\dot g^*_t - \mathcal{R}_{g_t} * \dot g^*_t + \dot g^*_t\,\mathrm{Ric}^*_{g_t}(\Omega)\Big]\\
&\quad - \mathcal{R}_{g_t}\nabla^{*\Omega}_{g_t}\dot g^*_t.
\end{aligned}$$

We pick now an arbitrary space time point (x_0, t_0) and let $(e_k)_k$ and ξ, η as in the proof of Lemma 23 with respect to g_{t_0}. We expand at the space time point (x_0, t_0) the term

$$\begin{aligned}\Big[\nabla_{T_X,g_t}(\mathcal{R}_{g_t} * \dot{g}_t^*)\Big](\xi,\eta) &= \nabla_{g_t,\xi}\Big[(\mathcal{R}_{g_t} * \dot{g}_t^*)\,\eta\Big] - \nabla_{g_t,\eta}\Big[(\mathcal{R}_{g_t} * \dot{g}_t^*)\,\xi\Big]\\ &= \nabla_{g_t,\xi}\,\mathcal{R}_{g_t}(\eta,e_k)\,\dot{g}_t^* e_k + \mathcal{R}_{g_t}(\eta,e_k)\nabla_{g_t,\xi}\,\dot{g}_t^* e_k\\ &\quad - \nabla_{g_t,\eta}\,\mathcal{R}_{g_t}(\xi,e_k)\,\dot{g}_t^* e_k - \mathcal{R}_{g_t}(\xi,e_k)\nabla_{g_t,\eta}\,\dot{g}_t^* e_k\,.\end{aligned}$$

Thus using the differential Bianchi identity we infer the equalities

$$\begin{aligned}\nabla_{T_X,g_t}(\mathcal{R}_{g_t} * \dot{g}_t^*) &= -\nabla_{g_t,e_k}\mathcal{R}_{g_t}\,\dot{g}_t^* e_k + \mathrm{Alt}\Big[\mathcal{R}_{g_t} * \Big(\nabla_{T_X,g_t} - \nabla_{g_t}\Big)\,\dot{g}_t^*\Big]\\ &= -\nabla_{g_t,e_k}\mathcal{R}_{g_t}\,\dot{g}_t^* e_k + \mathrm{Alt}\Big[\mathcal{R}_{g_t} \circledast \Big(\nabla_{T_X,g_t} - \nabla_{g_t}\Big)\,\dot{g}_t^*\Big]\\ &\quad + \nabla_{g_t}\dot{g}_t^* * \mathcal{R}_{g_t},\end{aligned}$$

by the identities (3.9) and (3.10). We infer the expression

$$\begin{aligned}\nabla_{T_X,g_t}\nabla^{*_\Omega}_{g_t}\nabla_{T_X,g_t}\dot{g}_t^* &= \nabla_{T_X,g_t}\Big[\Delta^{\Omega}_{g_t}\dot{g}_t^* + \dot{g}_t^*\,\mathrm{Ric}^*_{g_t}(\Omega)\Big]\\ &\quad + \mathrm{Alt}\Big[\mathcal{R}_{g_t} \circledast \Big(\nabla_{g_t} - \nabla_{T_X,g_t}\Big)\,\dot{g}_t^*\Big]\\ &\quad + \nabla_{g_t,e_k}\mathcal{R}_{g_t}\,\dot{g}_t^* e_k - \nabla_{g_t}\dot{g}_t^* * \mathcal{R}_{g_t} - \mathcal{R}_{g_t}\nabla^{*_\Omega}_{g_t}\dot{g}_t^*.\end{aligned}$$

This combined with the identity (3.15) and with the Bianchi type identity (3.11) implies the expression

$$\begin{aligned}\nabla_{T_X,g_t}\nabla^{*_\Omega}_{g_t}\nabla_{T_X,g_t}\dot{g}_t^* &= \nabla_{T_X,g_t}\Big[\Delta^{\Omega}_{g_t}\dot{g}_t^* + \dot{g}_t^*\,\mathrm{Ric}^*_{g_t}(\Omega)\Big]\\ &\quad + \mathrm{Alt}\Big[\mathcal{R}_{g_t} \circledast \Big(\nabla_{g_t} - \nabla_{T_X,g_t}\Big)\,\dot{g}_t^*\Big]\\ &\quad + \underline{\mathrm{div}}^{\Omega}_{g_t}\,[\mathcal{R}_{g_t}, \dot{g}_t^*] - \dot{g}_t^*\,\nabla_{T_X,g_t}\,\mathrm{Ric}^*_{g_t}(\Omega).\end{aligned}$$

This combined with the previous variation formula for $\nabla_{T_X,g_t}\mathrm{Ric}^*_{g_t}(\Omega)$ implies the required conclusion. □

19.3. An direct proof of the variation formula (2.3). We observe that (2.3) it is equivalent to the variation formula (2.4) that we show now. Let $(e_k)_k$ be a local tangent frame. Using the identity (3.7) we compute the variation

$$\begin{aligned}2\,\frac{d}{dt}\,\mathrm{Ric}^*_{g_t}\,\xi &= 2\,\frac{d}{dt}\,\mathcal{R}_{g_t}(\xi,e_k)\,g_t^{-1}e_k^*\\ &= [\mathcal{R}_{g_t}(\xi,e_k), \dot{g}_t^*]\,e_k - 2\,\mathcal{R}_{g_t}(\xi,e_k)\,\dot{g}_t^* e_k.\end{aligned}$$

We deduce the variation formula

$$2\,\frac{d}{dt}\,\mathrm{Ric}^*_{g_t} = -\dot{g}_t^*\,\mathrm{Ric}^*_{g_t} - \mathcal{R}_{g_t} * \dot{g}_t^*. \tag{19.2}$$

On the other hand using the identity (3.14) we can compute the variation of the Hessian

$$\begin{aligned}2\,\frac{d}{dt}\,\nabla^2_{g_t} f_t\,\xi &= 2\,\dot\nabla_{g_t,\xi}\nabla_{g_t} f_t - \nabla_{g_t,\xi}\left(\nabla^*_{g_t}\dot g^*_t + 2\,\dot g^*_t\,\nabla_{g_t} f_t\right)\\ &= \nabla_{g_t,\xi}\,\dot g^*_t\,\nabla_{g_t} f_t - \nabla_{g_t,\xi}\nabla^*_{g_t}\dot g^*_t\\ &\quad - 2\,\nabla_{g_t,\xi}\,\dot g^*_t\,\nabla_{g_t} f_t - 2\,\dot g^*_t\,\nabla^2_{g_t} f_t\,\xi\\ &= -\nabla_{g_t,\xi}\,\dot g^*_t\,\nabla_{g_t} f_t - \Delta_{g_t}\dot g^*_t\,\xi\\ &\quad + (\mathcal{R}_{g_t} * \dot g^*_t)\,\xi - \dot g^*_t\,\mathrm{Ric}^*_{g_t}\,\xi - 2\,\dot g^*_t\,\nabla^2_{g_t} f_t\,\xi,\end{aligned}$$

thanks to Lemma 24. We infer the variation identity

$$2\,\frac{d}{dt}\,\nabla^2_{g_t} f_t = -\Delta^{\Omega}_{g_t}\dot g^*_t + \mathcal{R}_{g_t} * \dot g^*_t - \dot g^*_t\,\mathrm{Ric}^*_{g_t} - 2\,\dot g^*_t\,\nabla^2_{g_t} f_t\,.$$

This combined with the variation formula (19.2) implies the formula (2.4) and thus the variation identity (2.3).

19.4. Basic differential identities. The results explained in this subsection are well-known. We include them here for readers convenience.

Lemma 27. *Let M be a differentiable manifold and let*

$$(\xi_t)_{t\geqslant 0} \subset C^\infty(M, T_M)\,,\quad (\alpha_t)_{t\geqslant 0} \subset C^\infty\left(M, (T^*_M)^{\otimes p}\otimes T_M^{\otimes r}\right)\,,$$

be smooth families and let $(\Phi_t)_{t\geqslant 0}$ be the flow of diffeomorphisms induced by the family $(\xi_t)_{t\geqslant 0}$, i.e.,

$$\frac{d}{dt}\,\Phi_t = \xi_t\circ\Phi_t\,,\quad \Phi_0 = \mathrm{Id}_M\,.$$

Then we have the derivation formula

$$\frac{d}{dt}\,\left(\Phi^*_t\alpha_t\right) = \Phi^*_t\left(\frac{d}{dt}\,\alpha_t + L_{\xi_t}\alpha_t\right),$$

Proof. We prove first the particular case

$$\frac{d}{dt}_{|t=0}\left(\Phi^*_t\alpha\right) = L_{\xi_0}\alpha\,, \tag{19.3}$$

where α is t-independent. For this purpose we consider the 1-parameter subgroup of diffeomorphisms $(\Psi_t)_{t\geqslant 0}$ induced by ξ_0, i.e.,

$$\frac{d}{dt}\,\Psi_t = \xi_0\circ\Psi_t,\quad \Psi_0 = \mathrm{Id}_M\ .$$

Let $\hat\Psi : \mathbb{R}_{\geqslant 0}\times M \longrightarrow M$ given by $\hat\Psi(t,x) = \Psi^{-1}_t(x)$ and observe the equalities

$$\frac{d}{dt}_{|t=0}\,\Psi^{-1}_t = -\xi_0 = \frac{d}{dt}_{|t=0}\,\Phi^{-1}_t\,, \tag{19.4}$$

We will note by ∂ the partial derivatives of the coefficients of the tensors with respect to a trivialization of the tangent bundle over an open set $U \subset M$. Let $v \in T_{M,x}^{\otimes p}$. Then

$$\begin{aligned}
(L_{\xi_0}\alpha)\cdot v &= \left(\frac{d}{dt}_{|t=0}\, \Psi_t^*\alpha\right)\cdot v \\
&= \frac{d}{dt}_{|t=0}\left[(d\Psi_t^{-1})^{\otimes r}\cdot(\alpha\circ\Psi_t)\right]\cdot(d\Psi_t)^{\otimes p}\cdot v \\
&= \left[\frac{d}{dt}_{|t=0}\,(\partial_x\hat{\Psi})^{\otimes r}(t,\Psi_t(x))\right]\cdot\alpha\cdot v \\
&\quad + \frac{d}{dt}_{|t=0}\left[(\alpha\circ\Psi_t)\cdot(d\Psi_t)^{\otimes p}\cdot v\right] \\
&= (\partial_t\,\partial_x\hat{\Psi})^{\otimes r}(0,x)\cdot\alpha\cdot v + \Big((\partial_x^2\,\hat{\Psi})^{\otimes r}(0,x)\cdot\xi_0^{\otimes r}(x)\Big)\cdot\alpha\cdot v \\
&\quad + \left(\frac{d}{dt}_{|t=0}\,\alpha(\Psi_t(x))\right)\cdot v + \alpha(x)\cdot(\partial_t\,\partial_x\Psi)^{\otimes p}(0,x)\cdot v \\
&= -(\partial_x\,\xi_0)^{\otimes r}(x)\cdot\alpha\cdot v \\
&\quad + (\partial_x\,\alpha\,(x)\cdot v)\cdot\xi_0(x) + \alpha(x)\cdot(\partial_x\,\xi_0)^{\otimes p}(x)\cdot v\,,
\end{aligned}$$

since the map

$$(\partial_x^2\,\hat{\Psi})^{\otimes r}(0,x) : S^2T_U^{\otimes r} \longrightarrow T_U^{\otimes r}\,,$$

is zero. Observe in fact the identity

$$\partial_x\Psi_0^{-1} = \mathrm{Id}_{T_U}\,.$$

Moreover the same computation and conclusion work for Φ_t thanks to (19.4). We infer the identity (19.3). We prove now the general case. We expand the time derivative

$$\begin{aligned}
\frac{d}{dt}\,\big(\Phi_t^*\alpha_t\big) &= \frac{d}{ds}_{|s=0}\,\Phi_{t+s}^*\,\alpha_{t+s} \\
&= \Phi_t^*\left(\frac{d}{dt}\,\alpha_t\right) + \frac{d}{ds}_{|s=0}\,\Phi_{t+s}^*\,\alpha_t \\
&= \Phi_t^*\left(\frac{d}{dt}\,\alpha_t\right) + \frac{d}{ds}_{|s=0}\,(\Phi_t^{-1}\Phi_{t+s})^*\,\Phi_t^*\,\alpha_t\,.
\end{aligned}$$

We set $\Phi_s^t := \Phi_t^{-1}\Phi_{t+s}$ and we observe the equalities

$$\begin{aligned}
\frac{d}{ds}_{|s=0}\,\Phi_s^t &= d\,\Phi_t^{-1}\cdot\frac{d}{ds}_{|s=0}\,\Phi_{t+s} \\
&= d\,\Phi_t^{-1}\cdot(\xi_t\circ\Phi_t) \\
&= \Phi_t^*\,\xi_t\,.
\end{aligned}$$

Then the identity (19.3) applied to the family $(\Phi_s^t)_s$ implies

$$\frac{d}{dt}\left(\Phi_t^* \alpha_t\right) = \Phi_t^* \left(\frac{d}{dt}\,\alpha_t\right) + L_{\Phi_t^* \xi_t} \Phi_t^*\, \alpha_t$$
$$= \Phi_t^* \left(\frac{d}{dt}\,\alpha_t + L_{\xi_t} \alpha_t\right). \qquad \square$$

Lemma 28. *Let (X, J) be an almost complex manifold and let N_J be the Nijenhhuis tensor. Then for any $\xi \in C^\infty(X, T_X)$ we have the identity*

$$L_\xi J = 2\, J \left(\bar{\partial}_{T_{X,J}} \xi - \xi \neg N_J\right). \tag{19.5}$$

Proof. Let $\eta \in C^\infty(X, T_X)$. Then

$$\bar{\partial}_{T_{X,J}} \xi\,(\eta) = [\eta^{0,1}, \xi^{1,0}]^{1,0} + [\eta^{1,0}, \xi^{0,1}]^{0,1},$$
$$N_J(\xi, \eta) = [\xi^{1,0}, \eta^{1,0}]^{0,1} + [\xi^{0,1}, \eta^{0,1}]^{1,0}.$$

and the conclusion follows by decomposing in type $(1, 0)$ and $(0, 1)$ the identity

$$(L_\xi J)\, \eta = [\xi, J\eta] - J[\xi, \eta]\ . \qquad \square$$

We observe now that if (X, J, ω) is a Kähler manifold and $u \in C^\infty(X, \mathbb{R})$, then we have the identities

$$\nabla_\omega u \neg \omega = -(d\,u) \cdot J = -\, i\, \partial_J u + i\, \bar{\partial}_J u,$$

and

$$L_{\nabla_\omega u}\, \omega = d\,(\nabla_\omega\, u \neg \omega) = 2\, i\, \partial_J \bar{\partial}_J u. \tag{19.6}$$

Lemma 29. *Let (X, J, g) be a Kähler manifold and let $u \in C^\infty(X, \mathbb{R})$. Then we have the decomposition formula*

$$\nabla_g\, d\,u = i\, \partial_J \bar{\partial}_J u\, (\cdot, J\cdot) + g\left(\cdot, \bar{\partial}_{T_{X,J}} \nabla_g\, u\, \cdot\right). \tag{19.7}$$

Proof. Let $\xi\,, \eta\,, \mu \in C^\infty(X, T_X)$. By definition of Lie derivative,

$$\xi\,.\, g(\eta, \mu) = (L_\xi\, g)(\eta, \mu) + g(L_\xi\, \eta, \mu) + g(\eta, L_\xi\, \mu)\ .$$

Let $\omega := g(J\cdot, \cdot)$ be the induced Kähler form. Then by using again the definition of Lie derivative we infer the equalities

$$\begin{aligned} \xi\,.\, g(\eta, \mu) &= \xi\,.\, \omega(\eta, J\mu) \\ &= (L_\xi\, \omega)(\eta, J\mu) + \omega(L_\xi\, \eta, J\mu) + \omega(\eta, L_\xi(J\mu)) \\ &= (L_\xi\, \omega)(\eta, J\mu) + g(L_\xi\, \eta, \mu) + \omega(\eta, (L_\xi\, J)\mu) + g(\eta, L_\xi\, \mu)\ . \end{aligned}$$

We deduce the identity

$$L_\xi\, g = L_\xi\, \omega\,(\cdot, J\cdot) + \omega\,(\cdot, L_\xi\, J\cdot).$$

We apply this identity to the vector field $\xi := \nabla_g\, u$. Then the conclusion follows from the identity

$$L_{\nabla_g u}\, g = 2\, \nabla_g\, d\,u\ ,$$

combined with (19.6), and (19.5). □

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(Nefton Pali) Université Paris Sud, Département de Mathématiques, Bâtiment 425 F91405 Orsay, France
nefton.pali@math.u-psud.fr

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