

Compactness of products of Hankel operators on convex Reinhardt domains in \mathbb{C}^2

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ABSTRACT. Let Ω be a piecewise smooth bounded convex Reinhardt domain in \mathbb{C}^2 . Assume that the symbols ϕ and ψ are continuous on $\bar{\Omega}$ and harmonic on the disks in the boundary of Ω . We show that if the product of Hankel operators $H_\psi^* H_\phi$ is compact on the Bergman space of Ω , then on any disk in the boundary of Ω , either ϕ or ψ is holomorphic.

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1. Introduction

This paper is a sequel to our two previous papers [ČŠ09, ČŠ10] on compactness of Hankel operators on Bergman spaces of domains in \mathbb{C}^n . In the first paper we studied compactness of a single Hankel operator with a smooth symbol on quite general domains. We note that in this paper smooth means C^∞ -smooth. We used $\bar{\partial}$ methods to relate the compactness property of Hankel operators to the behavior of the symbol on the analytic disks in the boundary of the domain. The most complete result is the following theorem in \mathbb{C}^2 . Here H_ϕ denotes the Hankel operator on the Bergman space $A^2(\Omega)$ with a symbol ϕ . Furthermore, $\partial\Omega$ and \mathbb{D} denote the boundary of Ω and the open unit disk in the complex plane, respectively.

Theorem 1 ([ČŠ09]). *Let Ω be a smooth bounded convex domain in \mathbb{C}^2 and $\phi \in C^\infty(\bar{\Omega})$. Then H_ϕ is compact on $A^2(\Omega)$ if and only if $\phi \circ f$ is holomorphic for any holomorphic mapping $f : \mathbb{D} \rightarrow \partial\Omega$.*

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In the second paper we studied compactness of products of two Hankel operators on the polydisk. Notable is the absence of $\bar{\partial}$ methods: the domain is simple enough to be treated by reducing the dimension by one. For simplicity, we state the main result in \mathbb{C}^2 only.

Theorem 2 ([ČŠ10]). *Let Ω be the bidisk in \mathbb{C}^2 and the symbols $\phi, \psi \in C(\bar{\Omega})$ such that $\phi \circ f$ and $\psi \circ f$ are harmonic for any holomorphic mapping $f : \mathbb{D} \rightarrow \partial\Omega$. Then $H_\psi^* H_\phi$ is compact on $A^2(\Omega)$ if and only if for any holomorphic function $f : \mathbb{D} \rightarrow \partial\Omega$, either $\phi \circ f$ or $\psi \circ f$ is holomorphic.*

In this paper we treat domains that are more general than a polydisk (see Theorem 3). A domain $\Omega \subset \mathbb{C}^n$ is called Reinhardt if $(z_1, \dots, z_n) \in \Omega$ and $\theta_1, \dots, \theta_n \in \mathbb{R}$ imply that $(e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n) \in \Omega$. Namely, the domain Ω is circular in each variable. The ball and the polydisk are the best known examples of Reinhardt domains.

The following theorem is the main result of our paper. As before, the analyticity of the symbols is a necessary condition for compactness of the product of Hankel operators, provided that their symbols are harmonic on the disks in the boundary.

Theorem 3. *Let Ω be a piecewise smooth bounded convex Reinhardt domain in \mathbb{C}^2 . Assume that the symbols $\phi, \psi \in C(\bar{\Omega})$ are such that $\phi \circ f$ and $\psi \circ f$ are harmonic for every holomorphic function $f : \mathbb{D} \rightarrow \partial\Omega$. If $H_\psi^* H_\phi$ is compact on $A^2(\Omega)$ then for every holomorphic function $f : \mathbb{D} \rightarrow \partial\Omega$ either $\phi \circ f$ or $\psi \circ f$ is holomorphic.*

The proof of Theorem 3 uses convexity and rotational symmetry of the domain in a significant way. If there is a disk Δ in the boundary of a convex Reinhardt domain Ω then there are disks in $\bar{\Omega}$ nearby Δ of at least the same size. Furthermore, these disks “converge” to Δ . This geometric property is an important ingredient in our proof.

Remark 1. Even though Theorem 1 is stated for symbols that are smooth up to the boundary and domains with smooth boundaries, the proof shows that the theorem is still true under reasonably weaker smoothness assumptions. In the case of the polydisk Le [Le10] studied compactness of Hankel operators with symbols continuous on the closure of the polydisk.

Remark 2. Products of Hankel operators can be viewed as semicommutators of Toeplitz operators. Several authors have studied compactness of these semicommutators on the unit disk \mathbb{D} and the polydisk \mathbb{D}^n . Zheng [Zhe89] characterized compact semicommutators of Toeplitz operators with symbols that are harmonic on \mathbb{D} . Later Ding and Tang [DT01], Choe, Koo, and Lee [CKL04], and Choe, Lee, Nam, and Zheng [CLNZ07] extended this result to semicommutators of Toeplitz operator acting on the Bergman space of \mathbb{D}^n with the assumption that the symbols are pluriharmonic functions on \mathbb{D}^n . Notice that the symbols in Theorem 2 are assumed to be continuous up to the boundary but pluriharmonic on the disks in the boundary of Ω only.

Remark 3. The class of domains to which Theorem 3 applies includes many more domains other than the bidisk. For example, it includes the intersection of Reinhardt domains such as $(\mathbb{D} \times \mathbb{D}) \cap B(0, (1 + \sqrt{2})/2)$ where $B(p, r)$ denotes the ball centered at p with radius r .

Remark 4. If there is no disk in the boundary of a convex domain then the $\bar{\partial}$ -Neumann operator is compact (see [FS98, Theorem 1.1] or [Str10, Theorem 4.26]); in turn, this implies that the Hankel operator with a symbol that is continuous on the closure of the domain is compact (see [Str10, Proposition 4.1]). Hence, if a bounded convex domain does not have a disk in the boundary then the product of Hankel operators with symbols continuous on the closure of the domain is compact. For more information about Reinhardt domains we refer the reader to [JP08, Kra01, Ran86].

2. Some background information and lemmas

Let Ω be a bounded domain in \mathbb{C}^n and $A^2(\Omega)$ denote the Bergman space, the set of holomorphic functions that are square integrable on Ω with respect to the Lebesgue measure V . Unless we integrate on a subdomain of Ω , the norm $\|\cdot\|_{L^2(\Omega)}$ is denoted by $\|\cdot\|$ and the complex inner product $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ by $\langle \cdot, \cdot \rangle$.

Let P^Ω denote the Bergman projection on Ω , the orthogonal projection from $L^2(\Omega)$ onto $A^2(\Omega)$. The Toeplitz and Hankel operators with symbol $\phi \in L^\infty(\Omega)$ are defined on $A^2(\Omega)$ by $T_\phi^\Omega f = P^\Omega(\phi f)$ and $H_\phi^\Omega f = \phi f - P^\Omega(\phi f)$, respectively. Notice that the range of H_ϕ^Ω is a subspace of the orthogonal complement of $A^2(\Omega)$ in $L^2(\Omega)$. Then one can define the product of two Hankel operators with symbols ψ and ϕ as $(H_\psi^\Omega)^* H_\phi^\Omega : A^2(\Omega) \rightarrow A^2(\Omega)$, where $(H_\psi^\Omega)^*$ denotes the Hilbert space adjoint of H_ψ^Ω . When it is clear from the context on which domain we are working on, we will omit the domain superscripts on the operators P, T_ϕ , and H_ϕ .

It is well known that this product can be written as a semicommutator of Toeplitz operators. Namely,

$$(1) \quad H_\psi^* H_\phi = T_{\bar{\psi}\phi} - T_{\bar{\psi}} T_\phi.$$

For more information about these operators we suggest the reader consult [Zhu07, Ax188].

We now present and prove several key lemmas that will be used in the proof of the main theorem. They represent our idea that geometry, analysis, and approximation intertwine in an interesting manner and they enable us to prove the main result in this paper.

The first lemma is simple and it allows us to rewrite the product of two Hankel operators in a different way than the semicommutator of Toeplitz operators.

Lemma 1. *Let Ω be a domain in \mathbb{C}^n and $\phi, \psi \in L^\infty(\Omega)$. Then*

$$H_\psi^* H_\phi = P M_{\bar{\psi}} H_\phi$$

where $M_{\bar{\psi}}$ denotes the product by $\bar{\psi}$.

Proof. Let $f, g \in A^2(\Omega)$. Then we have

$$\langle H_\psi^* H_\phi f, g \rangle = \langle H_\phi f, H_\psi g \rangle = \langle H_\phi f, \psi g \rangle = \langle \bar{\psi} H_\phi f, g \rangle = \langle P \bar{\psi} H_\phi f, g \rangle.$$

Therefore, $H_\psi^* H_\phi = P M_{\bar{\psi}} H_\phi$. \square

The next lemma gives us an important information about the disks in the boundary of complete Reinhardt domains in \mathbb{C}^2 . It shows that piecewise smooth bounded complete Reinhardt domains in \mathbb{C}^2 can have vertical or horizontal disks only. This will allow us to use the slicing method to approach the disks by horizontal and vertical slices of the domain itself.

Lemma 2. *Let Ω be a piecewise smooth bounded complete Reinhardt domain in \mathbb{C}^2 and let $F = (f, g) : \mathbb{D} \rightarrow \partial\Omega$ be a holomorphic function. Then either f or g is constant.*

Proof. Let $F(z) = (f(z), g(z))$ be an analytic disk in the boundary. If $|f(z)|$ and $|g(z)|$ are constant then F is constant. Therefore, there are no nontrivial disks on the singular part of the boundary.

Now assume that there is an analytic disk in the boundary away from singular points. Then we can assume that the domain is smooth and it is given by $\rho(|z|, |w|)$. By convexity if there is a disk then it must be an affine disk (see, for example, [ČŠ09, Lemma 2] and [FS98, Proposition 3.2]). So there exist $a, b, c, d \in \mathbb{C}$ such that the set $\{(a\xi + b, c\xi + d) \in \mathbb{C}^2 : \xi \in \mathbb{D}\}$ is a disk in the boundary. We may also assume that the disk does not intersect the coordinate axes. In other words, we may assume that $|a\xi + b| > 0$ and $|c\xi + d| > 0$. Computing the Laplacian of $r(\xi) = \rho(|a\xi + b|, |c\xi + d|)$ where $\xi \in \mathbb{D}$ and we get

$$0 = 4 \frac{\partial^2 r}{\partial \xi \partial \bar{\xi}}(\xi) = H_\rho(r(\xi); W) + \rho_x(r(\xi)) \frac{|a|^2}{|a\xi + b|} + \rho_y(r(\xi)) \frac{|c|^2}{|c\xi + d|}$$

where

$$W = \left(a \left(\frac{\overline{a\xi + b}}{a\xi + b} \right)^{1/2}, c \left(\frac{\overline{c\xi + d}}{c\xi + d} \right)^{1/2} \right)$$

and $H_\rho(p; X)$ is the (real) Hessian of ρ applied to the vector X at the point p . Let $(|p|, |q|)$ be a boundary point of

$$Z = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, \rho(x, y) < 0\}.$$

Then the rectangle $R_{(|p|, |q|)} \subset \mathbb{R}^2$ formed by $(0, 0)$, $(|p|, 0)$, $(0, |q|)$, and $(|p|, |q|)$ is inside Z and $(\rho_x(|p|, |q|), \rho_y(|p|, |q|))$ is normal to the boundary of Z at $(|p|, |q|)$. If $\rho_x(|p|, |q|) < 0$ and $\rho_y(|p|, |q|) > 0$ (or $\rho_x(|p|, |q|) > 0$ and $\rho_y(|p|, |q|) < 0$) then the tangential vector to ∂Z at $(|p|, |q|)$ has components

with the same sign. Then $R_{(|p|,|q|)} \cap \partial Z$ is nonempty which in turn implies that $R_{(|p|,|q|)} \setminus Z$ is nonempty. Similarly, if $\rho_x(|p|, |q|) < 0$ and $\rho_y(|p|, |q|) < 0$ then $R_{(|p|,|q|)}$ cannot be contained in Z . Hence, $\rho_x \geq 0, \rho_y \geq 0$ and $\rho_x + \rho_y > 0$, and $H_\rho(r(\xi); W) \geq 0$ for any $W \in \mathbb{C}^2$. Therefore, either $a = 0$ or $c = 0$. That is, the disk is either horizontal or vertical.

Assume that $F(z) = (f(z), g(z))$ is a non-trivial analytic disk through a singular point in the boundary. That is, F is nonconstant and there exists $p \in \mathbb{D}$ such that $F'(p) = 0$. Then by the previous part the smooth part of the disk is either horizontal or vertical. If it is horizontal then there exists an open set $U \subset \mathbb{D}$ such that $|g|$ is constant on U . The identity principle implies that g is constant on \mathbb{D} . Hence, the whole disk is horizontal. \square

As mentioned earlier, we use slicing of the domain and the resulting disks to approach horizontal or vertical disks in the boundary. The following lemma will enable us to do that in the sense that projections of these disks onto the complex plane approach the projection of the disk in the boundary. Even though this lemma is stated for horizontal disks, the result holds for vertical disks as well.

Lemma 3. *Let Ω be a bounded convex Reinhardt domain in \mathbb{C}^2 and*

$$\Delta_w = \{z \in \mathbb{C} : (z, w) \in \overline{\Omega}\}$$

for $w \in \mathbb{C}$. Assume that $\emptyset \neq \Delta_{w_0} \times \{w_0\} \subset \partial\Omega$ for some $w_0 \in \mathbb{C}$, $\{w_j\}$ is a sequence of complex numbers that converges to w_0 , and Δ_{w_j} is nonempty for all j . Then $\lim_{j \rightarrow \infty} r_j = r_0$ where r_j denotes the radius of the disk Δ_{w_j} for $j = 0, 1, 2, \dots$.

Proof. Since Ω is a convex Reinhardt domain it is also complete. Hence, all of these disks are centered at the origin and we want to prove that $\{r_j\}$ converges to r_0 , the radius of Δ_{w_0} . In addition, since the domain is also convex one can show that $r_j \geq r_0$ for $j \geq 1$. Hence $\liminf_{j \rightarrow \infty} r_j \geq r_0$

On the other hand, if $\limsup_{j \rightarrow \infty} r_j > r_0$ we can choose $p_k \in \Delta_{w_{j_k}}$ such that $|p_k| = r_{j_k}$ and $\lim_{k \rightarrow \infty} |p_k| = \limsup_{j \rightarrow \infty} r_j$. Then the sequence $\{(p_k, w_{j_k})\} \subset \partial\Omega$ has a subsequence that converges to a point $(p, w_0) \in \partial\Omega$. This means that $p \in \Delta_{w_0}$ and

$$\limsup_{j \rightarrow \infty} r_j = \lim_{k \rightarrow \infty} |p_k| = |p| \leq r_0.$$

Therefore, $\lim_{j \rightarrow \infty} r_j = r_0$. \square

The convergence of the disk in Lemma 3 brings the natural question of a convergence of the corresponding Bergman kernels and projections. Let K be a set in \mathbb{C}^n and T_K denote the characteristic function of K . That is, $T_K(z) = 1$ if $z \in K$ and $T_K(z) = 0$ otherwise. Also for a function f defined on a set U we let $E_U f$ denote the extension of f by 0 outside U .

Lemma 4. *Let $\psi \in L^2(\mathbb{C})$. Then $\lim_{r \rightarrow 1} \|E_{\mathbb{D}_r} P^{\mathbb{D}_r} \psi - E_{\mathbb{D}} P^{\mathbb{D}} \psi\|_{L^2(\mathbb{C})} = 0$.*

Proof. Since ψ is square integrable, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|r - 1| < \delta$ implies that $\|\psi\|_{L^2(\mathbb{D}_{1+\delta} \setminus \mathbb{D}_{1-\delta})} < \varepsilon/2$. Then

$$\left\| P^{\mathbb{D}_r}(T_{\mathbb{D}_r \setminus \mathbb{D}_{1-\delta}}\psi) \right\|_{L^2(\mathbb{D}_r)} + \left\| P^{\mathbb{D}}(T_{\mathbb{D} \setminus \mathbb{D}_{1-\delta}}\psi) \right\|_{L^2(\mathbb{D})} \leq 2\|\psi\|_{L^2(\mathbb{D}_{1+\delta} \setminus \mathbb{D}_{1-\delta})} \leq \varepsilon$$

for $|r - 1| < \delta$. Next the proof of the lemma will be completed by showing that

$$\left\| E_{\mathbb{D}_r} P^{\mathbb{D}_r}(T_{\mathbb{D}_{1-\delta}}\psi) - E_{\mathbb{D}} P^{\mathbb{D}}(T_{\mathbb{D}_{1-\delta}}\psi) \right\|_{L^2(\mathbb{C})} \rightarrow 0 \text{ as } r \rightarrow 1.$$

We define $G_r(z, w) = F^r(z, w) - F^1(z, w)$ for $(z, w) \in \mathbb{C} \times \mathbb{D}_{1-\delta}$, where

$$F^r(z, w) = \frac{T_{\mathbb{D}_r}(z)r^2}{(r^2 - z\bar{w})^2}$$

and $r > 1 - \delta$. We note that $\frac{r^2}{\pi(r^2 - z\bar{w})^2}$ is the Bergman kernel for \mathbb{D}_r . Then there exists $r_0 > 1$ such that $G_r \rightarrow 0$ uniformly on $\overline{\mathbb{D}_{r_0}} \times \mathbb{D}_{1-\delta}$ as $r \rightarrow 1$. For $1 - \delta < r < r_0$ we have

$$\begin{aligned} & \left\| E_{\mathbb{D}_r} P^{\mathbb{D}_r}(T_{\mathbb{D}_{1-\delta}}\psi) - E_{\mathbb{D}} P^{\mathbb{D}}(T_{\mathbb{D}_{1-\delta}}\psi) \right\|_{L^2(\mathbb{C})}^2 \\ &= \int_{\mathbb{C}} \left| \int_{\mathbb{D}_{1-\delta}} F^r(z, w)\psi(w)dV(w) - \int_{\mathbb{D}_{1-\delta}} F^1(z, w)\psi(w)dV(w) \right|^2 dV(z) \\ &\leq \int_{\mathbb{C}} \left(\int_{\mathbb{D}_{1-\delta}} |G_r(z, w)| |\psi(w)| dV(w) \right)^2 dV(z) \\ &\leq \|\psi\|_{L^2(\mathbb{D})}^2 \int_{\mathbb{D}_{r_0}} \int_{\mathbb{D}_{1-\delta}} |G_r(z, w)|^2 dV(w)dV(z). \end{aligned}$$

Since $G_r \rightarrow 0$ uniformly as $r \rightarrow 1$ we have

$$\left\| E_{\mathbb{D}_r} P^{\mathbb{D}_r}(T_{\mathbb{D}_{1-\delta}}\psi) - E_{\mathbb{D}} P^{\mathbb{D}}(T_{\mathbb{D}_{1-\delta}}\psi) \right\|_{L^2(\mathbb{C})} \rightarrow 0$$

as $r \rightarrow 1$. □

The lemma above and [Kra01, Lemma 1.4.1] imply the following corollary.

Corollary 1. *Let $\psi \in L^2(\mathbb{C})$ and K be a compact subset of \mathbb{D} . Then $\{P^{\mathbb{D}_r}\psi\}$ converges uniformly to $P^{\mathbb{D}}\psi$ on K as $r \rightarrow 1$.*

The following lemma is stated for bounded convex domains because these domains are the focus of our paper. However, similar ideas can be used for A^p spaces on starlike domains. This has been done for $A^p(\mathbb{D})$ in [DS04, Theorem 3, p.30].

Lemma 5. *Let U be a bounded convex domain in \mathbb{C} and $f \in A^2(U)$. Then for any $\varepsilon > 0$ there exists a holomorphic polynomial h such that $\|f - h\|_{L^2(U)} < \varepsilon$.*

Proof. Without loss of generality we may assume that U contains the origin. Let us define $f_r(z) = f(rz)$ for $r \in (0, 1)$ and assume that $\varepsilon > 0$ is given. Then $f_r \in A^2(U) \cap C(\bar{U})$ and one can show that there exists $0 < r < 1$ such that

$$\|f - f_{r_0}\| < \frac{\varepsilon}{2}.$$

This can be seen as follows: First there exists $0 < \delta < 1$ so that

$$\|f\|_{L^2(U \setminus \delta U)} < \frac{\varepsilon}{6}.$$

The uniform continuity of f on compact subsets of U implies that there exists $\frac{1}{2} < r < 1$ such that

$$\sup \left\{ |f(z) - f(rz)| : z \in \left(\frac{1+\delta}{2} \right) \bar{U} \right\} < \frac{\varepsilon}{6\sqrt{V(U)}}$$

where $V(U)$ denotes the volume of U . Then we have

$$\begin{aligned} \|f - f_r\| &\leq \|f - f_r\|_{L^2((\frac{1+\delta}{2})U)} + \|f\|_{L^2(U \setminus (\frac{1+\delta}{2})U)} + \|f_r\|_{L^2(U \setminus (\frac{1+\delta}{2})U)} \\ &\leq \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{1}{r} \|f\|_{L^2(U \setminus \delta U)} \\ &< \frac{2\varepsilon}{3}. \end{aligned}$$

On the other hand, Mergelyan’s theorem implies that there exists a holomorphic polynomial h such that

$$\sup \{ |f_r(z) - h(z)| : z \in \bar{U} \} < \frac{\varepsilon}{3\sqrt{V(U)}}.$$

Then we have

$$\|f - h\| \leq \|f - f_r\| + \|f_r - h\| \leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This completes the proof of Lemma 5. □

The next lemma shows that when concentric disks converge, then not only the kernels and the Bergman projections converge but also the products of Hankel operators converge “weakly”.

Lemma 6. For $r > 0$ let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$, f_1 and f_2 be entire functions, and $\phi, \psi \in C(\mathbb{C})$. Then

$$\lim_{r \rightarrow r_0} \left\langle H_\phi^{\mathbb{D}_r}(f_1), H_\psi^{\mathbb{D}_r}(f_2) \right\rangle_{\mathbb{D}_r} = \left\langle H_\phi^{\mathbb{D}_{r_0}}(f_1), H_\psi^{\mathbb{D}_{r_0}}(f_2) \right\rangle_{\mathbb{D}_{r_0}}.$$

Proof. First assume that $r_0 \leq r$. For any $0 < \delta < r_0$ we have

$$\begin{aligned} &\left| \left\langle H_\phi^{\mathbb{D}_r}(f_1), H_\psi^{\mathbb{D}_r}(f_2) \right\rangle_{\mathbb{D}_r} - \left\langle H_\phi^{\mathbb{D}_{r_0}}(f_1), H_\psi^{\mathbb{D}_{r_0}}(f_2) \right\rangle_{\mathbb{D}_{r_0}} \right| \\ &= \left| \left\langle \phi f_1, H_\psi^{\mathbb{D}_r}(f_2) \right\rangle_{\mathbb{D}_r} - \left\langle \phi f_1, H_\psi^{\mathbb{D}_{r_0}}(f_2) \right\rangle_{\mathbb{D}_{r_0}} \right| \end{aligned}$$

$$\begin{aligned}
&\leq \left| \langle \phi f_1, \psi f_2 \rangle_{\mathbb{D}_r \setminus \mathbb{D}_{r_0}} \right| + \left| \left\langle \phi f_1, P^{\mathbb{D}_r}(\psi f_2) \right\rangle_{\mathbb{D}_r} - \left\langle \phi f_1, P^{\mathbb{D}_{r_0}}(\psi f_2) \right\rangle_{\mathbb{D}_{r_0}} \right| \\
&\leq \left| \langle \phi f_1, \psi f_2 \rangle_{\mathbb{D}_r \setminus \mathbb{D}_{r_0}} \right| + \left| \left\langle \phi f_1, P^{\mathbb{D}_r}(\psi f_2) \right\rangle_{\mathbb{D}_{r_0-\delta}} - \left\langle \phi f_1, P^{\mathbb{D}_{r_0}}(\psi f_2) \right\rangle_{\mathbb{D}_{r_0-\delta}} \right| \\
&\quad + \left| \left\langle \phi f_1, P^{\mathbb{D}_r}(\psi f_2) \right\rangle_{\mathbb{D}_r \setminus \mathbb{D}_{r_0-\delta}} \right| + \left| \left\langle \phi f_1, P^{\mathbb{D}_{r_0}}(\psi f_2) \right\rangle_{\mathbb{D}_{r_0} \setminus \mathbb{D}_{r_0-\delta}} \right|.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
(2) \quad &\left| \left\langle H_\phi^{\mathbb{D}_r}(f_1), H_\psi^{\mathbb{D}_r}(f_2) \right\rangle_{\mathbb{D}_r} - \left\langle H_\phi^{\mathbb{D}_{r_0}}(f_1), H_\psi^{\mathbb{D}_{r_0}}(f_2) \right\rangle_{\mathbb{D}_{r_0}} \right| \\
&\leq \left| \langle \phi f_1, \psi f_2 \rangle_{\mathbb{D}_r \setminus \mathbb{D}_{r_0}} \right| + \left| \left\langle \phi f_1, P^{\mathbb{D}_r}(\psi f_2) - P^{\mathbb{D}_{r_0}}(\psi f_2) \right\rangle_{\mathbb{D}_{r_0-\delta}} \right| \\
&\quad + \|\phi f_1\|_{L^2(\mathbb{D}_r \setminus \mathbb{D}_{r_0-\delta})} \|\psi f_2\|_{L^2(\mathbb{D}_r)} + \|\phi f_1\|_{L^2(\mathbb{D}_{r_0} \setminus \mathbb{D}_{r_0-\delta})} \|\psi f_2\|_{L^2(\mathbb{D}_{r_0})}.
\end{aligned}$$

Then for $\varepsilon > 0$ one can choose $0 < \delta_1 < \min\{1, r_0\}$ so that

$$\|\phi f_1\|_{L^2(\mathbb{D}_{r_0+\delta_1} \setminus \mathbb{D}_{r_0-\delta_1})} \leq \varepsilon.$$

Furthermore, by Corollary 1 we can choose $0 < \delta_2 < \delta_1$ so that $r_0 \leq r \leq r_0 + \delta_2$ implies that

$$\left| P^{\mathbb{D}_r}(\psi f_2)(z) - P^{\mathbb{D}_{r_0}}(\psi f_2)(z) \right| \leq \varepsilon \text{ for } z \in \overline{\mathbb{D}_{r_0-\delta_1}}$$

and

$$\left| \langle \phi f_1, \psi f_2 \rangle_{\mathbb{D}_r \setminus \mathbb{D}_{r_0}} \right| \leq \varepsilon.$$

Therefore, for $r_0 \leq r \leq r_0 + \delta_2$ we have

$$\begin{aligned}
&\left| \left\langle H_\phi^{\mathbb{D}_r}(f_1), H_\psi^{\mathbb{D}_r}(f_2) \right\rangle_{\mathbb{D}_r} - \left\langle H_\phi^{\mathbb{D}_{r_0}}(f_1), H_\psi^{\mathbb{D}_{r_0}}(f_2) \right\rangle_{\mathbb{D}_{r_0}} \right| \\
&\leq \varepsilon \left(1 + \|\psi f_2\|_{L^2(\mathbb{D}_{r_0})} + \|\psi f_2\|_{L^2(\mathbb{D}_r)} \right) \\
&\quad + \left| \left\langle \phi f_1, P^{\mathbb{D}_r}(\psi f_2) - P^{\mathbb{D}_{r_0}}(\psi f_2) \right\rangle_{\mathbb{D}_{r_0-\delta_1}} \right| \\
&\leq \varepsilon \left(1 + \|\psi f_2\|_{L^2(\mathbb{D}_{r_0})} + \|\psi f_2\|_{L^2(\mathbb{D}_{r_0+1})} \right) \\
&\quad + \varepsilon r_0 \sqrt{\pi} \|\phi f_1\|_{L^2(\mathbb{D}_{r_0})}.
\end{aligned}$$

We note that by the Cauchy–Schwarz inequality we used the following inequality above

$$\left| \left\langle \phi f_1, P^{\mathbb{D}_r}(\psi f_2) - P^{\mathbb{D}_{r_0}}(\psi f_2) \right\rangle_{\mathbb{D}_{r_0-\delta}} \right| \leq \varepsilon r_0 \sqrt{\pi} \|\phi f_1\|_{L^2(\mathbb{D}_{r_0})}.$$

Therefore, there exists a constant $K > 0$ independent of r and ε so that

$$\left| \left\langle H_\phi^{\mathbb{D}_r}(f_1), H_\psi^{\mathbb{D}_r}(f_2) \right\rangle_{\mathbb{D}_r} - \left\langle H_\phi^{\mathbb{D}_{r_0}}(f_1), H_\psi^{\mathbb{D}_{r_0}}(f_2) \right\rangle_{\mathbb{D}_{r_0}} \right| \leq \varepsilon K_1$$

for $r_0 \leq r \leq r_0 + \delta_2$.

Similarly if $r \leq r_0$ equation (2) is valid for r and r_0 interchanged. For $\varepsilon > 0$ we choose $0 < \delta_3 < \min\{1, r_0/2\}$ such that

$$\|\phi f_1\|_{L^2(\mathbb{D}_{r_0} \setminus \mathbb{D}_{r_0-\delta_3})} < \varepsilon.$$

By Corollary 1 we choose $0 < \delta_4 < \frac{\delta_3}{2}$ so that so that $r_0 - \delta_4 < r \leq r_0$ implies that

$$\left| P^{\mathbb{D}_r}(\psi f_2)(z) - P^{\mathbb{D}_{r_0}}(\psi f_2)(z) \right| \leq \varepsilon \text{ for } z \in \overline{\mathbb{D}_{r_0-\frac{\delta_3}{2}}}$$

and

$$\left| \langle \phi f_1, \psi f_2 \rangle_{\mathbb{D}_{r_0} \setminus \mathbb{D}_r} \right| \leq \varepsilon.$$

Therefore, there exists a constant $K_2 > 0$ independent of r and ε such that $r_0 - \delta_4 \leq r \leq r_0$ implies that

$$\left| \left\langle H_{\phi}^{\mathbb{D}_r}(f_1), H_{\psi}^{\mathbb{D}_r}(f_2) \right\rangle_{\mathbb{D}_r} - \left\langle H_{\phi}^{\mathbb{D}_{r_0}}(f_1), H_{\psi}^{\mathbb{D}_{r_0}}(f_2) \right\rangle_{\mathbb{D}_{r_0}} \right| \leq \varepsilon K_2.$$

Thus, we have $\lim_{r \rightarrow r_0} \left\langle H_{\phi}^{\mathbb{D}_r}(f_1), H_{\psi}^{\mathbb{D}_r}(f_2) \right\rangle_{\mathbb{D}_r} = \left\langle H_{\phi}^{\mathbb{D}_{r_0}}(f_1), H_{\psi}^{\mathbb{D}_{r_0}}(f_2) \right\rangle_{\mathbb{D}_{r_0}}$. \square

3. Proof of Theorem 3

Proof of Theorem 3. Assume that $H_{\psi}^* H_{\phi}$ is a compact operator and there exists an analytic disk Δ in $\partial\Omega$ (if not we are done), and two symbols ϕ and ψ that are not holomorphic “along” Δ . Namely, there exists a holomorphic function $f : \mathbb{D} \rightarrow \Delta$ so that neither $\phi \circ f$ nor $\psi \circ f$ is holomorphic on \mathbb{D} . Since Ω is a convex Reinhardt bounded domain Lemma 2 implies that the disk Δ is either horizontal or vertical. So without loss of generality we may assume that Δ is horizontal and

$$\Omega = \bigcup_{w \in H} (\Delta_w \times \{w\})$$

where $H \subset \mathbb{C}, \Delta_w = \{z \in \mathbb{C} : (z, w) \in \Omega\}$ is a disk in \mathbb{C} centered at the origin, and $\Delta = \Delta_{w_0}$ for some $w_0 \in \partial H$. By using a linear holomorphic map, $(z, w) \rightarrow (z, e^{i\theta_0}(w - w_0))$ for some $\theta_0 \in \mathbb{R}$, we translate the domain Ω into $\{(z, w) \in \mathbb{C}^2 : \text{Im}(w) < 0\}$. Hence without loss of generality we may assume that $H \subset \{w \in \mathbb{C} : \text{Im}(w) < 0\}$ and $\Omega = \bigcup_{w \in H} (\Delta_w \times \{w\})$ where Δ_w 's are disks centered at the origin and $\Delta = \Delta_0$.

Let us extend $\phi(z, 0)$ and $\psi(z, 0)$ as continuous functions on \mathbb{C} and call the extensions $\phi_0(z)$ and $\psi_0(z)$. Since ϕ_0 and ψ_0 are harmonic and not holomorphic on Δ_0 , Theorem 5 in [Zhe89] (see also [AČ01, Corollary 6]) implies that the product $(H_{\psi_0}^{\Delta_0})^* H_{\phi_0}^{\Delta_0}$ is a nonzero operator. Then there exist $f_1, f_2 \in A^2(\Delta_0)$ such that

$$\int_{\Delta_0} H_{\phi_0}^{\Delta_0}(f_1)(z) \overline{H_{\psi_0}^{\Delta_0}(f_2)(z)} dV(z) \neq 0.$$

Then by Lemma 5 we can choose f_1 and f_2 to be holomorphic polynomials (of one variable).

For convenience, in the following calculations we will abuse the notation as follows: we will assume that ϕ_0, ψ_0, f_1 , and f_2 are functions of z only (or functions of (z, w) but independent of w). We remind the reader that in the computations below, the Bergman projection on the disk Δ_w is denoted by P^{Δ_w} and $H_\eta^{\Delta_w}(f) = \eta f - P^{\Delta_w}(\eta f)$ for $f \in A^2(\Delta_w)$ and $\eta \in L^\infty(\Delta_w)$. We note that functions $(z, w) \rightarrow P^{\Delta_w}(\eta f)(z)$ and $(z, w) \rightarrow H_\eta^{\Delta_w}(f)(z)$ are continuous on Ω . In case of the first function this can be seen as follows:

$$\begin{aligned} & |P^{\Delta_{w_0}}(\eta f)(z_0) - P^{\Delta_w}(\eta f)(z)| \\ & \leq |P^{\Delta_{w_0}}(\eta f)(z_0) - P^{\Delta_w}(\eta f)(z_0)| \\ & \quad + \int_{\Delta_w} |K_{\Delta_w}(z_0, \xi) - K_{\Delta_w}(z, \xi)| |\eta(\xi) f(\xi)| dV(\xi). \end{aligned}$$

As (z, w) goes to (z_0, w_0) in Ω , the first term on the right hand side goes to zero by Corollary 1 and the second term goes to zero because

$$\sup\{|K_{\Delta_w}(z_0, \xi) - K_{\Delta_w}(z, \xi)| : \xi \in \Delta_w\}$$

goes to zero. Also Fubini's Theorem implies that these functions are square integrable.

Let $g_j \in A^2(H)$ which will be specified later. For fixed $w \in H$ and any $z \in \Delta_w$

$$H_{\phi_0}^\Omega(f_1 g_j)(z, w) = \phi_0(z, w) f_1(z) g_j(w) - P^\Omega(\phi_0 f_1 g_j)(z, w)$$

and

$$H_{\phi_0(\cdot, w)}^{\Delta_w}(f_1)(z) = \phi_0(z, w) f_1(z) - P^{\Delta_w}(\phi_0(\cdot, w) f_1)(z)$$

imply that

$$H_{\phi_0}^\Omega(f_1 g_j)(z, w) - g_j(w) H_{\phi_0}^{\Delta_w}(f_1) = P^\Omega(\phi_0 f_1 g_j)(z, w) - g_j(w) P^{\Delta_w}(\phi_0 f_1)(z)$$

is holomorphic in z on Δ_w .

Using Lemma 1 in the first equality below we get

$$\begin{aligned} & \int_{\Delta_w} (H_{\psi_0}^\Omega)^* H_{\phi_0}^\Omega(f_1 g_j)(z, w) \overline{f_2(z)} dV(z) \\ & = \int_{\Delta_w} P^\Omega(\overline{\psi_0} H_{\phi_0}^\Omega(f_1 g_j))(z, w) \overline{f_2(z)} dV(z) \\ & = \int_{\Delta_w} \overline{\psi_0(z, w)} H_{\phi_0}^\Omega(f_1 g_j)(z, w) \overline{f_2(z)} dV(z) \\ & \quad - \int_{\Delta_w} (I - P^\Omega)(\overline{\psi_0} H_{\phi_0}^\Omega(f_1 g_j))(z, w) \overline{f_2(z)} dV(z) \end{aligned}$$

$$\begin{aligned}
 &= \int_{\Delta_w} H_{\phi_0}^\Omega(f_1g_j)(z, w) \overline{H_{\psi_0}^{\Delta_w}(f_2)(z)} dV(z) \\
 &\quad + \int_{\Delta_w} H_{\phi_0}^\Omega(f_1g_j)(z, w) \overline{P^{\Delta_w}(\psi_0f_2)(z)} dV(z) \\
 &\quad - \int_{\Delta_w} (I - P^\Omega)(\overline{\psi_0}H_{\phi_0}^\Omega(f_1g_j))(z, w) \overline{f_2(z)} dV(z) \\
 &= g_j(w) \int_{\Delta_w} H_{\phi_0}^{\Delta_w}(f_1)(z) \overline{H_{\psi_0}^{\Delta_w}(f_2)(z)} dV(z) \\
 &\quad + \int_{\Delta_w} H_{\phi_0}^\Omega(f_1g_j)(z, w) \overline{P^{\Delta_w}(\psi_0f_2)(z)} dV(z) \\
 &\quad - \int_{\Delta_w} (I - P^\Omega)(\overline{\psi_0}H_{\phi_0}^\Omega(f_1g_j))(z, w) \overline{f_2(z)} dV(z).
 \end{aligned}$$

If we multiply both sides by $\overline{g_j(w)}$ and integrate over H we get

$$\begin{aligned}
 &\langle H_{\phi_0}^\Omega(f_1g_j), H_{\psi_0}^\Omega(f_2g_j) \rangle \\
 &= \int_H |g_j(w)|^2 \int_{\Delta_w} H_{\phi_0}^{\Delta_w}(f_1)(z) \overline{H_{\psi_0}^{\Delta_w}(f_2)(z)} dV(z) dV(w) \\
 &\quad + \int_{\Omega} H_{\phi_0}^\Omega(f_1g_j)(z, w) \overline{P^{\Delta_w}(\psi_0f_2)(z)} \overline{g_j(w)} dV(z, w) \\
 &\quad - \int_{\Omega} (I - P^\Omega)(\overline{\psi_0}H_{\phi_0}^\Omega(f_1g_j))(z, w) \overline{f_2(z)} \overline{g_j(w)} dV(z, w).
 \end{aligned}$$

We note that the last integral on the right hand side above is zero. Hence, we have

$$\begin{aligned}
 (3) \quad &\langle H_{\phi_0}^\Omega(f_1g_j), H_{\psi_0}^\Omega(f_2g_j) \rangle \\
 &= \int_H |g_j(w)|^2 \int_{\Delta_w} H_{\phi_0}^{\Delta_w}(f_1)(z) \overline{H_{\psi_0}^{\Delta_w}(f_2)(z)} dV(z) dV(w) \\
 &\quad + \int_{\Omega} H_{\phi_0}^\Omega(f_1g_j)(z, w) \overline{P^{\Delta_w}(\psi_0f_2)(z)} \overline{g_j(w)} dV(z, w).
 \end{aligned}$$

Our next goal is to show that the second integral on the right hand side of (3) goes to zero while the first one does not as j goes to infinity.

Let h be an entire function on \mathbb{C} . Then

$$\begin{aligned}
 &\int_{\Omega} H_{\phi_0}^\Omega(f_1g_j)(z, w) \overline{P^{\Delta_w}(\psi_0f_2)(z)} \overline{g_j(w)} dV(z, w) \\
 &= \int_{\Omega} H_{\phi_0}^\Omega(f_1g_j)(z, w) \overline{h(z)} \overline{g_j(w)} dV(z, w) \\
 &\quad + \int_{\Omega} H_{\phi_0}^\Omega(f_1g_j)(z, w) \overline{(P^{\Delta_w}(\psi_0f_2)(z) - h(z))} \overline{g_j(w)} dV(z, w) \\
 &= \int_{\Omega} H_{\phi_0}^\Omega(f_1g_j)(z, w) \overline{(P^{\Delta_w}(\psi_0f_2)(z) - h(z))} \overline{g_j(w)} dV(z, w).
 \end{aligned}$$

Using the Cauchy–Schwarz inequality we have

$$\begin{aligned} & \int_{\Omega} |H_{\phi_0}^{\Omega}(f_1 g_j)(z, w) \overline{(P^{\Delta_w}(\psi_0 f_2)(z) - h(z))g_j(w)}| dV(z, w) \\ & \leq \|H_{\phi_0}^{\Omega}(f_1 g_j)\| \left(\int_{\Omega} |(P^{\Delta_w}(\psi_0 f_2)(z) - h(z))g_j(w)|^2 dV(z, w) \right)^{1/2}. \end{aligned}$$

Now we choose $g_j(w) = \frac{a_j}{w^{\alpha_j}}$ such that $a_j \rightarrow 0$, $\alpha_j \rightarrow 1^-$, and $\|g_j\|_H = 1$ as $j \rightarrow \infty$. Then one can show that

$$(4) \quad \|H_{\phi - \phi_0}^{\Omega}(g_j)\| \leq \|(\phi - \phi_0)g_j\| \rightarrow 0 \text{ as } j \rightarrow \infty$$

because g_j goes to 0 uniformly on any compact set away from Δ_0 and $\phi - \phi_0 = 0$ on Δ_0 .

Let $\varepsilon > 0$ be fixed. Then there exists a set $L_{\varepsilon} \Subset \Delta_0$ such that

$$\|\psi_0 f_2\|_{L^2(\Delta_0 \setminus L_{\varepsilon})} \leq \frac{\varepsilon}{2}.$$

Furthermore, Lemma 5 and [Kra01, Proposition 1.4.1] imply that there exists an entire function h such that

$$\|P^{\Delta_0}(\psi_0 f_2) - h\|_{L^2(\Delta_0)} \leq \varepsilon$$

and

$$\sup\{|P^{\Delta_0}(\psi_0 f_2)(z) - h(z)| : z \in \overline{L_{\varepsilon}}\} \leq \frac{\varepsilon}{2}.$$

Then by Lemma 3 we can choose $\delta_1 > 0$ such that $|w| < \delta_1$ implies that $L_{\varepsilon} \Subset \Delta_w$. Furthermore, δ_1 can be chosen so that

$$\|\psi_0 f_2\|_{L^2(\Delta_w \setminus \Delta_0)} + \|h\|_{L^2(\Delta_w \setminus \Delta_0)} \leq \frac{\varepsilon}{2}.$$

Finally, Lemma 3 and Corollary 1 imply that there exists $\delta_2 > 0$ such that

$$\sup\{|P^{\Delta_0}(\psi_0 f_2)(z) - P^{\Delta_w}(\psi_0 f_2)(z)| : z \in \overline{L_{\varepsilon}}\} \leq \frac{\varepsilon}{2},$$

for $|w| < \delta_2$ and Lemma 3 and Lemma 4 imply that there exists $\delta_3 > 0$ such that

$$\|E_{\Delta_w} P^{\Delta_w}(T_{L_{\varepsilon}} \psi_0 f_2) - E_{\Delta_0} P^{\Delta_0}(T_{L_{\varepsilon}} \psi_0 f_2)\|_{L^2(\mathbb{C})} \leq \varepsilon$$

for $|w| < \delta_3$.

If we put all these together we have the following: for $\varepsilon > 0$ there exist $\delta = \min\{\delta_1, \delta_2, \delta_3\} > 0$, a set $L_{\varepsilon} \Subset \Delta_0$, and an entire function h such that $|w| < \delta$ implies that:

- (i) $L_{\varepsilon} \Subset \Delta_w$, $\|\psi_0 f_2\|_{L^2(\Delta_w \setminus L_{\varepsilon})} \leq \varepsilon$, and $\|h\|_{L^2(\Delta_w \setminus \Delta_0)} \leq \varepsilon/2$,
- (ii) $\|P^{\Delta_0}(\psi_0 f_2) - h\|_{L^2(\Delta_0)} \leq \varepsilon$,
- (iii) $\sup\{|P^{\Delta_w}(\psi_0 f_2)(z) - h(z)| : z \in \overline{L_{\varepsilon}}\} \leq \varepsilon$,
- (iv) $\|E_{\Delta_w} P^{\Delta_w}(T_{L_{\varepsilon}} \psi_0 f_2) - E_{\Delta_0} P^{\Delta_0}(T_{L_{\varepsilon}} \psi_0 f_2)\|_{L^2(\mathbb{C})} \leq \varepsilon$.

Now we choose j_0 so that

$$|g_j(w)| < \varepsilon \left(1 + \int_{\Omega} |P^{\Delta_w}(\psi_0 f_2)(z) - h(z)|^2 dV(z, w) \right)^{-\frac{1}{2}}$$

for $|w| \geq \delta$ and $j \geq j_0$. Let us define $K_\delta = \cup_{|w| \geq \delta} \Delta_w \subset \Omega$. Then we have

$$\begin{aligned} & \int_{\Omega} |(P^{\Delta_w}(\psi_0 f_2)(z) - h(z))g_j(w)|^2 dV(z, w) \\ &= \int_{K_\delta} |(P^{\Delta_w}(\psi_0 f_2)(z) - h(z))g_j(w)|^2 dV(z, w) \\ & \quad + \int_{\Omega \cap (L_\varepsilon \times B(0, \delta))} |(P^{\Delta_w}(\psi_0 f_2)(z) - h(z))g_j(w)|^2 dV(z, w) \\ & \quad + \int_{\Omega \setminus (K_\delta \cup (L_\varepsilon \times B(0, \delta)))} |(P^{\Delta_w}(\psi_0 f_2)(z) - h(z))g_j(w)|^2 dV(z, w) \\ &\lesssim \sup\{|g_j(w)|^2 : |w| \geq \delta\} \int_{K_\delta} |P^{\Delta_w}(\psi_0 f_2)(z) - h(z)|^2 dV(z, w) \\ & \quad + \sup\{|P^{\Delta_w}(\psi_0 f_2)(z) - h(z)|^2 : z \in \overline{L_\varepsilon}, |w| \leq \delta\} \int_H |g_j(w)|^2 dV(w) \\ & \quad + \int_{|w| < \delta} |g_j(w)|^2 \int_{\Delta_w \setminus L_\varepsilon} |P^{\Delta_w}(\psi_0 f_2)(z) - h(z)|^2 dV(z) dV(w) \\ &\lesssim \varepsilon^2 + \int_{|w| < \delta} |g_j(w)|^2 \int_{\Delta_w \setminus L_\varepsilon} |P^{\Delta_w}(\psi_0 f_2)(z) - h(z)|^2 dV(z) dV(w). \end{aligned}$$

We note that (iii) is used in the last inequality. Then

$$\begin{aligned} & \int_{\Delta_w \setminus L_\varepsilon} |P^{\Delta_w}(\psi_0 f_2)(z) - h(z)|^2 dV(z) \\ &\lesssim \int_{\Delta_w \setminus L_\varepsilon} |P^{\Delta_w}((1 - T_{L_\varepsilon})\psi_0 f_2)(z)|^2 dV(z) \\ & \quad + \int_{\Delta_w \setminus L_\varepsilon} |P^{\Delta_w}(T_{L_\varepsilon}\psi_0 f_2)(z) - E_{\Delta_0}P^{\Delta_0}(T_{L_\varepsilon}\psi_0 f_2)(z)|^2 dV(z) \\ & \quad + \int_{\Delta_w \setminus L_\varepsilon} |E_{\Delta_0}P^{\Delta_0}(T_{L_\varepsilon}\psi_0 f_2)(z) - E_{\Delta_0}P^{\Delta_0}(\psi_0 f_2)(z)|^2 dV(z) \\ & \quad + \int_{\Delta_w \setminus L_\varepsilon} |E_{\Delta_0}P^{\Delta_0}(\psi_0 f_2)(z) - h(z)|^2 dV(z). \end{aligned}$$

Let $|w| < \delta$. Then by (i) we have

$$\|P^{\Delta_w}((1 - T_{L_\varepsilon})\psi_0 f_2)\|_{L^2(\Delta_w \setminus L_\varepsilon)}^2 \leq \|(1 - T_{L_\varepsilon})\psi_0 f_2\|_{L^2(\Delta_w)}^2 \leq \varepsilon^2$$

and by (iv) we have

$$\begin{aligned} & \int_{\Delta_w \setminus L_\varepsilon} |P^{\Delta_w}(T_{L_\varepsilon}\psi_0 f_2)(z) - E_{\Delta_0}P^{\Delta_0}(T_{L_\varepsilon}\psi_0 f_2)(z)|^2 dV(z) \\ & \leq \|E_{\Delta_w}P^{\Delta_w}(T_{L_\varepsilon}\psi_0 f_2) - E_{\Delta_0}P^{\Delta_0}(T_{L_\varepsilon}\psi_0 f_2)\|_{L^2(\mathbb{C})}^2 \\ & \leq \varepsilon^2. \end{aligned}$$

By (i) again and the fact that $\Delta_0 \subset \Delta_w$ we have

$$\begin{aligned} & \int_{\Delta_w \setminus L_\varepsilon} |E_{\Delta_0}P^{\Delta_0}(T_{L_\varepsilon}\psi_0 f_2)(z) - E_{\Delta_0}P^{\Delta_0}(\psi_0 f_2)(z)|^2 dV(z) \\ & \leq \|(1 - T_{L_\varepsilon})\psi_0 f_2\|_{L^2(\Delta_0)}^2 \leq \varepsilon^2. \end{aligned}$$

Furthermore, (i) and (ii) imply that

$$\begin{aligned} & \int_{\Delta_w \setminus L_\varepsilon} |E_{\Delta_0}P^{\Delta_0}(\psi_0 f_2)(z) - h(z)|^2 dV(z) \\ & = \int_{\Delta_0 \setminus L_\varepsilon} |P^{\Delta_0}(\psi_0 f_2)(z) - h(z)|^2 dV(z) + \int_{\Delta_w \setminus \Delta_0} |h(z)|^2 dV(z) \lesssim \varepsilon^2. \end{aligned}$$

Therefore, we have

$$\int_{\Delta_w \setminus L_\varepsilon} |P^{\Delta_w}(\psi_0 f_2)(z) - h(z)|^2 dV(z) \lesssim \varepsilon^2 \text{ for } |w| \leq \delta.$$

Furthermore, since $\int_H |g_j(w)|^2 dV(w) = 1$ and Ω is bounded there exists a constant $C > 0$ such that $\|g_j\| < C$. Therefore,

$$\int_{\Omega} H_{\phi_0}^\Omega(f_1 g_j)(z, w) \overline{P^{\Delta_w}(\psi_0 f_2)(z) g_j(w)} dV(z, w) \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Now we will show that the first integral on the right hand side of (3) stays away from zero as j goes to infinity. We remind the reader that f_1 and f_2 are holomorphic polynomials such that

$$\int_{\Delta_0} H_{\phi_0}^{\Delta_0}(f_1)(z) \overline{H_{\psi_0}^{\Delta_0}(f_2)(z)} dV(z) \neq 0.$$

Therefore, by Lemma 6, without loss of generality and by choosing a smaller $\delta > 0$, if necessary, we may assume that there exists $\beta > 0$ such that

$$\operatorname{Re} \left(\int_{\Delta_w} H_{\phi_0}^{\Delta_w}(f_1)(z) \overline{H_{\psi_0}^{\Delta_w}(f_2)(z)} dV(z) \right) > \beta$$

for $|w| < \delta$. The mass of g_j “accumulates” at the origin in the sense that

$$\int_H |g_j(w)|^2 dV(w) = 1$$

for all j while $g_j(w) \rightarrow 0$ as w stays away from Δ_0 . Then there exists j_0 so that $j \geq j_0$ implies that

$$\left| \int_{K_\delta} |g_j(w)|^2 H_{\phi_0}^{\Delta w}(f_1)(z) \overline{H_{\psi_0}^{\Delta w}(f_2)(z)} dV(z) dV(w) \right| < \frac{\beta}{4}.$$

On the other hand, there exists j_1 such that

$$\begin{aligned} \operatorname{Re} \left(\int_{\Omega \setminus K_\delta} |g_j(w)|^2 H_{\phi_0}^{\Delta w}(f_1)(z) \overline{H_{\psi_0}^{\Delta w}(f_2)(z)} dV(z, w) \right) \\ > \beta \int_{\{w \in H; |w| < \delta\}} |g_j(w)|^2 dV(w) > \frac{\beta}{2} \end{aligned}$$

for $j \geq j_1$. Therefore, for $j \geq \max\{j_0, j_1\}$ we have

$$\operatorname{Re} \left(\int_{\Omega} |g_j(w)|^2 H_{\phi_0}^{\Delta w}(f_1)(z) \overline{H_{\psi_0}^{\Delta w}(f_2)(z)} dV(z, w) \right) > \frac{\beta}{2}.$$

This shows that the first integral on the right hand side of (3) stays away from zero. Hence, by (3) again, $\langle H_{\phi_0}^\Omega(f_1 g_j), H_{\psi_0}^\Omega(f_2 g_j) \rangle$ does not converge to zero as j goes to infinity.

Now we will show that $\langle (H_\psi^\Omega)^* H_\phi^\Omega(f_1 g_j), f_2 g_j \rangle$ does not converge to zero which contradicts the assumption that $H_\psi^* H_\phi$ is compact.

$$\begin{aligned} \left| \langle (H_\psi^\Omega)^* H_\phi^\Omega(f_1 g_j), f_2 g_j \rangle \right| &= |\langle H_\phi^\Omega(f_1 g_j), H_\psi^\Omega(f_2 g_j) \rangle| \\ &\lesssim |\langle H_{\phi_0}^\Omega(f_1 g_j), H_{\psi_0}^\Omega(f_2 g_j) \rangle| \\ &\quad + \|(\phi - \phi_0) f_1 g_j\| \|\psi_0 f_2 g_j\| \\ &\quad + \|\phi_0 f_1 g_j\| \|(\psi - \psi_0) f_2 g_j\| \\ &\quad + \|(\phi - \phi_0) f_1 g_j\| \|(\psi - \psi_0) f_2 g_j\|. \end{aligned}$$

We note that by (4) the last three terms on the right hand side of the inequality above go to zero as j goes to ∞ and we just showed that the first term stays away from zero. Hence, $\langle (H_\psi^\Omega)^* H_\phi^\Omega(f_1 g_j), f_2 g_j \rangle$ does not converge to zero. \square

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