

# Divisors of Fourier coefficients of modular forms

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ABSTRACT. Let  $d(n)$  denote the number of divisors of  $n$ . In this paper, we study the average value of  $d(a(p))$ , where  $p$  is a prime and  $a(p)$  is the  $p$ -th Fourier coefficient of a normalized Hecke eigenform of weight  $k \geq 2$  for  $\Gamma_0(N)$  having rational integer Fourier coefficients.

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## 1. Introduction

Throughout the paper, let  $p, \ell$  be primes,  $\mathcal{H} = \{z \in \mathcal{C} \mid \Im(z) > 0\}$  be the upper half plane. Also let  $N \geq 1$  be a natural number and  $k \geq 2$  be an even integer. Let  $\pi(x)$  denote the usual prime counting function up to  $x$ . Let  $f$  be a normalized Hecke eigen cusp form of weight  $k$  for  $\Gamma_0(N)$  with Nebentypus  $\chi$ . Suppose that the Fourier expansion of  $f$  at  $i\infty$  is

$$f(z) = \sum_{n \geq 1} a(n)q^n,$$

where  $q = e^{2\pi iz}$ . In this paper, we assume that  $a(n)$  are rational integers. The second author and Kumar Murty [12] considered the average value of  $\nu(a(n))$ , where  $\nu(n)$  is the number of distinct prime divisors of  $n$ . In this paper, we investigate the sum  $\sum_{p \leq x} d(a(p))$ , where  $d(n) = \sum_{\delta|n} 1$ . An essential ingredient in our work is a technique that can be traced back

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to van der Corput [22] who majorized the divisor function by short sums. This technique was later refined by many authors. We use a refinement due to Friedlander and Iwaniec [7]. Our result can be thought of as a modular analogue of a result of Erdős [6] who considered the asymptotics of  $\sum_{n \leq x} d(F(n))$ , where  $F(x)$  is an irreducible polynomial with integral coefficients. Finding average value of divisors of arithmetic functions has a long history. Some of the relevant papers in this direction are [6], [7] and [10].

## 2. Preliminaries and statement of the result

For an integer  $\delta \geq 1$  and  $x \in \mathbb{R}$ , set

$$\begin{aligned}\pi^*(x, \delta) &= \#\{p \leq x \mid a(p) \equiv 0 \pmod{\delta}\}, \\ \pi(x, \delta) &= \#\{p \leq x \mid a(p) \neq 0, a(p) \equiv 0 \pmod{\delta}\}.\end{aligned}$$

As before, let  $f(z) = \sum_{n \geq 1} a(n)q^n$  be a normalized Hecke eigenform of weight  $k$  for  $\Gamma_0(N)$  with Nebentypus  $\chi$  and rational integer Fourier coefficients. For a prime  $\ell$ , let  $\mathbb{Z}_\ell$  denote the ring of  $\ell$ -adic integers and  $G := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ . By the work of Deligne [3] and as the Fourier coefficients of  $f$  are integers, there is a continuous representation

$$\rho_\delta : G \rightarrow \text{GL}_2\left(\prod_{\ell \mid \delta} \mathbb{Z}_\ell\right)$$

(where the product is over distinct prime divisors) for any positive integer  $\delta > 1$ . This representation is unramified outside the primes dividing  $\delta N$  and for  $p \nmid N\delta$ ,

$$\text{tr } \rho_\delta(\sigma_p) = a(p), \quad \det \rho_\delta(\sigma_p) = \chi(p)p^{k-1},$$

where  $\sigma_p$  is a Frobenius element of  $p$  in  $G$  and  $\mathbb{Z}$  is embedded diagonally in  $\prod_{\ell \mid \delta} \mathbb{Z}_\ell$ . Denote by  $\tilde{\rho}_\delta$  the reduction modulo  $\delta$  of  $\rho_\delta$ :

$$\tilde{\rho}_\delta : G \xrightarrow{\rho_\delta} \text{GL}_2\left(\prod_{\ell \mid \delta} \mathbb{Z}_\ell\right) \twoheadrightarrow \text{GL}_2(\mathbb{Z}/\delta).$$

Let  $H_\delta$  be the kernel of  $\tilde{\rho}_\delta$ ,  $K_\delta$  the sub field of  $\overline{\mathbb{Q}}$  fixed by  $H_\delta$  and  $G_\delta = \text{Gal}(K_\delta/\mathbb{Q})$ . Let  $C_\delta$  be the subset of  $\tilde{\rho}_\delta(G)$  consisting of elements of trace zero and let  $h(\delta) = |C_\delta|/|G_\delta|$ .

The condition  $a(p) \equiv 0 \pmod{\delta}$ , where  $(p, \delta N) = 1$  means that for any Frobenius element  $\sigma_p$  of  $p$ ,  $\tilde{\rho}_\delta(\sigma_p) \in C_\delta$ . Hence by the Chebotarev density theorem applied to  $K_\delta/\mathbb{Q}$ , we have

$$\pi^*(x, \delta) \sim \frac{|C_\delta|}{|G_\delta|} \pi(x) = h(\delta) \pi(x).$$

As  $C_\delta$  contains the image of complex conjugation, it is nonempty. Note that  $K_{\ell_1^{n_1}} \cap K_{\ell_2^{n_2}} = \mathbb{Q}$  for distinct primes  $\ell_1, \ell_2$  and natural numbers  $n_1, n_2$ . This implies that  $h(\delta) = \prod_{\ell^n \parallel \delta} h(\ell^n)$ , where  $\ell^n \parallel \delta$  means that  $\ell^n \mid \delta$  and  $\ell^{n+1} \nmid \delta$ .

Now suppose that the Generalized Riemann Hypothesis (GRH), i.e., the Riemann Hypothesis for all Artin  $L$ -series is true. Then by the works of Lagarias and Odlyzko [9], one can show that

$$\pi^*(x, \delta) = h(\delta)\pi(x) + O\left(\delta^3 x^{1/2} \log(\delta N x)\right).$$

An improved error term is available provided one also assumes the Artin holomorphy conjecture as proved by M. R. Murty, V. K. Murty and N. Saradha [13]. Moreover, if we define

$$Z(x) = \{p \leq x \mid a(p) = 0\}$$

then as mentioned in [12] one can show the following lemma from the works of Ribet [16] and Serre [20];

**Lemma 1.** *Suppose that  $f$  does not have complex multiplication. Then  $Z(x) \ll x/(\log x)^{3/2-\epsilon}$  for all  $\epsilon > 0$ . Further, suppose that GRH is true. Then  $Z(x) \ll x^{3/4}$ .*

If  $f$  has complex multiplication, then  $Z(x) \sim \frac{1}{2}\pi(x)$ . Now suppose that GRH is true. Then as noted by the second author and Kumar Murty [12], one has:

**Lemma 2.** *Suppose that  $f$  does not have complex multiplication and GRH is true. Then for  $x \geq 2$ ,*

$$\pi(x, \delta) = h(\delta)\pi(x) + O\left(\delta^3 x^{1/2} \log(\delta N x)\right) + O(x^{3/4}).$$

Also it follows from the works of Carayol [2], Momose [11], Ribet [15, 17], Serre [19] and Swinnerton-Dyer [21] that for  $\ell$  sufficiently large,

$$T_\ell := \text{Im } \rho_\ell = \left\{ \gamma \in \text{GL}_2(\mathbb{F}_\ell) \mid \det \gamma \in (\mathbb{F}_\ell^\times)^{k-1} \right\}.$$

In this paper, we prove:

**Theorem 3.** *Assume that GRH is true. Also, assume that  $f$  is a normalized Hecke eigen cusp form of weight  $k$  for  $\Gamma_0(N)$  with rational integer Fourier coefficients  $\{a(n)\}$ . Moreover, suppose that  $f$  does not have complex multiplication. We have*

$$(1) \quad x \ll \sum_{\substack{p \leq x \\ a(p) \neq 0}} d(a(p)) \ll x(\log x)^A,$$

where  $A$  is an absolute constant which depends on  $f$ .

**Remark 4.** It is worth noting that above theorem is true unconditionally when  $f$  is a normalized Hecke eigen cusp form of  $k = 2$ . Indeed, the estimate in Lemma 2 is unconditional if one is considering the case  $k = 2$ . For in

this case, the modular form corresponds to an elliptic curve by a celebrated theorem of Wiles [23] and subsequent work of Breuil, Conrad, Diamond and Taylor [1]. With this theorem in hand, the primes enumerated by  $Z(x)$  are precisely the supersingular primes. Indeed, based on a suggestion of the second author, Elkies (see p. 25 of [4] and [5]) has shown unconditionally that the number of supersingular primes is  $O(x^{3/4})$ . Thus, for  $k = 2$ , we can dispense with the GRH in Lemma 2.

In order to prove the theorem, the following lemmas play an important role. The first one was proved by Friedlander and Iwaniec [7] and for the proof of the second lemma we use Rankin's trick. But as Rankin [14] points out, it should really be called Ingham's trick since Ingham told Rankin about it.

**Lemma 5.** *Let  $m, r \geq 2$  and  $n \geq 1$ . Then*

$$d_r(n) \leq \sum_{\substack{\delta|n \\ \delta \leq n^{1/m}}} (2d(\delta))^{(r-1) \frac{m \log m}{\log 2}},$$

where

$$d_r(n) = \sum_{\substack{n_1 \cdots n_r = n \\ n_1, \dots, n_r \geq 1}} 1.$$

**Proof.** See [7].

**Lemma 6.** *Suppose  $b(n) \geq 0$  for  $n \geq 1$  and*

$$D(s) := \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$$

converges for  $s \in \mathbb{C}$  with  $\Re(s) > t \geq 0$ . Then

$$\sum_{n \leq x} b(n) \leq x^u D(u)$$

for any  $u, x \in \mathbb{R}$  with  $u > t$  and  $x \geq 1$ .

**Proof.** Note that for any real number  $u > t$ , we have

$$\sum_{n \leq x} b(n) \leq \sum_{n=1}^{\infty} b(n) \left(\frac{x}{n}\right)^u \leq x^u D(u). \quad \square$$

In most applications, we choose  $u = t + (1/\log x)$ .

### 3. A group theoretic estimate

For an odd prime  $\ell$ , let  $B_\ell := \mathrm{GL}_2(\mathbb{F}_\ell)$  and

$$A_\ell := \{\gamma \in B_\ell \mid \mathrm{tr} \gamma = 0\}.$$

The conjugacy classes of  $B_\ell$  are one of the following four types:

$$\begin{aligned} \alpha_a &:= \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, & \alpha_b &:= \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \\ \alpha_{a,\delta} &:= \begin{pmatrix} a & 0 \\ 0 & \delta \end{pmatrix}, \quad a \neq \delta, & \beta_{a,b} &:= \begin{pmatrix} a & \epsilon b \\ b & a \end{pmatrix}, \quad b \neq 0, \end{aligned}$$

where  $a, b, \delta \in \mathbb{F}_\ell^\times$  and  $\{1, \sqrt{\epsilon}\}$  is a basis for  $\mathbb{F}_\ell^2$  over  $\mathbb{F}_\ell$ . The number of elements in these classes are  $1, \ell^2 - 1, \ell^2 + \ell$  and  $\ell^2 - \ell$  respectively (see Fulton and Harris [8], page 68 for details). Hence the elements of  $A_\ell$  come from the conjugacy classes  $\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$  and  $\begin{pmatrix} 0 & \epsilon b \\ b & 0 \end{pmatrix}$ , where  $a, b \in \mathbb{F}_\ell^\times$ . Further, the elements  $\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$  and  $\begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}$  belong to the same class and the elements  $\begin{pmatrix} 0 & \epsilon b \\ b & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & -\epsilon b \\ b & 0 \end{pmatrix}$  belong to the same class. Therefore

$$|A_\ell| = [(\ell^2 + \ell)(\ell - 1) + (\ell^2 - \ell)(\ell - 1)]/2 = \ell^2(\ell - 1).$$

Also  $|B_\ell| = |\{\gamma \in A_\ell \mid \det \gamma \in \mathbb{F}_\ell^\times\}| = (\ell^2 - 1)(\ell^2 - \ell)$ . Further, we calculate the cardinality of the sets  $B_{\ell^n} := \text{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z})$  and

$$A_{\ell^n} := \{\gamma \in B_{\ell^n} \mid \text{tr } \gamma = 0\}$$

for all  $n \geq 1$ . Note that any  $\begin{pmatrix} a & b \\ c & \delta \end{pmatrix} \in B_\ell$  lifts to

$$\begin{pmatrix} a + \beta_1\ell & b + \beta_2\ell \\ c + \beta_3\ell & \delta + \beta_4\ell \end{pmatrix} \in B_{\ell^n},$$

where  $1 \leq \beta_1, \beta_2, \beta_3, \beta_4 \leq \ell^{n-1}$ ,  $\beta_i \in \mathbb{Z}$  for all  $1 \leq i \leq 4$ . Also any  $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in A_\ell$  lifts to

$$\begin{pmatrix} a + \beta_1\ell & b + \beta_2\ell \\ c + \beta_3\ell & -a - \beta_1\ell \end{pmatrix} \in A_{\ell^n}$$

for any choice of  $\beta_1, \beta_2, \beta_3 \in \mathbb{Z}$  with  $1 \leq \beta_1, \beta_2, \beta_3 \leq \ell^{n-1}$ . Since the maps  $B_{\ell^n} \rightarrow B_\ell$  and  $A_{\ell^n} \rightarrow A_\ell$  are surjective, it is easy to see from the above observations that  $|B_{\ell^n}| = \ell^{4(n-1)}|B_\ell|$  and  $|A_{\ell^n}| = \ell^{3(n-1)}|A_\ell|$ .

Next for an even integer  $k > 2$ , we calculate the cardinality of the sets

$$\begin{aligned} C_\ell &:= \{\gamma \in B_\ell \mid \det \gamma \in (\mathbb{F}_\ell^\times)^{k-1}\}, \\ D_\ell &:= \{\gamma \in C_\ell \mid \text{tr } \gamma = 0\}. \end{aligned}$$

Writing  $\text{gcd}(\ell - 1, k - 1) := d$ , one has

$$\begin{aligned} C_\ell &= \{\gamma \in B_\ell \mid \det \gamma \in (\mathbb{F}_\ell^\times)^d\} \\ &= \{\gamma \in B_\ell \mid (\det \gamma)^{(\ell-1)/d} \equiv 1 \pmod{\ell}\}. \end{aligned}$$

Consider the surjective group homomorphism

$$\phi : \text{GL}_2(\mathbb{F}_\ell) \rightarrow (\mathbb{F}_\ell^\times)^{(\ell-1)/d}$$

which sends  $\gamma \mapsto (\det \gamma)^{(\ell-1)/d}$ . From the previous discussions, it is clear that  $\ker \phi = C_\ell$ . Hence

$$|C_\ell| = \frac{|B_\ell|}{|(\mathbb{F}_\ell^\times)^{\frac{\ell-1}{d}}|}.$$

But

$$|(\mathbb{F}_\ell^\times)^{\frac{\ell-1}{d}}| = \frac{|\mathbb{F}_\ell^\times|}{|\{x \in \mathbb{F}_\ell^\times \mid x^{(\ell-1)/d} \equiv 1 \pmod{\ell}\}|} = \frac{\ell-1}{(\ell-1)/d} = d.$$

Therefore

$$|C_\ell| = \frac{(\ell^2-1)(\ell^2-\ell)}{d}.$$

The elements of  $D_\ell$  come from the conjugacy classes  $\alpha_{a,-a}$  with  $-a^2 \in (\mathbb{F}_\ell^\times)^{k-1}$  and  $\beta_{0,a}$  with  $-\epsilon a^2 \in (\mathbb{F}_\ell^\times)^{k-1}$ . Let  $g$  be the primitive root of  $\mathbb{F}_\ell^\times$ . We would like to find the cardinality of the sets

$$(2) \quad \{a \mid -a^2 \equiv w^{k-1} \pmod{\ell} \text{ for some } w \in \mathbb{F}_\ell^\times\}$$

and

$$(3) \quad \{a \mid -\epsilon a^2 \equiv w^{k-1} \pmod{\ell} \text{ for some } w \in \mathbb{F}_\ell^\times\}.$$

Write  $a = g^r$ ,  $-1 = g^{\frac{\ell-1}{2}}$  and  $w = g^s$ , where  $0 \leq r, s \leq \ell-1$ . Then the cardinality of (2) is equal to the number of solutions  $r$  for which

$$(4) \quad \frac{\ell-1}{2} + 2r \equiv s(k-1) \pmod{\ell-1},$$

where  $0 \leq s \leq \ell-1$ . This congruence has a solution  $\{r_0, s_0\}$  if and only if  $2r_0 \equiv -\frac{\ell-1}{2} \pmod{\ell-1}$ . Since  $(2, \ell-1) = 1$ , the last congruence has a unique solution in  $r_0$ . Hence the number of  $r$ 's which are solutions of (4) is  $\frac{\ell-1}{d}$ . Note that if  $a$  is in the set (2), then so is  $-a$  and that  $\alpha_{a,-a} = \alpha_{-a,a}$ . Hence

$$|\{a \mid -a^2 \equiv w^{k-1} \pmod{\ell} \text{ for some } w \in \mathbb{F}_\ell^\times\}| = \frac{\ell-1}{2d}.$$

Again writing  $a = g^r$ ,  $-\epsilon = g^{t_0}$  and  $w = g^s$ , where  $0 \leq r, s \leq \ell-1$  and solving the congruence

$$t_0 + 2r \equiv s(k-1) \pmod{\ell-1}$$

we show that the cardinality of the set (3) is  $\frac{\ell-1}{2d}$ . Hence

$$|D_\ell| = \frac{\ell-1}{2d}(\ell^2+\ell) + \frac{\ell-1}{2d}(\ell^2-\ell) = \frac{\ell^2(\ell-1)}{d}.$$

Finally, we calculate for  $n \geq 1$ , the cardinality of the sets

$$C_{\ell^n} := \{\gamma \in B_{\ell^n} \mid \gamma \pmod{\ell} \in C_\ell\}$$

and

$$D_{\ell^n} = \{\gamma \in C_{\ell^n} \mid \text{tr } \gamma = 0\}.$$

Clearly,  $|C_{\ell^n}| = \ell^{4(n-1)}|C_\ell|$  and  $|D_{\ell^n}| = \ell^{3(n-1)}|D_\ell|$ .

### 4. Proof of the theorem

**Proof.** Suppose that  $f$  is a normalized Hecke eigenform of weight  $k$  for  $\Gamma_0(N)$  and  $\delta$  is a large positive integer with the property that if  $p|\delta$ , then  $p \gg 1$ . It follows from the previous two sections that for such  $\delta$ , we have

$$(5) \quad h(\delta) = \prod_{\ell^n|\delta} h(\ell^n) = \prod_{\ell^n|\delta} \frac{\ell^{3(n-1)}\ell}{\ell^{4(n-1)}(\ell^2-1)} = \prod_{\ell^n|\delta} \frac{\ell}{\ell^{n-1}(\ell^2-1)}.$$

Clearly when  $\delta = \ell$  a prime,  $h(\ell) \asymp \frac{1}{\ell}$  for sufficiently large  $\ell$ . For a lower bound, note that

$$\begin{aligned} \sum_{\substack{p \leq x \\ a(p) \neq 0}} d(a(p)) &= \sum_{\substack{p \leq x \\ a(p) \neq 0}} \sum_{\delta|a(p)} 1 \geq \sum_{\substack{p \leq x \\ a(p) \neq 0}} \sum_{\substack{\delta < x^{1/12} \\ a(p) \equiv 0 \pmod{\delta}}} 1 \\ &\geq \sum_{\delta < x^{1/12}}^* \pi(x, \delta), \end{aligned}$$

where  $\sum^*$  varies over all those natural numbers  $\delta$  whose prime divisors are sufficiently large. Hence by Lemma 2, we have

$$\begin{aligned} \sum_{\substack{p \leq x \\ a(p) \neq 0}} d(a(p)) &\geq \pi(x) \sum_{\delta < x^{1/12}}^* h(\delta) + O\left(x^{1/2} \sum_{\delta < x^{1/12}}^* \delta^3 \log \delta\right) + O(x^{5/6}) \\ &= \pi(x) \sum_{\substack{\delta < x^{1/12} \\ \ell|\delta \iff \ell \gg 1}} h(\delta) + O(x^{5/6} \log x) \gg x. \end{aligned}$$

For an upper bound, we can use Lemma 5 to get

$$\sum_{\substack{p \leq x \\ a(p) \neq 0}} d(a(p)) \ll \sum_{\substack{p \leq x \\ a(p) \neq 0}} \sum_{\substack{\delta|a(p) \\ \delta \leq |a(p)|^{1/m}}} d(\delta)^{\frac{m \log m}{\log 2}}.$$

We choose  $m$  so that  $m > 7k$ . Write  $c = \frac{m \log m}{\log 2}$ . As  $|a(p)| < 2p^{k/2}$ , we have

$$\begin{aligned} \sum_{\substack{p \leq x \\ a(p) \neq 0}} d(a(p)) &\ll \sum_{\substack{p \leq x \\ a(p) \neq 0}} \sum_{\substack{\delta|a(p) \\ \delta < x^{1/12}}} d(\delta)^c \\ &= \sum_{\delta < x^{1/12}} d(\delta)^c \pi(x, \delta) \\ &= \sum_{\delta < x^{1/12}} d(\delta)^c \left\{ h(\delta)\pi(x) + O\left(\delta^3 x^{1/2} \log(\delta Nx)\right) + O(x^{3/4}) \right\} \end{aligned}$$

by using Lemma 2. Note that when  $\delta$  has small prime divisors, the value of  $h(\delta)$  is less than the value of the right hand side of (5). Hence for an upper bound we can use the right hand side of (5) for all values of  $\delta$ .

Consider the Dirichlet series

$$F(s) := \sum_{n \geq 1} \frac{d(n)^c h(n)}{n^s} = \zeta(s+1)^{2c} g(s),$$

where  $g(s)$  is analytic for  $\Re(s) \geq 0$ . Thus by Lemma 6, we have

$$\sum_{n \leq z} d(n)^c h(n) \leq z^u F(u),$$

for any real number  $u > 0$ . We choose  $u = 1/\log z$  so that

$$\sum_{n \leq z} d(n)^c h(n) \ll F\left(\frac{1}{\log z}\right).$$

Since

$$|\zeta(s)| \leq \frac{1}{s-1} + L,$$

where  $L$  is an absolute constant, we see easily

$$\sum_{n \leq z} d(n)^c h(n) \ll (\log z)^{2c}.$$

Again consider the Dirichlet series

$$G(s) := \sum_{n \geq 1} \frac{d(n)^c}{n^s} = \zeta(s)^{2c} g_1(s),$$

where  $g_1(s)$  is analytic for  $\Re(s) \geq 1$ . Thus by Lemma 6, we have

$$\sum_{n \leq z} d(n)^c \leq z^u G(u),$$

for any real number  $u > 1$ . We choose  $u = 1/\log z + 1$  so that

$$\sum_{n \leq z} d(n)^c \ll zG\left(\frac{1}{\log z} + 1\right)$$

and hence

$$\sum_{n \leq z} d(n)^c \ll z(\log z)^{2c}.$$

This implies that

$$\sum_{\substack{p \leq x \\ a(p) \neq 0}} d(a(p)) \ll x(\log x)^{2c-1} + O\left(x^{5/6}(\log x)^{2c+1}\right) \ll x(\log x)^{2c-1}.$$

This completes the proof of the theorem.  $\square$



## 5. Concluding remarks

We hasten to remark that the full strength of the GRH is not essential if one only wants an estimate of the form  $x(\log x)^A$  for some  $A$ . Indeed, if one assumes a quasi-GRH (that is, the assumption for any given  $\epsilon > 0$ , the Artin  $L$ -series have no zero in the region  $\Re(s) > 1 - \epsilon$ ), then one can deduce a result of the form

$$\sum_{\substack{p \leq x, \\ a(p) \neq 0}} d(a(p)) \ll x(\log x)^A,$$

for some  $A$  depending on  $\epsilon$ . This is not difficult to see. Indeed, a version of Lemma 1 with an estimate of the form  $x^{1-\epsilon}$  can easily be deduced under such a hypothesis. In addition, one can choose  $m$  appropriately in Lemma 5 so as to ensure that the subsequent sums in the proof of the main theorem can be reasonably estimated. It is also evident that the lower bound can be deduced from a version of the Chebotarev density theorem derived from a quasi-GRH. We leave the details to the reader. All of this analysis suggests the following question. Is it reasonable to expect that there exist constants  $B$  and  $v$  such that we have an asymptotic formula of the type

$$\sum_{p \leq x} d(a(p)) \sim Bx(\log x)^v$$

as  $x \rightarrow \infty$ ? Perhaps  $v = 0$ .

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