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Every strongly summable ultrafilter on $\bigoplus \mathbb{Z}_2$ is sparse

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Abstract. We investigate the possibility of the existence of nonsparse strongly summable ultrafilters on certain abelian groups. In particular, we show that every strongly summable ultrafilter on the countably infinite Boolean group is sparse. This answers a question of Hindman, Steprāns and Strauss.

Contents

1. Introduction

The concept of the Stone–Čech compactification of a semigroup has become one of central importance, and has been studied extensively. Throughout this paper, we think of the Stone–Čech compactification of a discrete abelian semigroup G as the set βG of all ultrafilters on G, where the point $x \in G$ is identified with the principal ultrafilter $\{A \subseteq G \mid x \in A\}$, and the basic open sets are those of the form $\bar{A} = \{p \in \beta G \mid A \in p\}$, for $A \subseteq G$. Then these sets are actually clopen, and $\bar{A}$ is really the closure in βG of the set A, regarded as a subset of βG under the aforementioned identification of points in G with principal ultrafilters. The semigroup operation $+$ on G is also extended by the formula

$$p + q = \{A \subseteq G \mid \{x \in G \mid \{y \in G \mid x + y \in A\} \in q\} \in p\}$$

which turns βG into a right topological semigroup, meaning that for each $p \in \beta G$ the mapping $(\cdot) + p : \beta G \longrightarrow \beta G$ is continuous (note that the extended operation $+$ need not be commutative, and, even if G is a group,

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elements $p \in G^* = \beta G \setminus G$ do not necessarily have inverses). The details of this construction (as well as a lot more information, along with applications) can be seen in [6]. In this paper, we will focus mainly on the case when G is a group.

The lowercase roman letters p, q, r are reserved for ultrafilters, while the uppercase roman letters A, B, C, D, with or without subscripts, will always denote subsets of the abelian group at hand. We will use the von Neumann natural numbers, i.e., a natural number n is viewed as the set $\{0, \dots, n-1\}$ (with 0 equal to $\varnothing$, the empty set); and ω will denote the set of finite ordinals, i.e., the set of natural numbers along with zero (thus the symbols $\in$ and $<$ mean the same when applied to natural numbers, 0, and to ω itself). The lowercase roman letters i, j, k, l, m, n, with or without subscript, will be reserved to denote elements of ω. The lowercase roman letters a, b, c, with or without subscript, will stand for elements of $[\omega]^{<\omega}$, i.e., for finite subsets of ω. The letters M and N, with or without subscripts, will in general be reserved for denoting (finite or infinite) subsets of ω. Given a subset $M \subseteq \omega$, $[M]^{<\omega}$ will denote the set of finite subsets of M, and $[M]^{\omega}$ denotes the set of infinite subsets of M. Of the groups that we study here, one of the most important ones is the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. When dealing with this group, we will identify its elements (which are cosets modulo $\mathbb{Z}$) with their unique representative t satisfying $0 \leq t < 1$. Therefore, when we refer to an element of $\mathbb{T}$ as a real number in $[0, 1)$, we really mean the coset of that number modulo $\mathbb{Z}$.

Definition 1.1. If G is an abelian semigroup, we say that an ultrafilter $p \in \beta G$ is *strongly summable* if it has a base of FS-sets, i.e., if for every $A \in p$ there exists a sequence $\vec{x} = \langle x_n \mid n < \omega \rangle$ such that $p \ni \mathrm{FS}(\vec{x}) \subseteq A$, where

$$\mathrm{FS}(\vec{x}) = \left\{ \sum_{n \in a} x_n \,\middle|\, a \in [\omega]^{<\omega} \setminus \{\varnothing\} \right\}$$

denotes the *set of finite sums of the sequence* $\vec{x}$.

Note that if a strongly summable ultrafilter is principal, then it must actually be 0. Nonprincipal strongly summable ultrafilters on $(\omega, +)$ were first constructed, under CH, by Neil Hindman in [3], although at that time the terminology was still not in use. Their importance at first came from the fact that they are examples of idempotents in $\beta\omega$, but among idempotents they are special in that the largest subgroup of $\omega^* = \beta\omega \setminus \omega$ containing one of them as the identity is just a copy of $\mathbb{Z}$. More concretely, [6, Th. 12.42] establishes that if $p \in \omega^*$ is a strongly summable ultrafilter, and $q, r \in \omega^*$ are such that $q + r = r + q = p$, then $q, r \in \mathbb{Z} + p$. In [4], the authors generalize some results previously only known to hold for ultrafilters on $\beta\omega$ or $\beta\mathbb{Z}$. In particular, they proved there that every strongly summable ultrafilter p on any abelian group G is an idempotent ([4, Th. 2.3]). And [4, Th. 4.6] states that if G can be embedded in $\mathbb{T}$, then whenever $q, r \in G^* = \beta G \setminus G$ are such that $q + r = r + q = p$, it must be the case that $q, r \in G + p$. It is possible to get a slightly stronger result if one strengthens the definition of strongly summable.

Definition 1.2. An ultrafilter $p \in \beta G$ is *sparse* if for every $A \in p$ there exist two sequences $\vec{x} = \langle x_n \mid n < \omega \rangle$, $\vec{y} = \langle y_n \mid n < \omega \rangle$, where $\vec{y}$ is a subsequence of $\vec{x}$ such that $\{x_n \mid n < \omega\} \setminus \{y_n \mid n < \omega\}$ is infinite, $\mathrm{FS}(\vec{x}) \subseteq A$, and $\mathrm{FS}(\vec{y}) \in p$.

Then obviously every sparse ultrafilter will be nonprincipal and strongly summable. And ([4, Th. 4.5]) if G can be embedded in $\mathbb{T}$ and $p \in G^*$ is sparse, then whenever $q, r \in G^*$ are such that $q + r = p$, it must be the case that $q, r \in G + p$.

In [5], the authors investigate the different kinds of abelian semigroups on which every nonprincipal strongly summable ultrafilter must be sparse. For example, every nonprincipal strongly summable ultrafilter $p \in \omega^*$ must actually be sparse (this follows from [5, Th. 3.2] together with either [6, Lemmas 12.20, 12.32] or [1, Lemmas 1A, 1C]). Thus the above result about p being expressible as a sum only trivially holds for all nonprincipal strongly summable ultrafilters on ω. More generally, [5, Th. 4.2] establishes that if S is a countable subsemigroup of $\mathbb{T}$, then every nonprincipal strongly summable ultrafilter on S is sparse. After, they build on this to prove a more general result.

Theorem 1.3 ([5], Th. 4.5)**.** *Let S be a countable subsemigroup of $\bigoplus_{n<\omega} \mathbb{T}$ and let p be a nonprincipal strongly summable ultrafilter on S. If*

$$\left\{ x \in S \;\middle|\; \pi_{\min(x)}(x) \neq \frac{1}{2} \right\} \in p,$$

then p must be sparse (here $\min(x)$ denotes the least i such that $\pi_i(x)$ is nonzero).

So, for example, this theorem, as well as the method for proving it, cannot be applied if p contains the set of $x \in \bigoplus_{n<\omega} \mathbb{T}$ all of whose nonzero entries equal $1/2$. This set is isomorphic to the countably infinite Boolean group $\bigoplus_{n<\omega} \mathbb{Z}_2$. While [5] was still a preprint, it contained the question of whether it is consistent with ZFC that there exists a nonprincipal nonsparse strongly summable ultrafilter on $\bigoplus_{n<\omega} \mathbb{Z}_2$. This question is answered in the negative in section 2, while section 3 gives a slight improvement of [5, Cor. 4.6].

2. Strongly summable ultrafilters in the Boolean group

By the Boolean group we mean the unique (up to isomorphism) countably infinite group all of whose nonidentity elements have order 2. This group is usually thought of as the direct sum of countably many copies of $\mathbb{Z}_2$. However, we will think of it as the group whose underlying set is $[\omega]^{<\omega}$, equipped with the symmetric difference $\triangle$ as the group operation. For $(a_n)_{n<\omega} \in \bigoplus_{n<\omega} \mathbb{Z}_2$, we can define the support of $(a_n)_{n<\omega}$ by

$$\operatorname{supp}(a_n)_{n<\omega} = \{n < \omega \mid a_n = 1\},$$

so that the mapping $(a_n)_{n<\omega} \longmapsto \operatorname{supp}(a_n)_{n<\omega}$ is an isomorphism from $\bigoplus_{n<\omega} \mathbb{Z}_2$ onto $[\omega]^{<\omega}$.

When dealing with FS-sets on this group, we will talk about sets instead of sequences. Thus, if $\vec{x} = \langle x_n \mid n < \omega \rangle$ is a sequence of elements of $[\omega]^{<\omega}$, and $X = \{x_n \mid n < \omega\}$ is the range of that sequence, then instead of $\mathrm{FS}(\vec{x})$ we will write $\mathrm{F}\triangle(X)$, the set of "finite symmetric differences", in order to emphasize that the elements of our group are sets and that their "sum" actually corresponds to taking symmetric differences, and using the fact that, even if the sequence $\vec{x}$ is not injective, that does not alter the resulting FS-set. This means that, for example, if $x_i = x_j$, and $i, j \in a$ for $i \neq j$, then $\sum_{k\in a} x_k = \sum_{k\in a\setminus\{i,j\}} x_k$, due to the fact that every element of our group at hand has order 2. We will use the uppercase roman letters X, Y, Z to denote infinite subsets of $[\omega]^{<\omega}$ whenever we are interested in considering their sets of finite symmetric differences. The main result of this section, and of this paper, is the following theorem.

Theorem 2.1. *Let p be a nonprincipal strongly summable ultrafilter on $[\omega]^{<\omega}$. Then, p is sparse.*

In order to prove this result, we need first of all a lemma which tells us that weakly summable ultrafilters in $[\omega]^{<\omega}$ have a property that is somewhat analogous to that of extending the Fréchet filter. Recall that an ultrafilter p on an abelian semigroup G is *weakly summable* if for every $A \in p$ there is a sequence $\vec{x}$ of elements of G such that $\mathrm{FS}(\vec{x}) \subseteq A$. Thus every strongly summable ultrafilter is weakly summable, and actually ([6, Th. 12.17]) every idempotent ultrafilter on an arbitrary semigroup is weakly summable (and in fact [6, Th. 12.17] an ultrafilter is weakly summable if and only if it is a closure point in βG of the set of idempotents). Notice that a principal weakly summable ultrafilter must be idempotent, in particular if G is a group then the only principal weakly summable ultrafilter is the one that corresponds to the identity element.

Lemma 2.2. *Let p be a weakly summable ultrafilter on $[\omega]^{<\omega}$. Then for any $n < \omega$, there exists an $A \in p$ such that $n \notin \bigcup A$.*

Proof. If p is principal, then $\{\varnothing\} \in p$ will do. Otherwise, let

$$A_0 = \{a \in [\omega]^{<\omega} \mid n \notin a\},$$
$$A_1 = [\omega]^{<\omega} \setminus A_0 = \{a \in [\omega]^{<\omega} \mid n \in a\}.$$

There is $j \in 2$ such that $A_j \in p$. But j cannot equal 1, for otherwise, since p is weakly summable, there would be an infinite set $X \subseteq [\omega]^{<\omega}$ such that $\mathrm{F}\triangle(X) \subseteq A_1$, so if $x, y \in X$ are two distinct elements, we have that $n \in x$

and $n \in y$, thus $n \notin x \triangle y \in \mathrm{F}\triangle(X) \subseteq A_1$, a contradiction. Therefore $A_0 \in p$, and certainly it is true that $n \notin \bigcup A_0$. □

Corollary 2.3. *If p is a weakly summable ultrafilter, then for any finite subset a of ω, there is $A \in p$ such that $\bigcup A$ is disjoint from a.*

Proof. If p is principal, then $\{\varnothing\} \in p$ will do. Otherwise, for each $n \in a$, choose $A_n \in p$ such that $n \notin \bigcup A_n$. Then $p \ni A = \bigcap_{n\in a} A_n$, and certainly this set is as required. □

Originally, the author had a much more involved proof for the previous corollary, whith ideas similar to those of [5, Th. 2.6] and [7, Th. 4], until he came up with the much simpler one that is presented above.

The fact that all elements of $[\omega]^{<\omega}$ have order 2 has some remarkable consequences, amongst which the following is relevant for our purposes.

Lemma 2.4. *Let $X \subseteq [\omega]^{<\omega}$. Then, $\mathrm{F}\triangle(\mathrm{F}\triangle(X)) = \mathrm{F}\triangle(X)$.*

Proof. The "$\supseteq$" part of the equality follows from the fact that $X \subseteq \mathrm{F}\triangle(X)$, and holds in any (semi)group. Now let us illustrate the "$\subseteq$" part with the case where we add two finite sums. Thus let $a, b \in [X]^{<\omega} \setminus \{\varnothing\}$ be distinct, and notice that, since every element in our group at hand has order two, the following holds:

$$\sum_{x\in a} x + \sum_{y\in b} y = \sum_{z\in a\triangle b} z \in \mathrm{F}\triangle(X),$$

and from this it is easy to conclude, by induction, the desired result. □

Now in order to prove our main result, namely Theorem 2.1, let p be a nonprincipal strongly summable ultrafilter on $[\omega]^{<\omega}$. We want to show that p is sparse, thus pick $A \in p$, and pick Z such that $p \ni \mathrm{F}\triangle(Z) \subseteq A$. We would like to find some sets X, Y such that $Y \subseteq X$, $\mathrm{F}\triangle(X) \subseteq A$, $\mathrm{F}\triangle(Y) \in p$ and $X \setminus Y$ is infinite. We will do so as follows.

Claim 2.5. *It is possible to find a Y such that $Y \subseteq \mathrm{F}\triangle(Z)$, $\mathrm{F}\triangle(Y) \in p$, and such that there are infinitely many $z \in Z$ with $z \notin Y$.*

Proof of Theorem 2.1 from Claim 2.5. Let $X = Y \cup Z$. Claim 2.5 guarantees that $X \setminus Y$ is infinite. Moreover $\mathrm{F}\triangle(Y) \in p$, and now by Lemma 2.4 we get that $\mathrm{F}\triangle(X) \subseteq \mathrm{F}\triangle(\mathrm{F}\triangle(Z)) = \mathrm{F}\triangle(Z) \subseteq A$, and we are done. □

Thus, the only thing that remains to be proved is Claim 2.5.

Proof of Claim 2.5. Consider the set $\limsup(Z)$ which contains exactly those $n < \omega$ such that $n \in z$ for infinitely many distinct $z \in Z$. Then if this set is nonempty, say $n \in \limsup(Z)$, we can use Lemma 2.2 to get a $B \in p$ such that $n \notin \bigcup B$. Since p is strongly summable, we can find a Y such that $p \ni \mathrm{F}\triangle(Y) \subseteq B \cap \mathrm{F}\triangle(Z)$. Then $Y \subseteq \mathrm{F}\triangle(Y) \subseteq \mathrm{F}\triangle(Z)$, and for each $z \in Z$ containing n (and by assumption there are infinitely many such) we have that $z \notin Y$, because otherwise we would have $n \in \bigcup B$ contradicting our choice of n and B.

The other case is when $\limsup(Z) = \varnothing$. In this case, let $M = \bigcup Z$. Then M is an infinite subset of ω, with the property that each $n \in M$ is contained in only finitely many $z \in Z$; and we will construct by recursion a very special subset of M. Start by letting $m_0 = \min(M)$, $Z_0 = \{z \in Z \mid m_0 \in z\}$ and $N_0 = \bigcup Z_0$. Then both Z_0, N_0 are finite and nonempty (although Z_0 is a subset of $[\omega]^{<\omega}$, whilst N_0 is a subset of ω). Now recursively define

$$m_{n+1} = \min\left(M \setminus \bigcup_{k \leq n} N_k\right),$$

$Z_{n+1} = \{z \in Z \mid m_{n+1} \in z\}$, and $N_{n+1} = \bigcup Z_{n+1}$. Then again Z_{n+1} is finite, nonempty, and disjoint from all previous Z_k. Also N_{n+1} is finite and nonempty, although the N_k need not be disjoint, and of course $m_{n+1} > m_n$. Now notice that $Z' = \bigcup_{k<\omega} Z_k$ is an infinite subset of Z, and if $z \in Z_k$, then $z \cap \{m_n \mid n < \omega\} = \{m_k\}$. Thus if we let $N = \{m_{2n} \mid n < \omega\}$, then for every $z \in Z'$, $z \cap N$ will be nonempty if and only if $z \in Z_k$ for some even index k. Let $B_0 = \{s \in [\omega]^{<\omega} \mid s \cap N = \varnothing\}$ and $B_1 = [\omega]^{<\omega} \setminus B_0$. Now notice that whenever $z \in Z_k$ for some $k \equiv i (\mathrm{mod}\ 2)$, we must have that $z \notin B_i$. Thus, if we let $i \in 2$ be such that $B_i \in p$, we will have that $z \notin B_i$ for all $z \in \bigcup_{n<\omega} Z_{2n+i}$, and there are infinitely many such. Now using the fact that p is strongly summable, just pick Y such that $p \ni \mathrm{F}\triangle(Y) \subseteq B_i \cap \mathrm{F}\triangle(Z)$. □

I am thankful to Juris Steprāns for pointing out an error in an earlier version of this proof, as well as to the anonymous referee for useful comments on it.

3. Existence of nonsparse strongly summable ultrafilters

In this section we will investigate a necessary condition for the existence of a nonsparse strongly summable ultrafilter on some abelian group G. This represents some partial progress towards answering [5, Question 4.12], and sheds some light on what an answer to that question might look like.

Let G be any abelian group, S a subsemigroup of G, and $p \in \beta G$ an ultrafilter such that $S \in p$. Then $p \upharpoonright S = p \cap \mathfrak{P}(S)$ will be an ultrafilter on S, and it is easy to see that $p \upharpoonright S$ is a nonprincipal ultrafilter if and only if p is. It is also reasonably straightforward to see that $p \upharpoonright S$ is strongly summable if and only if p is, and also that $p \upharpoonright S$ is sparse if and only if p is, because $A \in p$ if and only if $A \cap S \in p \upharpoonright S$. Now if G is any infinite abelian group, and $p \in \beta G$ is a strongly summable ultrafilter, then by definition, there is a sequence $\vec{x} = \langle x_n \mid n < \omega \rangle$ such that $\mathrm{FS}(\vec{x}) \in p$. If we let S denote the subsemigroup of G generated by $\{x_n \mid n < \omega\}$, then it must be the case that S is countable (and cancellative). Since $\mathrm{FS}(\vec{x}) \subseteq S$, then $S \in p$, and thus $q = p \upharpoonright S$ will be strongly summable. Moreover the question of whether p is sparse reduces to the question of whether q is sparse. Thus, when investigating the possibility of a strongly summable ultrafilter being nonsparse on an arbitrary abelian group, we may as well focus our attention on strongly summable ultrafilters on countable cancellative abelian semigroups.

Now if S is a countable cancellative abelian semigroup, then it can be embedded in a countable abelian group H (just in the same way that $(\mathbb{N}, +)$ can be embedded into $(\mathbb{Z}, +)$, or $(\mathbb{Z}\setminus\{0\}, \cdot)$ into $(\mathbb{Q}\setminus\{0\}, \cdot)$). And it is a well-known result (see, e.g., [2, Th. 24.1], [8, 4.1.6], or [9, Th. 9.23]) that every abelian group can be embedded in a divisible group; moreover, each divisible group is a direct sum of copies of $\mathbb{Q}$ and of quasicyclic groups ([2, Th. 23.1], [8, 4.1.5], or [9, Th. 9.14]). Since $\mathbb{Q}$, as well as all quasicyclic groups, can certainly be embedded in $\mathbb{T}$, the conclusion is that every countable abelian group H can be embedded in the direct sum of countably many circle groups $G = \bigoplus_{n<\omega} \mathbb{T}$. (From now on, G will denote that group). Thus we can think of S as a subset of G, and if $q \in \beta S$ is an ultrafilter, then by letting p be the filter on G generated by q, we will actually get an ultrafilter. Moreover $S \in p$, and $q = p \restriction S$, so q will be strongly summable if and only if p is. And again, the question of whether q is sparse reduces to the question of whether p is sparse. Therefore, the whole investigation of whether there is a nonsparse strongly summable ultrafilter on some abelian group (or abelian cancellative semigroup) reduces to the question of whether there exists a nonsparse strongly summable ultrafilter on G.

Our starting point will be the following Theorem of Hindman, Steprāns and Strauss.

Theorem 3.1 ([5], Cor. 4.6)**.** *Let p be a nonsparse nonprincipal strongly summable ultrafilter on G. Then p contains the set of elements of G whose order is some power of* 2*.*

It is not hard to see that the set of elements of G whose order is a power of two is exactly $H = \bigoplus_{n<\omega} \mathbb{T}[2^\infty]$, where

$$\mathbb{T}[2^\infty] = \left\{ t \in \mathbb{T} \,\middle|\, (\exists m, n \in \omega) \left(t = \frac{m}{2^n} \right) \right\}$$

is the quasicyclic 2-group (also known as the Prüfer group of type 2^∞). From now on we will focus on strongly summable ultrafilters on that group (which we will keep denoting by H). We will also be using the groups

$$\mathbb{T}[2^n] = \left\{ t \in \mathbb{T} \,\middle|\, (\exists m \in \omega) \left(t = \frac{m}{2^n} \right) \right\} \cong \mathbb{Z}_{2^n},$$

the isomorphism being given by $\frac{m}{2^n} \longmapsto m$ (whenever we refer to a number $l \in \mathbb{Z}$ as an element of $\mathbb{Z}_k$, we really mean its coset modulo k). Notice that if $n < m < \omega$, then $\mathbb{T}[2^n] \subseteq \mathbb{T}[2^m]$, and $\mathbb{T}[2^\infty] = \bigcup\limits_{n<\omega} \mathbb{T}[2^n]$.

Before stating our first lemma, we need to recall a definition.

Definition 3.2 ([5], Def. 3.1)**.** A sequence $\vec{x}$ on an abelian semigroup S is said to satisfy *strong uniqueness of finite sums* if for each $a, b \in [\omega]^{<\omega} \setminus \{\varnothing\}$:

- If $\sum_{k\in a} x_k = \sum_{k\in b} x_k$ then $a = b$.
- If $\sum_{k\in a} x_k + \sum_{k\in b} x_k \in \mathrm{FS}(\vec{x})$, then $a \cap b = \varnothing$.

By [5, Th. 3.2], if p is a nonprincipal ultrafilter, and for each $A \in p$ there is a sequence $\vec{x}$ satisfying strong uniqueness of finite sums such that $p \ni \mathrm{FS}(\vec{x}) \subseteq A$, then p is sparse.

Lemma 3.3. *Let p be a nonsparse strongly summable ultrafilter on H. Then for each $n < \omega$, the set $B_n = \pi_n^{-1}[\mathbb{T}[2]] = \{x \in H \mid \pi_n(x) \in \{0, 1/2\}\} \in p$.*

Proof. We proceed by contraposition, so let us assume that there is $n < \omega$ such that $B_n \notin p$, and, essentially without loss of generality, let us also assume that $\{x \in H \mid \pi_n(x) \in (0, 1/2)\} \in p$. Pick $j \in 3$ such that $X_j \in p$, where

$$X_j = \left\{ x \in H \;\middle|\; \pi_n(x) \in \bigcup_{m<\omega} \left[\frac{1}{2^{3m+j+2}}, \frac{1}{2^{3m+j+1}}\right) \right\},$$

i.e., thinking of $\pi_n(x)$ as a number in $(0, 1/2)$ written in binary notation, its first digits will be 0.0 and then there will be an infinite string of zeroes and ones. Then $x \in X_j$ if and only if the first such nonzero digit appears in a position that is congruent with j modulo 3. Let $C \in p$, and let $\vec{x}$ be a sequence of elements of H satisfying $p \ni \mathrm{FS}(\vec{x}) \subseteq C \cap X_j$. Note that if $l \neq k$ and for some m, we have that $\frac{1}{2^{3m+j+2}} \leq \pi_n(x_l) < \frac{1}{2^{3m+j+1}}$ and $\frac{1}{2^{3m+j+2}} \leq \pi_n(x_k) < \frac{1}{2^{3m+j+1}}$, then $\frac{1}{2^{3m+j+1}} \leq \pi_n(x_l + x_k) < \frac{1}{2^{3m+j}}$ and so $x_l + x_k \notin X_j$, which is impossible. Thus there is at most one $\pi_n(x_k)$ in each interval $\left[\frac{1}{2^{3m+j+2}}, \frac{1}{2^{3m+j+1}}\right)$ (the positions of the first nonzero digits of distinct $\pi_n(x_k)$ are distinct), so we may assume that the sequence $\vec{x}$ is arranged in such a way that $k < l$ implies $\pi_n(x_k) > \pi_n(x_l)$ ($\vec{x}$ is arranged in increasing order of its first nonzero digit in the n-th projection). Consequently, for each $k < \omega$ we have that $\pi_n(x_k) > 4\pi_n(x_{k+1})$ and therefore $\pi_n(x_k) > 3\sum_{l=k+1}^{\infty} \pi_n(x_l)$. This is easily seen to imply that for $a, b \in [\omega]^{<\omega} \setminus \{\varnothing\}$ and for $\varepsilon : a \longrightarrow \{1, 2\}, \delta : b \longrightarrow \{1, 2\}$, if $\sum_{k\in a} \varepsilon(k)\pi_n(x_k) = \sum_{k\in b} \delta(k)\pi_n(x_k)$ then $a = b$ and $\varepsilon = \delta$. And of course this implies that for $a, b \in [\omega]^{<\omega} \setminus \{\varnothing\}$ and for $\varepsilon : a \longrightarrow \{1, 2\}, \delta : b \longrightarrow \{1, 2\}$, if $\sum_{k\in a} \varepsilon(k)x_k = \sum_{k\in b} \delta(k)x_k$ then $a = b$ and $\varepsilon = \delta$. The latter statement in turn easily implies that the sequence $\vec{x}$ satisfies strong uniqueness of finite sums, hence, by [5, Th. 3.2], p must be sparse. □

The following lemma is stated in more generality than will actually be needed. Notice that we can recover Lemma 2.2 as a particular case of it.

Lemma 3.4. *Let p be a weakly summable ultrafilter on H, and assume that for some $n < \omega$ there is an $A \in p$ such that $\pi_n[A]$ is finite. Then $\{x \in G \mid \pi_n(x) = 0\} \in p$.*

Proof. Enumerate the finite set $\pi_n[A] = \{g_0, \dots, g_{k-1}\}$ and choose $i < k$ such that $A_i = \{x \in A \mid \pi_n(x) = g_i\} \in p$. Since p is weakly summable, we can pick a sequence $\vec{x}$ of elements of G such that $\mathrm{FS}(\vec{x}) \subseteq A_i$. But then, for example, $x_0, x_1, x_0 + x_1 \in A_i$, thus $g_i = \pi_n(x_0 + x_1) = \pi_n(x_0) + \pi_n(x_1) = g_i + g_i$ and this implies that $g_i = 0$. □

Corollary 3.5. *Let p be a nonsparse strongly summable ultrafilter on H. Then for each $n < \omega$, $\{x \in H \mid \pi_n(x) = 0\} \in p$.*

Proof. Just put together Lemmas 3.3 and 3.4. □

In what follows we will use [5, Th. 4.5], which says that if p is a nonprincipal, strongly summable ultrafilter on a subsemigroup S of $G = \bigoplus_{n<\omega} \mathbb{T}$, and if

$$\{x \in S \setminus \{0\} \mid \pi_{\min(x)}(x) \neq 1/2\} \in p$$

(where $\min(x)$ denotes the least n such that $\pi_n(x) \neq 0$), then there exists an $X \in p$ such that any sequence $\vec{x}$ with $\mathrm{FS}(\vec{x}) \subseteq X$ satisfies strong uniqueness of finite sums (and in particular p is sparse). Now assume that p is a nonprincipal, nonsparse strongly summable ultrafilter on H. Notice that p cannot contain the set $\{x \in G \mid (\forall n < \omega)(\pi_n(x) \in \{0, 1/2\})\} = \bigoplus_{n<\omega} \mathbb{T}[2]$, because this set is a copy of $\bigoplus_{n<\omega} \mathbb{Z}_2$ and hence if p contains it, that would induce a strongly summable ultrafilter q on the Boolean group, which by Theorem 2.1 must be sparse and therefore p will also be sparse. Hence p must contain the set $C = \{x \in G \mid (\exists n < \omega)(\pi_n(x) \notin \{0, 1/2\})\}$. For $x \in C$, let $\rho(x)$ denote the least n such that $\pi_n(x) \notin \{0, 1/2\}$.

Lemma 3.6. *Let p be a strongly summable ultrafilter on H. If*

$$\{x \in C \mid \pi_{\rho(x)}(x) \notin \{1/4, 3/4\}\} \in p$$

then p is sparse.

Proof. Consider the morphism $\varphi : H \longrightarrow H$ given by $\varphi(x) = 2x$, whose kernel is exactly $\bigoplus_{n<\omega} \mathbb{T}[2]$. Since the latter is not an element of p, then $\varphi(p) = (\beta\varphi)(p)$ (i.e., the image of p under the continuous extension of $\varphi : H \longrightarrow \beta H$ to βH, which is given by $\{A \subseteq H \mid \varphi^{-1}[A] \in p\}$) is a nonprincipal ultrafilter. Moreover, since p is strongly summable, by [5, Lemma 4.4], so is $\varphi(p)$. Now notice that for $x \in H \setminus \ker(\varphi) = C$, we have $\rho(x) = \min(\varphi(x))$. Thus $\varphi(p)$ contains the set $\{x \in H \setminus \{0\} \mid \pi_{\min(x)}(x) \neq 1/2\}$, since its preimage under φ is exactly $\{x \in C \mid \pi_{\rho(x)}(x) \notin \{1/4, 3/4\}\}$. Therefore by [5, Th. 4.5], there is a set $X \in \varphi(p)$ such that whenever $\mathrm{FS}(\vec{y}) \subseteq X$, y must satisfy strong uniqueness of finite sums. Now for $A \in p$, we can pick a sequence $\vec{x}$ such that $p \ni \mathrm{FS}(\vec{x}) \subseteq A \cap \varphi^{-1}[X]$. Then if we let $\vec{y}$ be the sequence given by $y_n = \varphi(x_n)$, we get that $\mathrm{FS}(\vec{y}) = \varphi[\mathrm{FS}(\vec{x})] \subseteq X$, thus $\vec{y}$ satisfies strong uniqueness of finite sums. It is not hard to see that this implies that $\vec{x}$ satisfies strong uniqueness of finite sums as well, thus p has a basis of sets of the form $\mathrm{FS}(\vec{x})$ for sequences $\vec{x}$ satisfying strong uniqueness of finite sums. Therefore by [5, Th. 3.2], p is sparse. □

Now we are ready to state the main result of this section.

Theorem 3.7. *Assume that there exists a nonsparse strongly summable ultrafilter p on H. Then there exists a sequence $\vec{n} = \langle n_i \mid i < \omega \rangle$ of natural numbers such that $\bigoplus_{n<\omega} \mathbb{T}[2^{n_i}] \in p$. In particular, if there exists a (nonprincipal) nonsparse strongly summable ultrafilter on some abelian cancellative semigroup, then there exists one on $\bigoplus_{n<\omega} \mathbb{Z}_{2^{n_i}}$, for some sequence $\vec{n}$.*

Proof. Let p be a nonprincipal, nonsparse strongly summable ultrafilter on H. As was pointed out above, p cannot contain the set

$$\{x \in G \mid (\forall n < \omega)(\pi_n(x) \in \{0, 1/2\})\} = \bigoplus_{n<\omega} \mathbb{T}[2],$$

hence $C = \{x \in G \mid (\exists n < \omega)(\pi_n(x) \notin \{0, 1/2\})\} \in p$. Moreover by Lemma 3.6, $C_0 = \{x \in C \mid \pi_{\rho(x)}(x) \in \{1/4, 3/4\}\} \in p$. Now $C_0 = C_1 \cup C_3$, where $C_i = \{x \in C_0 \mid \pi_{\rho(x)}(x) = i/4\}$. Essentially without loss of generality, we can assume that $C_1 \in p$. Now choose a sequence $\vec{x}$ with $p \ni \mathrm{FS}(\vec{x}) \subseteq C_1$, and for $i < \omega$ let $M_i = \{n < \omega \mid \rho(x_n) = i\}$ (so $n \in M_i$ implies $\pi_i(x_n) = 1/4$).

Claim 3.8. *For each $i < \omega$, $|M_i| \leq 2$.*

Proof of Claim. Assume, by way of contradiction, that there are pairwise distinct $n, m, k \in M_i$. Let $x = x_n + x_m + x_k$. For $j < i$, we have that $\pi_j(x) \in \{0, 1/2\}$, because $\pi_j(x_n), \pi_j(x_m), \pi_j(x_k) \in \{0, 1/2\}$. On the other hand, $\pi_i(x_n) = \pi_i(x_m) = \pi_i(x_k) = 1/4$ thus $\pi_i(x) = 3/4$, so $\rho(x) = i$ and $x \in C_3$, which is a contradiction. □

Thus, by rearranging the sequence if necessary, we may assume that $i < j$ and $n \in M_i, m \in M_j$ implies that $n < m$. Equivalently, $n < m$ implies that $\rho(x_n) \leq \rho(x_m)$, where the inequality is strict if $m > n + 1$.

Claim 3.9. *Let $n < m < \omega$ and assume that $i = \rho(x_n) < \rho(x_m)$ (which may or may not hold if $m = n + 1$, but must hold if $m > n + 1$). Then $\pi_i(x_m) = 0$.*

Proof of Claim. Let $x = x_n + x_m$. For $j < i$, we since $\pi_j(x_n), \pi_j(x_m) \in \{0, 1/2\}$ we have that $\pi_j(x) \in \{0, 1/2\}$. On the other hand, $\pi_i(x_n) = 1/4$ while $\pi_i(x_m) \in \{0, 1/2\}$, so $\pi_i(x) \in \{1/4, 3/4\}$. Hence $\rho(x) = i$, now since $x \in C_1$, whe must have $\pi_i(x) = 1/4$, and the only way that this can happen is if $\pi_i(x_m) = 0$. □

Claim 3.10. *For every $i < \omega$, the set $\{\pi_i(x_n) \mid n < \omega\}$ is finite.*

Proof of Claim. Let $i < \omega$. We have two cases according to whether M_i is nonempty or not.

If $M_i \neq \varnothing$, then by Claim 3.8, we know that $|M_i| \leq 2$. Thus we can let $k = \min(M_i)$ and $k' = \max(M_i)$ (so that k' equals either k or $k + 1$, $M_i = \{k, k'\}$, and $\pi_i(x_k) = \pi_i(x_{k'}) = 1/4$). Now Claim 3.9 yields $\pi_i(x_n) = 0$

for $n > k'$, therefore

$$\begin{aligned}\{\pi_i(x_n) \mid n < \omega\} &= \{\pi_i(x_n) \mid n < k\} \cup \{\pi_i(x_n) \mid n \in \{k, k'\}\} \\ &\quad \cup \{\pi_i(x_n) \mid n > k\} \\ &= \{\pi_i(x_n) \mid n < k\} \cup \{1/4\} \cup \{0\}\end{aligned}$$

which is finite.

Now if $M_i = \varnothing$, then Claim 3.8 guarantees that there are only finitely many integers l such that $\rho(x_l) < i$, so let k be the greatest such integer if some exists, or $k = 0$ otherwise (equivalently $k = \max(M_j)$ where j is the greatest integer less that i for which $M_j \neq \varnothing$, if such a j exists, or $k = 0$ otherwise). Thus we know that for $n > k$, $\pi_i(x_n) \in \{0, 1/2\}$. Therefore

$$\begin{aligned}\{\pi_i(x_n) \mid n < \omega\} &= \{\pi_i(x_n) \mid n \leq k\} \cup \{\pi_i(x_n) \mid n > k\} \\ &\subseteq \{\pi_i(x_n) \mid n \leq k\} \cup \{0, 1/2\}\end{aligned}$$

which is finite as well. □

Notice that, if F is a finite subset of $\mathbb{T}[2^\infty]$, then the subgroup of the latter generated by the former must be $\mathbb{T}[2^n]$ for suitable n. Namely, if $F = \{\frac{n_0}{2^{m_0}}, \dots, \frac{n_k}{2^{m_k}}\}$, where $2 \nmid n_i$ for $i \leq k$, and $m = \max\{m_0, \dots, m_k\}$, then F generates the subgroup $\mathbb{T}[2^m]$. Hence by Claim 3.10, for each $i < \omega$ we can choose $n_i \in \mathbb{N}$ such that the subgroup of $\mathbb{T}[2^\infty]$ generated by $\{\pi_i(x_n) \mid n < \omega\}$ is $\mathbb{T}[2^{n_i}]$. In this way we construct the sequence $\vec{n} = \langle n_i \mid i < \omega\rangle$ of natural numbers which satisfies that $p \ni \mathrm{FS}(x) \subseteq \bigoplus_{n<\omega} \mathbb{T}[2^{n_i}]$. □

Recall that $\mathbb{T}[2^n] \cong \mathbb{Z}_{2^n}$; and that, if $n < m$, then $\mathbb{T}[2^n] \subseteq \mathbb{T}[2^m]$. From this, it is not hard to see that the sequence $\vec{n}$ from the previous theorem has to be unbounded. For if that sequence was bounded, say by n, then we would have that $\bigoplus_{i<\omega} \mathbb{T}[2^{n_i}] \subseteq \bigoplus_{i<\omega} \mathbb{T}[2^n]$. Hence if p is the ultrafilter yielding $\vec{n}$, p would contain $\bigoplus_{i<\omega} \mathbb{T}[2^n]$, thus inducing a (nonprincipal) nonsparse strongly summable ultrafilter q on its isomorphic copy $\bigoplus_{i<\omega} \mathbb{Z}_{2^n}$. But this cannot happen, more generally, for every $n \geq 2$ every strongly summable ultrafilter q on $G(n) = \bigoplus_{i<\omega} \mathbb{Z}_n$ is sparse. The case when $n = 2$ is just Theorem 2.1, and for $n \geq 3$, pick $0 < k < n$ such that $A_k \in p$, where A_k is the set consisting of those $x \in G(n)$ whose first nonzero coordinate equals k. It is then easy to see that, for $A \in p$, if $\vec{x}$ is a sequence of elements of $G(n)$ such that $\mathrm{FS}(\vec{x}) \subseteq A \cap A_k$, then for distinct i, j the indices of the first nonzero coordinates of x_i and x_j must be different (otherwise the first nonzero coordinate of $x_i + x_j$ would be $2k \neq k$, so we would have $x_i + x_j \notin A_k$, which is absurd). This in turn implies that the sequence $\vec{x}$ satisfies the strong uniqueness of finite sums, and thus by [5, Th. 3.2] the desired conclusion follows.

Therefore, since every sequence $\vec{n}$ given by the theorem must be unbounded, one might be tempted to think that every such sequence should tend to infinity very quickly, but this is really not the case, as the following corollary shows. I am thankful to Andreas Blass for pointing this out to me.

Corollary 3.11. *If p is a nonprincipal nonsparse strongly summable ultrafilter on H, then there is an injective homomorphism $\varphi : H \longrightarrow H$ sending p to an ultrafilter q (which must necessarily be also nonprincipal nonsparse strongly summable) containing the set $\bigoplus_{n<\omega} \mathbb{T}[2^n]$. In particular, if there is a nonprincipal nonsparse strongly summable ultrafilter on some abelian cancellative semigroup, then there is one on $\bigoplus_{n<\omega} \mathbb{Z}_{2^n}$.*

Proof. Given the sequence $\vec{n}$ from Theorem 3.7, create a new strictly increasing sequence $\vec{m} = \langle m_i \mid i < \omega \rangle$ by letting m_0 be the least k such that $n_0 < k$, and recursively letting m_{i+1} be the least k with $\max\{n_{i+1}, m_i\} < k$. Then we can define the embedding $\varphi : H \longrightarrow H$ by letting $\varphi(x)$ be the element of H whose m_i-th coordinate is exactly the i-th coordinate of x and whose k-th coordinate is zero whenever $k \notin \{m_i \mid i < \omega\}$. Clearly φ is injective. Now notice that for $x \in \bigoplus_{i<\omega} \mathbb{T}[2^{n_i}]$, since by construction $n_i < m_i$, we have for every $i < \omega$ that $\pi_{m_i}(\varphi(x)) = \pi_i(x) \in \mathbb{T}[2^{n_i}] \subseteq \mathbb{T}[2^{m_i}]$, and of course for $k \notin \{m_i \mid i < \omega\}$ we have that $\pi_k(\varphi(x)) = 0 \in \mathbb{T}[2^k]$. Thus $\varphi(x) \in \bigoplus_{n<\omega} \mathbb{T}[2^n]$, hence $\varphi\left[\bigoplus_{i<\omega} \mathbb{T}[2^{n_i}]\right] \subseteq \bigoplus_{n<\omega} \mathbb{T}[2^n]$ and so the latter is an element of $\varphi(p)$, and thus the result follows. □

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