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Motivic Hopf elements and relations

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Abstract. We use Cayley–Dickson algebras to produce Hopf elements η, ν, and σ in the motivic stable homotopy groups of spheres, and we prove the relations $\eta\nu = 0$ and $\nu\sigma = 0$ by geometric arguments. Along the way we develop several basic facts about the motivic stable homotopy ring.

Contents

1. Introduction

The work of Morel and Voevodsky [MV, V] has shown how to construct from the category Sm/k of smooth schemes over a commutative ring k a corresponding motivic stable homotopy category. This comes to us as the homotopy category of a model category of motivic symmetric spectra [Ho, J]. Among the motivic spectra are certain "spheres" $S^{p,q}$ for all $p, q \in \mathbb{Z}$, so that for any motivic spectrum X one obtains the bi-graded stable homotopy groups $\pi_{*,*}(X) = \oplus_{p,q}[S^{p,q}, X]$. This paper deals with the construction of some elements and relations in the motivic stable homotopy ring $\pi_{*,*}(S)$, where S is the sphere spectrum.

Classically there are two ways of trying to compute stable homotopy groups. First there was the hands-on approach of Hopf, Toda, Whitehead,

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and others, where one constructs explicit elements and explicit relations. Of course this is difficult and painstaking. Later on, Serre's thesis, and its ultimate realization in the Adams spectral sequence, greatly reduced the difficulties in calculation—but at the expense of computing the homotopy groups of the completions $S^{\wedge}_p$, not S itself. Fortunately these things are closely related: $\pi_j(S^{\wedge}_p)$ is just the p-completion of $\pi_j(S)$.

In the motivic setting the analog of the Adams spectral sequence was explored in [DI2] over an algebraically closed field. For other fields the computations are much more challenging, even for fields like $\mathbb{R}$ and $\mathbb{Q}$. The motivic Adams–Novikov spectral sequence over algebraically closed fields was considered in [HKO]. Recent work of Ormsby–Østvaer studies related issues over the p-adic fields $\mathbb{Q}_p$ [OØ1].

In the present paper our goal is to explore a little of the "hands-on" approach of Hopf, Toda, and Whitehead to motivic homotopy groups. That is to say, our goal is to construct very explicit elements of these groups and to demonstrate some relations that they satisfy. Most of our results work over an arbitrary base; equivalently, they work over the universal base $\mathbb{Z}$. But in practice it is often useful to assume that the base k is either a field or the integers $\mathbb{Z}$. Occasionally we will restrict to the case of a field, for purposes of exposition.

In comparison to Adams spectral sequence computations, the hands-on constructions considered in this paper are very grueling. The ratio of effort versus payoff is fairly large. For this reason we give a few remarks about the motivation for pursuing this line of inquiry.

A drawback of the Adams spectral sequence methods is that the spectral sequences converge only to the homotopy groups of a suitable completion $S^{\wedge}_H$, based on the choice of a prime p. Unlike the classical case, appropriate finiteness theorems are not available and *a priori* there can be a significant difference between the motivic homotopy groups themselves, their completions, and the homotopy groups of the completed sphere spectrum. The motivic Adams spectral sequences compute highly interesting objects, regardless of their exact relationship to the motivic stable homotopy groups. For example, one can use these motivic spectral sequences to learn about classical and equivariant stable homotopy theory, even without identifying the motivic completion $S^{\wedge}_H$ precisely. But while these techniques lead to interesting results, one still cannot help but wonder about the nature of the "true" motivic homotopy groups.

One important aspect of the motivic stable homotopy groups is that they act as operations on every (generalized) motivic cohomology theory. And although it is tautological, it is useful to keep in mind that the motivic stable homotopy ring $\pi_{*,*}(S)$ *equals* the ring of universal motivic cohomology operations. Studying this ring thereby gives us potential tools relevant to algebraic K-theory and algebraic cobordism, for example. In contrast, studying the motivic homotopy groups of the Adams completions only give tools relevant to completed versions of K-theory and cobordism.

Another drawback of the motivic Adams spectral sequence approach is that it only applies in situations where one has very detailed information about the structure of the motivic Steenrod algebra. This information is available when working over an essentially smooth scheme over a field whose characteristic is different from the chosen prime p [HKØ]. However, it is not clear whether these results can be extended to schemes that are not defined over a field, such as $\operatorname{Spec}\mathbb{Z}$. Moreover, when p equals the characteristic of the base field the structure of the motivic Steenrod algebra is likely to be more complicated.

1.1. Background. It follows from Morel's connectivity theorem [M3] that the motivic stable homotopy groups of spheres vanish in a certain range: $\pi_{p,q}(S) = 0$ for $p < q$. The group $\oplus_p \pi_{p,p}(S)$ is called the "0-line", and was completely determined by Morel. It will be useful to briefly review this.

Recall that $S^{1,1} = (\mathbb{A}^1 - 0)$. For each $a \in k^\times$ let $\rho_a \colon S^{0,0} \to S^{1,1}$ be the map that sends the basepoint to 1 and the nonbasepoint to a. This gives a homotopy element ρ_a in $\pi_{-1,-1}(S)$. We write ρ for ρ_{-1} because, as we will see, this element plays a special role.

Furthermore, performing the Hopf construction (cf. Appendix C) on the multiplication map $(\mathbb{A}^1 - 0) \times (\mathbb{A}^1 - 0) \to (\mathbb{A}^1 - 0)$ gives a map $\eta \colon S^{3,2} \to S^{1,1}$, and therefore a corresponding element η in $\pi_{1,1}(S)$. Finally, let

$$\epsilon \colon S^{1,1} \wedge S^{1,1} \to S^{1,1} \wedge S^{1,1}$$

be the twist map. It represents an element in $\pi_{0,0}(S)$.

Morel's theorem [M1] is the following.

Theorem 1.2 (Morel [M2])**.** *Let k be a perfect field whose characteristic is not 2. The ring $\oplus_n \pi_{n,n}(S)$ is the free associative algebra generated by the elements η and ρ_a (for all a in $k^\times$) subject to the following relations:*

- (i) *$\eta\rho_a = \rho_a\eta$ for all a in $k^\times$.*
- (ii) *$\rho_a \cdot \rho_{1-a} = 0$ for all $a \in k - \{0, 1\}$.*
- (iii) *$\eta^2\rho + 2\eta = 0$.*
- (iv) *$\rho_{ab} = \rho_a + \rho_b + \eta\rho_a\rho_b$, for all $a, b \in k^\times$.*
- (v) *$\rho_1 = 0$.*

Additionally, one has $\epsilon = -1 - \rho\eta$.

The relations in Theorem 1.2 have a number of algebraic consequences, some of which are interesting for their own sakes. For example, it follows through a lengthy chain of manipulations that $\rho_a\rho_b = \epsilon\rho_b\rho_a$ [M4, Lemma 2.7(3)]. This is a special case of a more general formula from Proposition 2.5 concerning commutativity in the motivic stable homotopy ring.

There is a map of symmetric monoidal categories

$$\operatorname{Ho}(\mathcal{S}pectra) \to \operatorname{Ho}(\mathcal{M}ot\mathcal{S}pectra)$$

that sends a spectrum to the corresponding "constant presheaf". The technical details are unimportant here, only that this gives a map $\pi_n(S) \to \pi_{n,0}(S)$ from the classical stable homotopy groups to their motivic analogs. For an element $\theta \in \pi_n(S)$ let us write θ_{top} for its image in $\pi_{n,0}(S)$. So, for example, we have the elements η_{top} in $\pi_{1,0}(S)$, ν_{top} in $\pi_{3,0}(S)$, and σ_{top} in $\pi_{7,0}(S)$.

At this point our exposition has reached the limit of what is available in the literature. No complete computation has been made of any stable motivic homotopy group $\pi_{p,q}(S)$ for $p > q$. (For some computations of unstable homotopy groups, though, see [AF]; also, after the present paper was circulated the paper [OØ2] computed the group $\pi_{1,0}(S)$ over certain ground fields.)

1.3. Statements of results. Using a version of Cayley–Dickson algebras we construct elements ν in $\pi_{3,2}(S)$ and σ in $\pi_{7,4}(S)$. Taken together with η in $\pi_{1,1}(S)$ we call these the *motivic Hopf elements*. There is also a zeroth Hopf element: classically this is 2 in $\pi_0(S)$, but in the motivic context it turns out to be better to take this to be $1-\epsilon$ in $\pi_{0,0}(S)$ (we will see why momentarily).

Morel shows in [M2] that the relation $\epsilon\eta = \eta$ follows from commutativity of the multiplication map $\mu\colon (\mathbb{A}^1 - 0) \times (\mathbb{A}^1 - 0) \to \mathbb{A}^1 - 0$. We offer the general philosophy that properties of the higher Cayley–Dickson algebras should give rise to relations amongst the Hopf elements. Teasing out such relations from the properties of the algebras is a tricky business, though. In this paper we prove the following generalization of Morel's result:

Theorem 1.4. $(1-\epsilon)\eta = \eta\nu = \nu\sigma = 0$.

The three relations fit an evident pattern: the product of two successive Hopf elements is zero. We call this the "null-Hopf relation". Notice that the pattern of three equations provides some motivation for regarding $1-\epsilon$ as the zeroth Hopf map. The following corollary is worth recording:

Corollary 1.5. $\epsilon\nu = -\nu$.

Proof. $\epsilon\nu = (-1-\rho\eta)\nu = -\nu$, since $\eta\nu = 0$. □

As a long-term goal it would be nice to completely determine the subalgebra of $\pi_{*,*}(S)$ generated by the motivic Hopf elements, the elements ρ_a, and the image of $\pi_*(S) \to \pi_{*,0}(S)$. These constitute the part of $\pi_{*,*}(S)$ that is "easy to write down". Completion of this goal seems far away, however.

There are other evident geometric sources for maps between spheres. One class of examples are the nth power maps $P_n\colon (\mathbb{A}^1 - 0) \to (\mathbb{A}^1 - 0)$. These give elements of $\pi_{0,0}(S)$, and we completely identify these elements in Theorem 1.6 below. Another group of examples are the diagonal maps $\Delta_{p,q}\colon S^{p,q} \to S^{p,q} \wedge S^{p,q}$. In classical topology these are all null-homotopic, and most of them are null motivically as well. There is, however, an exception when $p = q$:

Theorem 1.6.

(a) *For $n \geq 0$ the diagonal map $\Delta\colon S^{n,n} \to S^{n,n} \wedge S^{n,n}$ represents ρ^n in $\pi_{-n,-n}(S)$.*
(b) *For $p > q \geq 0$ the diagonal map $\Delta\colon S^{p,q} \to S^{p,q} \wedge S^{p,q}$ is null homotopic.*
(c) *For n in $\mathbb{Z}$, the nth power map $P_n\colon (\mathbb{A}^1 - 0) \to (\mathbb{A}^1 - 0)$ represents*

$$\begin{cases} \frac{n}{2}(1-\epsilon) & \text{if } n \text{ is even,} \\ 1 + \frac{n-1}{2}(1-\epsilon) & \text{if } n \text{ is odd.} \end{cases}$$

The various facts in the above theorem are useful in a variety of circumstances, but there is a specific reason for including them in the present paper: all three parts play a role in the proof of the null-Hopf relation from Theorem 1.4.

1.7. Next Steps. This paper does not exhaust the possibilities of the "hands-on" approach to motivic stable homotopy groups over $\operatorname{Spec}\mathbb{Z}$. An obvious next step is to consider generalizations of the classical relation $12\nu = \eta^3$ in $\pi_3(S)$.

This formula as written cannot possibly hold motivically, since the left side belongs to $\pi_{3,2}(S)$ while the right side belongs to $\pi_{3,3}(S)$. An obvious substitution is to ask whether 12ν equals $\eta^2\eta_{\mathrm{top}}$ in $\pi_{3,2}(S)$. One might speculate that the 12ν should be replaced by $6(1-\epsilon)\nu$, but these expressions are already known to be equal in $\pi_{3,2}(S)$ by Corollary 1.5.

Another possible extension concerns Toda brackets. Classically, the Toda bracket $\langle \eta, 2, \nu^2 \rangle$ in $\pi_8(S)$ contains an element called "ϵ" that is a multiplicative generator for the stable homotopy ring. (Beware that this bracket has indeterminacy generated by $\eta\sigma$.) Motivically, we can form the Toda bracket $\langle \eta, 1-\epsilon, \nu^2 \rangle$ in $\pi_{8,5}(S)$ and obtain a motivic generalization over $\operatorname{Spec}\mathbb{Z}$. (There is a notational conflict here because ϵ is used in the motivic context for the twist map in $\pi_{0,0}(S)$.)

There is much more to say about Toda brackets in this context, but we will leave the details for future work.

1.8. Organization of the paper. There is a certain amount of technical machinery needed for the paper, and this has all been deposited into three appendices. The body of the paper has been written assuming knowledge of these appendices, but the most efficient way to read the paper might be to first ignore them, referring back only as needed for technical details. Appendix A deals with stable splittings of smash products inside of Cartesian products. Appendix B deals with joins and also certain issues of "canonical isomorphisms" in homotopy theory. Finally, Appendix C treats the Hopf construction and related issues; there is a key idea of "melding" two pairings together, and a recondite formula for the Hopf construction of such a melding (Proposition C.10). This formula is perhaps the most important technical element in our proof of the null-Hopf relation.

Concerning the main body of the paper, Section 2 reviews basic facts about the motivic stable homotopy category and the ring $\pi_{*,*}(S)$. An important issue here is the precise definition of what it means for a map in the stable homotopy category to represent an element of $\pi_{*,*}(S)$, and also formulas for dealing with the "motivic signs" that inevitably arise in calculations.

Section 3 deals with diagonal maps and power maps, and there we prove Theorem 1.6. Section 4 reviews the necessary material about Cayley–Dickson algebras and defines the motivic Hopf elements η, ν, and σ. Finally, in Section 5 we prove the null-Hopf relation of Theorem 1.4.

1.9. Notation. We remark that the symbols χ and p, when applied to maps, have a special meaning in this paper. Maps called p are always the projection from a Cartesian to a smash product, and maps called χ are certain stable splittings for these projections. See Appendix A for details.

2. Preliminaries

This section describes certain foundational issues and conventions regarding the motivic stable homotopy category and the motivic stable homotopy ring $\pi_{*,*}(S)$.

2.1. Basic setup. Fix a commutative ring k (in practice this will usually be $\mathbb{Z}$ or a field). Let Sm/k denote the category of smooth schemes over $\operatorname{Spec} k$. The category of *motivic spaces* is the category of simplicial presheaves $sPre(Sm/k)$. This category carries various Quillen-equivalent model structures that represent unstable $\mathbb{A}^1$-homotopy theory, but for the purposes of this paper we will mostly use the injective model structure developed in [MV]. It is very convenient that all objects are cofibrant in this structure. We will usually shorten "motivic spaces" to just "spaces" for the rest of the paper.

Most of the paper actually restricts to the setting of pointed motivic spaces. This is the associated model category

$$sPre(Sm/k)_* = (*\downarrow sPre(Sm/k))$$

of motivic spaces under $*$.

As explained in [J], one can stabilize the category of pointed motivic spaces to form a model catgory of motivic symmetric spectra. We write $\mathcal{M}otSpectra$ for this category. Our aim in this paper is to work in the homotopy category $\operatorname{Ho}(\mathcal{M}otSpectra)$ as much as possible, and this is where all of our theorems take place. As is usual in homotopy theory, however, a certain amount of work necessarily has to take place at the model category level.

It is useful to be able to compare the motivic homotopy category to the classical homotopy category of topological spaces, and there are a couple of ways to do this. The "constant presheaf" functor is the left adjoint in a Quillen pair $s\mathcal{S}et \to sPre(Sm/k)$ (when we write Quillen pairs we draw an arrow in the direction of the left adjoint). This stabilizes to a similar Quillen pair between symmetric spectra categories

$$\mathcal{S}pectra \xrightarrow{c} \mathcal{M}ot\mathcal{S}pectra.$$

Alternatively, if the base ring k is embedded in $\mathbb{C}$ then we can 'realize' our motivic spaces as ordinary topological spaces, and likewise realize motivic spectra as ordinary spectra. Unfortunately this doesn't work well at the model category level if we use the injective model structure, as we do not get Quillen pairs. For these comparison purposes it is more convenient to use the flasque model structure of [I]. We will not need the details in the present paper, only the fact that this can be done; we occasionally refer to topological realization in a passing comment.

2.2. Spheres and the ring $\pi_{*,*}(S)$. We begin with the two objects $S^{1,0}$ and $S^{1,1}$ in $\mathrm{Ho}\,(\mathcal{M}ot\mathcal{S}pectra)$. Here $S^{1,0} = \Sigma^\infty S^1$, where S^1 is the "simplicial circle", i.e., the constant presheaf with value S^1. Likewise, $S^{1,1}$ is the suspension spectrum of the representable presheaf $(\mathbb{A}^1 - 0)$, which has basepoint given by the rational point 1 in $(\mathbb{A}^1 - 0)$.

Let us fix motivic spectra $S^{-1,0}$ and $S^{-1,-1}$ together with isomorphisms (in the homotopy category) $a_1\colon S^{-1,0} \wedge S^{1,0} \to S^{0,0}$ and $a_2\colon S^{-1,-1} \wedge S^{1,1} \to S^{0,0}$. There is some choice involved in these isomorphisms, as they can be varied by an arbitrary self-homotopy equivalence of the spectrum $S^{0,0}$. For a_1 it is convenient to fix the corresponding isomorphism $a\colon S^{-1} \wedge S^1 \to S^0$ in $\mathcal{S}pectra$ and then let a_1 be the image of a under the derived functor of c. For a_2 it is perhaps best to fix a choice once and for all over $\mathrm{Spec}\,\mathbb{Z}$, and to insist that the topological realization of a_2 is a; this is not strictly necessary, however.

For each integer n, define

$$S^{n,0} = \begin{cases} (S^{1,0})^{\wedge(n)} & \text{if } n \geq 0, \\ (S^{-1,0})^{\wedge(-n)} & \text{if } n < 0, \end{cases}$$

$$S^{n,n} = \begin{cases} (S^{1,1})^{\wedge(n)} & \text{if } n \geq 0, \\ (S^{-1,-1})^{\wedge(-n)} & \text{if } n < 0. \end{cases}$$

Finally, for integers p and q, define

$$S^{p,q} = (S^{1,0})^{\wedge(p-q)} \wedge (S^{1,1})^{\wedge(q)}.$$

We will need the following important result from [D]: for any $(p_1, q_1), \dots, (p_n, q_n)$ in $\mathbb{Z}^2$ and $(p_1', q_1'), \dots, (p_k', q_k')$ in $\mathbb{Z}^2$ such that $\sum_i p_i = \sum_i p_i'$ and $\sum_i q_i = \sum_i q_i'$, there is a uniquely–distinguished "canonical isomorphism"

$$\phi\colon S^{p_1,q_1} \wedge \cdots \wedge S^{p_n,q_n} \to S^{p_1',q_1'} \wedge \cdots \wedge S^{p_k',q_k'}$$

in the homotopy category of motivic spectra. These canonical isomorphisms have the properties that:

- If ϕ and ϕ' are canonical then so are $\phi \wedge \phi'$ and ϕ^{-1}.
- Identity maps are all canonical, as are the maps a_1 and a_2.
- The unit maps $S^{p,q} \wedge S^{0,0} \cong S^{p,q}$ and $S^{0,0} \wedge S^{p,q} \cong S^{p,q}$ are canonical.
- Any composition of canonical maps is canonical.

See [D, Remark 1.9] for a complete discussion. We will always denote these canonical morphisms by the symbol ϕ. (Note: In this paper we systematically suppress all associativity isomorphisms; but if we were not suppressing them, they would also be canonical).

Define $\pi_{p,q}(S) = [S^{p,q}, S^{0,0}]$ and write $\pi_{*,*}(S)$ for $\oplus_{p,q}\pi_{p,q}(S)$. If $f \in \pi_{a,b}(S)$ and $g \in \pi_{c,d}(S)$ define $f \cdot g$ to be the composite

$$S^{a+c,b+d} \xrightarrow{\phi} S^{a,b} \wedge S^{c,d} \xrightarrow{f \wedge g} S^{0,0} \wedge S^{0,0} \cong S^{0,0}.$$

By [D, Proposition 6.1(a)] this product makes $\pi_{*,*}(S)$ into an associative and unital ring, where the subring $\pi_{0,0}(S)$ is central.

2.3. Representing elements of $\pi_{*,*}(S)$. Let $f\colon S^{a,b} \to S^{p,q}$. We write $|f|$ for $(a-p, b-q)$, i.e., the bidegree of the motivic stable homotopy element that f will represent. There are two ways to obtain an element of $\pi_{a-p,b-q}(S)$ from f, which we will denote $[f]_l$ and $[f]_r$. Let $[f]_l$ be the composite

$$S^{a-p,b-q} \xrightarrow{\phi} S^{a,b} \wedge S^{-p,-q} \xrightarrow{f \wedge \mathrm{id}} S^{p,q} \wedge S^{-p,-q} \xrightarrow{\phi} S^{0,0}$$

and let $[f]_r$ be the composite

$$S^{a-p,b-q} \xrightarrow{\phi} S^{-p,-q} \wedge S^{a,b} \xrightarrow{\mathrm{id} \wedge f} S^{-p,-q} \wedge S^{p,q} \xrightarrow{\phi} S^{0,0}.$$

It is proven in [D, Section 6.2] that $[gf]_r = [g]_r \cdot [f]_r$, whereas $[gf]_l = [f]_l \cdot [g]_l$. In this paper we will never use $[f]_l$, and so we will just write $[f] = [f]_r$.

For each $a, b, p, q \in \mathbb{Z}$ let $t_{(a,b),(p,q)}$ denote the composition

$$S^{a+p,b+q} \xrightarrow{\phi} S^{a,b} \wedge S^{p,q} \xrightarrow{t} S^{p,q} \wedge S^{a,b} \xrightarrow{\phi} S^{a+p,b+q}$$

where t is the twist isomorphism for the smash product. Write

$$\tau_{(a,b),(p,q)} = [t_{(a,b),(p,q)}] \in \pi_{0,0}(S).$$

It is easy to see that $\tau_{1,0} = -1$, as this formula holds in $\mathrm{Ho}\,(\mathcal{S}pectra)$ and one just pushes it into $\mathrm{Ho}\,(\mathcal{M}ot\mathcal{S}pectra)$ via the functor c. Let $\epsilon = \tau_{1,1}$. The following formula is then a special case of [D, Proposition 6.6]:

$$\tau_{(a,b),(p,q)} = (-1)^{(a-b)\cdot(p-q)} \cdot \epsilon^{b \cdot q}. \tag{2.4}$$

Note that $\tau\colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to \pi_{0,0}(S)^\times$ is bilinear.

The elements $\tau_{(a,b),(p,q)}$ arise in various formulas related to commutativity of the smash product. For example, the following is from [D, Proposition 1.18]:

Proposition 2.5 (Graded-commutativity)**.** *Let $f \in \pi_{a,b}(S)$ and $g \in \pi_{c,d}(S)$. Then*

$$fg = gf \cdot \tau_{(a,b),(c,d)} = gf \cdot (-1)^{(a-b)(c-d)} \cdot \epsilon^{bd}.$$

Remark 2.6. If $f\colon S^{a,b}\to S^{p,q}$ then we may consider the two maps $f\wedge \mathrm{id}_{r,s}$ and $\mathrm{id}_{r,s}\wedge f$. All three of these maps represent elements in $\pi_{*,*}(S)$, but not necessarily the same ones. The following two facts are proven in [D, Proposition 6.11]:

$$\text{(i)} \qquad [\mathrm{id}_{r,s}\wedge f]=[f],$$

$$\text{(ii)} \qquad [f\wedge \mathrm{id}_{r,s}]=\tau_{|f|,(r,s)}\cdot[f]=\tau_{(a-p,b-q),(r,s)}[f] = (-1)^{(a-p-b+q)\cdot(r-s)}\epsilon^{(b-q)s}[f].$$

A useful special case says that if $f\colon S^{a,b}\to S^{a,b}$ then

$$[f]=[\mathrm{id}_{r,s}\wedge f]=[f\wedge \mathrm{id}_{r,s}].$$

If $g\colon S^{r,s}\to S^{t,u}$ then combining (i) and (ii) we obtain

$$\text{(iii)} \quad [f\wedge g]=[(f\wedge \mathrm{id}_{t,u})\circ(\mathrm{id}_{a,b}\wedge g)]=[f\wedge \mathrm{id}_{t,u}]\cdot[\mathrm{id}_{a,b}\wedge g]=[f]\cdot[g]\cdot\tau_{|f|,(t,u)}.$$

2.7. Homotopy spheres. We will often study maps $f\colon X\to Y$ where X and Y are homotopy equivalent to motivic spheres but not actual spheres themselves. In this case one can obtain a corresponding element $[f]$ of $\pi_{*,*}(S)$, but only after making specific choices of orientations for X and Y.

To make this precise, let us say that a *homotopy sphere* is a motivic spectrum X that is isomorphic to some sphere $S^{p,q}$ in the motivic stable homotopy category. An *oriented homotopy sphere* is a motivic spectrum X together with a *specified* isomorphism $X\to S^{p,q}$ in the motivic stable homotopy category.

A given homotopy sphere has many orientations. The set of orientations is in bijective correspondence with the set of multiplicative units inside $\pi_{0,0}(S)$. We call this set of units the *motivic orientation group*, which depends on the base scheme in general. Note that the analog in classical topology is the group $\mathbb{Z}/2=\{-1,1\}$. By Morel's Theorem, over perfect fields whose characteristic is not 2, the motivic orientation group is the group of units in the Grothendieck–Witt ring $GW(k)$. A formulaic description of this group seems to be unknown, but we do not actually need to know anything specific about it for the content of this paper. Nevertheless, understanding this group is a curious problem and so we do offer the remark below:

Remark 2.8 (Motivic orientations)**.** Recall that $GW(k)$ is obtained by quotienting the free algebra on symbols $\langle a\rangle$ for $a\in k^\times$ by the relations:

(1) $\langle a\rangle\langle b\rangle=\langle ab\rangle$.
(2) $\langle a^2\rangle=1$.
(3) $\langle a\rangle+\langle b\rangle=\langle a+b\rangle+\langle ab(a+b)\rangle$.

The elements $\langle a\rangle$ are clearly units in $GW(k)$, and so one obtains a group map $\mathbb{Z}/2\times\left[k^\times/(k^\times)^2\right]\to GW(k)^\times$ by sending the generator of $\mathbb{Z}/2$ to -1 and the element $[a]$ of $k^\times/(k^\times)^2$ to $\langle a\rangle$.

The map $\mathbb{Z}/2 \times \left[k^\times/(k^\times)^2\right] \to GW(k)^\times$ is an isomorphism in some cases like $k = \mathbb{R}$ and $k = \mathbb{Q}_p$ for $p \equiv 1 \bmod 4$, but not in other cases like $k = \mathbb{Q}_p$ for $p \equiv 3 \bmod 4$.

Remark 2.9 (Suspension data)**.** Here is a fundamental difficulty that occurs even in classical homotopy theory: given two objects A and B that are models for the suspension of an object X, there is no canonical isomorphism between A and B in the homotopy category. In some sense, the problem boils down to the fact that if we are just handed a model of ΣX then we are likely to see two cones on X glued together, but we do not know which is the "top" cone and which is the "bottom". Mixing the roles of the two cones tends to alter maps by a factor of -1. So when talking about models for ΣX it is important to have the two cones distinguished. We define *suspension data* for X to be a diagram $[C_+X \longleftarrow X \longrightarrow C_-X]$ where both maps are cofibrations and both C_+X and C_-X are contractible. We call C_+X the *top cone* and C_-X the *bottom cone*. Choices of suspension data will appear throughout the paper, starting in Remark 2.10(2) below. See Appendix B.1 for more discussion of this and related issues.

Remark 2.10 (Induced orientations on constructions)**.** If X and Y are homotopy spheres then constructions like suspension, smash product, and the join (see Appendix B) yield other homotopy spheres. If X and Y are oriented then these constructions inherit orientations in a specified way:

(1) If X and Y have orientations $X \to S^{p,q}$ and $Y \to S^{a,b}$, then $X \wedge Y$ has an induced orientation

$$X \wedge Y \longrightarrow S^{p,q} \wedge S^{a,b} \xrightarrow{\phi} S^{p+a,q+b},$$

where the second map is the canoncal isomorphism.

(2) If X has an orientation $X \to S^{p,q}$ and $C_+X \longleftarrow X \longrightarrow C_-X$ constitutes suspension data for X, then $C_+X \amalg_X C_-X$ has an induced orientation

$$C_+X \amalg_X C_-X \longrightarrow S^{1,0} \wedge X \longrightarrow S^{1,0} \wedge S^{p,q} \xrightarrow{\phi} S^{p+1,q},$$

where the first map is the canonical isomorphism in the homotopy category (see Section B.1).

(3) Suppose that $X \to S^{p,q}$ and $Y \to S^{a,b}$ are orientations. Then the join $X * Y$ (see Appendix B) has an induced orientation

$$X * Y \longrightarrow S^{1,0} \wedge X \wedge Y \xrightarrow{\simeq} S^{1,0} \wedge S^{p,q} \wedge S^{a,b} \xrightarrow{\phi} S^{p+a+1,q+b},$$

where the first map is the canonical isomorphism in the homotopy category from Lemma B.5.

A map $f\colon X \to Y$ between homotopy spheres does not by itself yield an element of $\pi_{*,*}(S)$. But once X and Y are oriented we obtain the composite

$$S^{a,b} \xrightarrow{\cong} X \longrightarrow Y \xrightarrow{\cong} S^{p,q}.$$

If $\tilde{f}$ denotes this composite, then $[\tilde{f}]$ gives an element of $\pi_{a-p,b-q}(S)$. In the future we will just denote this element by $[f]$, by abuse of notation.

There is an important case in which one does not have to worry about orientations:

Lemma 2.11. *Let X be a homotopy sphere and $f\colon X \to X$ be a self-map. Then f represents a well-defined element of $\pi_{0,0}(S)$ that is independent of the choice of orientation on X.*

Proof. Choose any two orientations $g\colon X \longrightarrow S^{p,q}$ and $h\colon X \longrightarrow S^{p,q}$. The diagram

$$\begin{array}{ccccccc} S^{p,q} & \xrightarrow{g^{-1}} & X & \xrightarrow{f} & X & \xrightarrow{g} & S^{p,q} \\ {\scriptstyle hg^{-1}}\downarrow & & & & & & \downarrow{\scriptstyle hg^{-1}} \\ S^{p,q} & \xrightarrow{h^{-1}} & X & \xrightarrow{f} & X & \xrightarrow{h} & S^{p,q} \end{array}$$

commutes in the stable homotopy category. This shows that the elements of $\pi_{0,0}(S)$ represented by the top and bottom rows are related by conjugation by the element hg^{-1} of $\pi_{0,0}(S)$. But the ring $\pi_{0,0}(S)$ is commutative, so conjugation by hg^{-1} acts as the identity. □

Example 2.12. The following examples specify standard orientations for the models of spheres that we commonly encounter.

(1) $\mathbb{P}^1$ based at $[1:1]$ or $[0:1]$.

By the *standard affine covering diagram* of $\mathbb{P}^1$ we mean

$$U_1 \leftarrow U_1 \cap U_2 \rightarrow U_2$$

where U_1 (resp., U_2) is the open subscheme of points $[x:y]$ such that $x \neq 0$ (resp., $y \neq 0$). There is an evident isomorphism of diagrams

$$\begin{array}{ccccc} U_1 & \longleftarrow & U_1 \cap U_2 & \longrightarrow & U_2 \\ {\scriptstyle\cong}\downarrow & & {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ \mathbb{A}^1 & \xleftarrow{i} & (\mathbb{A}^1 - 0) & \xrightarrow{\text{inv}} & \mathbb{A}^1 \end{array}$$

where i is the inclusion and inv sends a point x to x^{-1}. (For example, $U_1 \to \mathbb{A}^1$ sends $[x:y]$ to $\frac{y}{x}$). The bottom row of the diagram is suspension data for $(\mathbb{A}^1 - 0)$. As in Remark 2.10, this gives an orientation to $\mathbb{A}^1 \amalg_{(\mathbb{A}^1-0)} \mathbb{A}^1$. The canonical map $\mathbb{A}^1 \amalg_{(\mathbb{A}^1-0)} \mathbb{A}^1 \to \mathbb{P}^1$ is a weak equivalence, which gives an orientation on $\mathbb{P}^1$ as well.

(2) $(\mathbb{A}^n - 0)$ based at $(1,1,\dots,1)$ or at $(1,0,\dots,0)$.

We orient $(\mathbb{A}^n - 0)$ as the join of $(\mathbb{A}^1 - 0)$ and $(\mathbb{A}^{n-1} - 0)$. That is to say, $\left[(\mathbb{A}^1 - 0) \times \mathbb{A}^{n-1}\right] \amalg_{(\mathbb{A}^1-0)\times(\mathbb{A}^{n-1}-0)} \left[\mathbb{A}^1 \times (\mathbb{A}^{n-1} - 0)\right]$ has an orientation using Remark 2.10 and induction. The canonical map

$$\left[\left[(\mathbb{A}^1 - 0) \times \mathbb{A}^{n-1}\right] \amalg_{(\mathbb{A}^1-0)\times(\mathbb{A}^{n-1}-0)} \left[\mathbb{A}^1 \times (\mathbb{A}^{n-1} - 0)\right]\right] \to (\mathbb{A}^n - 0)$$

is a weak equivalence, which gives an orientation on $(\mathbb{A}^n - 0)$.

(3) The split unit sphere S_{2n-1} based at $(1, 0, \dots, 0)$.

Let $\mathbb{A}^{2n}$ have coordinates $x_1, y_1, \dots, x_n, y_n$ and let $S_{2n-1} \hookrightarrow \mathbb{A}^{2n}$ be the closed subvariety defined by $x_1 y_1 + \cdots + x_n y_n = 1$. The quadratic form on the left of this equation is called the *split quadratic form*, and S_{2n-1} is called the unit sphere with respect to this split form. Let $\pi\colon S_{2n-1} \to (\mathbb{A}^n - 0)$ be the map

$$(x_1, y_1, \dots, x_n, y_n) \mapsto (x_1, x_2, \dots, x_n).$$

This is a Zariski-trivial bundle with fibers $\mathbb{A}^{n-1}$, and so π is a weak equivalence (this follows from a standard argument, for example using the techniques of [DI1, 3.6–3.9]). The standard orientation on $(\mathbb{A}^n - 0)$ therefore induces an orientation on S_{2n-1} via π.

Note that there are other weak equivalences $S_{2n-1} \to (\mathbb{A}^n - 0)$, such as $(x_1, y_1, \dots, x_n, y_n) \mapsto (x_1, x_2, \dots, x_{n-1}, y_n)$. These maps can induce different orientations on S_{2n-1}.

3. Diagonal maps and power maps

Le X be an unstable, oriented, homotopy sphere that is equivalent to $S^{p,q}$ for some $p \geq q \geq 0$. The diagonal $\Delta_X\colon X \to X \wedge X$ represents an element in $\pi_{-p,-q}(S)$. When $X = S^{p,q}$ we write $\Delta_{p,q} = \Delta_{S^{p,q}}$. Our goal in this section is the following result.

Theorem 3.1. *Let $p \geq q \geq 0$. The element $[\Delta_{p,q}]$ in $\pi_{-p,-q}(S)$ is zero if $p > q$, and it is ρ^q if $p = q$.*

In classical algebraic topology these diagonal maps are all null (except for Δ_{S^0}) because of the following lemma.

Lemma 3.2. *If X is a simplicial suspension then Δ_X is null.*

Proof. We assume that $X = S^{1,0} \wedge Z$ and we consider the commutative diagram

$$\begin{array}{ccc} S^{1,0} \wedge Z & & \\ \downarrow{\scriptstyle \Delta_X} & \searrow^{\Delta_{1,0} \wedge \Delta_Z} & \\ S^{1,0} \wedge Z \wedge S^{1,0} \wedge Z & \xrightarrow{1 \wedge T \wedge 1} & S^{1,0} \wedge S^{1,0} \wedge Z \wedge Z. \end{array}$$

Since the horizontal map is an isomorphism in the homotopy category, it is sufficient to check that $\Delta_{1,0}$ is null. But this is true in the homotopy category of $s\mathcal{S}et_*$, and therefore is true in pointed motivic spaces—the latter follows using the left Quillen functor $c\colon s\mathcal{S}et_* \to sPre_*(Sm/k)$. □

Our next goal will be to show that $[\Delta_{1,1}] = \rho$ in $\pi_{-1,-1}(S)$. We will exhibit an explicit geometric homotopy, but to do this we will need to suspend so that we are looking at maps $S^{2,1} \to S^{3,2}$ instead of $S^{1,1} \to S^{2,2}$. The point is that $(\mathbb{A}^2 - 0)$ gives a convenient geometric model for $S^{3,2}$ (see Example 2.12(2)).

We start by considering the two maps

$$\Delta_{1,1}, \mathrm{id} \wedge \rho \colon (\mathbb{A}^1 - 0) \longrightarrow (\mathbb{A}^1 - 0) \wedge (\mathbb{A}^1 - 0)$$

(for the second, we implicitly identify the space $\mathbb{A}^1 - 0$ in the domain with $(\mathbb{A}^1 - 0) \wedge S^{0,0}$). We will model the suspensions of these two maps via conveniently chosen suspension data.

Let C be the pushout $[(\mathbb{A}^1 - 0) \times \mathbb{A}^1] \amalg_{(\mathbb{A}^1-0)\times\{1\}} [\mathbb{A}^1 \times \{1\}]$; by left properness, C is contractible (recall that all constructions occur in the presheaf category). Below we will provide several maps $C \to \mathbb{A}^2 - 0$, so note that to specify such a map it suffices to give a polynomial formula $(x,t) \mapsto f(x,t) = (f_1(x,t), f_2(x,t))$ with the "formal" properties that $f(x,t) \neq (0,0)$ whenever $x \neq 0$, and $f(x,1) \neq (0,0)$ for all x. Rigorously, this amounts to the ideal-theoretic conditions that $f_1, f_2 \in k[x,t]$, $x \in \mathrm{Rad}(f_1, f_2)$ and $(f_1(x,1), f_2(x,1)) = k[x]$.

Let $D = C \amalg_{(\mathbb{A}^1-0)} C$ where $(\mathbb{A}^1 - 0)$ is embedded in both copies of C as the presheaf $(\mathbb{A}^1 - 0) \times \{0\}$. Note that $[C \leftarrowtail (\mathbb{A}^1 - 0) \rightarrowtail C]$ is suspension data for $(\mathbb{A}^1 - 0)$, and so D is a model for the suspension of $(\mathbb{A}^1 - 0)$. Let us adopt the notation where we use (x,t) for the coordinates in the first copy of C, and (y,s) for the coordinates in the second copy of C. Heuristically, D consists of two kinds of points (x,t) and (y,s), which are identified when $s = t = 0$ and $x = y$. Let us also use (z,w) for the coordinates on the target $(\mathbb{A}^2 - 0)$.

Define the map $\delta\colon D \to (\mathbb{A}^2 - 0)$ by the following formulas:

$$(x,t) \mapsto (x, (1-t)x + t) = (1-t)(x,x) + t(x,1),$$
$$(y,s) \mapsto ((1-s)y + s, y) = (1-s)(y,y) + s(1,y).$$

The reader can verify that these formulas do specify two maps $C \to (\mathbb{A}^2 - 0)$ that agree on $(\mathbb{A}^1 - 0) \times \{0\}$, and hence determine a map $D \to (\mathbb{A}^2 - 0)$ as claimed. In a moment we will show that δ gives a model for $\Sigma\Delta_{1,1}$ in the motivic stable homotopy category.

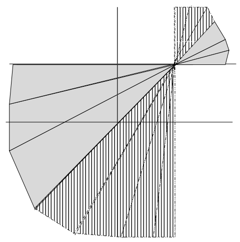
Let us also define a map $R\colon D \to (\mathbb{A}^2 - 0)$ by the formulas:

$$(x,t) \mapsto (x, -1 + 2t),$$
$$(y,s) \mapsto (y, -1).$$

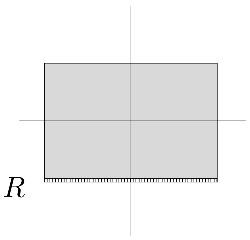
Once again, these formulas do specify two maps $C \to (\mathbb{A}^2 - 0)$ that agree on $(\mathbb{A}^1 - 0) \times \{0\}$, and hence determine a map $D \to (\mathbb{A}^2 - 0)$. We will show that R gives a model for $\Sigma(\mathrm{id} \wedge \rho)$.

Because the formulas are rather unenlightening, we give pictures that depict the maps δ and R. Each picture shows a map $D \to (\mathbb{A}^2 - 0)$, with the two different shadings representing the image of the "top" and "bottom" halves of D, i.e., the two copies of C. Note that the pictures have been drawn as if the second coordinate on C were an interval $[0,1]$ instead of $\mathbb{A}^1$. Really, the two shaded regions should each stretch out infinitely in both directions; but this would produce a picture with too much overlap to be useful.

δ



R



In the picture for δ the reader should note the diagonal $(\mathbb{A}^1-0) \hookrightarrow (\mathbb{A}^2-0)$, and the picture should be interpreted as giving two deformations of the diagonal: one deformation swings the punctured-line about $(1,1)$ until it becomes the $w=1$ punctured-line (at which time it can be "filled in" to an $\mathbb{A}^1$, not just an $\mathbb{A}^1-0$). The second swings the punctured line in the other direction until it becomes $z=1$, and again is filled in to an $\mathbb{A}^1$ at that time. This is our map $\delta\colon D \to (\mathbb{A}^2-0)$.

The picture for R is simpler to interpret. We map $\mathbb{A}^1-0$ to $\mathbb{A}^2-0$ via $x \mapsto (x,-1)$; one deformation moves this vertically up to $x \mapsto (x,1)$ and then fills it in to a map from $\mathbb{A}^1$, whereas the second deformation leaves it constant and then fills it in. This gives us two maps $C \to \mathbb{A}^2-0$ which patch together to define $R\colon D \to \mathbb{A}^2-0$.

Having introduced δ and R, our next step is to show that they represent the elements $[\Delta_{1,1}]$ and ρ in $\pi_{-1,-1}(S)$.

Lemma 3.3.

(a) *The map* $\delta\colon D \to (\mathbb{A}^2-0)$ *represents* $[\Delta_{1,1}]$ *in* $\pi_{-1,-1}(S)$.
(b) *The map* $R\colon D \to (\mathbb{A}^2-0)$ *represents* ρ *in* $\pi_{-1,-1}(S)$.

Proof. For δ, consider the diagram

$$\begin{array}{ccccc} & & \big[(\mathbb{A}^1-0)\wedge\mathbb{A}^1\big] \amalg_{(\mathbb{A}^1-0)\wedge(\mathbb{A}^1-0)} \big[\mathbb{A}^1\wedge(\mathbb{A}^1-0)\big] & & \\ & \nearrow^{\delta'} & \uparrow & \searrow & \\ C\amalg_{(\mathbb{A}^1-0)} C & & \big[(\mathbb{A}^1-0)\times\mathbb{A}^1\big]\amalg_{(\mathbb{A}^1-0)\times(\mathbb{A}^1-0)}\big[\mathbb{A}^1\times(\mathbb{A}^1-0)\big] & \longrightarrow & \frac{\mathbb{A}^2-0}{\mathbb{A}^1\times 1\cup 1\times\mathbb{A}^1} \\ & \searrow_{\delta} & \downarrow & \nearrow & \\ & & \mathbb{A}^2-0. & & \end{array}$$

Here δ' takes the first copy of C to $(\mathbb{A}^1-0)\wedge\mathbb{A}^1$ via $(x,t) \mapsto (x,(1-t)x+t)$, and takes the second copy of C to $\mathbb{A}^1\wedge(\mathbb{A}^1-0)$ via the formula $(y,s) \mapsto ((1-s)y+s,y)$. The smash products are important; they allow us to define δ' on the two copies of $\mathbb{A}^1\times\{1\}$ in the two copies of C. Note that the outer parallelogram obviously commutes.

The five unlabeled arrows are all weak equivalences between homotopy spheres. Orient each of the spheres in the following way:

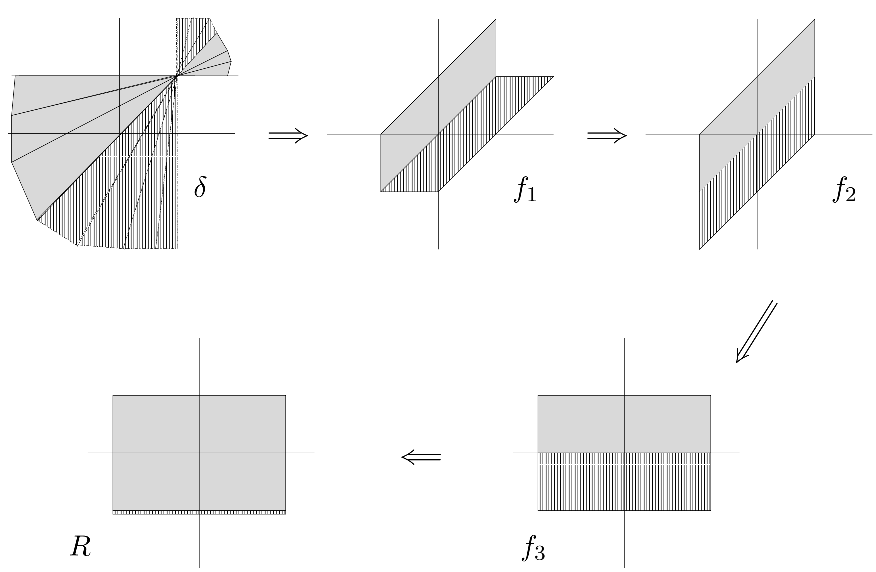
- The top sphere is oriented as the suspension of $(\mathbb{A}^1 - 0) \wedge (\mathbb{A}^1 - 0)$.
- The middle sphere is oriented as the join $(\mathbb{A}^1 - 0) * (\mathbb{A}^1 - 0)$.
- $\mathbb{A}^2 - 0$ is oriented in the standard way.
- $(\mathbb{A}^2 - 0)/(\mathbb{A}^1 \times 1 \cup 1 \times \mathbb{A}^1)$ is oriented so that the projection map from $\mathbb{A}^2 - 0$ is orientation-preserving.

The five weak equivalences are then readily checked to be orientation-preserving. This part of the argument only serves to verify that the standard orientations on the top and bottom spaces (in the middle column) match up when we map to $[\mathbb{A}^2 - 0]/(\mathbb{A}^1 \times 1 \cup 1 \times \mathbb{A}^1)$.

The commutativity of the outer parallelogram now implies that δ and δ' represent the same element of $\pi_{-1,-1}(S)$. Since δ' is clearly a model for the suspension of $\Delta_{1,1}$, this completes the proof that $[\delta] = [\Delta_{1,1}]$.

The same argument shows that R is a model for the suspension of $\mathrm{id} \wedge \rho$. Here, we use a map R' that takes the first copy of C to $\mathbb{A}^1 \wedge (\mathbb{A}^1 - 0)$ via $(x, t) \mapsto (x, -1 + 2t)$, and takes the second copy of C to $(\mathbb{A}^1 - 0) \wedge \mathbb{A}^1$ via $(y, s) \mapsto (y, -1)$. □

Our final step is to show that δ and R are homotopic. We will give a series of homotopies that deforms δ to R. Since the formulas are again rather unenlightening we begin by giving a sequence of pictures that depict three intermediate stages. Each is a map $D \to (\mathbb{A}^2 - 0)$; we will then give four homotopies showing how to deform each picture to the next. These pictures will not make complete sense until one compares to the formulas in the arguments below, but it is nevertheless useful to see the pictures ahead of time.



These pictures (and the explicit formulas in the proof below) can seem unmotivated. It is useful to know that each homotopy is a standard straight-line homotopy. If one has the idea of deforming δ to R, and the only tool one is allowed to use is a straight-line homotopy, a bit of stumbling around quickly leads to the above chain of maps; there is nothing deep here.

Lemma 3.4. *The maps δ and R are homotopic as unbased maps $D \to (\mathbb{A}^2 - 0)$.*

Proof. We will give a sequence of maps $H\colon D \times \mathbb{A}^1 \to (\mathbb{A}^2 - 0)$, each giving an $\mathbb{A}^1$-homotopy from H_0 to H_1. These will assemble into a chain of homotopies from δ to R. Note that $D \times \mathbb{A}^1$ is isomorphic to

$$(C \times \mathbb{A}^1) \amalg_{(\mathbb{A}^1-0)\times\mathbb{A}^1} (C \times \mathbb{A}^1)$$

and that $C \times \mathbb{A}^1$ is isomorphic to

$$\big[(\mathbb{A}^1 - 0) \times \mathbb{A}^1 \times \mathbb{A}^1\big] \amalg_{(\mathbb{A}^1-0)\times\{1\}\times\mathbb{A}^1} \big[\mathbb{A}^1 \times \{1\} \times \mathbb{A}^1\big].$$

To specify a map $C \times \mathbb{A}^1 \to (\mathbb{A}^2 - 0)$, it suffices to give a polynomial formula $(x, t, u) \mapsto f(x, t, u) = (f_1(x, t, u), f_2(x, t, u))$ with the "formal" properties that $f(x, t, u) \neq (0, 0)$ whenever $x \neq 0$, and $f(x, 1, u) \neq (0, 0)$ for all x and u. Rigorously, this amounts to the ideal-theoretic conditions that $f_1, f_2 \in k[x, t, u]$, $x \in \mathrm{Rad}(f_1, f_2)$ and $(f_1(x, 1, u), f_2(x, 1, u)) = k[x, u]$.

Here are three maps $D \to (\mathbb{A}^2 - 0)$:

$$\begin{aligned} f_1\colon &\quad (x,t) \mapsto (x, x+t), &\quad (y,s) \mapsto (y+s, y),\\ f_2\colon &\quad (x,t) \mapsto (x, x+t), &\quad (y,s) \mapsto (y, y-s),\\ f_3\colon &\quad (x,t) \mapsto (x, t), &\quad (y,s) \mapsto (y, -s), \end{aligned}$$

and here are four homotopies:

$$H_1 : \begin{cases} (x,t,u) \mapsto (x, (1-t+ut)x + t), \\ (y,s,u) \mapsto ((1-s+us)y + s, y). \end{cases}$$

$$H_2 : \begin{cases} (x,t,u) \mapsto (x, x+t), \\ (y,s,u) \mapsto (y + (1-u)s, y - us). \end{cases}$$

$$H_3 : \begin{cases} (x,t,u) \mapsto (x, x+t-ux), \\ (y,s,u) \mapsto (y, y-s-uy). \end{cases}$$

$$H_4 : \begin{cases} (x,t,u) \mapsto (x, (1-u)t + u(2t-1)), \\ (y,s,u) \mapsto (y, (u-1)s - u). \end{cases}$$

We leave it to the reader to verify that each formula really does define a map $D \times \mathbb{A}^1 \to (\mathbb{A}^2 - 0)$, and that these give $\mathbb{A}^1$-homotopies

$$\delta \simeq f_1 \simeq f_2 \simeq f_3 \simeq R. \qquad \square$$

Proposition 3.5. *The diagonal map $(\mathbb{A}^1 - 0) \to (\mathbb{A}^1 - 0) \wedge (\mathbb{A}^1 - 0)$ represents ρ in $\pi_{-1,-1}(S)$.*

Proof. Consider the two maps $\delta, R\colon D \to \mathbb{A}^2 - 0$. These are based maps if D and $\mathbb{A}^2 - 0$ are both given the basepoint $(1,1)$ (in the case of D, choose the point $(1,1)$ in the first copy of C). Since δ and R are unbased homotopic, Lemma 3.6 below yields that $\Sigma^\infty \delta = \Sigma^\infty R$ in the stable homotopy category. We therefore obtain $[\Delta] = [\delta] = [R] = \rho$, with the first and last equalities by Lemma 3.3. ☐

Lemma 3.6. *Let X and Y be pointed motivic spaces, and let $f, g\colon X \to Y$ be two maps. If f and g are homotopic as unbased maps then $\Sigma^\infty f = \Sigma^\infty g$ in the motivic stable homotopy category.*

Proof. For any motivic space A, let $C_u A = [A \times c(\Delta^1)]/[A \times \{1\}]$ denote the unbased simplicial cone on a space A, and let $\Sigma_u A$ be the unbased suspension functor

$$\Sigma_u A = (C_u A) \amalg_{A \times \{0\}} (C_u A).$$

Equip $\Sigma_u A$ with the basepoint given by the "cone point" in the first copy of $C_u A$. When A is pointed, let ΣA be the usual based simplicial suspension, i.e. $\Sigma A = (\Sigma_u A)/(\Sigma_u *)$. Note that the projection $\Sigma_u A \to \Sigma A$ is a natural based motivic weak equivalence.

Applying Σ_u and Σ to f and g, and then stabilizing via $\Sigma^\infty(-)$, yields the diagram

$$\begin{array}{ccc} \Sigma^\infty(\Sigma_u X) & \underset{\Sigma^\infty(\Sigma_u g)}{\overset{\Sigma^\infty(\Sigma_u f)}{\rightrightarrows}} & \Sigma^\infty(\Sigma_u Y) \\ \simeq \downarrow & & \downarrow \simeq \\ \Sigma^\infty(\Sigma X) & \underset{\Sigma^\infty(\Sigma g)}{\overset{\Sigma^\infty(\Sigma f)}{\rightrightarrows}} & \Sigma^\infty(\Sigma Y). \end{array}$$

Since f and g are unbased homotopic, $\Sigma_u f$ and $\Sigma_u g$ are based homotopic. Hence $\Sigma^\infty(\Sigma_u f) = \Sigma^\infty(\Sigma_u g)$ in the stable homotopy category. The above diagram then shows that $\Sigma^\infty(\Sigma f) = \Sigma^\infty(\Sigma g)$, and hence $\Sigma^\infty f = \Sigma^\infty g$. ☐

Our identification of $[\Delta_{1,1}]$ with ρ gives a nice geometric explanation for the following relation in $\pi_{*,*}(S)$.

Corollary 3.7. *In $\pi_{*,*}(S)$, there is the relation $\epsilon\rho = \rho\epsilon = \rho$.*

Proof. We already know from Proposition 2.5 that ϵ is central, as it lies in $\pi_{0,0}(S)$. Using the model for ϵ as the twist map on $S^{1,1}$, note that $\epsilon\Delta_{1,1} = \Delta_{1,1}$ as maps $S^{1,1} \to S^{1,1} \wedge S^{1,1}$ ☐

We can finally conclude the proof of the main result of this section.

Proof of Theorem 3.1. If $p > q$ then $S^{p,q}$ is a simplicial suspension, and so $\Delta_{p,q}$ is null by Lemma 3.2. For the case where $p = q$ we consider the commutative diagram

$$\begin{array}{ccc} S^{q,q} & = & S^{1,1}\wedge\cdots\wedge S^{1,1} \\ \downarrow{\scriptstyle \Delta_{q,q}} & & \downarrow{\scriptstyle \Delta_{1,1}\wedge\cdots\wedge\Delta_{1,1}} \\ S^{q,q}\wedge S^{q,q} & \xrightarrow{T} & (S^{1,1}\wedge S^{1,1})\wedge\cdots\wedge(S^{1,1}\wedge S^{1,1}) \end{array}$$

where T is an appropriate composition of twist and associativity maps, involving $\binom{q}{2}$ twists. Note that $[T]=\epsilon^{\binom{q}{2}}$, using Remark 2.6. The diagram then gives

$$\rho^q=[\Delta_{1,1}]^q=[T]\cdot[\Delta_{q,q}]=\epsilon^{\binom{q}{2}}[\Delta_{q,q}]$$

using Proposition 3.5 for the first equality. Rearranging gives

$$[\Delta_{q,q}]=\epsilon^{\binom{q}{2}}\rho^q=\rho^q,$$

using $\epsilon\rho=\rho$ in the final step. □

The following result about arbitrary motivic homotopy ring spectra is a direct consequence of our work above.

Corollary 3.8. *Let E be a motivic homotopy ring spectrum, i.e., a monoid in the motivic stable homotopy category. Write $\bar\rho$ for the image of ρ under the unit map $\pi_{*,*}(S)\to\pi_{*,*}(E)$. For each $n\geq 0$, there is an isomorphism $E^{*,*}(S^{n,n})\cong E^{*,*}\oplus E^{*,*}x$ as $E^{*,*}$-modules, where x is a generator of bidegree (n,n). The ring structure is completely determined by graded commutativity in the sense of Proposition 2.5 together with the fact that $x^2=\bar\rho^n x$.*

In other words, the ring $E^{*,*}(S^{n,n})$ is an ϵ-graded-commutative $E^{*,*}$-algebra on one generator x of bidegree (n,n), subject to the single relation $x^2=\bar\rho^n x$.

Proof. The statement about $E^{*,*}(S^{n,n})$ as an $E^{*,*}$-module is formal; the generator x is the map $S^{n,n}\cong S^{n,n}\wedge S^{0,0}\xrightarrow{\mathrm{id}\wedge u}S^{n,n}\wedge E$, where $u\colon S^{0,0}\to E$ is the unit map. The graded commutativity of $E^{*,*}(S^{n,n})$ is by [D, Remark 6.14]. It only remains to calculate x^2, which is the composite

$$\begin{array}{l} S^{n,n}\xrightarrow{\Delta}S^{n,n}\wedge S^{n,n}\xrightarrow{x\wedge x}(S^{n,n}\wedge E)\wedge(S^{n,n}\wedge E)\xrightarrow{1\wedge T\wedge 1}S^{n,n}\wedge S^{n,n}\wedge E\wedge E \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\downarrow{\scriptstyle \phi\wedge\mu} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad S^{2n,2n}\wedge E. \end{array}$$

It is useful to write this as $h(x\wedge x)\Delta$ where $h=(\phi\wedge\mu)(1\wedge T\wedge 1)$.

Let $f\colon S^{0,0}\to S^{n,n}$ be a map representing ρ^n. Then the class $\bar\rho^n$ in $E^{n,n}$ is represented by the composite

$$S^{0,0}\xrightarrow{f}S^{n,n}\cong S^{n,n}\wedge S^{0,0}\xrightarrow{\mathrm{id}\wedge u}S^{n,n}\wedge E,$$

which can also be written as $x\circ f$. So $\bar\rho^n\cdot x$ is represented by the composite

$$S^{n,n}\cong S^{0,0}\wedge S^{n,n}\xrightarrow{xf\wedge x}(S^{n,n}\wedge E)\wedge(S^{n,n}\wedge E)\xrightarrow{h}S^{2n,n}\wedge E$$

Note that the first two maps in this composite may be written as

$$S^{n,n} \cong S^{0,0} \wedge S^{n,n} \xrightarrow{f\wedge 1} S^{n,n} \wedge S^{n,n} \xrightarrow{x\wedge x} (S^{n,n} \wedge E) \wedge (S^{n,n} \wedge E).$$

Comparing our representation of x^2 to that of $\bar{\rho}^n \cdot x$, to show that they are equal it will suffice to prove that $\Delta\colon S^{n,n} \to S^{n,n} \wedge S^{n,n}$ represents the same homotopy class as $S^{n,n} \cong S^{0,0} \wedge S^{n,n} \xrightarrow{f\wedge \mathrm{id}_{n,n}} S^{n,n} \wedge S^{n,n}$. That is, we must verify that $[\Delta] = [f \wedge \mathrm{id}_{n,n}]$.

But $[f \wedge \mathrm{id}_{n,n}]$ equals $\epsilon^n[f] = \epsilon^n \rho^n = \rho^n$ by Remark 2.6 and Corollary 3.7. This equals $[\Delta_{n,n}]$ by Theorem 3.1. □

Remark 3.9. When E is mod 2 motivic cohomology, Corollary 3.8 is essentially [V1, Lemma 6.8]. The proof in [V1] uses special properties about motivic cohomology (specifically, the isomorphism between certain motivic cohomology groups and Milnor K-theory). Our argument avoids these difficult results. In some sense, our proof is a universal argument that reflects the spirit of Grothendieck's original "motivic" philosophy.

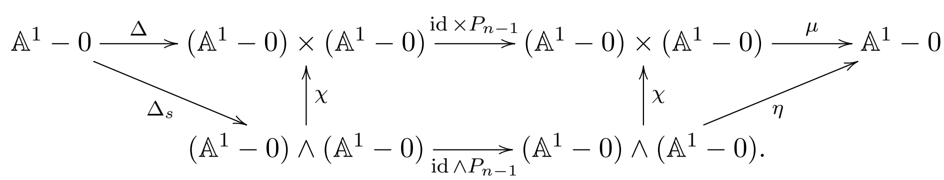
3.10. Power maps. The following result is a simple corollary of Theorem 3.1. In the remainder of the paper we will only need to use the case $n = -1$, which could be proven more directly, but it seems natural to include the entire result.

Proposition 3.11. *For any integer n, let $P_n\colon (\mathbb{A}^1 - 0) \to (\mathbb{A}^1 - 0)$ be the nth power map $z \mapsto z^n$. In $\pi_{0,0}(S)$ one has*

$$[P_n] = \begin{cases} \frac{n}{2}(1-\epsilon) & \text{if } n \text{ is even,} \\ 1 + \frac{n-1}{2}(1-\epsilon) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We will prove that $[P_n] = 1 - \epsilon[P_{n-1}]$; multiplication by ϵ then yields that $[P_{n-1}] = \epsilon - \epsilon[P_n]$. The main result follows by induction (in the positive and negative directions) starting with the trivial base case $n = 0$.

Consider the following diagram:



Here χ is the stable splitting of the map from the Cartesian product to the smash product—see Appendix A. This diagram is commutative except for the left triangle, where we have the relation

$$\Delta = \chi\Delta_s + j(\pi_1 + \pi_2)\Delta \tag{3.12}$$

by Lemma A.14, where π_1 and π_2 are the two projections $(\mathbb{A}^1 - 0) \times (\mathbb{A}^1 - 0) \to (\mathbb{A}^1 - 0) \vee (\mathbb{A}^1 - 0)$ and $j : (\mathbb{A}^1 - 0) \vee (\mathbb{A}^1 - 0) \to (\mathbb{A}^1 - 0) \times (\mathbb{A}^1 - 0)$ is the usual inclusion of the wedge into the product. Note that P_n is the composition along the top of the diagram.

We now compute that

$$\begin{aligned}[P_n] &= [\mu(\mathrm{id} \times P_{n-1})\Delta] \\ &= [\mu(\mathrm{id} \times P_{n-1})(\chi\Delta_s + j\pi_1\Delta + j\pi_2\Delta)] \\ &= [\mu(\mathrm{id} \times P_{n-1})\chi\Delta_s] + [\mu(\mathrm{id} \times P_{n-1})j\pi_1\Delta] + [\mu(\mathrm{id} \times P_{n-1})j\pi_2\Delta] \\ &= [\eta(\mathrm{id} \wedge P_{n-1})\Delta_s] + [\mathrm{id}] + [P_{n-1}] \\ &= [\eta][P_{n-1}][\Delta_s] + 1 + [P_{n-1}].\end{aligned}$$

In the last equality we have used that $[\mathrm{id} \wedge P_{n-1}] = [P_{n-1}]$, by Remark 2.6.

Now use that $[\Delta_s] = \rho$, elements of $\pi_{0,0}(S)$ commute, and that $\eta\rho = -(1 + \epsilon)$ (see the last statement in Theorem 1.2). The above equation becomes $[P_n] = -(1 + \epsilon)[P_{n-1}] + 1 + [P_{n-1}]$, or $[P_n] = 1 - \epsilon[P_{n-1}]$. □

4. Cayley–Dickson algebras and Hopf maps

In this section we introduce the particular Cayley–Dickson algebras needed for our work. We then apply the general Hopf construction from Appendix C to define motivic Hopf elements η, ν, and σ in $\pi_{*,*}(S)$. We also review some basic properties of η, due to Morel.

4.1. Fundamentals. We begin by reviewing the notion of generalized Cayley–Dickson algebras from [A] and [Sch]. For momentary convenience, let k be a field not of characteristic 2; we will explain below in Remark 4.3 how to deal with the integers and fields of characteristic 2.

An *involutive algebra* is a k-vector space A equipped with a (possibly nonassociative) unital bilinear pairing $A \times A \to A$ and a linear anti-automorphism $(-)^* \colon A \to A$ whose square is the identity, and such that

$$x + x^* = 2t(x) \cdot 1_A, \qquad xx^* = x^*x = n(x) \cdot 1_A$$

for some linear function $t \colon A \to k$ and some quadratic form $n \colon A \to k$. Given such an algebra together with a γ in $k^\times$, one can form the *Cayley–Dickson double* of A with respect to γ. This is the algebra $D_\gamma(A)$ whose underlying vector space is $A \oplus A$ and where the multiplication and involution are given by the formulas

$$(a, b) \cdot (c, d) = (ac - \gamma d^*b, da + bc^*), \qquad (a, b)^* = (a^*, -b).$$

It is easy to check that this is again an involutive algebra, with $t(a, b) = t(a)$ and $n(a, b) = n(a) + \gamma n(b)$. We will sometimes write $D(A)$ for $D_\gamma(A)$, when the constant γ is understood.

Because the Cayley–Dickson doubling process yields a new involutive algebra, it can be repeated. Let $\underline{\gamma} = (\gamma_0, \gamma_1, \gamma_2, \ldots)$ be a sequence of elements in $k^\times$. Start with $A_0 = k$ with the trivial involution, and inductively define $A_i = D_{\gamma_{i-1}}(A_{i-1})$. This gives a sequence of Cayley–Dickson algebras $A_0, A_1, A_2, \ldots$, where A_n has dimension 2^n over k. This sequence depends on the choice of $\underline{\gamma}$. Here are some well-known properties of these Cayley–Dickson algebras:

(1) A_1 is commutative, associative, and normed in the sense that $n(xy) = n(x)n(y)$ for all x and y in A_1.
(2) A_2 is associative and normed (but noncommutative in general).
(3) A_3 is normed (but noncommutative and nonassociative in general).

The standard example of these algebras occurs with $k = \mathbb{R}$ and $\gamma_i = 1$ for all i. This data gives $A_0 = \mathbb{R}$, $A_1 = \mathbb{C}$, $A_2 = \mathbb{H}$, and $A_3 = \mathbb{O}$ (as well as more complicated algebras at later stages). In each of these algebras, the norm form $n(x)$ is the usual sum-of-squares form on the underlying real vector space, under an appropriate choice of basis.

Because motivic homotopy theory takes schemes, rather than rings, as its basic objects, we will often make the trivial change in point-of-view from Cayley–Dickson *algebras* to "Cayley–Dickson *varieties*". If A is a Cayley–Dickson algebra over k of dimension 2^n, then the associated variety is isomorphic to the affine space $\mathbb{A}^{2^n}$, equipped with the corresponding bilinear map $\mathbb{A}^{2^n} \times \mathbb{A}^{2^n} \to \mathbb{A}^{2^n}$ and involution $\mathbb{A}^{2^n} \to \mathbb{A}^{2^n}$. We will write A both for the algebra and for the associated affine space. Let $S(A)$ denote the closed subvariety of A defined by the equation $n(x) = 1$. We call this subvariety the "unit sphere" inside of A, although the word "sphere" should be loosely interpreted. If A is normed, then we obtain a pairing

$$S(A) \times S(A) \to S(A).$$

The rest of this section will exploit these pairings in motivic homotopy theory.

4.2. Cayley–Dickson algebras with split norms. In motivic homotopy theory, the affine quadrics $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ are not models for motivic spheres unless the ground field contains a square root of -1. This limits the usefulness of the classical Cayley–Dickson algebras (where $\gamma_i = 1$ for all i), at least as far as producing elements in $\pi_{*,*}(S)$. Instead, we will focus on the sequence of Cayley–Dickson algebras corresponding to $\underline{\gamma} = (-1, 1, 1, 1, \cdots)$. From now on let A_i denote the ith algebra in this sequence. We will shortly see that the norm form in A_i is, under a suitable choice of basis, equal to the split form; therefore $S(A_i)$ is a model for a motivic sphere (see Example 2.12(3)).

We start by analyzing A_1. This is $\mathbb{A}^2$ equipped with the mutiplication $(a, b)(c, d) = (ac + bd, da + bc)$ and involution $(a, b)^* = (a, -b)$. With the change of basis $(a, b) \mapsto (a + b, a - b)$, we can write the multiplication as $(a, b)(c, d) = (ac, bd)$ and the involution as $(a, b)^* = (b, a)$. In this new basis, $(1, 1)$ is the identity element of A_1, and the norm form is $n(a, b) = ab$. We will abandon the "old" Cayley–Dickson basis for A_1 and from now on always use this new basis (in essence, we simply forget that A_1 came to us as $D(A_0)$).

Observe that the unit sphere $S(A_1)$ is the subvariety of $\mathbb{A}^2$ defined by $xy = 1$, which is isomorphic to $(\mathbb{A}^1 - 0)$. So $S(A_1)$ is a model for $S^{1,1}$.

Remark 4.3. If 2 is not invertible in k, then we cannot perform the same change-of-basis when analyzing A_1. This case includes the integers and fields of characteristic 2. Instead, we can simply ignore A_0 altogether and rather *define* A_1 to be the ring $k\times k$, together with $A_2 = D_1(A_1)$ and $A_3 = D_1(A_2)$. This is just a small shift in perspective.

The next algebra A_2 is $D_1(A_1)$, which is $\mathbb{A}^4$ with the following multiplication:

$$\begin{aligned}&(a_1, a_2, b_1, b_2) \cdot (c_1, c_2, d_1, d_2)\\ &= (ac - d^*b, da + bc^*)\\ &= (a_1c_1 - d_2b_1, a_2c_2 - d_1b_2, d_1a_1 + b_1c_2, d_2a_2 + b_2c_1).\end{aligned}$$

The involution is $(a_1, a_2, b_1, b_2)^* = (a_2, a_1, -b_1, -b_2)$, and the norm form is readily checked to be

$$n(a_1, a_2, b_1, b_2) = a_1a_2 + b_1b_2.$$

This is the split quadratic form on $\mathbb{A}^4$.

The algebra $A_3 = D_1(A_2)$ has underlying variety $\mathbb{A}^8$. We will not write out the formulas for multiplication and involution here, although they are easy enough to deduce. The norm form on A_3 is once again the split form.

The algebras A_1, A_2, and A_3 all have normed multiplications, in the sense that $n(xy) = n(x)n(y)$ for all x and y.

It is useful to regard A_1, A_2, and A_3 as analogs of the classical algebras $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. We call them the "split complex numbers", the "split quaternions", and the "split octonions", respectively. One should not take the comparisons too seriously: for example, A_1 has zero divisors whereas $\mathbb{C}$ is a field. Nevertheless, they are normed algebras that turn out to play roles in motivic homotopy that are entirely analogous to the roles that $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ play in ordinary homotopy theory. We will adopt the notation

$$\begin{aligned}A_{\mathbb{C}} &= A_1, & A_{\mathbb{H}} &= A_2, & A_{\mathbb{O}} &= A_3;\\ S_{\mathbb{C}} &= S(A_{\mathbb{C}}), & S_{\mathbb{H}} &= S(A_{\mathbb{H}}), & S_{\mathbb{O}} &= S(A_{\mathbb{O}}).\end{aligned}$$

The multiplications in $A_{\mathbb{C}}$, $A_{\mathbb{H}}$, and $A_{\mathbb{O}}$ restrict to pairings $S_{\mathbb{C}} \times S_{\mathbb{C}} \to S_{\mathbb{C}}$, $S_{\mathbb{H}} \times S_{\mathbb{H}} \to S_{\mathbb{H}}$, and $S_{\mathbb{O}} \times S_{\mathbb{O}} \to S_{\mathbb{O}}$. Note that

$$S_{\mathbb{C}} \simeq S^{1,1}, \qquad S_{\mathbb{H}} \simeq S^{3,2}, \qquad S_{\mathbb{O}} \simeq S^{7,4}.$$

More generally, $S(A_n)$ is a model for $S^{2^n-1,2^{n-1}}$ for all n.

Recall the isomorphism $S_{\mathbb{C}} \xrightarrow{\simeq} \mathbb{A}^1 - 0$, via $(a_1, a_2) \mapsto a_1$. This provides an orientation on $S_{\mathbb{C}}$. Under this isomorphism, the product $S_{\mathbb{C}} \times S_{\mathbb{C}} \to S_{\mathbb{C}}$ coincides with the usual multiplication map on $(\mathbb{A}^1 - 0)$.

We orient $S_{\mathbb{H}}$ via the weak equivalence $S_{\mathbb{H}} \xrightarrow{\simeq} (\mathbb{A}^2 - 0)$ that sends $(a_1, a_2, b_1, b_2) \mapsto (a_1, b_1)$, using the standard orientation on $(\mathbb{A}^2 - 0)$ from Example 2.12(2). Similarly, we orient $S_{\mathbb{O}}$ via the analogous weak equivalence $S_{\mathbb{O}} \xrightarrow{\simeq} (\mathbb{A}^4 - 0)$.

Remark 4.4. The algebra $A_{\mathbb{H}}$ is isomorphic to the algebra of 2×2 matrices, via the isomorphism

$$(a_1, a_2, b_1, b_2) \mapsto \begin{bmatrix} a_1 & b_1 \\ -b_2 & a_2 \end{bmatrix}.$$

This is easy to check using the formula for multiplication in $A_{\mathbb{H}}$. Under this isomorphism, the conjugate of a matrix A corresponds to the classical adjoint of A, i.e.,

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto M^* = \operatorname{adj} M = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The norm form is equal to the determinant, and consequently one has

$$S_{\mathbb{H}} \cong SL_2.$$

The pairing $S_{\mathbb{H}} \times S_{\mathbb{H}} \to S_{\mathbb{H}}$ is just the usual product on SL_2.

Remark 4.5. The 'splitting' of the algebra $A_{\mathbb{C}}$ gives us coordinates having the property that the multiplication rule does not mix the two coordinates: that is, $(a, b)(c, d) = (ac, bd)$. This nonmixing property propagates somewhat into $A_{\mathbb{H}}$, and we will need to use this at a key stage below. Let $\omega\colon A_{\mathbb{H}} \to \mathbb{A}^2$ be the map $(a_1, a_2, b_1, b_2) \mapsto (a_1, b_2)$. Then the diagram

$$\begin{array}{ccc} A_{\mathbb{H}} \times A_{\mathbb{H}} & \xrightarrow{\mu} & A_{\mathbb{H}} \\ {\scriptstyle \mathrm{id}\times\omega}\downarrow & & \downarrow{\scriptstyle \omega} \\ A_{\mathbb{H}} \times \mathbb{A}^2 & \xrightarrow{\mu'} & \mathbb{A}^2 \end{array}$$

commutes, where μ' is given by $(a_1, a_2, b_1, b_2) * (x, y) = (a_1x - yb_1, ya_2 + b_2x)$. In words, for all u and v in $A_{\mathbb{H}}$, the first and last coordinates of uv only depend on the first and last coordinates of v. It is somewhat more intuitive to see this using the isomorphism of Remark 4.4, where the map μ' corresponds to the usual action of matrices on column vectors (up to some sporadic signs).

4.6. The Hopf maps. The basic definition of the motivic Hopf maps now proceeds just as in classical homotopy theory. The reader should review Appendix C at this point, for the definition and properties of the Hopf construction.

Definition 4.7.

(1) The first Hopf map η is the element of $\pi_{1,1}(S)$ represented by the Hopf construction of the multiplication map $S_{\mathbb{C}} \times S_{\mathbb{C}} \to S_{\mathbb{C}}$.
(2) The second Hopf map ν is the element of $\pi_{3,2}(S)$ represented by the Hopf construction of the multiplication map $S_{\mathbb{H}} \times S_{\mathbb{H}} \to S_{\mathbb{H}}$.
(3) The third Hopf map σ is the element of $\pi_{7,4}(S)$ represented by the Hopf construction of the multiplication map $S_{\mathbb{O}} \times S_{\mathbb{O}} \to S_{\mathbb{O}}$.

The following result and its proof are due to Morel [M2, Lemma 6.2.3]:

Lemma 4.8. *The elements η and $\eta\epsilon$ are equal in $\pi_{1,1}(S)$.*

Proof. Multiplication on $S_{\mathbb{C}}$ is commutative. Recall that ϵ is represented by the twist map on $S_{\mathbb{C}} \wedge S_{\mathbb{C}}$. The diagram

$$\begin{array}{ccccc} S_{\mathbb{C}} \wedge S_{\mathbb{C}} & \xrightarrow{\chi} & S_{\mathbb{C}} \times S_{\mathbb{C}} & \xrightarrow{\mu} & S_{\mathbb{C}} \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle =} \\ S_{\mathbb{C}} \wedge S_{C} & \xrightarrow[\chi]{} & S_{\mathbb{C}} \times S_{\mathbb{C}} & \xrightarrow[\mu]{} & S_{\mathbb{C}} \end{array}$$

commutes by Lemma A.9, where μ is multiplication and T is the twist map. The horizontal compositions represent η. □

The above lemma deduces an identity involving the Hopf elements as a consequence of the commutativity of $A_{\mathbb{C}}$. The point of the next section will be to deduce some other identities from deeper properties of the Cayley–Dickson algebras.

Remark 4.9. The above proof works unstably, but only after three simplicial suspensions. As explained in Appendix A.1, the map χ exists as an unstable map $\Sigma(S^{1,1} \wedge S^{1,1}) \to \Sigma(S^{1,1} \times S^{1,1})$, which gives a model for η in the unstable group $\pi_{3,2}(S^{2,1})$. But the left square in the diagram is only guaranteed to commute after an additional *two* suspensions, by Lemma A.9. The necessity of some of these suspensions is demonstrated by the fact that classically one does not have $\eta = -\eta$ in $\pi_3(S^2)$.

The next result is also due to Morel [M2, Lemma 6.2.3]. We include this result and its proof only for didactic purposes. The proof demonstrates how one must be careful with orientations, canonical isomorphisms, and commutativity relations.

Proposition 4.10. *Let $\mathbb{P}^1$ and $(\mathbb{A}^2 - 0)$ be oriented as in Example 2.12. The usual projection $\pi\colon (\mathbb{A}^2 - 0) \to \mathbb{P}^1$ represents the element η in $\pi_{1,1}(S)$.*

Proof. We have the diagram

$$\begin{array}{ccccc} S_{\mathbb{C}} \wedge S_{\mathbb{C}} & \xrightarrow{\chi} & S_{\mathbb{C}} \times S_{\mathbb{C}} & \xrightarrow{f} & S_{\mathbb{C}} \\ {\scriptstyle (-)^{-1} \wedge \mathrm{id}}\downarrow & & {\scriptstyle (-)^{-1} \times \mathrm{id}}\downarrow & & {\scriptstyle =}\downarrow \\ S_{\mathbb{C}} \wedge S_{\mathbb{C}} & \xrightarrow[\chi]{} & S_{\mathbb{C}} \times S_{\mathbb{C}} & \xrightarrow[\mu]{} & S_{\mathbb{C}}, \end{array}$$

where μ is multiplication and f is the map $(x, y) \mapsto x^{-1}y$. By definition, η is the composition along the bottom of the diagram. Recall that the inverse map on $S_{\mathbb{C}}$ represents ϵ, by Proposition 3.11; therefore $(-)^{-1} \wedge \mathrm{id}$ represents ϵ as well, using Remark 2.6. Since η equals $\eta\epsilon$ by Lemma 4.8, it suffices to show that the composition along the top of the diagram is equivalent to π, i.e., that π is the Hopf construction on f.

Let $U_1 \longleftarrow U_1 \cap U_2 \longrightarrow U_2$ be the standard affine cover of $\mathbb{P}^1$, as in Example 2.12(1). Taking the preimage under π gives a cover of $(\mathbb{A}^2 - 0)$, so we get the diagram

$$(4.11)\qquad \begin{array}{ccccc} \pi^{-1}U_1 & \longleftarrow & \pi^{-1}(U_1 \cap U_2) & \longrightarrow & \pi^{-1}U_2 \\ \downarrow & & \downarrow & & \downarrow \\ U_1 & \longleftarrow & U_1 \cap U_2 & \longrightarrow & U_2, \end{array}$$

where $(\mathbb{A}^2 - 0)$ and $\mathbb{P}^1$ are the homotopy pushouts of the top and bottom rows respectively.

Diagram (4.11) is isomorphic to the diagram

$$(4.12)\qquad \begin{array}{ccccc} (\mathbb{A}^1 - 0) \times \mathbb{A}^1 & \xleftarrow{i} & (\mathbb{A}^1 - 0) \times (\mathbb{A}^1 - 0) & \xrightarrow{i} & \mathbb{A}^1 \times (\mathbb{A}^1 - 0) \\ {\scriptstyle (x,y)\mapsto x^{-1}y}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle (x,y)\mapsto xy^{-1}} \\ \mathbb{A}^1 & \xleftarrow{i} & (\mathbb{A}^1 - 0) & \xrightarrow{\text{inv}} & \mathbb{A}^1 \end{array}$$

where all maps labelled i are the inclusions. This new diagram in turn maps, via a natural weak equivalence, to the diagram

$$(4.13)\qquad \begin{array}{ccccc} (\mathbb{A}^1 - 0) & \xleftarrow{\pi_1} & (\mathbb{A}^1 - 0) \times (\mathbb{A}^1 - 0) & \xrightarrow{\pi_2} & (\mathbb{A}^1 - 0) \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ * & \longleftarrow & (\mathbb{A}^1 - 0) & \longrightarrow & *. \end{array}$$

Diagram (4.13) induces a map on homotopy pushouts of the rows, which is equal to the Hopf construction $H(f)$ on f by definition (see Appendix C). So we have produced a zig-zag of equivalences between π and $H(f)$. The weak equivalences in this zig-zag turn out to be orientation-preserving, which follows by the definition of our standard orientations in Example 2.12. It follows that $[\pi] = [H(f)]$. $\square$

Remark 4.14 (Nontriviality of η, ν, and σ)**.** It is worth pointing out that if our base k is a field of characteristic not equal to 2 then none of η, ν, and σ are equal to the zero element. Here it is useful to work unstably: since the splitting χ exists after one suspension, η, ν, and σ can be modelled by unstable maps $S^{3,2} \to S^{2,1}$, $S^{7,4} \to S^{4,2}$, and $S^{15,8} \to S^{8,4}$. A completely routine modification of the standard argument from [SE, Lemma 1.5.3] shows that the homotopy cofibers of these maps have nontrivial cup products in their mod 2 motivic cohomology—more precisely, the maps have Hopf invariant one in the usual sense that the square of the generator in bidegree $(2n, n)$ equals the generator in dimension $(4n, 2n)$, for $n = 1, 2, 4$ in the three respective cases. Properties of the motivic Steenrod squares then show the existence of the expected Steenrod operations in the cohomology of the cofibers, which proves that the maps are not stably trivial.

The assumption that the base is a field not of characteristic 2 is because it is in that setting that we know the necessary results about the Steenrod operations in motivic cohomology. It of course follows that η, ν, and σ are nonzero over $\mathbb{Z}$ as well. One can presumably use motivic $\mathbb{F}_3$-cohomology to detect ν and σ over fields of characteristic 2. We do not know whether η is nonzero over fields of characteristic 2.

When the base is a field of characteristic zero, another approach is to reduce to the case $k \hookrightarrow \mathbb{C}$ and then apply the topological realization from motivic homotopy theory to classical homotopy theory. The motivic elements η, ν, and σ all map to elements of Hopf invariant one.

5. The null-Hopf relation

The goal of this section is to prove with geometric arguments that $\eta\nu$ and $\nu\sigma$ are both zero. The proofs for these two results follow essentially the same pattern, but in the case of $\nu\sigma = 0$ one part of the argument develops some complications that require a nonobvious workaround. Our approach in this section will be to first concentrate on the $\eta\nu = 0$ proof, so that the reader can see the basic strategy of what is happening. Then we repeat most of the steps for the case of the $\nu\sigma = 0$ argument, explaining what the differences are.

We begin our work by returning to Cayley–Dickson algebras:

Lemma 5.1. *Let A be an associative involutive k-algebra, let γ be in $k^\times$, and let t be an element of A having norm 1. The map $\theta_t \colon D_\gamma(A) \to D_\gamma(A)$ given by $\theta_t(a,b) = (a, tb)$ is an involution-preserving endomorphism of the Cayley–Dickson double $D_\gamma(A)$. In particular, θ_t is norm-preserving.*

Proof. Verify that θ_t is an involution-preserving endomorphism directly with the formulas for $D_\gamma(A)$ given at the beginning of Section 4.1. Since $n(x) = xx^*$, it follows that θ_t also preserves the norm. □

From Lemma 5.1, we find that there is a pairing

$$\theta \colon S(A) \times S(D_\gamma A) \to S(D_\gamma A)$$

given by $\theta(t,x) = \theta_t(x)$. In particular, this yields maps

$$\alpha \colon S_{\mathbb{C}} \times S_{\mathbb{H}} \to S_{\mathbb{H}} \quad \text{and} \quad \beta \colon S_{\mathbb{H}} \times S_{\mathbb{O}} \to S_{\mathbb{O}}.$$

Note that these maps commute with multiplication in the sense that

$$\alpha(t, ab) = \alpha(t,a)\alpha(t,b) \qquad \text{and} \qquad \beta(a, xy) = \beta(a,x)\beta(a,y).$$

In other words, the diagram
(5.2)

$$\begin{array}{ccccc} S_{\mathbb{C}} \times S_{\mathbb{H}} \times S_{\mathbb{H}} & \xrightarrow{\Delta\times 1\times 1} & S_{\mathbb{C}} \times S_{\mathbb{C}} \times S_{\mathbb{H}} \times S_{\mathbb{H}} & \xrightarrow{1\times T\times 1} & S_{\mathbb{C}} \times S_{\mathbb{H}} \times S_{\mathbb{C}} \times S_{\mathbb{H}} \\ & & & & \downarrow{\scriptstyle \alpha\times\alpha} \\ \downarrow{\scriptstyle 1\times\mu} & & & & S_{\mathbb{H}} \times S_{\mathbb{H}} \\ & & & & \downarrow{\scriptstyle \mu} \\ S_{\mathbb{C}} \times S_{\mathbb{H}} & & \xrightarrow[\alpha]{} & & S_{\mathbb{H}}, \end{array}$$

commutes, where Δ and T are the evident diagonal and twist maps. A similar diagram commutes for $S_{\mathbb{H}}$, $S_{\mathbb{O}}$, and β.

Lemma 5.3. *The Hopf construction on α represents η.*

Proof. Recall the orientation-preserving weak equivalence $\pi\colon S_{\mathbb{H}} \to (\mathbb{A}^2 - 0)$ that takes (a_1, a_2, b_1, b_2) to (a_1, b_1), as well as the isomorphism $p\colon S_{\mathbb{C}} \to (\mathbb{A}^1 - 0)$ that sends (t_1, t_2) to t_1. We have a commutative diagram

$$\begin{array}{ccccc} S_{\mathbb{C}} \wedge S_{\mathbb{H}} & \xrightarrow{\chi} & S_{\mathbb{C}} \times S_{\mathbb{H}} & \xrightarrow{\alpha} & S_{\mathbb{H}} \\ \simeq\downarrow & & \simeq\downarrow & & \downarrow\simeq \\ (\mathbb{A}^1 - 0) \wedge (\mathbb{A}^2 - 0) & \xrightarrow[\chi]{} & (\mathbb{A}^1 - 0) \times (\mathbb{A}^2 - 0) & \xrightarrow[\alpha']{} & \mathbb{A}^2 - 0, \end{array}$$

where $\alpha'\colon (\mathbb{A}^1 - 0) \times (\mathbb{A}^2 - 0) \to (\mathbb{A}^2 - 0)$ is given by $(t, (x, y)) \mapsto (x, ty)$. The diagram shows that the Hopf constructions $H(\alpha)$ and $H(\alpha')$ represent the same map in $\pi_{*,*}(S)$, so we will now focus on the latter.

Recall that we have fixed an isomorphism (in the homotopy category) between $(\mathbb{A}^2 - 0)$ and the join $(\mathbb{A}^1 - 0) * (\mathbb{A}^1 - 0)$. Under this isomorphism, α' coincides with the melding $\pi_2 \# \mu$, where π_2 and μ are the projection and multiplication maps $(\mathbb{A}^1 - 0) \times (\mathbb{A}^1 - 0) \to (\mathbb{A}^1 - 0)$. See Appendix C.6 for the definition of $\pi_2 \# \mu$.

Proposition C.10 gives a formula for $H(\pi_2 \# \mu)$. But $H(\pi_2)$ is null by Lemma C.2, and so that formula simplifies to just

$$[H(\alpha')] = [H(\pi_2 \# \mu)] = \tau_{(1,1),(1,1)} \cdot [H(\mu)] = \epsilon [H(\mu)] = \epsilon\eta = \eta.$$

The element $\tau_{(1,1),(1,1)}$ is computed by Equation (2.4), and the last equality is by Lemma 4.8. □

The next result is the desired null-Hopf relation.

Proposition 5.4. $\eta\nu = 0$.

Proof. We will examine what happens when both routes around Diagram (5.2) are precomposed with the splitting map $\chi\colon S_{\mathbb{C}} \wedge S_{\mathbb{H}} \wedge S_{\mathbb{H}} \to S_{\mathbb{C}} \times S_{\mathbb{H}} \times S_{\mathbb{H}}$. Note that throughout this proof we work in the stable category.

We will begin with the lower-left composition. To analyze $\alpha(1 \times \mu)\chi$, use the commutative diagram

$$\begin{array}{ccccc}
S_{\mathbb{C}} \wedge S_{\mathbb{H}} \wedge S_{\mathbb{H}} & & & & \\
{\scriptstyle 1\wedge\chi}\downarrow & \searrow^{1\wedge H(\mu)} & & & \\
S_{\mathbb{C}} \wedge (S_{\mathbb{H}} \times S_{\mathbb{H}}) & \xrightarrow[1\wedge\mu]{} & S_{\mathbb{C}} \wedge S_{\mathbb{H}} & & \\
{\scriptstyle \chi}\downarrow & & \downarrow{\scriptstyle \chi} & \searrow^{H(\alpha)} & \\
S_{\mathbb{C}} \times S_{\mathbb{H}} \times S_{\mathbb{H}} & \xrightarrow[1\times\mu]{} & S_{\mathbb{C}} \times S_{\mathbb{H}} & \xrightarrow[\alpha]{} & S_{\mathbb{H}}.
\end{array}$$

The square commutes because χ is natural, and the two triangles commute by definition of the Hopf construction. The left vertical composite equals χ by Remark A.20. Recall from Lemma 5.3 that $H(\alpha) = \eta$, and of course $H(\mu) = \nu$ by definition. So we have that

$$[\alpha(1 \times \mu)\chi] = [H(\alpha)] \cdot [1 \wedge H(\mu)] = [H(\alpha)] \cdot [H(\mu)] = \eta\nu.$$

The second equality uses Remark 2.6(i).

Next we analyze what happens when we compose

$$\chi\colon S_{\mathbb{C}} \wedge S_{\mathbb{H}} \wedge S_{\mathbb{H}} \to S_{\mathbb{C}} \times S_{\mathbb{H}} \times S_{\mathbb{H}}$$

with the top-right part of Diagram (5.2). We will obtain zero, which will finish the proof. This is mostly an application of Proposition C.10, where the maps $f\colon S_{\mathbb{C}} \times S_{\mathbb{H}} \to S_{\mathbb{H}}$ and $g\colon S_{\mathbb{C}} \times S_{\mathbb{H}} \to S_{\mathbb{H}}$ are both equal to α.

First recall from Corollary A.13(b) that the identity map on $S_{\mathbb{H}} \times S_{\mathbb{H}}$ can be written as $\mathrm{id}_{S_{\mathbb{H}} \times S_{\mathbb{H}}} = \chi p + j_1\pi_1 + j_2\pi_2$ where π_1 and π_2 are the two projections $S_{\mathbb{H}} \times S_{\mathbb{H}} \to S_{\mathbb{H}}$; $j_1, j_2\colon S_{\mathbb{H}} \to S_{\mathbb{H}} \times S_{\mathbb{H}}$ are the two inclusions as horizontal and vertical slices; and p is the projection from the product to the smash product. The composite of interest can therefore be written as a sum of three composites of the form

$$S_{\mathbb{C}} \wedge S_{\mathbb{H}} \wedge S_{\mathbb{H}} \xrightarrow{\chi} S_{\mathbb{C}} \times S_{\mathbb{H}} \times S_{\mathbb{H}} \xrightarrow{h} S_{\mathbb{H}} \times S_{\mathbb{H}} \xrightarrow{u} S_{\mathbb{H}}$$

where h denotes the composition $S_{\mathbb{C}} \times S_{\mathbb{H}} \times S_{\mathbb{H}} \to S_{\mathbb{H}} \times S_{\mathbb{H}}$ along the top-right part of Diagram (5.2), and u is one of $\mu\chi p$, $\mu j_1\pi_1 = \pi_1$, and $\mu j_2\pi_2 = \pi_2$. But in the latter two cases the composites are clearly null; in the case of π_1, for example, this follows from the diagram

$$\begin{array}{ccccc}
S_{\mathbb{C}} \wedge S_{\mathbb{H}} \wedge S_{\mathbb{H}} & \xrightarrow{\chi} & S_{\mathbb{C}} \times S_{\mathbb{H}} \times S_{\mathbb{H}} & \xrightarrow{\pi_1 h} & S_{\mathbb{H}} \\
& \searrow^{\cdots} & \downarrow & \nearrow_{\alpha} & \\
& & S_{\mathbb{C}} \times S_{\mathbb{H}} \times * & &
\end{array}$$

and the fact that the dotted composite is null by the defining properties of χ (Proposition A.12).

So it remains to analyze the composite

$$S_{\mathbb{C}} \wedge S_{\mathbb{H}} \wedge S_{\mathbb{H}} \xrightarrow{\chi} S_{\mathbb{C}} \times S_{\mathbb{H}} \times S_{\mathbb{H}} \xrightarrow{h} S_{\mathbb{H}} \times S_{\mathbb{H}} \xrightarrow{p} S_{\mathbb{H}} \wedge S_{\mathbb{H}} \xrightarrow{\chi} S_{\mathbb{H}} \wedge S_{\mathbb{H}} \xrightarrow{\mu} S_{\mathbb{H}}.$$

This is equal to $H(\mu) \circ H(\alpha\#\alpha)$, using Lemma C.9 for the second factor. Proposition C.10 says that $[H(\alpha\#\alpha)]$ equals

$$\tau_{(1,1),(3,2)}[H(\alpha)] + \tau_{(1,1),(3,2)}[H(\alpha)] + (\tau_{(1,1),(3,2)})^2[H(\alpha)] \cdot [H(\alpha)] \cdot [\Delta_{S^{1,1}}].$$

Now use $[\Delta_{S^{1,1}}] = \rho$ from Theorem 3.1; $[H(\alpha)] = \eta$ from Lemma 5.3; and $\tau_{(1,1),(3,2)} = 1$ by Equation (2.4). We obtain $H(\alpha\#\alpha) = 2\eta + \eta^2\rho$, which equals zero by Theorem 1.2(ii). □

We next duplicate the above arguments to prove the analogous Hopf relation $\nu\sigma = 0$, using the pairing $\beta\colon S_{\mathbb{H}} \times S_{\mathbb{O}} \to S_{\mathbb{O}}$. This time we go through the steps in reverse order, saving what is now the hardest step for last.

Proposition 5.5. $\nu\sigma = 0$.

Proof. The proof is very similar to the proof of Proposition 5.4. One starts with the commutative diagram analogous to Diagram (5.2) showing that β respects multiplication, and then precomposes the two routes around the diagram with χ. By exactly the same arguments as before, the composition along the bottom-left part of the diagram gives $[H(\beta)] \cdot \sigma$, and composition along the top-right part of the diagram gives

$$\sigma \cdot \Big[\tau_{(3,2),(7,4)}[H(\beta)] + \tau_{(3,2),(7,4)}[H(\beta)] + \big(\tau_{(3,2),(7,4)}\big)^2[\Delta_{S^{3,2}}]\Big]$$

(using Proposition C.10). But here the diagonal map is equal to zero by Theorem 3.1, because $S^{3,2}$ is a simplicial suspension. Using that

$$\tau_{(3,2),(7,4)} = -1$$

by Equation (2.4), our formula becomes

$$[H(\beta)] \cdot \sigma = \sigma \cdot [-2H(\beta)] = 2[H(\beta)] \cdot \sigma,$$

We have used graded-commutativity from Proposition 2.5 in the second equality. This shows that $[H(\beta)] \cdot \sigma = 0$. Finally, use that $[H(\beta)] = -\nu$ by Lemma 5.6 below. □

Our next goal is to compute the Hopf construction $H(\beta)$. Recall that the pairing $\beta\colon S_{\mathbb{H}} \times S_{\mathbb{O}} \to S_{\mathbb{O}}$ sends $[t,(x,y)] \mapsto (x,ty)$. The idea is to realize $S_{\mathbb{O}}$ as the join of two copies of $S^{3,2}$, corresponding to the two coordinates x and y. Under this equivalence, β becomes the melding $\pi_2\#\mu$ (Section C.6), where π_2 and μ are the projection and multiplication maps $S_{\mathbb{H}} \times S_{\mathbb{H}} \to S_{\mathbb{H}}$. Proposition C.10 then shows that $[H(\beta)] = \tau_{(3,2),(3,2)}\nu = -\nu$.

The difficulty comes in realizing $S_{\mathbb{O}}$ as a join, in a way that is compatible with the β-action by $S_{\mathbb{H}}$. To understand the problem, it is useful to review how this would work in classical topology. Let S be the unit sphere inside the classical octonions $\mathbb{O}$, consisting of pairs $(x,y) \in \mathbb{H} \times \mathbb{H}$ with $|x|^2 + |y|^2 = 1$. Let $U_1 \subseteq S$ be the set of pairs where $x \neq 0$, and let $U_2 \subseteq S$ be the set of pairs where $y \neq 0$. There are evident projections $q_1 \colon U_1 \to S^3$ and $q_2 \colon U_2 \to S^3$ given by $q_1(x,y) = \frac{x}{|x|}$ and $q_2(x,y) = \frac{y}{|y|}$. The diagram

$$\begin{array}{ccccc} U_1 & \longleftarrow & U_1 \cap U_2 & \longrightarrow & U_2 \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_1 \times q_2} & & \downarrow{\scriptstyle q_2} \\ S^3 & \xleftarrow{\pi_1} & S^3 \times S^3 & \xrightarrow{\pi_2} & S^3 \end{array}$$

is commutative; the homotopy colimit of the top row is S, and the homotopy colimit of the bottom row is the join $S^3 * S^3$. All of the vertical maps are homotopy equivalences. Moreover, if we let $S(\mathbb{H}) = S^3$ act on $S^3 \times S^3$ trivially on the first factor and by left multiplication on the second factor, then the $S(\mathbb{H})$-actions on S and $S^3 \times S^3$ are compatible with respect to the maps in the above diagram. This identifies $S(\mathbb{H}) \times S \to S$ with the melding of the two evident $S(\mathbb{H})$-actions on S^3.

Unfortunately, the above argument does not work in the motivic setting. We do not have square roots, so we cannot normalize vectors; likewise, the homotopies that show the vertical maps in the diagram to be equivalences all use square roots. So the above simple argument breaks down in several spots.

We get around these difficulties by using a special property of the split quaternions $A_{\mathbb{H}}$. Basically, we use the splitting to reduce the action to a different model of the same sphere, where it is easier to see the melding.

Lemma 5.6. *The Hopf construction on β represents $-\nu$.*

Proof. Recall the pairing $\mu' \colon A_{\mathbb{H}} \times \mathbb{A}^2 \to \mathbb{A}^2$ given by

$$(a_1, a_2, b_1, b_2) * (x, y) = (a_1 x - y b_1, y a_2 + b_2 x)$$

from Remark 4.5, as well as the commutative diagram

$$\begin{array}{ccc} A_{\mathbb{H}} \times A_{\mathbb{H}} & \xrightarrow{\mu} & A_{\mathbb{H}} \\ {\scriptstyle \mathrm{id} \times \omega}\downarrow & & \downarrow{\scriptstyle \omega} \\ A_{\mathbb{H}} \times \mathbb{A}^2 & \xrightarrow{\mu'} & \mathbb{A}^2. \end{array}$$

Note that μ' restricts to give $S_{\mathbb{H}} \times (\mathbb{A}^2 - 0) \to \mathbb{A}^2 - 0$, and ω restricts to give an equivalence $S_{\mathbb{H}} \to \mathbb{A}^2 - 0$.

Consider now the commutative diagram

$$\begin{array}{ccc} S_{\mathbb{H}} \times S_{\mathbb{O}} & \xrightarrow{\beta} & S_{\mathbb{O}} \\ \downarrow{\scriptstyle \mathrm{id} \times \omega'} & & \downarrow{\scriptstyle \omega'} \\ S_{\mathbb{H}} \times (\mathbb{A}^4 - 0) & \xrightarrow{\beta'} & \mathbb{A}^4 - 0 \end{array}$$

where $\omega'(x,y) = (\omega(x),\omega(y))$ and $\beta'(t,(u,v)) = (u, t*v)$ for u and v in $\mathbb{A}^2$. The vertical maps are weak equivalences by the argument from Example 2.12(3). So the Hopf constructions for β and β' represent the same element of $\pi_{*,*}(S)$.

We know how to identify the variety $\mathbb{A}^4 - 0$ as the join $(\mathbb{A}^2-0)*(\mathbb{A}^2-0)$ (Example 2.12(2)). Under this identification, the pairing β' is the melding $\pi_2\#\mu'$ where π_2 and μ' are the projection and multiplication maps

$$\mathbb{S}_{\mathbb{H}} \times (\mathbb{A}^2 - 0) \to (\mathbb{A}^2 - 0).$$

Proposition C.10 now yields the formula

$$[H(\beta')] = [H(\pi_2\#\mu')] = \tau_{(3,2),(3,2)}[H(\mu')] = -[H(\mu')]$$

using that $H(\pi_2) = 0$ from Lemma C.2. Finally, we turn to the commutative square

$$\begin{array}{ccc} S_{\mathbb{H}} \times S_{\mathbb{H}} & \xrightarrow{\mu} & S_{\mathbb{H}} \\ {\scriptstyle \mathrm{id}\times\omega}\downarrow & & \downarrow{\scriptstyle \omega} \\ S_{\mathbb{H}} \times (\mathbb{A}^2 - 0) & \xrightarrow{\mu'} & \mathbb{A}^2 - 0. \end{array}$$

The vertical maps are equivalences, so $H(\mu')$ and $H(\mu)$ represent the same element of $\pi_{*,*}(S)$. Since $H(\mu)$ is equal to ν by definition, we have

$$[H(\beta)] = [H(\beta')] = -[H(\mu')] = -[H(\mu)] = -\nu. \qquad \square$$

Remark 5.7. In the proof of Lemma 5.6, we have not established that $\omega'\colon S_{\mathbb{O}} \to (\mathbb{A}^4 - 0)$ and $\omega\colon S_{\mathbb{H}} \to (\mathbb{A}^2 - 0)$ are orientation-preserving. By Proposition C.4, this issue is irrelevant because the homotopy elements represented by Hopf constructions are independent of these orientations.

Appendix A. Stable splittings of products

These appendices develop certain technical homotopy-theoretic constructions that are used in the body of the paper. Appendices A and B, as well as the first part of Appendix C, build on ideas that appear in the papers of Morel [M1, M2]. Our aim here is to offer additional details that are necessary for our proofs. This applies, in particular, to the proof of Proposition C.10, which is the most important technical tool for the paper. These appendices are largely structured with the goal of providing a comprehensible proof of Proposition C.10.

A.1. Generalities. For most of the applications in this paper, it suffices to work in the stable category of motivic spectra. This is also true for the splittings we are about to discuss, and for the Hopf construction developed in Appendix C. However, for didactic reasons we are briefly going to work *unstably* and be careful about the number of suspensions required at various stages.

Let $j\colon A \hookrightarrow B$ be a cofibration of pointed motivic spaces, and let $p\colon B \to B/A$ be the quotient map. Suppose that there is a map $\alpha\colon \Sigma B \to \Sigma A$ that splits Σj in the homotopy category, i.e., $\alpha(\Sigma j) \simeq \mathrm{id}_{\Sigma A}$. For any pointed object X, we have an exact sequence

$$\cdots \longleftarrow [B/A, X]_* \longleftarrow [\Sigma A, X]_* \overset{(\Sigma j)^*}{\longleftarrow} [\Sigma B, X]_* \overset{(\Sigma p)^*}{\longleftarrow} [\Sigma(B/A), X]_* \leftarrow \cdots$$

of sets. Then α^* is a splitting for $(\Sigma j)^*$, so $(\Sigma j)^*$ is surjective. It follows that we have an exact sequence of groups

$$1 \longleftarrow [\Sigma A, X]_* \overset{(\Sigma j)^*}{\longleftarrow} [\Sigma B, X]_* \overset{(\Sigma p)^*}{\longleftarrow} [\Sigma(B/A), X]_* \longleftarrow 1.$$

Because α is not necessarily the suspension of a map $B \to A$, the map α^* is not necessarily a group homomorphism. So the exact sequence is not necessarily split-exact. Although the groups in the above sequence need not be abelian, we will still write $+$ for the group operation and 0 for the identity. When X equals ΣB, the element $\mathrm{id}_{\Sigma B} - (\Sigma j)\alpha$ of $[\Sigma B, \Sigma B]_*$ belongs to the kernel of $(\Sigma j)^*$ and is therefore in the image of $(\Sigma p)^*$.

Definition A.2. The map $\chi\colon \Sigma(B/A) \to \Sigma B$ is the unique map in the homotopy category of pointed spaces such that $\mathrm{id}_{\Sigma B}$ equals $(\Sigma j)\alpha + \chi(\Sigma p)$.

Lemma A.3. *The map χ satisfies:*

(1) $(\Sigma p)\chi = \mathrm{id}_{\Sigma(B/A)}$.
(2) $\alpha\chi = 0$.

Proof. Let X be $\Sigma(B/A)$. Compute that $(\Sigma p)^*((\Sigma p)\chi - \mathrm{id}_{\Sigma B/A})$ is zero. Since $(\Sigma p)^*$ is one-to-one, we get that $(\Sigma p)\chi - \mathrm{id}_{\Sigma B/A}$ is zero.

For the second, let X be ΣA. Compute that $(\Sigma p)^*(\alpha\chi)$ is zero. Since $(\Sigma p)^*$ is one-to-one, we get that $\alpha\chi$ is zero. □

We now suspend once more to obtain the sequence

$$0 \longleftarrow [\Sigma^2 A, X]_* \overset{(\Sigma^2 j)^*}{\longleftarrow} [\Sigma^2 B, X]_* \overset{(\Sigma^2 p)^*}{\longleftarrow} [\Sigma^2(B/A), X]_* \longleftarrow 0. \tag{A.4}$$

This is now a short exact sequence of abelian groups, and it *is* split-exact because the map $(\Sigma\alpha)^*$ is a group homomorphism.

With at least two suspensions, we have the following converse to Lemma A.3.

Lemma A.5. *Let $i \geq 2$. Suppose $x\colon \Sigma^i(B/A) \to \Sigma^i B$ is such that:*

(1) *$(\Sigma^i p)x$ equals $\mathrm{id}_{\Sigma^i(B/A)}$.*
(2) *$(\Sigma^{i-1}\alpha)x$ equals zero.*

Then Σx equals $\Sigma^i \chi$ in $[\Sigma^{i+1}(B/A), \Sigma^{i+1} B]_*$.

Proof. We simply compute:

$$\begin{aligned}\Sigma x = \mathrm{id} \circ (\Sigma x) &= [(\Sigma^i j)(\Sigma^{i-1}\alpha) + (\Sigma^{i-1}\chi)(\Sigma^i p)](\Sigma x)\\ &= (\Sigma^i j)(\Sigma^{i-1}\alpha)(\Sigma x) + (\Sigma^{i-1}\chi)(\Sigma^i p)(\Sigma x)\\ &= \Sigma^i \chi,\end{aligned}$$

where the second equality comes from Definition A.2 and the fourth equality comes from the given properties of x. In the third equality we have used $(A+B)(\Sigma x) = A(\Sigma x) + B(\Sigma x)$; note that the analogous formula without the suspension does not hold in general. □

Remark A.6. We will often apply Lemma A.5 in the case $i = \infty$, where the statement yields that the stable homotopy class of χ is characterized by the given two properties.

A.7. Splittings of products. Now let X and Y be pointed spaces, and specialize to the cofiber sequence

$$X \vee Y \xrightarrow{j} X \times Y \xrightarrow{p} X \wedge Y. \tag{A.8}$$

Let $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ be the two projection maps. Let α be the homotopy class $\Sigma\pi_1 + \Sigma\pi_2 \colon \Sigma(X \times Y) \to \Sigma X \vee \Sigma Y$, defined using the group structure on $[\Sigma(X \times Y), \Sigma(X \vee Y)]_*$. The composition $\alpha(\Sigma j)$ is the identity (up to homotopy), i.e. $\Sigma\pi_1 + \Sigma\pi_2$ splits Σj.

By Definition A.2, we obtain a map $\chi \colon \Sigma(X \wedge Y) \to \Sigma(X \times Y)$, uniquely defined up to based homotopy. This map is a splitting for Σp and satisfies $(\Sigma\pi_1 + \Sigma\pi_2)\chi = 0$. Moreover, Lemma A.5 says that χ is completely characterized by these criteria, up to suspension. When necessary for clarity, we will write $\chi(X, Y)$ for the map $\chi \colon \Sigma(X \wedge Y) \to \Sigma(X \times Y)$.

The definition of χ shows that it is natural in X and Y. That is, if Z and W are also pointed and $f \colon X \to Z$ and $g \colon Y \to W$ are two based maps, then the diagram

$$\begin{array}{ccc} \Sigma(X \wedge Y) & \xrightarrow{\chi} & \Sigma(X \times Y) \\ {\scriptstyle \Sigma(f\wedge g)}\downarrow & & \downarrow{\scriptstyle \Sigma(f\times g)} \\ \Sigma(Z \wedge W) & \xrightarrow[\chi]{} & \Sigma(Z \times W) \end{array}$$

commutes in the based homotopy category. This is a routine argument, boiling down to the fact that projections and inclusions are natural.

In several cases we will need to understand the compatibility of χ with the twist maps where one interchanges the roles of X and Y:

Lemma A.9. *The diagram*

$$\begin{array}{ccc} \Sigma(X \times Y) & \xrightarrow{\Sigma T_\times} & \Sigma(Y \times X) \\ {\scriptstyle \chi(X,Y)}\uparrow & & \uparrow{\scriptstyle \chi(Y,X)} \\ \Sigma(X \wedge Y) & \xrightarrow[\Sigma T_\wedge]{} & \Sigma(Y \wedge X) \end{array}$$

commutes (up to homotopy) after two suspensions, where $T_\times$ and $T_\wedge$ are the evident twist maps.

Proof. Let f denote the composite

$$\Sigma(X \wedge Y) \xrightarrow{\Sigma T_\wedge} \Sigma(Y \wedge X) \xrightarrow{\chi(Y,X)} \Sigma(Y \times X) \xrightarrow{\Sigma T_\times} \Sigma(X \times Y).$$

We want to show that $\Sigma^2 f$ equals $\Sigma^2 \chi(X,Y)$. By Lemma A.5, it suffices to prove that $(\Sigma\pi_1 + \Sigma\pi_2)f = 0$ and $(\Sigma p)f = \mathrm{id}_{\Sigma(X\wedge Y)}$. These follow from naturality of the twist maps and the corresponding properties of $\chi(Y,X)$. $\square$

Remark A.10. Lemma A.9 is almost true after one suspension, but we need the extra suspension because of the restriction $i \geq 2$ from Lemma A.5.

A.11. Stable considerations. From now on we disregard the suspensions required for the careful statement of unstable results. That is, we work in the stable category of spectra. When X is a pointed space, we will often abuse notation and write X again for $\Sigma^\infty X$. Also, it will be convenient to now let χ denote the desuspension of the splitting $\Sigma(X \wedge Y) \to \Sigma(X \times Y)$ produced in the last section. So χ is now a map $X \wedge Y \to X \times Y$.

Proposition A.12 below is a stable version of Lemma A.5, with a slight strengenthing due to stability.

Proposition A.12. *The map χ is the unique stable homotopy class $X \wedge Y \to X \times Y$ such that $p\chi$ is the identity on $X \wedge Y$ and $\pi_1\chi = \pi_2\chi = 0$ is zero.*

Proof. Lemma A.5 implies that χ is the unique stable homotopy class such that $p\chi = \mathrm{id}_{X\wedge Y}$ and $(\pi_1 + \pi_2)\chi = 0$. It suffices for us to show that the second condition is equivalent to $\pi_1\chi = \pi_2\chi = 0$. Clearly the latter implies the former, since $(\pi_1+\pi_2)\chi = \pi_1\chi + \pi_2\chi$ (and note that this uses stability). Conversely, if $(\pi_1+\pi_2)\chi = 0$ then multiplying by π_1 on the left gives $\pi_1\chi = 0$ since the composite $\pi_1\pi_2$ is null. Similarly, left multiplication by π_2 gives $\pi_2\chi = 0$. $\square$

Corollary A.13. *In the stable homotopy category, we have:*

(a) $\pi_1 + \pi_2 + p\colon X \times Y \to X \vee Y \vee (X \wedge Y)$ *is an isomorphism, and* $j \vee \chi$ *is a homotopy inverse.*
(b) $j(\pi_1 + \pi_2) + \chi p$ *is the identity on* $X \times Y$.

Proof. For part (a), use that $(\pi_1 + \pi_2)j = \mathrm{id}_{X\vee Y}$; $pj = 0$; $(\pi_1 + \pi_2)\chi = 0$; and $p\chi = \mathrm{id}_{X\wedge Y}$. For part (b), $j(\pi_1+\pi_2)+\chi p$ is $\mathrm{id}_{X\times Y}$ by Definition A.2. $\square$

Let $\Delta_\times\colon X \to X \times X$ and $\Delta_\wedge\colon X \to X \wedge X$ be the evident diagonal maps.

Lemma A.14. *Let X be any pointed motivic space. Then the two maps $\Delta_\times$ and $\chi\Delta_\wedge + j(\pi_1 + \pi_2)\Delta_\times$ are equal maps $X \to X \times X$.*

Proof. Start with $j(\pi_1 + \pi_2) + \chi p = 1$ from Corollary A.13, and apply $\Delta_\times$ on the right. Finally, note that $\Delta_\wedge = p\Delta_\times$. $\square$

Later we will need the following calculation of χ in a specific example.

Lemma A.15. *Let $S^{0,0}$ consist of the two points 1 and -1, where 1 is the basepoint. Let i, j, and k be the based maps $S^{0,0} \to S^{0,0} \times S^{0,0}$ that take -1 to $(1, -1)$, $(-1, 1)$, and $(-1, -1)$ respectively. Then $\chi\colon S^{0,0} \wedge S^{0,0} \to S^{0,0} \times S^{0,0}$ is stably equal to the composition of the isomorphism $S^{0,0} \wedge S^{0,0} \xrightarrow{\cong} S^{0,0}$ with the map $k - i - j$.*

Proof. We apply Proposition A.12. The result follows from the observations that:

(1) pi and pj are zero, whereas pk is the identity.
(2) $\pi_1 i$ is zero, but $\pi_1 j$ and $\pi_1 k$ are the identity.
(3) $\pi_2 j$ is zero, but $\pi_2 i$ and $\pi_2 k$ are the identity. □

A.16. Higher splittings. Given based spaces $X_1, \ldots, X_n$ and a subset $S = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$, write $X^{\times S}$ for $X_{i_1} \times \cdots \times X_{i_k}$ (where $i_1 < i_2 < \cdots < i_k$). Also, write $X^{\wedge S}$ for $X_{i_1} \wedge \cdots \wedge X_{i_k}$. Write $p_S\colon X_1 \times \cdots \times X_n \to X^{\wedge S}$ for the composition

$$X_1 \times \cdots \times X_n \xrightarrow{\pi} X^{\times S} \xrightarrow{p} X^{\wedge S},$$

where π is the evident projection.

Proposition A.17. *The map $\sum_S p_S\colon X_1 \times \cdots \times X_n \to \bigvee_S X^{\wedge S}$ is a weak equivalence in the stable category, where the sum and wedge range over all nonempty subsets S of $\{1, \ldots, n\}$.*

Proof. This follows from induction and part (a) of Corollary A.13. □

Proposition A.17 allows us to make the following definition.

Definition A.18. Let $\chi_S\colon X^{\wedge S} \to X_1 \times \cdots \times X_n$ be the unique homotopy class of maps such that:

(1) $p_S \chi_S = \mathrm{id}$.
(2) $p_T \chi_S = 0$ for all $T \neq S$.

Definition A.18 generalizes the properties of the 2-fold splitting that $p\chi$ is the identity, while $\pi_1 \chi$ and $\pi_2 \chi$ are both zero. We will usually write just χ instead of χ_S. Just as for the 2-fold splittings, the maps χ_S are natural in the objects $X_1, \ldots, X_n$.

Analogously to Lemma A.9, Proposition A.19 shows that the higher splittings respect permutations of the factors.

Proposition A.19. *If σ is a permutation of $\{1, \ldots, n\}$, then the diagram*

$$\begin{array}{ccc} X_1 \wedge \cdots \wedge X_n & \longrightarrow & X_{\sigma 1} \wedge \cdots \wedge X_{\sigma n} \\ \chi \downarrow & & \chi \downarrow \\ X_1 \times \cdots \times X_n & \longrightarrow & X_{\sigma 1} \times \cdots \times X_{\sigma n} \end{array}$$

is commutative, where the horizontal maps are permutations of factors.

Proof. The proof is similar to the proof of Lemma A.9. □

Now let $X_1, \dots, X_n$ be formal symbols, and let w be a parenthesized word made form these symbols using the two operations $\times$ and $\wedge$. For example, w might be $(X_1 \times X_3) \wedge (X_4 \times X_2)$. Let w' be a word obtained from w by changing one $\times$ symbol to a $\wedge$ symbol, e.g. $w' = (X_1 \wedge X_3) \wedge (X_4 \times X_2)$. We can regard both w and w' as functors $\mathrm{Ho}\,(\mathcal{C})^n \to \mathrm{Ho}\,(\mathcal{C})$, and we let p denote the evident natural transformation $w \to w'$. In our example, p is more precisely $p_{X_1,X_3} \wedge (\mathrm{id}_{X_4} \times \mathrm{id}_{X_2})$. There is also an evident natural transformation $w' \to w$ made from maps of the form χ_S, and we denote this just by χ.

Remark A.20. One has the following "coherence results" for the maps χ and p:

(i) Given any two sequences of maps

$$w = w_1 \xrightarrow{\chi} w_2 \xrightarrow{\chi} \cdots \xrightarrow{\chi} w_r = v$$

and

$$w = w_1' \xrightarrow{\chi} w_2' \xrightarrow{\chi} \cdots \xrightarrow{\chi} w_s = v$$

with the same source and target, the two composite natural transformations are equal.

(ii) Given any two sequences of maps

$$w = w_1 \xrightarrow{p} w_2 \xrightarrow{p} \cdots \xrightarrow{p} w_r = v$$

and

$$w = w_1' \xrightarrow{p} w_2' \xrightarrow{p} \cdots \xrightarrow{p} w_s = v$$

with the same source and target, the two composite natural transformations are equal.

(iii) Let "χ-map" now refer to any composition as in (i), and "p-map" refer to any composition as in (ii). From now on, if a map is labelled as χ or p it means it belongs to one of these classes.

Let α be a composition as in (i) and let β be a composition as in (ii), and assume that the last word of α coincides with the first word in β (so that $\beta\alpha$ makes sense). Moreover, assume that the "spots" which β turns from $\times$ to $\wedge$ form a subset of the "spots" which α turns from $\wedge$ to $\times$. Then $\beta\alpha$ is a χ-map. (This generalizes the splitting property $p\chi = \mathrm{id}$).

We will not give proofs for the claims in Remark A.20, since proving them in the stated generality involves an unpleasant amount of bookkeeping. In all three cases, the proofs boil down to Proposition A.17. When we use Remark A.20 in the context of this paper, it will always be in cases where only three or four maps are involved. In those cases, it is easy enough to check the claims by hand.

Example A.21. Here are some examples to demonstrate the use of Remark A.20.

(1) The compositions

$$X \wedge Y \wedge Z \xrightarrow{1\wedge\chi(Y,Z)} X \wedge (Y \times Z) \xrightarrow{\chi(X,Y\times Z)} X \times Y \times Z$$

$$X \wedge Y \wedge Z \xrightarrow{\chi(X,Y)\wedge 1} (X \times Y) \wedge Z \xrightarrow{\chi(X\times Y,Z)} X \times Y \times Z$$

are both equal to $\chi(X, Y, Z)$ in the homotopy category.

(2) The composition

$$W \wedge X \wedge Y \wedge Z \xrightarrow{\chi(W,X)\wedge\chi(Y,Z)} (W\times X) \wedge (Y\times Z) \xrightarrow{\chi(W\times X,Y\times Z)} W\times X\times Y\times Z$$

is equal to $\chi(W, X, Y, Z)$ in the homotopy category.

(3) The triangle

$$\begin{array}{ccc} X \wedge Y \wedge Z & \xrightarrow{\chi} & X \times Y \times Z \\ & {\scriptstyle \chi(X,Y)\wedge 1}\searrow & \downarrow{\scriptstyle p} \\ & & (X \times Y) \wedge Z \end{array}$$

commutes for any objects X, Y, and Z.

Appendix B. Joins and other homotopically canonical constructions

In the next two sections we will need to deal with several homotopical constructions. In each situation, the output is not just a single object but rather a whole contractible category of objects. Things become complicated when we want to identify the outputs of different multi-layered constructions as being essentially the same. We start with a brief review of the machinery needed to handle these kinds of situations.

B.1. Canonical constructions. Suppose that $\mathcal{M}$ is a model category, I is a small category, and $X\colon I \to \mathcal{M}$ is a diagram. Homotopy theorists are faced with the troublesome fact that there is not a single homotopy colimit for X; rather, there are many different models for the homotopy colimit, but they are all weakly equivalent to each other. The trouble really begins when one needs to *choose* a weak equivalence between two different models, and then use this to perform further constructions. One cannot choose the weak equivalence arbitrarily and expect to obtain consistent results later on.

As an elementary example, suppose A and B are models for $\operatorname{hocolim}_I X$, and choose a weak equivalence $A \to B$. If at some later stage one similarly chooses a weak equivalence $B \to A$, then the composition $A \to B \to A$ may or may not be homotopic to the identity.

One way to control these issues is as follows. In good cases, one can define a model category structure on the category of diagrams $\mathcal{M}^I$, where the weak equivalences and fibrations are the objectwise ones [H, Theorem 11.6.1]. Let $\operatorname{Cof}(X)$ be the category of cofibrant approximations to X in this model structure. This category is contractible [H, Theorem 14.6.2], and the colimit of any object in $\mathrm{Cof}(X)$ gives a model for $\mathrm{hocolim}_I X$. The image of the composition

$$\mathrm{Cof}(X) \xrightarrow{\mathrm{colim}} \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$$

is a contractible groupoid; so for any two objects A and B in the image, there is a unique isomorphism $A \to B$ that is also in the image. Note that there might be many different isomorphisms in $\mathrm{Ho}(\mathcal{M})$ from A to B, but only *one* of them lies in the image of the above composite. In this sense there is a "homotopically canonical" isomorphism between A and B.

The considerations of the previous paragraph give a solution to our problem, but it is not a simple one. Given two models A and B for $\mathrm{hocolim}_I X$, we only can get our hands on the canonical homotopy equivalence between them by finding diagrams D and D' in $\mathrm{Cof}(X)$ and specifying A and B as the colimit of these diagrams—in essence, one must specify *why* A and B are models for $\mathrm{hocolim}_I X$, and only then does one get the comparison map.

A simple example demonstrates what is happening here. The suspension of a topological space X is a homotopy colimit for the diagram $* \longleftarrow X \longrightarrow *$. Given two spaces A and B that happen to have the homotopy type of ΣX, there is not a unique way to get a weak equivalence $A \to B$, even up to homotopy. However, if one specifies a decomposition into "top" and "bottom" cones for both A and B, then one *does* obtain a comparison map that is unique up to homotopy. The choice of top and bottom cones gives a diagram $[C_+ \longleftarrow X \longrightarrow C_-]$ that is a cofibrant model for $* \longleftarrow X \longrightarrow *$.

Now suppose that I and J are two small categories, and let $X\colon I \to \mathcal{M}$ and $Y\colon J \to \mathcal{M}$ be two diagrams. We think of $\mathrm{hocolim}_I X$ as a contractible groupoid inside of $\mathrm{Ho}(\mathcal{M})$, and likewise for $\mathrm{hocolim}_J Y$. Let A and B be specific models for these two homotopy colimits, and let $A \to B$ be a map. For any other models $\hat{A}$ and $\hat{B}$ for the two homotopy colimits, we immediately obtain corresponding maps $\hat{A} \to \hat{B}$ in $\mathrm{Ho}(\mathcal{M})$. Namely, we have the composite $\hat{A} \cong A \longrightarrow B \cong \hat{B}$, where the first and last isomorphisms are the ones from the respective contractible groupoids. In this case, we say that the maps $A \to B$ and $\hat{A} \to \hat{B}$ are "canonically isomorphic".

Often in practice, we have a certain *procedure* for producing a map between models for $\mathrm{hocolim}_I X$ and $\mathrm{hocolim}_J Y$. We want to know that this procedure, applied to A and B, or applied to $\hat{A}$ and $\hat{B}$, gives maps that are canonically isomorphic—i.e., conjugate to each other via the contractible groupoids for $\mathrm{hocolim}_I X$ and $\mathrm{hocolim}_J Y$. This is often the case, but it is something that needs to be *checked*; it is not automatic.

Unfortunately, there is no known efficient and carefree way to keep track of all of these kinds of compatibilities. While a direct approach works in simple arguments, this becomes harder in multi-layered constructions. We will see some examples in the remainder of these appendices. One precise but cumbersome technique is to work always at the level of diagrams, i.e., work in $\mathcal{M}^I$ and related categories as much as possible, rather than work in Ho $(\mathcal{M})$. This has the effect of pinning down precise models and precise maps between models. We follow this approach in most of our arguments.

As one specific place in which these ideas will be applied, consider the study of suspensions in a model category. Let I be the pushout category $0 \longleftarrow 1 \longrightarrow 2$. The category $\mathcal{M}^I$ has a model structure where the weak equivalences are objectwise and where the cofibrant objects are diagrams

$$X_0 \longleftarrow X_1 \longrightarrow X_2$$

such that each X_i is cofibrant and both maps are cofibrations.

Definition B.2. Let X be any object of $\mathcal{M}$. *Suspension data* for X is a cofibrant diagram $C_+ \longleftarrow QX \longrightarrow C_-$ where C_+ and C_- are contractible, together with a weak equivalence $QX \to X$.

See also Remark 2.9 for a discussion of suspension data. This is the same as specifying a cofibrant replacement for $* \longleftarrow X \longrightarrow *$ in $\mathcal{M}^I$. Every collection of suspension data gives rise to a model for the suspension of X, namely the pushout $C_+ \amalg_{QX} C_-$.

In many places in mathematics one learns how to handle technical details and then immediately starts to leave them in the background, seemingly ignored. Our discussion of canonical constructions is one of these instances. While there are real issues that require attention whenever such constructions appear, giving *complete* details in proofs quickly becomes an obstruction rather than an aid to comprehension; such details are best left to the reader. Certain canonical equivalences will be ubiquitous throughout the rest of these appendices—a clear example is in the statement of Lemma C.9, but most often the equivalences are appearing with less acknowledgement. The present section was meant to provide a kind of "global" acknowledgement that this is what is going on.

B.3. Joins.

We now construct the join of two objects and establish a connection between the join and the splitting maps χ from Appendix A.

Definition B.4. Given objects X and Y, the join $X * Y$ is the homotopy colimit of the diagram $X \longleftarrow X \times Y \longrightarrow Y$.

Note that the diagram

$$\begin{array}{ccccc} X & \longleftarrow & X \times Y & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & X \times Y & \longrightarrow & * \end{array}$$

yields a canonical map $X * Y \xrightarrow{\gamma} \Sigma(X \times Y)$.

Lemma B.5. *The composite*

$$X * Y \xrightarrow{\gamma} \Sigma(X \times Y) \xrightarrow{\Sigma p} \Sigma(X \wedge Y)$$

is a weak equivalence.

Proof. Let $X \rightarrowtail CX$ and $Y \rightarrowtail CY$ be cofibrations with contractible target. Consider the diagram

$$\begin{array}{ccccc} X \vee CY & \leftarrowtail & X \vee Y & \rightarrowtail & CX \vee Y \\ \downarrow & & \downarrow & & \downarrow \\ X \times CY & \leftarrowtail & X \times Y & \rightarrowtail & CX \times Y \\ \downarrow & & \downarrow & & \downarrow \\ X \wedge CY & \leftarrowtail & X \wedge Y & \rightarrowtail & CX \wedge Y. \end{array}$$

The pushout of each row is a model for the homotopy pushout, because of the horizontal cofibrations. The pushout of the top row is $CX \vee CY$; the pushout of the middle row is a model for $X * Y$, and the pushout of the last row is a model for $\Sigma(X \wedge Y)$ because both $X \wedge CY$ and $CX \wedge Y$ are contractible.

The columns of the diagram are homotopy cofiber sequences, so taking homotopy pushouts of each row gives a new homotopy cofiber sequence

$$CX \vee CY \rightarrowtail X * Y \to \Sigma(X \wedge Y).$$

But $CX \vee CY$ is contractible, so the second map is a weak equivalence.

There is an evident weak equivalence between the two diagrams

$$\begin{array}{ccccc} X \times CY & \longleftarrow & X \times Y & \longrightarrow & CX \times Y \\ \downarrow & & \downarrow & & \downarrow \\ X \wedge CY & \longleftarrow & X \wedge Y & \longrightarrow & CX \wedge Y \end{array} \qquad \begin{array}{ccccc} X & \longleftarrow & X \times Y & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & X \wedge Y & \longrightarrow & * \end{array}$$

mapping the diagram on the left to the one on the right. On taking homotopy pushouts of the rows, the first diagram gives the weak equivalence of the previous paragraph, while the second diagram gives $(\Sigma p)\gamma$. □

Proposition B.6 below shows that γ is a model for χ, once we identify $X * Y$ with $\Sigma(X \wedge Y)$.

Proposition B.6. *The map χ is equal to the composition*

$$\Sigma(X \wedge Y) \xrightarrow{\simeq} X * Y \xrightarrow{\gamma} \Sigma(X \times Y),$$

where the first map is the homotopy inverse to $(\Sigma p)\gamma$.

Proof. Let χ' be the composition under consideration. By Proposition A.12, it suffices to show that $(\Sigma p)\chi'$ is the identity on $\Sigma(X \wedge Y)$ and that $(\Sigma \pi_1)\chi' = (\Sigma \pi_2)\chi' = 0$. The first of these is immediate from the definition of χ'. For the second, observe that $(\Sigma \pi_1)\gamma$ and $(\Sigma \pi_2)\gamma$ are both zero. For example, in the first case this follows from the diagram

$$\begin{array}{ccccc} X & \longleftarrow & X \times Y & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & X & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & X & \longrightarrow & *. \end{array}$$

Upon taking homotopy colimits of the rows, the diagram induces $(\Sigma\pi_1)\gamma$, but the homotopy colimit of the middle row is contractible. □

We will also need the following simple result:

Proposition B.7. *Let* $D\colon I \to sPre(Sm/k)$ *be a diagram of motivic spaces, and let* X *be a fixed motivic space. Then there is a canonical equivalence between* $\operatorname{hocolim}_I[X * D_i]$ *and* $X * (\operatorname{hocolim}_I D)$*.*

Proof. Let J be the pushout indexing category $1 \longleftarrow 0 \longrightarrow 2$ and let $\overline{D}\colon I \times J \to sPre(Sm/k)$ be the evident diagram where

$$\overline{D}(i,1) = X, \quad \overline{D}(i,0) = D(i) \times X, \quad \overline{D}(i,2) = D(i)$$

(the maps in $\overline{D}$ are the obvious ones). A standard result in the theory of homotopy colimits gives canonical equivalences between $\operatorname{hocolim} \overline{D}$, $\operatorname{hocolim}_I[\operatorname{hocolim}_J D]$, and $\operatorname{hocolim}_J[\operatorname{hocolim}_I D]$, where the latter two expressions indicate the evident iterated homotopy colimits along slices of $I \times J$. Now just observe that $\operatorname{hocolim}_I[\operatorname{hocolim}_J D] = \operatorname{hocolim}_I(X * D_i)$ and $\operatorname{hocolim}_J[\operatorname{hocolim}_I D]$ is canonically equivalent to $X * [\operatorname{hocolim}_I D]$; the latter uses that homotopy colimits commute with products by a fixed space, which is a standard property of simplicial presheaf categories (following from the analogous result for $s\mathcal{S}et$). □

Appendix C. The Hopf construction

We now describe and study the Hopf construction.

Definition C.1. Let X, Y, and Z be pointed spaces, and let $h\colon X \times Y \to Z$ be a pointed map. The *Hopf construction* of h is the map $H(h)\colon X * Y \to \Sigma Z$ obtained by taking homotopy colimits of the rows of the diagram

$$\begin{array}{ccccc} X & \longleftarrow & X \times Y & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & Z & \longrightarrow & *. \end{array}$$

We often regard $H(h)$ as a map $\Sigma(X \wedge Y) \to \Sigma Z$ (or as just a map $X \wedge Y \to Z$) using the standard equivalence $X * Y \simeq \Sigma(X \wedge Y)$ from Lemma B.5.

Lemma C.2 below is a simple example of the Hopf construction.

Lemma C.2. *Let* $\pi_1\colon X \times Y \to X$ *and* $\pi_2\colon X \times Y \to Y$ *be the evident projection maps. The maps* $H(\pi_1)$ *and* $H(\pi_2)$ *are trivial.*

Proof. Consider the diagram from the end of the proof of Proposition B.6. The map $H(\pi_1)$ is obtained by taking the homotopy colimits of the rows, but the homotopy colimit of the middle row is contractible.

The argument for $H(\pi_2)$ is identical. □

Using the model for χ given in Proposition B.6, we can give an alternative model for the Hopf construction on $h\colon X \times Y \to Z$:

Proposition C.3. *Let X, Y, and Z be pointed spaces, and let $h\colon X\times Y \to Z$ be a pointed map. The Hopf construction $H(h)$ equals the composite*

$$X * Y \xrightarrow{\simeq} \Sigma(X \wedge Y) \xrightarrow{\chi} \Sigma(X \times Y) \xrightarrow{\Sigma h} \Sigma Z,$$

where the first map is the weak equivalence from Lemma B.5.

Proof. The diagram

$$\begin{array}{ccccc} X & \longleftarrow & X \times Y & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & X \times Y & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle h} & & \downarrow \\ * & \longleftarrow & Z & \longrightarrow & *. \end{array}$$

shows that $H(h)$ is equal to $(\Sigma h)\gamma$, where γ is from Section B.3. By Proposition B.6, the composition $X * Y \xrightarrow{\simeq} \Sigma(X \wedge Y) \xrightarrow{\chi} \Sigma(X \times Y)$ is equal to γ. □

The following simple result will save us a bit of trouble in the body of the paper.

Proposition C.4. *Let $f\colon X\times Y \to Y$ be a pointed map, where X and Y are oriented homotopy spheres. Then $[H(f)]$ does not depend on the orientation of Y.*

Proof. Let $\sigma\colon Y \to Y$ be an automorphism in the homotopy category, and consider the diagram

$$\begin{array}{ccccc} X \wedge Y & \xrightarrow{\chi} & X \times Y & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle 1\wedge\sigma} & & \downarrow{\scriptstyle 1\times\sigma} & & \downarrow{\scriptstyle \sigma} \\ X \wedge Y & \xrightarrow{\chi} & X \times Y & \xrightarrow{g} & Y \end{array}$$

where $g = \sigma f(1 \times \sigma^{-1})$. We must show that $[H(f)] = [H(g)]$. But the diagram yields

$$[\sigma] \cdot [H(f)] = [H(g)] \cdot [1 \wedge \sigma] = [H(g)] \cdot [\sigma]$$

using Remark 2.6 in the last step. Since $[\sigma]$ is central and invertible, $[H(f)] = [H(g)]$. □

For the following proposition, regard $S^{0,0}$ as the group scheme $\mathbb{Z}/2$. This result is not needed in the present paper, but we include it for future reference.

Proposition C.5. *The Hopf construction on the product map*

$$\mu\colon S^{0,0} \times S^{0,0} \to S^{0,0}$$

represents the element -2 *in* $\pi_{0,0}(S)$.

Proof. The composition

$$\Sigma S^{0,0} \xrightarrow{\cong} \Sigma(S^{0,0} \wedge S^{0,0}) \xrightarrow{\chi} \Sigma(S^{0,0} \times S^{0,0}) \xrightarrow{\Sigma\mu} \Sigma S^{0,0}$$

is the Hopf construction on μ, which is equal to $(\Sigma\mu)(\Sigma k - \Sigma i - \Sigma j)$, where i, j, and k are as in Lemma A.15. This is evidently the same as $\Sigma(\mu k) - \Sigma(\mu i) - \Sigma(\mu j)$. But μk is null, and both μi and μj are the identity maps. It follows that $(\Sigma\mu)(\Sigma k - \Sigma i - \Sigma j)$ equals -2. □

C.6. Hopf constructions of meldings. Let X, Y_1, Y_2, Z_1, and Z_2 be pointed spaces and let $f_1\colon X \times Y_1 \to Z_1$ and $f_2\colon X \times Y_2 \to Z_2$ be two based maps. Consider the diagram

$$\text{(C.7)} \qquad \begin{array}{ccccc} X \times Y_1 & \longleftarrow & X \times Y_1 \times Y_2 & \longrightarrow & X \times Y_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle \alpha(f_1,f_2)} & & \downarrow{\scriptstyle f_2} \\ Z_1 & \longleftarrow & Z_1 \times Z_2 & \longrightarrow & Z_2, \end{array}$$

where the horizontal maps are the evident projections and $\alpha(f_1, f_2)$ is the composite

$$X \times Y_1 \times Y_2 \xrightarrow{\Delta\times 1\times 1} X \times X \times Y_1 \times Y_2 \xrightarrow{1\times T\times 1} X \times Y_1 \times X \times Y_2 \xrightarrow{f_1\times f_2} Z_1 \times Z_2.$$

Definition C.8. The *melding* $f_1\#f_2$ of the pairings f_1 and f_2 is the pairing

$$X \times (Y_1 * Y_2) \to Z_1 * Z_2$$

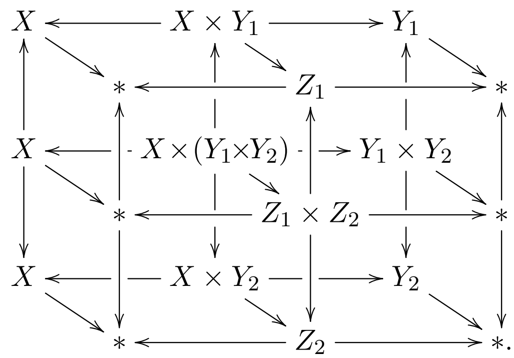
obtained by taking homotopy pushouts of the rows of Diagram (C.7).

Lemma C.9 below gives a tool for computing Hopf constructions of meldings.

Lemma C.9. *The Hopf construction* $H(f_1\#f_2)\colon X * (Y_1 * Y_2) \to \Sigma(Z_1 * Z_2)$ *is canonically identified with the double suspension of the composite*

$$X \wedge Y_1 \wedge Y_2 \xrightarrow{\chi} X \times Y_1 \times Y_2 \xrightarrow{\alpha(f_1,f_2)} Z_1 \times Z_2 \xrightarrow{p} Z_1 \wedge Z_2.$$

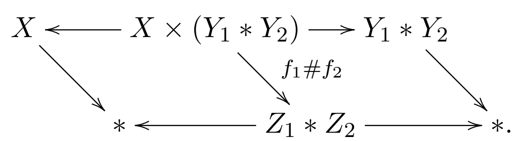
Proof. This is an exercise in the manipulation of homotopy colimits. Consider the rectangular box



All of the horizontal and vertical maps are the evident projections, while the three maps in the middle column coming out of the page are f_1, $\alpha(f_1, f_2)$, and f_2.

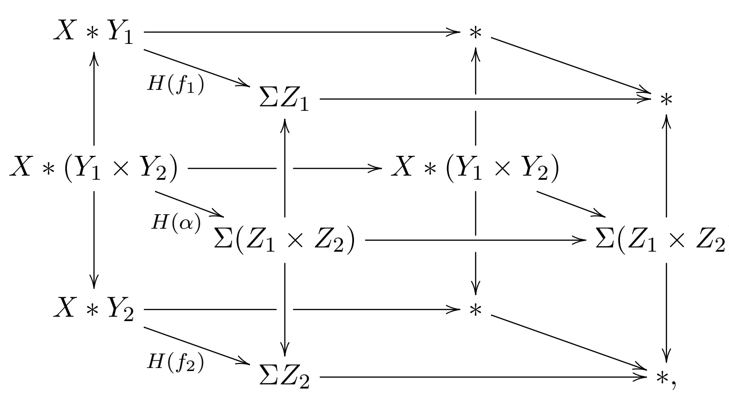
Let P_f be the front 3×3 diagram, and let P_b be the back 3×3 diagram. The whole diagram is a natural transformation $P_b \to P_f$. We will compute the induced map $g : \operatorname{hocolim} P_b \to \operatorname{hocolim} P_f$ in two different ways.

We can calculate the homotopy colimit of a 3×3 grid in two ways: first take homotopy pushouts of the rows, and then take the homotopy pushout of the resulting column; or first take the homotopy pushouts of the columns, and then take the homotopy pushout of the resulting row. If we first take homotopy colimits of the columns, then we obtain the diagram



Now take homotopy colimits along the rows to yield $H(f_1 \# f_2)$.

On the other hand, if we first take homotopy colimits along the rows, then we obtain the left face of the diagram



where α is $\alpha(f_1, f_2)$. Taking the homotopy pushouts of the columns and applying the evident canonical isomorphisms, we get the commutative diagram

$$\begin{array}{ccc} \Sigma^2 X \wedge Y_1 \wedge Y_2 & \xrightarrow{\Sigma^2\left(1\wedge\chi\right)} & \Sigma^2 X \wedge (Y_1 \times Y_2) \\ \downarrow{\scriptstyle \tilde{g}} & & \downarrow{\scriptstyle \Sigma H(\alpha)} \\ \Sigma^2 Z_1 \wedge Z_2 & \xrightarrow[\chi]{} & \Sigma^2 (Z_1 \times Z_2). \end{array}$$

Note that Proposition B.7 has been used for one of the columns. Here $\tilde{g}$ is canonically equivalent to the map $g\colon \operatorname{hocolim} P_b \to \operatorname{hocolim} P_f$ that we are trying to understand, and the horizontal maps have been identified with the help of Proposition B.6.

In the following computation we leave out several suspension operators. The above parallelogram allows us to write

$$\tilde{g} = p\chi\tilde{g} = p \circ H(\alpha) \circ (1 \wedge \chi) = p \circ \big(\alpha \circ \chi\big) \circ (1 \wedge \chi) = p\alpha\chi,$$

where the various χ symbols mean slightly different things and we are using Remark A.20 to make sense of this.

The upshot of this argument is that the following maps are canonically identified: $H(f_1 \# f_2)$, g, and $\tilde{g} = p\alpha\chi$. This completes the proof. □

Before stating the next result we introduce a piece of notation. If X is an oriented homotopy sphere then write $|X|$ for the unique pair $(p, q) \in \mathbb{Z}^2$ such that $X \simeq S^{p,q}$ (uniqueness follows, for example, by base-extending to a field and then using motivic cohomology calculations). Recall that if X and Y are oriented homotopy spheres then the twist map $T\colon X \wedge Y \to Y \wedge X$ represents an element $\tau_{|X|,|Y|} = [T]$ in $\pi_{0,0}(S)$. As discussed in Section 2.3, if $X \simeq S^{p,q}$ and $Y \simeq S^{s,t}$ then $\tau_{|X|,|Y|} = \tau_{(p,q),(s,t)} = (-1)^{(p-q)(s-t)} \cdot \epsilon^{qt}$. These elements are central in $\pi_{*,*}(S)$ because every element of $\pi_{0,0}(S)$ is central.

In analyses that involve extensive sign calculations, it is very convenient to drop the absolute value signs and write $\tau_{X,Y}$ for $\tau_{|X|,|Y|}$. In fact we carry this to an extreme: if the name of a homotopy sphere appears inside a subscript for a τ-expression, it is to be interpreted as the associated bidegree. For example, $\tau_{X+Y-Z,W}$ is shorthand for $\tau_{|X|+|Y|-|Z|,|W|}$. In practice this never leads to any confusion. Since $\tau_{(-),(-)}$ is bilinear and takes its value in the 2-torsion subgroup of the multiplicative group $\pi_{0,0}(S)^\times$, we can also write formulas like

$$\tau_{X+Y-Z,W} = \tau_{X,W}\tau_{Y,W}\tau_{Z,W}^{-1}\tau_{X,W} = \tau_{X,W}\tau_{Y,W}\tau_{Z,W}\tau_{X,W}.$$

The following proposition gives a key formula used in the paper. The complexity of the signs is unfortunate, but there seems to be no avoiding this. The lack of symmetry in the signs on the first two terms is tied to the asymmetry in the signs for $[\mathrm{id}_{r,s} \wedge f]$ and $[f \wedge \mathrm{id}_{r,s}]$ appearing in Remark 2.6.

Proposition C.10. *Suppose given oriented homotopy spheres X, Y_1, Y_2, Z_1, and Z_2, together with pointed maps $f\colon X\times Y_1 \to Z_1$ and $g\colon X\times Y_2 \to Z_2$. Let f^* and g^* denote the composites*

$$Y_1 \cong * \times Y_1 \longrightarrow X\times Y_1 \xrightarrow{f} Z_1 \qquad \textit{and} \qquad Y_2 \cong * \times Y_2 \longrightarrow X\times Y_2 \xrightarrow{g} Z_2.$$

Then $[H(f\#g)]$ equals

$$\tau_{X+Y_1-Z_1,Z_2}[H(f)]\cdot[g^*] + \tau_{X,Y_1}\tau_{Y_1-Z_1,Z_2}[f^*]\cdot[H(g)] \\ + \tau_{X,Y_2}\tau_{X+Y_1-Z_1,Z_2}[H(f)]\cdot[H(g)]\cdot[\Delta_X].$$

Proof. By Lemma C.9 the map $H(f\#g)$ can be modelled by the composite

$$\begin{array}{c} X\wedge Y_1\wedge Y_2 \xrightarrow{\chi} X\times Y_1\times Y_2 \xrightarrow{\Delta_\times \times 1} X\times X\times Y_1\times Y_2 \xrightarrow{1\times T\times 1} X\times Y_1\times X\times Y_2 \\ \downarrow{\scriptstyle f\times g} \\ Z_1\wedge Z_2 \xleftarrow{p} Z_1\times Z_2. \end{array}$$

We use the fact that $\Delta_\times \times 1 = (j_1\times 1) + (j_2\times 1) + (\chi\Delta_\wedge \times 1)$ from Lemma A.14. So our composite is the sum of three pieces, which we analyze separately.

The j_1-composite: This piece is the composition along the top right in the diagram

$$\begin{array}{ccccc} X\wedge Y_1\wedge Y_2 & \xrightarrow{\chi} & X\times Y_1\times Y_2 & \xrightarrow{f\times g^*} & Z_1\times Z_2 \\ & \searrow^{\chi\wedge \mathrm{id}} & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ & & (X\times Y_1)\wedge Y_2 & \xrightarrow[f\wedge g^*]{} & Z_1\wedge Z_2. \end{array}$$

The composite along the bottom is $H(f)\wedge g^*$. Remark 2.6(iii) yields the formula $[H(f)\wedge g^*] = [H(f)]\cdot[g^*]\cdot \tau_{X+Y_1-Z_1,Z_2}$.

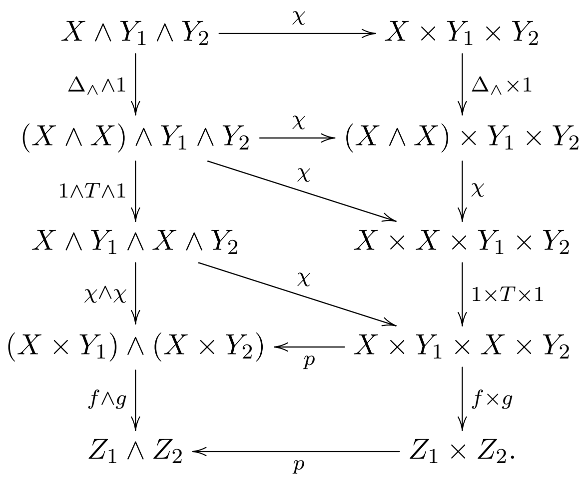
The j_2-composite: This piece is the composition along the top right in the diagram

$$\begin{array}{ccccccc} X\wedge Y_1\wedge Y_2 & \xrightarrow{\chi} & X\times Y_1\times Y_2 & \xrightarrow{T\times 1} & Y_1\times X\times Y_2 & \xrightarrow{f^*\times g} & Z_1\times Z_2 \\ \downarrow{\scriptstyle T\wedge 1} & & \nearrow^{\chi} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ Y_1\wedge X\wedge Y_2 & & \xrightarrow{1\wedge\chi} & & Y_1\wedge(X\times Y_2) & \xrightarrow{f^*\wedge g} & Z_1\wedge Z_2. \end{array}$$

The upper left region commutes by Proposition A.19, and the middle region commutes by Remark A.20. The composition along the bottom left of the diagram is $(f^*\wedge H(g))(T\wedge 1)$. Remark 2.6(iii) yields the formula

$$[f^*\wedge H(g)][T\wedge 1] = [f^*]\cdot[H(g)]\cdot\tau_{Y_1-Z_1,Z_2}\cdot\tau_{X,Y_1}.$$

The $\chi\Delta_\wedge$*-composite*: Here we examine the diagram



The parallelogram in the center commutes by Proposition A.19, and the adjacent triangles commute by Remark A.20. The composition along the top, right, and bottom is the composite in which we are interested, whereas the composition along the left is $[H(f) \wedge H(g)] \cdot \tau_{X,Y_1} \cdot [\Delta_\wedge \wedge 1]$. By Remark 2.6(ii), we have

$$[\Delta_\wedge \wedge 1] = [\Delta_\wedge] \cdot \tau_{-X,Y_1+Y_2} = [\Delta_\wedge] \cdot \tau_{X,Y_1}\tau_{X,Y_2},$$

and by Remark 2.6(iii) we have

$$[H(f) \wedge H(g)] = [H(f)] \cdot [H(g)] \cdot \tau_{X+Y_1-Z_1,Z_2}.$$

Putting everything together now gives that our $\chi\Delta_\wedge$-composite equals

$$[H(f)] \cdot [H(g)] \cdot \tau_{X+Y_1-Z_1,Z_2} \cdot \tau_{X,Y_1} \cdot [\Delta_\wedge] \cdot \tau_{X,Y_1}\tau_{X,Y_2}.$$

Note that the two τ_{X,Y_1} terms cancel, leading to the expression given in the statement of the proposition. □

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