

Cohomological Hasse principle and resolution of quotient singularities

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ABSTRACT. In this paper we study weight homology of singular schemes. Weight homology is an invariant of a singular scheme defined in terms of hypercoverings of resolution of singularities. Our main result is a McKay principle for weight homology of quotient singularities, i.e., we describe weight homology of a quotient scheme in terms of weight homology of an equivariant scheme. Our method is to reduce the geometric McKay principle for weight homology to Kato’s cohomological Hasse principle for arithmetic schemes. The McKay principle for weight homology implies the McKay principle for the homotopy type of the dual complex of the exceptional divisors of a resolution of a quotient singularity. As a consequence we show that the dual complex is contractible for isolated quotient singularities.

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Introduction

Quite a few examples have been observed which show that an arithmetic method can play a significant role for a geometric question. In this paper we present such a new example. The geometric question concerns the dual (or configuration) complex of the exceptional divisors of a resolution of a quotient singularity X/G , where X is a quasi-projective smooth scheme over a perfect field k endowed with an action of a finite group G . Let Z be the singular locus of X/G . Assume furthermore that there exists a resolution of singularities $g : \tilde{Y} \rightarrow X/G$ such that g is proper birational and an isomorphism outside Z and $E = g^{-1}(Z)_{\text{red}}$ is a simple normal crossing divisor on the smooth scheme \tilde{Y} . The dual complex $\Gamma(E)$ is a CW -complex (which is a Δ -complex in the sense of [Hat, Section 2.1]) whose a -simplices correspond to the connected components of

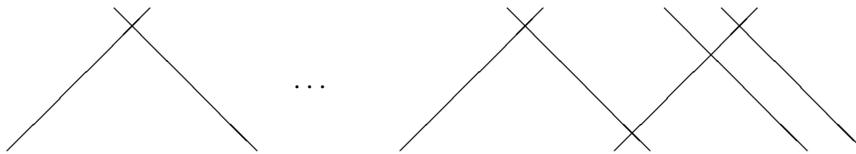
$$E^{[a]} = \coprod_{1 \leq i_0 < i_1 < \dots < i_a \leq N} E_{i_0} \cap \dots \cap E_{i_a}$$

where E_1, \dots, E_N are the irreducible components of E .

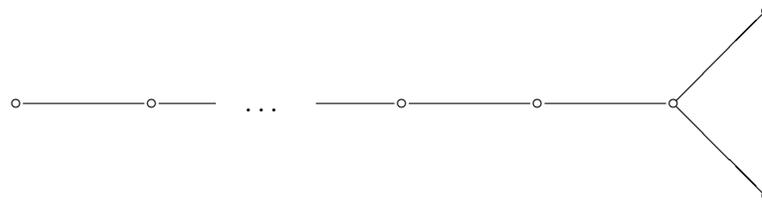
A model case is a Klein quotient singularity \mathbb{C}^2/G , where $G \subset \text{SL}_2(\mathbb{C})$ acts linearly on \mathbb{C}^2 . The origin $0 \in \mathbb{C}^2/G$ is the unique singular point of \mathbb{C}^2/G . Let $g : \tilde{Y} \rightarrow \mathbb{C}^2/G$ be the minimal resolution. The irreducible components of the exceptional locus $E = g^{-1}(0)$ are rational curves that form a configuration expressed by the Dynkin diagrams. For example, if we take a binary dihedral

$$G = \langle \sigma, \tau \mid \sigma^n = 1, \tau^2 = -1, \tau\sigma\tau = -\sigma^{-1} \rangle,$$

the irreducible components of E form a configuration looking as



consisting of $(n + 2)$ rational curves. The configuration complex $\Gamma(E)$ is



with $(n + 2)$ vertices and $(n + 1)$ edges.

For general $G \subset \mathrm{SL}_2(\mathbb{C})$ McKay observed a mysterious coincidence of $\Gamma(E)$ and the so-called McKay graph which is computed in terms of representations of G . The higher dimensional generalization of this fact, called McKay correspondence, is now a fertile land of algebraic geometry. For more details on this, we refer the readers to an excellent exposition by M. Reid [Re]. Here we just quote the following principle:

McKay Principle. *Let G be a finite group and X be a quasi-projective smooth G -scheme over a field k (i.e., a quasi-projective smooth scheme over k endowed with an action of G), and $g : \tilde{Y} \rightarrow X/G$ be a resolution of singularities of X/G . Then the answer to any well posed question about the geometry of \tilde{Y} is the G -equivariant geometry of X .*

In this paper we investigate McKay principle for the homotopy type of $\Gamma(E)$, which is known to be an invariant of X/G independent of a choice of a resolution $\tilde{Y} \rightarrow X/G$ by a theorem of Stepanov and its generalizations ([St], [ABW1], [Pa], [Th]).

Let us fix the setup for our main result. Let X and G be as before and $\pi : X \rightarrow X/G$ be the projection. Fix a closed reduced subscheme $S \subset X/G$ which is projective over k and contains the singular locus $(X/G)_{\mathrm{sing}}$ of X/G . Let $T = \pi^{-1}(S)_{\mathrm{red}}$ be the reduced part of $\pi^{-1}(S)$. Assume that we are given the following datum:

- a proper birational morphism $g : \tilde{Y} \rightarrow X/G$ such that \tilde{Y} is smooth, $E_S = g^{-1}(S)_{\mathrm{red}}$ is a simple normal crossing divisor on \tilde{Y} and g is an isomorphism over $X/G - S$;
- a proper birational G -equivariant morphism $f : \tilde{X} \rightarrow X$ in $\mathcal{C}_{G/k}$ such that \tilde{X} is a smooth G -scheme, $E_T = f^{-1}(T)_{\mathrm{red}}$ is a G -strict simple normal crossing divisor on \tilde{X} (cf. Definition 1.5) and f is an isomorphism over $X - T$.

$$\begin{array}{ccccccc}
 \pi^{-1}(S)_{\mathrm{red}} = T & \longrightarrow & X & \xleftarrow{f} & \tilde{X} & \longleftarrow & E_T = f^{-1}(T)_{\mathrm{red}} \\
 & & \downarrow & & \downarrow \pi & & \\
 S & \longrightarrow & X/G & \xleftarrow{g} & \tilde{Y} & \longleftarrow & E_S = g^{-1}(S)_{\mathrm{red}}
 \end{array}$$

Note that we do not assume the existence of a morphism $E_T \rightarrow E_S$. By definition G acts on $\Gamma(E_T)$ and we can form a CW-complex $\Gamma(E_T)/G$.

Theorem 0.1 (Theorem 11.1). *We assume that $\mathrm{ch}(k) = 0$ or that k is perfect and canonical resolution of singularities in the sense of [BM] holds over k . In the homotopy category of CW-complexes, there exists a canonical map*

$$\phi : \Gamma(E_T)/G \rightarrow \Gamma(E_S)$$

which induces isomorphisms on the homology and fundamental groups:

$$H_a(\Gamma(E_T)/G) \xrightarrow{\cong} H_a(\Gamma(E_S)) \quad \text{for } \forall a \in \mathbb{Z},$$

$$\pi_1(\Gamma(E_T)/G) \xrightarrow{\cong} \pi_1(\Gamma(E_S)).$$

By using basic theorems in algebraic topology (Whitehead and Hurewicz), Theorem 0.1 implies the following:

Corollary 0.2. *Let the assumption be as in Theorem 0.1.*

- (1) *If $\Gamma(E_T)/G$ is simply connected, ϕ is a homotopy equivalence.*
- (2) *If $\Gamma(E_T)/G$ is contractible, $\Gamma(E_S)$ is contractible.*
- (3) *If T is smooth (e.g., $\dim(T) = 0$ which means that $(X/G)_{\text{sing}}$ is isolated), $\Gamma(E_S)$ is contractible.*

We will deduce the following variant of Corollary 0.2(3).

Corollary 0.3 (Corollary 11.4). *Let A be a complete regular local ring containing \mathbb{Q} and let G be a finite group acting on A . Set $X = \text{Spec}(A)$ and assume that X/G has an isolated singularity $s \in X/G$. Let $g : \tilde{Y} \rightarrow X/G$ be a proper morphism such that g is an isomorphism outside s and $E_s = g^{-1}(s)_{\text{red}}$ is a simple normal crossing divisor in the regular scheme \tilde{Y} . Then the topological space $\Gamma(E_s)$ is contractible.*

Here we recall some known results on the contractibility of the dual complex of the exceptional divisor of a resolution of singularities. Let (Y, S) be an isolated singularity over algebraically closed field of characteristic 0. We recall the following implications for (Y, S) :

$$\text{finite quotient} \Rightarrow \text{KLT} \Rightarrow \text{rational},$$

where *KLT* stands for Kawamata log terminal. Let $g : \tilde{Y} \rightarrow Y$ be proper birational such that \tilde{Y} smooth, $E_S = g^{-1}(S)_{\text{red}}$ is a simple normal crossing divisor and g is an isomorphism over $Y - S$. The following facts are known:

- If (Y, S) is rational, $H_*(\Gamma(E_S))$ is torsion ([ABW2]). The proof uses a weight argument in Hodge theory.
- There exists a rational singularity (Y, S) such that $\Gamma(E_S)$ has the homotopy type of $\mathbb{P}_{\mathbb{R}}^2$, in particular $\pi_1(\Gamma(E_S)) = H_1(\Gamma(E_S)) = \mathbb{Z}/2\mathbb{Z}$ ([Pa]).
- If (Y, S) is KLT, $\pi_1(\Gamma(E_S)) = 1$. This is a consequence of a theorem of Kollár and Takayama ([Ko], [Ta]).

Recently, the intriguing question whether $\Gamma(E_S)$ is contractible if (Y, S) is KLT has been answered positively in [FKX].

Now we explain the main idea of the proof of Theorem 0.1. The proof of the assertion on the fundamental group relies on a geometric interpretation of the fundamental group of the dual complex of a simple normal crossing divisor. More generally, for any locally noetherian scheme E , we associate a *CW*-complex $\Gamma(E)$ to E in the same way as the case of simple normal crossing divisors and show a natural isomorphism

$$\pi_1(\Gamma(E)) \simeq \pi_1^{cs}(E),$$

where the right-hand side denotes the classifying group of *cs-coverings* $E' \rightarrow E$, which is by definition a morphism of schemes such that any point $x \in E$ has a Zariski open neighborhood $U \subset E$ such that $E' \times_E U \simeq \coprod U$ with a possibly infinite coproduct (see §10).

In order to show the assertion of Theorem 0.1 on the homology groups, we need introduce the equivariant weight homology. Let $\mathcal{C}_{\text{eq}/k}$ be the category of pairs (X, G) of a finite group G and a G -scheme X which is quasi-projective over k (see §1 for the definition of morphisms in $\mathcal{C}_{\text{eq}/k}$). For simplicity we ignore the p -torsion if $\text{ch}(k) = p > 0$ and work over $\Lambda := \mathbb{Z}[\frac{1}{p}]$ (we can extend the following results to the case $\Lambda = \mathbb{Z}$ assuming appropriate form of resolution of singularities). Let Mod_Λ be the category of Λ -modules. An equivariant homology theory $H = \{H_a\}_{a \geq 0}$ with values in Λ -modules on $\mathcal{C}_{\text{eq}/k}$ is a sequence of functors:

$$H_a(-) : \mathcal{C}_{\text{eq}/k} \rightarrow \text{Mod}_\Lambda \quad (a \in \mathbb{Z}_{\geq 0})$$

which are covariant for proper morphisms and contravariant for strict open immersions (a strict morphism in $\mathcal{C}_{\text{eq}/k}$ means a G -equivariant morphism for some fixed G) such that if $i : Y \hookrightarrow X$ is a strict closed immersion in $\mathcal{C}_{\text{eq}/k}$, with open complement $j : V \hookrightarrow X$ there is a long exact sequence called localization sequence (see Definition 1.1)

$$\dots \xrightarrow{\partial} H_a(Y) \xrightarrow{i_*} H_a(X) \xrightarrow{j^*} H_a(V) \xrightarrow{\partial} H_{a-1}(Y) \rightarrow \dots$$

A main input is the following result (see Theorem 1.4).

Theorem 0.4. *For a Λ -module M , there exists a homology theory on $\mathcal{C}_{\text{eq}/k}$ with values in Λ -modules:*

$$(X, G) \mapsto H_a^W(X, G; M) \quad (a \in \mathbb{Z}_{\geq 0})$$

called equivariant weight homology with coefficient M satisfying the following condition: Let E be a projective G -scheme over k which is a G -strict simple normal crossing divisor on a smooth G -scheme over k (see Definition 1.5 for G -strict). Then we have

$$H_*^W(E, G; M) \simeq H_*(\Gamma(E)/G)$$

which is computed as the homology of the complex:

$$\dots \rightarrow M^{\pi_0(E^{[a]})/G} \xrightarrow{\partial} M^{\pi_0(E^{[a-1]})/G} \xrightarrow{\partial} \dots \xrightarrow{\partial} M^{\pi_0(E^{[1]})/G}.$$

In particular, if X is a projective smooth G -scheme over k , we have

$$H_a^W(X, G; M) \cong \begin{cases} M^{X^{(0)}/G} & \text{for } a = 0, \\ 0 & \text{for } a \neq 0, \end{cases}$$

where $X^{(0)}$ is the set of the generic points of X .

In case G is the trivial group e , we write $H_a^W(X; M)$ for $H_a^W(X, G; M)$.

The construction of $H_*^W(X, G; M)$ hinges on a descent argument due to Gillet–Soulé ([GS1], [GS2]) and Jannsen [J], going back to Deligne [SGA4] Exposé V^{bis}. If X is proper over k , we take a Λ -admissible hyperenvelope (X_\bullet, G) (see Definition 2.4) which is a certain simplicial object in $\mathcal{C}_{\text{eq}/k}$ such that X_n are smooth projective over k for all n , and then define $H_*^W(X, G; M)$ as homology groups of the complex

$$\dots \rightarrow M^{\pi_0(X_n)/G} \xrightarrow{\partial} M^{\pi_0(X_{n-1})/G} \xrightarrow{\partial} \dots \xrightarrow{\partial} M^{\pi_0(X_0)/G}.$$

The existence of hyperenvelopes relies on equivariant resolution of singularities. In case $\text{ch}(k) = 0$ it follows from canonical resolution of singularities [BM]. In case $\text{ch}(k) > 0$ the construction depends on Gabber’s refinement of de Jong’s alteration theorem [II].

Theorem 0.1 is deduced from the following theorem (see Theorem 1.7 and Corollary 1.9 in §1).

Theorem 0.5 (McKay principle for weight homology). *For $(X, G) \in \mathcal{C}_{\text{eq}/k}$ let $\pi : X \rightarrow X/G$ be the projection viewed as a morphism $(X, G) \rightarrow (X/G, e)$ in $\mathcal{C}_{\text{eq}/k}$, where e is the trivial group. Then the induced map*

$$\pi_* : H_a^W(X, G; M) \rightarrow H_a^W(X/G; M)$$

is an isomorphism for all $a \in \mathbb{Z}$. In particular, if X is projective smooth over k , we have $H_a^W(X/G; M) = 0$ for $a \neq 0$.

Note that the last statement is nontrivial since X/G may be singular even though X is smooth.

For $k = \bar{k}$ and M uniquely divisible Theorem 0.5 follows from the yoga of weights. To see this let us for simplicity assume $k = \mathbb{C}$. In terms of the weight filtration on singular cohomology with compact support, see [D2], one has

$$H_a^W(X, G; M) = \text{Hom}_{\mathbb{Q}}(W_0 H_c^a(X_{\text{an}}, \mathbb{Q})^G, M),$$

where the upper index G stands for G -invariants. It is not difficult to check that

$$H_c^a((X/G)_{\text{an}}, \mathbb{Q}) \xrightarrow{\cong} H_c^a(X_{\text{an}}, \mathbb{Q})^G$$

is an isomorphism, from which Theorem 0.5 follows in this special case.

Our basic strategy to show Theorem 0.5 in general is to introduce another (arithmetic) homology theory

$$(X, G) \mapsto KH_a(X, G; M) \quad (a \in \mathbb{Z}_{\geq 0})$$

called equivariant Kato homology with coefficient M as a replacement for the weight zero part of singular cohomology with compact support above. Here we assume that M is a torsion Λ -module. As in case of the weight homology, we simply write $KH_a(X, G; M) = KH_a(X; M)$ in case $G = e$. For this homology theory McKay principle is rather easily shown:

Proposition 0.6 (Proposition 8.3). *For $(X, G) \in \mathcal{C}_{\text{eq}/k}$, $\pi : X \rightarrow X/G$ induces an isomorphism*

$$\pi_* : KH_a(X, G; M) \xrightarrow{\cong} KH_a(X/G; M) \text{ for all } a \in \mathbb{Z}.$$

Theorem 0.5 follows from Proposition 0.6 and the following.

Theorem 0.7 (Theorem 9.1). *There exists a map of homology theories on $\mathcal{C}_{\text{eq}/k}$:*

$$KH_*(X, G; M) \rightarrow H_*^W(X, G; M)$$

which is an isomorphism if k is a purely inseparable extension of a finitely generated field and $M = \Lambda_\infty := \mathbb{Q}/\Lambda$.

The proof of Theorem 0.7 relies on Theorem 0.9 below on the cohomological Hasse principle. The cohomological Hasse principle originally formulated as conjectures by K.Kato [K1], concerns a certain complex of Bloch–Ogus type on an arithmetic scheme X (which means a scheme of finite type over \mathbb{Z}):

$$(0.1) \quad \begin{aligned} \dots \xrightarrow{\partial} \bigoplus_{x \in X_{(a)}} H^{a+1}(x, \mathbb{Z}/n\mathbb{Z}(a)) \xrightarrow{\partial} \bigoplus_{x \in X_{(a-1)}} H^a(x, \mathbb{Z}/n\mathbb{Z}(a-1)) \xrightarrow{\partial} \dots \\ \dots \xrightarrow{\partial} \bigoplus_{x \in X_{(1)}} H^2(x, \mathbb{Z}/n\mathbb{Z}(1)) \xrightarrow{\partial} \bigoplus_{x \in X_{(0)}} H^1(x, \mathbb{Z}/n\mathbb{Z}) \end{aligned}$$

Here $X_{(a)}$ denotes the set of points $x \in X$ such that $\dim \overline{\{x\}} = a$ with the closure $\overline{\{x\}}$ of x in X , and the term in degree a is the direct sum of the Galois cohomology $H^{a+1}(x, \mathbb{Z}/n\mathbb{Z}(a))$ of the residue fields $\kappa(x)$ for $x \in X_{(a)}$ (for the coefficients $\mathbb{Z}/n\mathbb{Z}(a)$, see [KeS, Lemma 1.5]. If $(n, \text{ch}(\kappa(x))) = 1$, $\mathbb{Z}/n\mathbb{Z}(a)$ is the Tate twist of n -roots of unity). The homology groups of the above complex is denoted by $KH_a(X; \mathbb{Z}/n\mathbb{Z})$ and called the Kato homology of X (in case X has a component flat over \mathbb{Z} , one need modify the definition in order to take into account contributions of \mathbb{R} -valued points of X but we ignore it). Kato [K1] conjectured

Conjecture 0.8. *Let X be a proper smooth scheme over a finite field, or a regular scheme proper flat over \mathbb{Z} . Then*

$$KH_a(X; \mathbb{Z}/n\mathbb{Z}) = 0 \text{ for } a \neq 0.$$

In case $\dim(X) = 1$ the Kato conjecture is equivalent to the Hasse principle for the Brauer group of a global field (i.e., a number field or a function field of a curve over a finite field), a fundamental fact in number theory.

A generalization of the Kato homology to the (nonequivariant) case over a finitely generated field was introduced in [JS]. The equivariant Kato homology $KH_a(X, G; M)$ is its equivariant version (it agrees with $KH_a(X; \mathbb{Z}/n\mathbb{Z})$ in case $G = e$ and $M = \mathbb{Z}/n\mathbb{Z}$). As for nonequivariant case, the following result has been shown.

Theorem 0.9 ([KeS]). *Assume that k is a purely inseparable extension of a finitely generated field. For a proper smooth scheme X over k and for a prime to $\ell \neq \text{ch}(k)$, we have*

$$KH_a(X; \mathbb{Q}_\ell/\mathbb{Z}_\ell) = \varinjlim_n KH_a(X; \mathbb{Z}/\ell^n\mathbb{Z}) = 0 \quad \text{for } a \neq 0.$$

If k is finite, the same holds by replacing $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ by $\mathbb{Z}/n\mathbb{Z}$ with n prime to $\text{ch}(k)$.

Key ingredients of the proof of Theorem 0.9 are Deligne's theorem [D1] on the Weil conjecture and Gabber's refinement of de Jong's alteration.

Theorem 0.7 is shown by using Theorem 0.9. As a corollary of Theorem 0.7 and Proposition 0.6, we get the following extension of Theorem 0.9 to a singular case.

Corollary 0.10 (Corollary 9.3). *Let k be a purely inseparable extension of a finitely generated field. Let X be a proper smooth scheme with an action of a finite group G over k . For a prime to $\ell \neq \text{ch}(k)$, we have*

$$KH_a(X/G; \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0 \quad \text{for } a \neq 0.$$

If k is finite, the same holds by replacing $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ by $\mathbb{Z}/n\mathbb{Z}$ with n prime to $\text{ch}(k)$.

The paper is organized as follows. In §1 we explain the properties of the equivariant weight homology theory and state McKay principle for the equivariant weight homology and its corollaries. The construction of the equivariant weight homology theory occupies four section §2 through §5. The sections §6 and §7 are preliminaries for the construction of the equivariant Kato homology and the proof of its basic properties given in §8. In §9 a comparison theorem of the equivariant Kato homology and the weight homology is proved. McKay principle for the equivariant weight homology stated in §1 is deduced from the comparison theorem. In §10 we relate the fundamental group of the dual complex to the classifying group of cs -coverings. Using this and the McKay principle stated in §1, we prove the main theorem 0.1 in §11.

Recently, A. Thuillier observed that one can deduce the main results of our paper using the theory of Berkovich spaces as developed in his article [Th]. This Berkovich approach has the advantage that it does not use resolution of singularities.

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1. Equivariant weight homology

By k we denote a field and let Λ be either \mathbb{Z} or $\mathbb{Z}[1/p]$, where p is the exponential characteristic of k . Let $\mathcal{C}_{\text{eq}/k}$ be the category of pairs (X, G) of a finite group G and a G -scheme X which is quasi-projective over k (i.e., $X \in \mathcal{C}/k$ equipped with a left action of G over k). For objects (X, G) and (Y, H) of $\mathcal{C}_{\text{eq}/k}$, the morphisms $(X, G) \rightarrow (Y, H)$ are pairs (ϕ, f) of a group homomorphism $\phi : G \rightarrow H$ and a map of G -schemes $f : X \rightarrow Y$, where we endow Y with the induced G -action. Note that the category $\mathcal{C}_{\text{eq}/k}$ has fibre products. We will often write just X for an object $(X, G) \in \mathcal{C}_{\text{eq}/k}$ when it is clear from the context that X is endowed with a group-action.

We say (ϕ, f) is strict if $G = H$ and ϕ is the identity. For fixed G , let $\mathcal{C}_{G/k} \subset \mathcal{C}_{\text{eq}/k}$ be the subcategory of G -schemes and strict morphisms. A morphism in $\mathcal{C}_{G/k}$ is simply denoted by $f : X \rightarrow Y$. Notice that in case $G = e$ the trivial group, $\mathcal{C}_{G/k}$ is identified with \mathcal{C}/k , i.e., the category of quasi-projective schemes over k .

A strict open immersion (resp. strict closed immersion, resp. strict proper morphism) in $\mathcal{C}_{\text{eq}/k}$ means an open immersion (resp. closed immersion, resp. proper morphism) in $\mathcal{C}_{G/k}$ for some G . An equivariant simplicial scheme is a simplicial object in $\mathcal{C}_{G/k}$ for some G .

Definition 1.1. Let $\mathcal{C}_{\text{eq}/k^*}$ be the category with the same objects as $\mathcal{C}_{\text{eq}/k}$ and with morphisms in $\mathcal{C}_{\text{eq}/k}$ whose underlying morphisms of schemes are proper. Let Λ be a commutative ring. An equivariant homology theory $H = \{H_a\}_{a \geq 0}$ with values in Λ -modules on $\mathcal{C}_{\text{eq}/k}$ is a sequence of covariant functors:

$$H_a(-) : \mathcal{C}_{\text{eq}/k^*} \rightarrow \text{Mod}_\Lambda$$

satisfying the following conditions:

- (i) For each strict open immersion $j : V \hookrightarrow X$ in $\mathcal{C}_{\text{eq}/k}$, there is a map $j^* : H_a(X) \rightarrow H_a(V)$, associated to j in a functorial way.
- (ii) If $i : Y \hookrightarrow X$ is a strict closed immersion in $\mathcal{C}_{\text{eq}/k}$, with open complement $j : V \hookrightarrow X$, there is a long exact sequence (called localization sequence)

$$\dots \xrightarrow{\partial} H_a(Y) \xrightarrow{i_*} H_a(X) \xrightarrow{j^*} H_a(V) \xrightarrow{\partial} H_{a-1}(Y) \longrightarrow \dots$$

(The maps ∂ are called the connecting morphisms.)

- (iii) The sequence in (ii) is functorial with respect to proper morphisms or strict open immersions in $\mathcal{C}_{\text{eq}/k}$, in the following sense. Consider a commutative diagram in $\mathcal{C}_{\text{eq}/k}$

$$\begin{array}{ccccc} (U', G') & \xrightarrow{j'} & (X', G') & \xleftarrow{i'} & (Z', G') \\ \downarrow f_U & & \downarrow f & & \downarrow f_Z \\ (U, G) & \xrightarrow{j} & (X, G) & \xleftarrow{i} & (Z, G), \end{array}$$

such that the squares of the underlying schemes are cartesian, and i (resp. i') is a strict closed immersion and j (resp. j') is its open complement.

If $G = G'$ and f is a strict open immersion, the following diagram is commutative.

$$\begin{CD} H_a(Z, G) @>>> H_a(X, G) @>>> H_a(U, G) @>>> H_{a+1}(Z, G) \\ @V f_Z^* VV @V f^* VV @V f_U^* VV @V f_Z^* VV \\ H_a(Z', G) @>>> H_a(X', G) @>>> H_a(U', G) @>>> H_{a+1}(Z', G) \end{CD}$$

If f is proper, the following diagram is commutative.

$$\begin{CD} H_a(Z', G') @>>> H_a(X', G') @>>> H_a(U', G') @>>> H_{a+1}(Z', G') \\ @V (fz)^* VV @V f^* VV @V (fU)^* VV @V (fz)^* VV \\ H_a(Z, G) @>>> H_a(X, G) @>>> H_a(U, G) @>>> H_{a+1}(Z, G) \end{CD}$$

A morphism between homology theories H and H' is a morphism $\phi : H \rightarrow H'$ of functors on $\mathcal{C}_{\text{eq}/k*}$, which is compatible with the long exact sequences from (ii).

Definition 1.2. We call $(X, G) \in \mathcal{C}_{\text{eq}/k}$ primitive if G acts transitively on the set of irreducible components of X . Let $\mathcal{S}_{\text{eq}/k} \subset \mathcal{C}_{\text{eq}/k}$ (resp. $\mathcal{S}_{G/k} \subset \mathcal{C}_{G/k}$) be the full subcategory of objects whose underlying schemes are smooth projective over k and $\mathcal{S}_{\text{eq}/k}^{\text{prim}} \subset \mathcal{S}_{\text{eq}/k}$ (resp. $\mathcal{S}_{G/k}^{\text{prim}} \subset \mathcal{S}_{G/k}$) be the full subcategories of primitive objects.

Let p be the exponential characteristic of k . Let \tilde{k} be the perfection of k . We consider the following condition on resolution of singularities:

(RS)_{eq} For $(X, G) \in \mathcal{C}_{\text{eq}/\tilde{k}}$ with X reduced, there are $(X', G) \in \mathcal{C}_{\text{eq}/\tilde{k}}$ with X' smooth over \tilde{k} and a strict, projective, birational morphism $f : X' \rightarrow X$ such that f is an isomorphism over the regular locus X_{reg} of X and that the reduced part of $f^{-1}(X - X_{\text{reg}})$ is a simple normal crossing divisor on X' .

Remark 1.3.

- (1) The condition **(RS)_{eq}** includes as a special case $G = e$ (nonequivariant) resolution of singularities proved by Hironaka [H] assuming $\text{ch}(k) = 0$. Under the same assumption **(RS)_{eq}** is a consequence of canonical resolution of singularities shown in [BM].
- (2) In case $\text{ch}(k) > 0$, **(RS)_{eq}** would be false in general if we did not work over the perfection \tilde{k} of k .

Theorem 1.4 (See Construction 5.2). *Assume the following conditions:*

(★) $\Lambda = \mathbb{Z} \begin{bmatrix} 1 \\ p \end{bmatrix}$, or $\Lambda = \mathbb{Z}$ and **(RS)_{eq}** holds.

For a Λ -module M , there exists a homology theory on $\mathcal{C}_{\text{eq}/k}$ with values in Λ -modules:

$$(X, G) \mapsto H_a^W(X, G; M)$$

called equivariant weight homology. It satisfies:

- (i) There are canonical isomorphisms for $(X, G) \in \mathcal{S}_{\text{eq}/k}$

$$H_a^W(X, G; M) \cong \begin{cases} M^{X^{(0)}/G} & \text{for } a = 0, \\ 0 & \text{for } a \neq 0, \end{cases}$$

compatible with pushforward in $\mathcal{S}_{\text{eq}/k}$, where $X^{(0)}$ is the set of the generic points of X .

- (ii) $M \mapsto H_*^W(-; M)$ is a covariant functor from Λ -modules to the category of equivariant homology theories.
- (iii) For a short exact sequence of Λ -modules

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

there is a natural long exact sequence

$$\cdots \rightarrow H_a^W(-; M_1) \rightarrow H_a^W(-; M_2) \rightarrow H_a^W(-; M_3) \rightarrow H_{a-1}^W(-; M_1) \rightarrow \cdots$$

- (iv) For $(X, G) \in \mathcal{C}_{\text{eq}/k}$ there is a natural convergent spectral sequence

$$E_{a,b}^2 = \text{Tor}_a^\Lambda(M, H_b^W(X, G; \Lambda)) \implies H_{a+b}^W(X, G; M).$$

- (v) $H_a^W(X, G; \Lambda)$ is a finitely generated Λ -module for all $a \geq 0$.
- (vi) For extensions of fields $k \subset k'$ there are natural pushforward maps

$$\tau_{k'/k} : H_a^W(X_{k'}, G; M) \rightarrow H_a^W(X, G; M),$$

where the left-hand side weight homology is relative to the base field k' . They satisfy:

- (a) If k' is purely inseparable over k , $\tau_{k'/k}$ is an isomorphism.
- (b) For fixed $(X, G) \in \mathcal{C}_{\text{eq}/k}$ and $a \geq 0$, and for a directed system of fields $\{k_\alpha\}_{\alpha \in I}$ with $k_\alpha \subset k_{\alpha'}$ for $\alpha' \succeq \alpha$ and $\varinjlim_\alpha k_\alpha = k'$ the inverse system

$$\{H_a^W(X_{k_\alpha}, G; M)\}_\alpha$$

becomes stationary for big α and equal to $H_a^W(X_{k'}, G; M)$.

If $G = e$ is trivial, we write $H_a^W(X; M)$ for $H_a^W(X, G; M)$.

In §4 and §5 we give a descent construction of the equivariant weight homology. In the nonequivariant case this is due to Gillet–Soulé ([GS1], [GS2]) and Jannsen [J], relying on ideas of Deligne [SGA4] Exposé V^{bis}. If $\text{ch}(k) > 0$ our construction depends on Gabber’s refinement of de Jong’s alteration theorem [II], see Theorem 2.2.

Definition 1.5. A G -stable simple normal crossing divisor E on a smooth scheme over k with a G -action is G -strict if for any irreducible component E_i of E and for any $g \in G$, $g(E_i) = E_i$ or $E_i \cap g(E_i) = \emptyset$.

Example 1.6. Let $X \in \mathcal{C}_{G/k}$ be a G -strict simple normal crossing divisor (Definition 1.5). Then $H_*^W(X, G; M)$ is the homology of the complex:

$$\dots \rightarrow M^{\pi_0(X^{[a]}/G)} \xrightarrow{\partial} M^{\pi_0(X^{[a-1]}/G)} \xrightarrow{\partial} \dots \xrightarrow{\partial} M^{\pi_0(X^{[0]}/G)}.$$

Here X_1, \dots, X_N are the G -primitive components of X and

$$X^{[a]} = \coprod_{1 \leq i_0 < \dots < i_a \leq N} X_{i_0, \dots, i_a} \quad (X_{i_0, \dots, i_a} = X_{i_0} \cap \dots \cap X_{i_a}),$$

and $\pi_0(-)$ denotes the set of connected components, and the differentials ∂ are obvious alternating ones. Note that assuming equivariant resolution of singularities one can essentially calculate H^W by these recipes up to extensions of groups.

One of the main results of this paper is the following. Our proof in Section 9 uses arithmetic results due to Jannsen and the authors on Kato’s cohomological Hasse principle ([J], [JS] and [KeS]). Recall that p is the exponential characteristic of k .

Theorem 1.7 (McKay principle for weight homology). *Assume the condition (★) of Theorem 1.4. For $(X, G) \in \mathcal{C}_{\text{eq}/k}$ let $\pi : X \rightarrow X/G$ be the projection viewed as a morphism $(X, G) \rightarrow (X/G, e)$ in $\mathcal{C}_{\text{eq}/k}$. Then the induced map*

$$\pi_* : H_a^W(X, G; M) \rightarrow H_a^W(X/G; M)$$

is an isomorphism for all $a \in \mathbb{Z}$.

Corollary 1.8. *Let the assumption be as in 1.7. Assume $(X, G) \in \mathcal{S}_{\text{eq}/k}^{\text{prim}}$. Then*

$$H_a^W(X/G; M) = \begin{cases} M & a = 0, \\ 0 & a \neq 0. \end{cases}$$

Corollary 1.8 is a direct consequence of Theorem 1.7 and Theorem 1.4(i).

Corollary 1.9. *Let the assumption be as in Theorem 1.7. Let $(X, G) \in \mathcal{C}_{\text{eq}/k}$ and assume X is smooth over k . Let $S \subset X/G$ be a reduced closed subscheme which contains the singular locus of X/G . Let $f : Y \rightarrow X/G$ be a proper birational morphism such that Y is smooth over k and f is an isomorphism outside S . Assume:*

- (i) S is proper over k (e.g., S is isolated).
- (ii) There exist a strict open immersion $j : X \rightarrow \bar{X}$ in $\mathcal{C}_{G/k}$ and an open immersion $Y \hookrightarrow \bar{Y}$ with \bar{X} and \bar{Y} projective and smooth over k , and a commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & \bar{Y} \\ f \downarrow & & \bar{f} \downarrow \\ X/G & \longrightarrow & \bar{X}/G \end{array}$$

where \bar{f} is proper birational.

Then the map

$$H_a^W(f^{-1}(S); M) \xrightarrow{\sim} H_a^W(S; M)$$

is an isomorphism for all $a \in \mathbb{Z}$. In particular, if S is smooth over k , $H_a^W(f^{-1}(S); M) = 0$ for $a \neq 0$.

Remark 1.10. The assumption (ii) always holds if $\text{ch}(k) = 0$ thanks to [H] and [BM]. Assuming that k is perfect, the assumption (ii) is a consequence of $(\mathbf{RS})_{\text{eq}}$.

Proof of Corollary 1.9, assuming Theorem 1.7. We may assume that (\bar{X}, G) is primitive and \bar{Y} is connected. Put $\bar{S} = S \cup (\bar{X}/G - X/G) \subset \bar{X}/G$. By the localization sequence for weight homology we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} H_{a+1}^W(\bar{Y}) & \longrightarrow & H_{a+1}^W(\bar{Y} - f^{-1}(\bar{S})) & \longrightarrow & H_a^W(f^{-1}(\bar{S})) & \longrightarrow & H_a^W(\bar{Y}) \\ \downarrow & & \downarrow \simeq & & \downarrow & & \downarrow \\ H_{a+1}^W(\bar{X}/G) & \longrightarrow & H_{a+1}^W(\bar{X}/G - \bar{S}) & \longrightarrow & H_a^W(\bar{S}) & \longrightarrow & H_a^W(\bar{X}/G). \end{array}$$

In the diagram we suppressed the coefficients M for simplicity of notation. The second vertical isomorphism holds since $\bar{Y} - f^{-1}(\bar{S}) \cong \bar{X}/G - \bar{S}$. By Corollary 1.8 we get

$$H_a^W(\bar{Y}, M) \xrightarrow{\sim} H_a^W(\bar{X}/G, M) \xrightarrow{\sim} \begin{cases} M & a = 0, \\ 0 & a \neq 0. \end{cases}$$

Thus, from the above diagram we deduce an isomorphism

$$H_a^W(f^{-1}(\bar{S}); M) \xrightarrow{\sim} H_a^W(\bar{S}; M)$$

By the assumption (i), \bar{S} is the disjoint sum of S and $(\bar{X}/G - X/G)$. Hence the above isomorphism induces the desired isomorphism of Corollary 1.9. \square

From the proof of Theorem 1.7 we also deduce the boundedness of H^W in Section 9.

Proposition 1.11. For $X \in \mathcal{C}_{G/k}$ and for a Λ -module M we have under condition (\star) of Theorem 1.4

$$H_a^W(X, G; M) = 0$$

if $a \geq \dim(X) + 1$.

Proposition 1.11 would follow immediately from Theorem 1.4 under the assumption of a strong form of equivariant resolution of singularities, which we do not have in positive characteristic at the moment. See [GS1, Sec. 2.5] for an analogous argument. Our proof of Proposition 1.11 relies on equivariant Kato homology and is of arithmetic nature (see its proof in the last part of §9).

2. Admissible envelopes

Let the notation be as in Section 1. We define Λ -admissible envelopes of equivariant schemes generalizing an idea of Gillet [Gi]. Then we state an equivariant alteration theorem due to Gabber and sketch a proof. This allows us to construct smooth hyperenvelopes of equivariant simplicial schemes. Recall that p denotes the exponential characteristic of the base field k and that Λ is either \mathbb{Z} or $\mathbb{Z}[\frac{1}{p}]$.

Definition 2.1 ([GS1, 1.4.1]). A morphism $f : (X, G) \rightarrow (Y, G)$ in $\mathcal{C}_{G/k}$ is called a Λ -admissible envelope if the underlying morphism $X \rightarrow Y$ is proper and it satisfies the following condition.

If $\Lambda = \mathbb{Z}$, for a primitive G -equivariant scheme $(\mathrm{Spec}(L), G)$ where L is a finite product of fields, the induced map $X(L) \rightarrow Y(L)$ on the the sets of G -equivariant points of X and Y with values in $(\mathrm{Spec}(L), G)$ is surjective.

If $\Lambda \neq \mathbb{Z}$, for each prime number $\ell \neq p$ and for each G -equivariant point $P \in Y(L)$ with values in $(\mathrm{Spec}(L), G)$, there is a finite G -equivariant extension $(\mathrm{Spec}(L'), G) \rightarrow (\mathrm{Spec}(L), G)$ of degree prime to ℓ and a G -equivariant point $P' \in X(L')$ mapping to $P \in Y(L) \hookrightarrow Y(L')$.

Clearly the system of Λ -admissible envelopes is closed under compositions and base changes.

For the construction of admissible envelopes we need some sort of resolution of singularities over the base field k . For this consider the following condition:

- (**G**) $_{\ell}$ For a prime number ℓ and $(X, G) \in \mathcal{C}_{\mathrm{eq}/k}$ with X reduced, there are $(X', G) \in \mathcal{C}_{\mathrm{eq}/k}$ with X' smooth over k and a strict, surjective, projective morphism $f : X' \rightarrow X$ which is finite of degree prime to ℓ over any maximal point of X .

Theorem 2.2 (Gabber). *If k is perfect, (**G**) $_{\ell}$ holds whenever $\ell \neq p$.*

In case $G = e$ this result due to Gabber is shown in [II]. Gabber communicated the following proof [Ga] to us in the general case.

Proof. Without loss of generality X is G -primitive. Let X_1 be an irreducible component of X and put $G_1 = \{g \in G \mid g(X_1) = X_1\}$. Let G_{ℓ} be an ℓ -Sylow subgroup of G_1 and denote by $D \hookrightarrow Y = X_1/G_{\ell}$ the locus over which G_{ℓ} does not act freely on X_1 . According to [II, Theorem 1.3], there is a dominant generically finite morphism $g : Y' \rightarrow Y$ of degree prime to ℓ such that Y' is smooth and $D' = g^{-1}(D)$ is a simple normal crossing divisor on Y' . Let Y_1 be the normalization of Y' in $k(Y') \otimes_{k(Y)} k(X_1)$. As Y_1 is tamely ramified over Y' , we use Abhyankar's lemma [SGA1, XIII Proposition 5.1] in order to find étale locally over Y' an embedding of Y_1 into a Kummer étale extension of the log scheme (Y', D') . By [II, Proposition 6.6(c)] (Y_1, D_1) is log regular where D_1 is the reduced preimage of D' . Using results of Kato, Niziol and Gabber [K2] and [N] on resolution of log regular schemes, we

find an equivariant resolution of singularities $(X'_1, G_\ell) \rightarrow (Y_1, G_\ell)$. If we let (X', G) be the equivariant smooth scheme given by $X' = G \times_{G_\ell} X'_1$, the morphism $f : (X', G) \rightarrow (X, G)$ is as demanded in $(\mathbf{G})_\ell$. \square

Proposition 2.3. *Assume the condition (\star) of Theorem 1.4 and further that k is perfect. Then, for any $(X, G) \in \mathcal{C}_{G/k}$ with X reduced, there is a Λ -admissible envelope $f : (X', G) \rightarrow (X, G)$ with X' smooth over k .*

Proof. In case $\Lambda = \mathbb{Z}$ and $(\mathbf{RS})_{\text{eq}}$ holds, the assertion is proved by the same argument as [GS1, Lemma 2]. We prove the assertion in case $\Lambda = \mathbb{Z}[\frac{1}{p}]$. We use the induction on $d = \dim(X)$. Start with an arbitrary prime $\ell \neq p$. According to Theorem 2.2, there is (X'_ℓ, G) with X'_ℓ smooth over k and a surjective proper morphism $f_\ell : (X'_\ell, G) \rightarrow (X, G)$ which is finite of degree prime to ℓ over each maximal point of X . For every prime ℓ' in the set

$$L = \{\ell' \text{ prime} \mid \ell' \neq p \text{ and } \ell' \mid \deg(f_\ell)\},$$

choose $(X'_{\ell'}, G)$ with $X'_{\ell'}$ smooth over k and a surjective proper morphism $f_{\ell'} : (X'_{\ell'}, G) \rightarrow (X, G)$ which is finite of degree prime to ℓ' over each maximal point of X . Consider the object

$$X'_{\text{gen}} = X'_\ell \amalg \prod_{\ell' \in L} X'_{\ell'}$$

in $\mathcal{C}_{G/k}$. Then $f_{\text{gen}} : (X'_{\text{gen}}, G) \rightarrow (X, G)$ is a Λ -admissible envelope over a dense open G -equivariant subscheme $U \subset X$. Now by induction there is a Λ -admissible envelope $f_{\text{sp}} : (Y', G) \rightarrow (Y, G)$ with Y' smooth over k , where $Y = X - U$ with reduced structure. The morphism

$$f = f_{\text{gen}} \amalg f_{\text{sp}} : X'_d \amalg Y' \rightarrow X$$

is the envelope we are looking for. \square

In the next section Λ -admissible equivariant hyperenvelopes will play a central role.

Definition 2.4. A Λ -admissible hyperenvelope is an equivariant morphism of simplicial schemes $f : X_\bullet \rightarrow Y_\bullet$ in $\mathcal{C}_{\text{eq}/k}^\Delta$ such that the morphism

$$X_a \rightarrow (\text{cosk}_{a-1}^{Y_\bullet} \text{sk}_{a-1} X_\bullet)_a$$

is a Λ -admissible envelope for any $a \geq 0$.

Note that the system of Λ -admissible hyperenvelopes of equivariant simplicial schemes is closed under composition and base change. Indeed, this can be deduced from the characterization of hypercovers given in [GS2, Lem. 2.6] and the analogous statement for Λ -admissible envelopes.

Combining Proposition 2.3 and Theorem 2.2 with the proof of [GS1, Lemma 2], we deduce:

Proposition 2.5. *Assume the condition (\star) of Theorem 1.4 and further that k is perfect. Then for $X_\bullet \in \mathcal{C}_{G/k}^\Delta$ there is a Λ -admissible hyperenvelope $Y_\bullet \rightarrow X_\bullet$ in \mathcal{C}_G^Δ with Y_a/k smooth for all $a \geq 0$.*

3. Descent construction of homology theories

In this section we explain how to extend equivariant homology functors defined for smooth projective varieties to all varieties, assuming that they satisfy a certain descent property. The construction is an equivariant version of Gillet–Soulé [GS1, §2]

Let $C_+(\text{Mod}_\Lambda)$ be the category of homological complexes of Λ -modules vanishing in negative degrees. By definition an equivariant simplicial scheme (X_\bullet, G) is a simplicial object in $\mathcal{C}_{G/k}$ for a fixed finite group G . We say that (X_\bullet, G) has proper face maps if the underlying morphisms of the face maps are proper.

Definition 3.1. A homology functor

$$\Phi : \mathcal{C}_{\text{eq}/k} \rightarrow C_+(\text{Mod}_\Lambda)$$

is a covariant functor on $\mathcal{C}_{\text{eq}/k}$ which is also contravariant with respect to strict open immersions and which satisfies the following conditions:

- (i) If $i : Y \rightarrow X$ is a strict closed immersion in $\mathcal{C}_{\text{eq}/k}$ with strict open complement $j : V \rightarrow X$ the composition

$$\Phi(Y) \xrightarrow{i_*} \Phi(X) \xrightarrow{j^*} \Phi(V)$$

vanishes as a map of complexes.

- (ii) In the situation (i) the induced homomorphism

$$\text{cone}[\Phi(Y) \xrightarrow{i_*} \Phi(X)] \longrightarrow \Phi(V)$$

is a quasi-isomorphism.

- (iii) Consider a diagram in $\mathcal{C}_{\text{eq}/k}$

$$\begin{array}{ccc} (V', G') & \xrightarrow{j'} & (X', G') \\ \downarrow f_V & & \downarrow f \\ (V, G) & \xrightarrow{j} & (X, G), \end{array}$$

such that the diagram of the underlying schemes is cartesian, and f and f_V are proper, and j and j' are strict open immersions. Then the following square commutes:

$$\begin{array}{ccc} \Phi(X', G') & \xrightarrow{j'^*} & \Phi(V', G') \\ (f_V)_* \downarrow & & \downarrow f_* \\ \Phi(X, G) & \xrightarrow{j^*} & \Phi(V, G). \end{array}$$

(iv) For (X, G) and (Y, G) in $\mathcal{S}_{\text{eq}/k}$, the natural morphism of complexes

$$\Phi(X) \oplus \Phi(Y) \xrightarrow{\sim} \Phi\left(X \coprod Y\right)$$

is an isomorphism.

We also use the notion of a homology functor on $\mathcal{C}_{G/k}$ for a fixed finite group G , whose definition is analogous. Using total complexes one extends a homology functor $\Phi : \mathcal{C}_{\text{eq}/k} \rightarrow C_+(\text{Mod}_\Lambda)$ to a functor

$$\Phi : \mathcal{C}_{\text{eq}/k*}^\Delta \rightarrow C_+(\text{Mod}_\Lambda),$$

where $\mathcal{C}_{\text{eq}/k*}^\Delta$ is the category of equivariant simplicial schemes with proper face maps and proper morphisms.

Definition 3.2. For a homology functor Φ , the functors

$$H(\Phi) = \{H_a(\Phi)\}_{a \geq 0}$$

with

$$H_a(\Phi) : \mathcal{C}_{\text{eq}/k} \rightarrow \text{Mod}_\Lambda ; (X, G) \rightarrow H_a(\Phi(X, G))$$

form a homology theory on $\mathcal{C}_{\text{eq}/k}$ (this is obvious from the definition). We call $H(\Phi)$ the homology theory associated to Φ .

Definition 3.3. A pre-homology functor

$$F : \mathcal{S}_{\text{eq}/k} \rightarrow C_+(\text{Mod}_\Lambda)$$

is a covariant functor on $\mathcal{S}_{\text{eq}/k}$ which satisfies the following condition: For (X, G) and (Y, G) in $\mathcal{S}_{\text{eq}/k}$, the natural morphism of complexes

$$F(X) \oplus F(Y) \xrightarrow{\sim} F\left(X \coprod Y\right)$$

is an isomorphism.

Definition 3.4 ([GS1]).

- (1) Let $\Lambda\mathcal{S}_{G/k}$ be the category with the same objects as $\mathcal{S}_{G/k}$ and with morphisms

$$\text{Hom}_{\Lambda\mathcal{S}_{G/k}}(X, Y) = \Lambda\text{Hom}_{\mathcal{S}_{G/k}}(X, Y),$$

the free Λ -module on the set $\text{Hom}_{\mathcal{S}_{G/k}}(X, Y)$. It is easy to check that $\Lambda\mathcal{S}_{G/k}$ is a Λ -additive category and the coproduct $X \oplus Y$ of $X, Y \in \text{Ob}(\mathcal{S}_{G/k})$ is given by $X \coprod Y$.

- (2) For a simplicial object in $\mathcal{S}_{G/k}$:

$$X_\bullet : \quad \cdots X_2 \begin{array}{c} \xrightarrow{\delta_0} \\ \xleftarrow{s_0} \end{array} X_1 \begin{array}{c} \xrightarrow{\delta_0} \\ \xleftarrow{s_0} \end{array} X_0, \\ \quad \quad \quad \begin{array}{c} \xleftarrow{s_1} \\ \xrightarrow{\delta_1} \end{array} \\ \quad \quad \quad \xrightarrow{\delta_2}$$

we define a homological complex in $\Lambda\mathcal{S}_{G/k}$:

$$\Lambda X_{\bullet} : \cdots \rightarrow X_n \xrightarrow{\partial_n} X_{n-1} \rightarrow \cdots, \quad \partial_n = \sum_{j=0}^n (-1)^j \delta_j.$$

This gives a functor

$$\mathcal{S}_{G/k}^{\Delta} \rightarrow C_+(\Lambda\mathcal{S}_{G/k}),$$

where $C_+(\Lambda\mathcal{S}_{G/k})$ is the category of homological complexes in $\Lambda\mathcal{S}_{G/k}$.

Let $F : \mathcal{S}_{G/k} \rightarrow C_+(\text{Mod}_{\Lambda})$ be a pre-homology functor. Using total complexes, one extends F to a functor

$$(3.1) \quad F : C_+(\Lambda\mathcal{S}_{G/k}) \rightarrow C_+(\text{Mod}_{\Lambda}).$$

For a simplicial object X_{\bullet} in $\mathcal{S}_{G/k}$, we write $F(X_{\bullet}) = F(\Lambda X_{\bullet})$.

Consider the following descent conditions for a homology functor Φ on $\mathcal{C}_{\text{eq}/k}$ (resp. a pre-homology functor F on $\mathcal{S}_{\text{eq}/k}$). For a homology functor Φ , we let $\Phi|_{\mathcal{S}_{\text{eq}/k}}$ denote the pre-homology functor obtained by restricting Φ to the subcategory $\mathcal{S}_{\text{eq}/k}$ of $\mathcal{C}_{\text{eq}/k}$.

(D)_Φ For any finite group G and for any Λ -admissible equivariant hyperenvelope of simplicial schemes $f : X_{\bullet} \rightarrow Y_{\bullet}$ in $\mathcal{C}_{G/k}^{\Delta}$, the map

$$f_* : \Phi(X_{\bullet}) \rightarrow \Phi(Y_{\bullet})$$

is a quasi-isomorphism.

(D)_F For any finite group G and for any $X_* \in C_+(\Lambda\mathcal{S}_{G/k})$ with the property that $\Phi|_{\mathcal{S}_{\text{eq}/k}}(X_*)$ is acyclic for any homology functor Φ satisfying **(D)_Φ**, we have that $F(X_*)$ is acyclic.

Lemma 3.5.

- (1) If a homology functor Φ satisfies **(D)_Φ**, then $F = \Phi|_{\mathcal{S}_{\text{eq}/k}}$ satisfies **(D)_F**.
- (2) Let $f : X_{\bullet} \rightarrow Y_{\bullet}$ be a Λ -admissible hyperenvelope in $\mathcal{C}_{G/k}$ such that X_{\bullet} and Y_{\bullet} are objects in $\mathcal{S}_{G/k}$. If a pre-homology functor F satisfies **(D)_F**, then $f_* : F(X_{\bullet}) \rightarrow F(Y_{\bullet})$ is quasi-isomorphism.

Proof. (1) is obvious. We prove (2). Put $C_* = \text{cone}(\Lambda X_{\bullet} \xrightarrow{\Lambda f} \Lambda Y_{\bullet})$ in $C_+(\Lambda\mathcal{S}_{G/k})$. For a homology functor Φ , we have

$$\Phi|_{\mathcal{S}_{\text{eq}/k}}(C_*) = \text{cone}(\Phi(X_{\bullet}) \xrightarrow{f_*} \Phi(Y_{\bullet})).$$

Hence, if Φ satisfies **(D)_Φ**, $\Phi|_{\mathcal{S}_{\text{eq}/k}}(C_*)$ is acyclic since f_* is a quasi-isomorphism. Thus **(D)_F** implies

$$F(C_*) = \text{cone}(F(X_{\bullet}) \xrightarrow{f_*} F(Y_{\bullet}))$$

is acyclic, which proves the desired conclusion of Lemma 3.5. \square

The heuristic idea for the following construction is that there should be a universal extension up to quasi-isomorphisms of a pre-homology functor F satisfying $(\mathbf{D})_F$ to something similar to a homology functor Φ^F satisfying $(\mathbf{D})_{\Phi^F}$. However we do not know how to make this precise, but we can at least construct the expected homology theory associated to Φ^F . The construction we explain below is essentially due to Gillet–Soulé [GS1].

Construction 3.6. We now assume the following:

- (i) k is perfect.
- (ii) The condition (\star) of Theorem 1.4 holds.

Given a pre-homology functor F with values in Λ -modules satisfying $(\mathbf{D})_F$, we construct an equivariant homology theory H^F (see Definition 1.1) with the following properties:

- (1) There is a canonical isomorphism of homology groups

$$H_a^F|_{\mathcal{S}_{\text{eq}/k}} \cong H_a(F) \quad \text{for all } a \in \mathbb{Z}$$

compatible with equivariant pushforwards.

- (2) For $X_\bullet \in \mathcal{S}_{G/k}^\Delta$ and $X \in \mathcal{C}_{G/k}$ proper and for a Λ -admissible equivariant hyperenvelope $X_\bullet \rightarrow X$, there is a natural descent spectral sequence

$$E_{a,b}^1 = H_b^F(X_a, G) \implies H_{a+b}^F(X, G).$$

- (3) If $F = \Phi|_{\mathcal{S}_{\text{eq}/k}}$ for a homology functor Φ satisfying $(\mathbf{D})_\Phi$, we have a canonical isomorphism of homology theories $H^F \cong H(\Phi)$.
- (4) Let $F \rightarrow F'$ be a morphism of pre-homology functors satisfying $(\mathbf{D})_F$ and $(\mathbf{D})_{F'}$ with associated homology theories H^F and $H^{F'}$. Then there is a canonical morphism $H^F \rightarrow H^{F'}$ of homology theories. Furthermore, if for every $(X, G) \in \mathcal{S}_{\text{eq}/k}$, the map $F(X, G) \rightarrow F'(X, G)$ is a quasi-isomorphism, the induced morphism of homology theories $H^F \rightarrow H^{F'}$ is an isomorphism.
- (5) Let F'' be $\text{cone}(F \rightarrow F')$ for a morphism of pre-homology functors $F \rightarrow F'$ satisfying $(\mathbf{D})_F$ and $(\mathbf{D})_{F'}$ and let $H^F, H^{F'}$ and $H^{F''}$ be the associated homology theories. Then for $(X, G) \in \mathcal{S}_{\text{eq}/k}$ there is a long exact sequence

$$\cdots \rightarrow H_a^F(X, G) \rightarrow H_a^{F'}(X, G) \rightarrow H_a^{F''}(X, G) \rightarrow H_{a-1}^F(X, G) \rightarrow \cdots$$

compatible with equivariant proper pushforward and strict equivariant open pullback.

- (6) Assume that for every $(X, G) \in \mathcal{S}_{\text{eq}/k}$ and every $a \in \mathbb{Z}$, the complex $F(X, G)$ consists of finitely generated Λ -modules. Then the homology groups $H_a^F(X, G)$ are finitely generated Λ -modules for every $(X, G) \in \mathcal{C}_{\text{eq}/k}$.

- (7) Let M be a Λ -module. Assume that for every $(X, G) \in \mathcal{S}_{\text{eq}/k}$, the complex $F(X, G)$ consists of flat Λ -modules. There is a natural spectral sequence

$$E_{a,b}^2 = \text{Tor}_a^R(M, H_b^F(X, G)) \implies H_{a+b}^{F \otimes_\Lambda M}(X, G),$$

where $F \otimes_\Lambda M(X, G) = F(X, G) \otimes_\Lambda M$.

Now we describe how this construction is accomplished. The nonequivariant case is explained in detail by Gillet–Soulé [GS1, §2] and the equivariant case works similarly, which we explain briefly.

We start with a pre-homology functor F satisfying $(\mathbf{D})_F$. For $(X, G) \in \mathcal{C}_{\text{eq}/k}$ choose an equivariant compactification $j : (X, G) \rightarrow (\overline{X}, G)$, where \overline{X} is projective over k and j is a strict open immersion with dense image. It exists by a construction explained in the first part of Section 6. Let $i : (Y, G) \hookrightarrow (\overline{X}, G)$ be the strict closed immersion of the (reduced) complement of X in \overline{X} . By Proposition 2.5 we can choose a commutative diagram in $\mathcal{C}_{G/k}^\Delta$

$$(3.2) \quad \begin{array}{ccc} (Y_\bullet, G) & \xrightarrow{i^\Delta} & (\overline{X}_\bullet, G) \\ \downarrow f & & \downarrow g \\ (Y, G) & \xrightarrow{i} & (\overline{X}, G). \end{array}$$

where $(Y_\bullet, G), (\overline{X}_\bullet, G) \in \mathcal{S}_{G/k}^\Delta$ and f, g are Λ -admissible equivariant hyperenvelopes. Then we set

$$H_a^F(X, G) = H_a(\text{cone}[F(Y_\bullet) \xrightarrow{i_*^\Delta} F(\overline{X}_\bullet)]).$$

It can be shown that the homology groups $H_a^F(X, G)$ are independent of the choices made, i.e., are unique up to canonical isomorphism, and satisfy the axioms of an equivariant homology theory.

We sketch a proof of independence. Let us first fix the compactification $(X, G) \rightarrow (\overline{X}, G)$. Choose a different commutative diagram in $\mathcal{C}_{G/k}^\Delta$

$$(3.3) \quad \begin{array}{ccc} (Y'_\bullet, G) & \xrightarrow{i'^\Delta} & (\overline{X}'_\bullet, G) \\ \downarrow f' & & \downarrow g' \\ (Y, G) & \xrightarrow{i} & (\overline{X}, G). \end{array}$$

where $(Y'_\bullet, G), (\overline{X}'_\bullet, G) \in \mathcal{S}_{G/k}^\Delta$ and f', g' are Λ -admissible equivariant hyperenvelopes. Thanks to Lemma 3.5(2), the method of [GS1, §2.2] produces a canonical isomorphism

$$\text{cone}[F(Y_\bullet) \rightarrow F(\overline{X}_\bullet)] \cong \text{cone}[F(Y'_\bullet) \rightarrow F(\overline{X}'_\bullet)] \text{ in } D_+(\text{Mod}_\Lambda).$$

As for independence from compactifications, consider another compactification $(X, G) \rightarrow (\overline{X}', G)$ with complement (Y', G) . By a standard trick, see [GS1, §2.3], one can assume that there is a commutative diagram

$$(3.4) \quad \begin{array}{ccc} (Y', G) & \longrightarrow & (\overline{X}', G) \\ \pi \downarrow & & \downarrow \pi \\ (Y, G) & \longrightarrow & (\overline{X}, G). \end{array}$$

Choose smooth Λ -admissible hyperenvelopes mapping to (3.4)

$$(3.5) \quad \begin{array}{ccc} (Y'_\bullet, G) & \longrightarrow & (\overline{X}'_\bullet, G) \\ \pi_\bullet \downarrow & & \downarrow \pi_\bullet \\ (Y_\bullet, G) & \longrightarrow & (\overline{X}_\bullet, G), \end{array}$$

where all terms are in $\mathcal{S}_{G/k}^\Delta$. We are ought to show that the induced map

$$\text{cone}[F(Y'_\bullet) \rightarrow F(\overline{X}'_\bullet)] \xrightarrow{\pi_*} \text{cone}[F(Y_\bullet) \rightarrow F(\overline{X}_\bullet)]$$

is a quasi-isomorphism. This follows from the following.

Claim 3.7. *Let Ξ denote the total complex of the diagram in $C_+(\Lambda\mathcal{S}_{G/k})$ obtained by applying the functor in Definition 3.4(2) to the diagram (3.5). Then*

$$F(\Xi) \simeq 0 \quad \text{in } D_+(\text{Mod}_\Lambda).$$

Proof of Claim. For any homology functor Φ satisfying $(\mathbf{D})_\Phi$ we have

$$\begin{aligned} \Phi_{|\mathcal{S}_{\text{eq}/k}}(\Xi) &\simeq \text{Tot} \begin{pmatrix} \Phi(Y'_\bullet) & \rightarrow & \Phi(\overline{X}'_\bullet) \\ \downarrow & & \downarrow \\ \Phi(Y_\bullet) & \rightarrow & \Phi(\overline{X}_\bullet) \end{pmatrix} \simeq \text{Tot} \begin{pmatrix} \Phi(Y') & \rightarrow & \Phi(\overline{X}') \\ \downarrow & & \downarrow \\ \Phi(Y) & \rightarrow & \Phi(\overline{X}) \end{pmatrix} \\ &\simeq 0 \quad \text{in } D_+(\text{Mod}_\Lambda), \end{aligned}$$

where the first quasi-isomorphism is obvious from the definition, the second follows from $(\mathbf{D})_\Phi$, and the last follows from Definition 3.1(ii). By $(\mathbf{D})_F$, this implies the claim. \square

4. Descent criterion for homology functors

In this section and the next we will explain two basic descent theorems which give criteria for a homology functor Φ (resp. a pre-homology functor F) to satisfy the descent condition $(\mathbf{D})_\Phi$ (resp. $(\mathbf{D})_F$) from §3. This will then allow us to apply the methods of Section 3 in order to construct equivariant weight homology theory.

We say that an equivariant homology functor (Definition 3.1)

$$\Phi : \mathcal{C}_{\text{eq}/k} \rightarrow C_+(\text{Mod}_\Lambda)$$

has *quasi-finite flat pullback* if the following conditions hold.

- For a strict quasi-finite flat morphism $f : (X, G) \rightarrow (Y, G)$ there is a functorial pullback $f^* : \Phi(Y, G) \rightarrow \Phi(X, G)$ which coincides with the usual open pullback if f is an open immersion.
- Consider a commutative diagram in $\mathcal{C}_{G/k}$

$$\begin{array}{ccc} (V', G) & \xrightarrow{j'} & (X', G) \\ \downarrow f_V & & \downarrow f \\ (V, G) & \xrightarrow{j} & (X, G), \end{array}$$

such that the diagram of the underlying schemes is cartesian, and f and f_V are proper, and j and j' are quasi-finite flat. Then the following diagram commutes:

$$\begin{array}{ccc} \Phi(X') & \xrightarrow{j^*} & \Phi(V') \\ f_* \downarrow & & \downarrow f_* \\ \Phi(X) & \xrightarrow{j^*} & \Phi(V). \end{array}$$

- For a strict finite flat morphism $f : (X, G) \rightarrow (Y, G)$ of degree d , the composition

$$f_* \circ f^* : \Phi(Y, G) \rightarrow \Phi(Y, G)$$

is multiplication by d .

Our first fundamental descent theorem reads:

Theorem 4.1. *Consider an equivariant homology functor Φ on $\mathcal{C}_{G/k}$, which we assume to have quasi-finite flat pullback if $\Lambda \neq \mathbb{Z}$. Then Φ satisfies $(\mathbf{D})_\Phi$.*

Proof. The proof of Theorem 4.1 is practically the same as that of [GS2, Theorem 3.4 and Theorem 3.9] (see also [GS2, §5.6]). One just has to add a G -action everywhere. The details are left to the readers. \square

5. Descent criterion for pre-homology functors and weight homology

We explain a descent criterion for equivariant pre-homology functors (Definition 3.3) in terms of equivariant Chow motives. We start with explaining the construction of the latter.

By the naive equivariant Chow group of $(X, G) \in \mathcal{C}_{\text{eq}/k}$, one usually means the Chow group of the quotient variety $\text{CH}_a(X/G)$ ($a \geq 0$). The problem with this definition is that there is no intersection product and no flat pullback on naive Chow groups. Edidin, Graham and Totaro ([EG], [To]) suggested a definition of refined Chow groups. Choose a k -linear representation $G \times V \rightarrow V$ such that for all $1 \neq g \in G$ the codimension in V of

$$\text{Fix}(g, V) = \{v \in V \mid g \cdot v = v \text{ and } g|_{k(v)} = \text{id}\}$$

is greater than the dimension of X . Here we view V as an equivariant affine space. Now define

$$\mathrm{CH}_a(X, G) = \mathrm{CH}_{a+d_V}(X \times V/G) \quad (d_V = \dim(V)),$$

where we endow $X \times V$ with the diagonal G -action. In [EG] it is shown that this is independent of choices of V . For (X, G) primitive of dimension d , we write

$$\mathrm{CH}^a(X, G) = \mathrm{CH}_{d-a}(X, G).$$

In what follows we consider covariant Chow motives with coefficients in Λ . We do not care about pseudo-abelian envelopes nor noneffective motives. Fix a finite groups G . The objects of our category of Chow motives, denoted by $\mathrm{Chow}_{G/k}$, are the objects of $\mathcal{S}_{G/k}$. For $(X, G) \in \mathcal{S}_{G/k}$ we write $\mathcal{M}(X, G)$ for the induced object in $\mathrm{Chow}_{G/k}$. For $X, Y \in \mathcal{S}_{G/k}$, we let the morphisms in the category of Chow motives be

$$\mathrm{Hom}_{\mathrm{Chow}_{G/k}}(\mathcal{M}(X, G), \mathcal{M}(Y, G)) = \bigoplus_{i \in I} \mathrm{CH}_{d_i}(X_i \times Y, G) \otimes_{\mathbb{Z}} \Lambda.$$

where $X = \coprod_{i \in I} X_i$ with $(X_i, G) \in \mathcal{S}_{G/k}$ primitive of dimension d_i . Compositions are defined as compositions of correspondences. Clearly \mathcal{M} defines a covariant functor from $\mathcal{S}_{G/k}$ to $\mathrm{Chow}_{G/k}$ by associating to a morphism $f : X \rightarrow Y$ its graph in $X \times Y$. It extends in an obvious way to a Λ -linear functor

$$(5.1) \quad \mathcal{M} : \Lambda\mathcal{S}_{G/k} \rightarrow \mathrm{Chow}_{G/k},$$

where $\Lambda\mathcal{S}_{G/k}$ is defined in Definition 3.4(1). Clearly the functor \mathcal{M} in (5.1) induces a canonical Λ -linear functor

$$(5.2) \quad \mathcal{M} : C_+(\Lambda\mathcal{S}_{G/k}) \rightarrow C_+(\mathrm{Chow}_{G/k})$$

which induces a canonical Λ -linear functor

$$(5.3) \quad \mathcal{M} : K_+(\Lambda\mathcal{S}_{G/k}) \rightarrow K_+(\mathrm{Chow}_{G/k}),$$

where $K_+(\cdot)$ represents the homotopy category of $C_+(\cdot)$ in each case.

We can now state our fundamental descent theorem for pre-homology functors.

Theorem 5.1. *Given a pre-homology functor*

$$F : \mathcal{S}_{\mathrm{eq}/k} \rightarrow C_+(\mathrm{Mod}_\Lambda),$$

assume that for any finite group G , there is an additive Λ -linear functor

$$\tilde{F}_G : \mathrm{Chow}_{G/k} \rightarrow C_+(\mathrm{Mod}_\Lambda)$$

such that the restriction $F|_{\mathcal{S}_{G/k}}$ of F to $\mathcal{S}_{G/k}$ is the composition of \tilde{F}_G with $\mathcal{S}_{G/k} \rightarrow \Lambda\mathcal{S}_{G/k} \xrightarrow{\mathcal{M}} \mathrm{Chow}_{G/k}$. Then F satisfies the descent property $(\mathbf{D})_F$.

Construction 5.2 (Construction of weight homology). Fix a Λ -module M . For $(X, G) \in \mathcal{S}_{\text{eq}/k}$ we set

$$(5.4) \quad F_M^W(X, G) = \text{Hom}_{\mathbb{Z}}(\text{CH}^0(X, G), M)[0] \simeq M^{X^{(0)}/G}[0] \in C_+(\text{Mod}_{\Lambda}),$$

where $X^{(0)}$ is the set of the generic points of X . It is easy to see that F_M^W is a pre-homology functor such that $(F_M^W)|_{\mathcal{S}_{G/k}}$ extends to a Λ -linear functor on $\text{Chow}_{G/k}$. Therefore Theorem 5.1 shows that F_M^W satisfies the descent property **(D)** $_{F_M^W}$.

We now assume that the condition **(★)** of Theorem 1.4 holds. If one assumes further that k is perfect, we can apply Construction 3.6 to F_M^W to get a homology theory on $\mathcal{C}_{\text{eq}/k}$:

$$(X, G) \mapsto \{H_a^W(X, G; M)\}_{a \geq 0}.$$

If k is not perfect field k , we define

$$(5.5) \quad H_*^W(X, G; M) = H_*^W(X \otimes_k \tilde{k}, G; M),$$

where \tilde{k} is the perfection of k . This homology theory is called *weight homology with coefficient M* .

It is clear from Construction 3.6 that Theorem 1.4(i) through (v) hold. To define the map $\tau_{k'/k}$ in (vi), we argue as follows. By (5.5), we may replace k and k' by their perfections to assume that those fields are perfect. Let $\iota : \mathcal{S}_{\text{eq}/k} \rightarrow \mathcal{S}_{\text{eq}/k'}$ be the base change functor $X \rightarrow X \otimes_k k'$. Write F and F' for the pre-homology functors (5.4) on $\mathcal{S}_{\text{eq}/k}$ and $\mathcal{S}_{\text{eq}/k'}$ respectively. Then we have a morphism $F' \circ \iota \rightarrow F$ of pre-homology functors on $\mathcal{S}_{G/k}$, induced by natural maps

$$\text{CH}^0(X, G) \rightarrow \text{CH}^0(X \otimes_k k', G) \quad \text{for } (X, G) \in \mathcal{S}_{\text{eq}/k}.$$

By Construction 3.6(4), this induces a morphism of corresponding homology theories

$$H^{F' \circ \iota} = H^{F'} \circ \iota \rightarrow H^F$$

which gives the desired map $\tau_{k'/k}$. Theorem 1.4(vi)(a) is obvious and (b) is easily shown by using (v).

To prepare the proof of Theorem 5.1 we have to introduce a rigidified version of the equivariant Gersten complex for algebraic K -theory $[Q]$. In fact one could also use the Gersten complex for Milnor K -theory. Fix a sequence of linear G -representations V_j ($j \geq 0$) over k , $d_{V_j} = \dim_k V_j$, with linear surjective equivariant transition maps $V_j \rightarrow V_{j-1}$ such that for any $g \neq 1 \in G$, the codimension of $\text{Fix}(g, V_j)$ in V_j converges to infinity as $j \rightarrow \infty$. For $X \in \mathcal{C}_{\text{eq}/k}$ define

$$R_{q,i}(X, G) = \varinjlim_j \bigoplus_{\substack{x \in (X \times V_j)/G \\ \dim \overline{\{x\}} = q+i+d_{V_j}}} K_i(k(x)) \otimes_{\mathbb{Z}} \Lambda,$$

where the limit is taken over flat pullbacks. For fixed $q \in \mathbb{Z}$ we get a homological complex of Λ -modules:

$$R_q(X, G) : \cdots \xrightarrow{\partial} R_{q,i}(X, G) \xrightarrow{\partial} R_{q,i-1}(X, G) \xrightarrow{\partial} \cdots \xrightarrow{\partial} R_{q,0}(X, G),$$

where $R_{q,i}(X, G)$ sits in degree i and ∂ are boundary maps arising from localization theory for algebraic K -theory [Q]. These complexes form homology functors

$$R_q : \mathcal{C}_{\text{eq}/k} \rightarrow C_+(\text{Mod}_\Lambda)$$

which have quasi-finite flat pullbacks [Q]. Hence we deduce the following proposition from Theorem 4.1.

Proposition 5.3. *For any $q \in \mathbb{Z}$ the descent property $(\mathbf{D})_\Phi$ is satisfied for $\Phi = R_q$.*

Recall that the prehomology functor $(R_q)|_{\mathcal{S}_{\text{eq}/k}}$ defined as the restriction of R_q to $\mathcal{S}_{\text{eq}/k}$ extends to a canonical functor (cf. (3.1))

$$C_+(\Lambda\mathcal{S}_{G/k}) \rightarrow C_+(\text{Mod}_\Lambda).$$

which we denote by R_q for simplicity. With almost verbatim the same proof as for Theorem 1 of [GS1] we obtain the following (cf. (5.3)):

Proposition 5.4. *For $X_* \in K_+(\Lambda\mathcal{S}_{G/k})$ with $R_q(X_*)$ acyclic for all $q \geq 0$ we have*

$$\mathcal{M}(X_*) = 0 \in K_+(\text{Chow}_{G/k}).$$

Proof of Theorem 5.1. Consider $X_* \in C_+(\Lambda\mathcal{S}_{G/k})$ such that $\Phi|_{\mathcal{S}_{\text{eq}/k}}(X_*)$ is acyclic for every homology functor Φ satisfying $(\mathbf{D})_\Phi$. In particular, by Proposition 5.3 $R_q(X_*)$ is acyclic for every $q \in \mathbb{Z}$. By Proposition 5.4 this implies that $\mathcal{M}(X_*) = 0$ in $K_+(\text{Chow}_{G/k})$. Since F factors through $\text{Chow}_{G/k}$ by assumption, we conclude that $F(X_*) = 0$ in $K_+(\text{Mod}_\Lambda)$. \square

6. Equivariant cohomology with compact support

Let the notation be as in §1. For $X \in \mathcal{C}_{G/k}$, we consider G -sheaves \mathcal{F} of torsion Λ -module on $X_{\text{ét}}$, which means that G acts in compatible way with the action of G on X . The G -sheaves of torsion Λ -modules on (X, G) form an abelian category $\text{Shv}_G(X, \Lambda)$. For a morphism $f : X \rightarrow Y$ in $\mathcal{C}_{G/k}$, the direct image functor f_* on sheaves extends to the functor

$$(6.1) \quad f_* : \text{Shv}_G(X, \Lambda) \rightarrow \text{Shv}_G(Y, \Lambda).$$

For a morphism $(\phi, f) : (X, G) \rightarrow (Y, H)$ in $\mathcal{C}_{\text{eq}/k}$, the pullback functor f^* on sheaves extends to the functor

$$(6.2) \quad (\phi, f)^* : \text{Shv}_H(Y) \rightarrow \text{Shv}_G(X, \Lambda).$$

For $\mathcal{F} \in \text{Shv}_G(X, \Lambda)$, we define its equivariant cohomology groups with compact support. First we choose an equivariant compactification

$$j : X \hookrightarrow \overline{X}$$

where j is an open immersion in $\mathcal{C}_{G/k}$ and \overline{X} is proper over k . Such j always exists: Take a (not necessarily equivariant) compactification $j' : X \hookrightarrow Z$ and consider the map

$$\tau : X \hookrightarrow \overbrace{Z \times_k \cdots \times_k Z}^{|G| \text{ times}}$$

induced by $j' \cdot g$ for all $g \in G$. Then one takes \overline{X} to be the closure of the image of τ and j to be the induced immersion. Global sections on \overline{X} form a functor

$$\Gamma(\overline{X}, -) : \mathrm{Shv}_G(\overline{X}) \rightarrow \mathrm{Mod}_{\Lambda[G]} ; \mathcal{F} \rightarrow \Gamma(\overline{X}, \mathcal{F}).$$

with its derived functor

$$R\Gamma(\overline{X}, -) : D^+(\mathrm{Shv}_G(\overline{X})) \rightarrow D^+(\mathrm{Mod}_{\Lambda[G]}).$$

Let

$$\Gamma(G, -) : \mathrm{Mod}_{\Lambda[G]} \rightarrow \mathrm{Mod}_{\Lambda} ; M \rightarrow M^G$$

be the functor of taking the G -invariants of $\Lambda[G]$ -modules and

$$R\Gamma(G, -) : D^+(\mathrm{Mod}_{\Lambda[G]}) \rightarrow D^+(\mathrm{Mod}_{\Lambda})$$

be its derived functor.

We define equivariant cohomology groups by

$$H^n(X, G; \mathcal{F}) = H^n(R\Gamma(G, R\Gamma(X, \mathcal{F}))) \quad \text{for } n \in \mathbb{Z}$$

and we define equivariant cohomology groups with compact support as

$$(6.3) \quad H_c^n(X, G; \mathcal{F}) = H^n(R\Gamma(G, R\Gamma(\overline{X}, j_! \mathcal{F}))) \quad \text{for } n \in \mathbb{Z}.$$

We have a convergent spectral sequence

$$(6.4) \quad E_2^{a,b} = H^a(G, H_c^b(X, \mathcal{F})) \Rightarrow H_c^{a+b}(X, G; \mathcal{F}).$$

Lemma 6.1. *Let $(X, G) \in \mathcal{C}_{\mathrm{eq}/k}$ with G acting trivially on X . Then there is an isomorphism of Λ -modules*

$$H^n(X, G; \mathcal{F}) = H^n(X, B_G(\mathcal{F})).$$

Here B_G denotes the cohomological Bar resolution functor

$$B_G(-) : \mathrm{Shv}_G(X) \rightarrow C^+(\mathrm{Shv}(X)).$$

Let $i : Z \rightarrow X$ be a closed immersion in $\mathcal{C}_{G/k}$ and $j : U \rightarrow X$ be its open complement. For $\mathcal{F} \in \mathrm{Shv}_G(X, \Lambda)$ we have an exact sequence in $\mathrm{Shv}_G(X, \Lambda)$

$$0 \rightarrow j_! j^* \mathcal{F} \rightarrow \mathcal{F} \rightarrow i_* i^* \mathcal{F} \rightarrow 0$$

which induces a long exact sequence

$$(6.5) \quad \cdots \xrightarrow{\delta} H_c^n(U, G; j^* \mathcal{F}) \xrightarrow{j^*} H_c^n(X, G; \mathcal{F}) \xrightarrow{i^*} H_c^n(Z, G; i^* \mathcal{F}) \\ \xrightarrow{\delta} H_c^{n+1}(U, G; j^* \mathcal{F}) \rightarrow \cdots$$

Take $f : (X, G) \rightarrow (Y, H)$ in $\mathcal{C}_{\text{eq}/k}$ and choose a commutative diagram in $\mathcal{C}_{\text{eq}/k}$

$$(6.6) \quad \begin{array}{ccc} (X, G) & \xrightarrow{j_X} & (\overline{X}, G) \\ \downarrow f & & \downarrow \overline{f} \\ (Y, H) & \xrightarrow{j_Y} & (\overline{Y}, H) \end{array}$$

where j_X and j_Y are strict open immersions, and \overline{X} and \overline{Y} are proper over k . Such a choice is always possible by the above construction.

Assume $G = H$, f is strict and the underlying morphism $f : X \rightarrow Y$ is flat and quasi-finite. If p is invertible in Λ we construct the pushforward map

$$(6.7) \quad f_* : H_c^n(X, G; f^* \mathcal{F}) \rightarrow H_c^n(Y, G; \mathcal{F}).$$

By SGA XVIII Theorem 2.9, we have the trace morphism $Rf_! f^* \mathcal{F} \rightarrow \mathcal{F}$. By applying $(j_Y)_!$, it induces a map $R\overline{f}_* j_{X!} f^* \mathcal{F} \rightarrow j_{Y!} \mathcal{F}$, which gives rise to a G -equivariant map

$$R\Gamma(\overline{X}, j_{X!} f^* \mathcal{F}) \rightarrow R\Gamma(\overline{Y}, j_{Y!} \mathcal{F}).$$

This induces the map (6.7) by applying $R\Gamma(G, -)$ and taking cohomology.

Assume the underlying morphism $f : X \rightarrow Y$ is proper. Then we construct the pullback map

$$(6.8) \quad f^* : H_c^n(Y, H; \mathcal{F}) \rightarrow H_c^n(X, G; f^* \mathcal{F}).$$

By the properness of f the map $X \rightarrow Y \times_{\overline{Y}} \overline{X}$ induced from (6.6) is an isomorphism and we have the base change isomorphism $\overline{f}^* j_{Y!} \mathcal{F} \xrightarrow{\cong} j_{X!} f^* \mathcal{F}$ by SGA2 XVII 5.2.1. It gives rise to an equivariant map

$$f^* : R\Gamma(\overline{Y}, j_{Y!} \mathcal{F}) \rightarrow R\Gamma(\overline{X}, j_{X!} f^* \mathcal{F})$$

which induces

$$R\Gamma(H, R\Gamma(\overline{Y}, j_{Y!} \mathcal{F})) \rightarrow R\Gamma(G, R\Gamma(\overline{Y}, j_{Y!} \mathcal{F})) \xrightarrow{f^*} R\Gamma(G, R\Gamma(\overline{X}, j_{X!} f^* \mathcal{F})),$$

where the first map is induced by $G \rightarrow H$. This induces the map (6.8). If f is a strict closed immersion, it is easy to see that f^* coincides with the pullback map in (6.5).

For an extension k'/k of fields and for $(X, G) \in \mathcal{C}_{\text{eq}/k}$, put

$$X' = X \times_{\text{Spec}(k)} \text{Spec}(k') \in \mathcal{C}_{G/k'}$$
 with $f : X' \rightarrow X$ the natural map,

where the action of G on X' is induced from that on X via the base change. We construct the base change map

$$(6.9) \quad \iota_{k'/k} : H_c^n(X, G; \mathcal{F}) \rightarrow H_c^n(X', G; f^* \mathcal{F}).$$

For a G -equivariant compactification $j_X : X \hookrightarrow \overline{X}$, we have a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{j_{X'}} & \overline{X}' \\ \downarrow f & & \downarrow \overline{f} \\ X & \xrightarrow{j_X} & \overline{X} \end{array}$$

where $\overline{X}' = \overline{X} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$ with $\overline{f} : \overline{X}' \rightarrow \overline{X}$ the natural map. It gives rise to the base change isomorphism $\overline{f}^* j_{X'} \mathcal{F} \xrightarrow{\cong} j_{X'}^* f^* \mathcal{F}$, which induces the map (6.9) by the same argument as before.

Lemma 6.2. *Let the notation be as above.*

(1) *Consider a commutative diagram in $\mathcal{C}_{\mathrm{eq}/k}$*

$$\begin{array}{ccccc} (U', G') & \xrightarrow{j'} & (X', G') & \xleftarrow{i'} & (Z', G') \\ \downarrow f_U & & \downarrow f & & \downarrow f_Z \\ (U, G) & \xrightarrow{j} & (X, G) & \xleftarrow{i} & (Z, G) \end{array}$$

such that the squares of the underlying schemes are cartesian, and i (resp. i') is a strict closed immersion and j (resp. j') is its open complement. Take $\mathcal{F} \in \mathrm{Shv}_G(X, \Lambda)$ and put $\mathcal{F}' = f^ \mathcal{F}$.*

If f is proper, the following diagram is commutative.

$$\begin{array}{ccccccc} H_c^n(U, G; j^* \mathcal{F}) & \longrightarrow & H_c^n(X, G; \mathcal{F}) & \longrightarrow & H_c^n(Z, G; i^* \mathcal{F}) & \longrightarrow & H_c^{n+1}(U, G; j^* \mathcal{F}) \\ f^* \downarrow & & f^* \downarrow & & f^* \downarrow & & f^* \downarrow \\ H_c^n(U', G'; j'^* \mathcal{F}') & \longrightarrow & H_c^n(X', G'; \mathcal{F}') & \longrightarrow & H_c^n(Z', G'; i'^* \mathcal{F}') & \longrightarrow & H_c^{n+1}(U', G'; j'^* \mathcal{F}') \end{array}$$

If $G = G'$ and f is strict, flat and quasi-finite and if

$$\mathcal{F} \in \mathrm{Shv}_G(Y, \Lambda)$$

is the limit of the subsheaves annihilated by integers prime to $\mathrm{ch}(k)$, the following diagram is commutative.

$$\begin{array}{ccccccc} H_c^n(U', G; j'^* \mathcal{F}') & \longrightarrow & H_c^n(X', G; \mathcal{F}') & \longrightarrow & H_c^n(Z', G; i'^* \mathcal{F}') & \longrightarrow & H_c^{n+1}(U', G; j'^* \mathcal{F}') \\ f_* \downarrow & & f_* \downarrow & & f_* \downarrow & & f_* \downarrow \\ H_c^n(U, G; j^* \mathcal{F}) & \longrightarrow & H_c^n(X, G; \mathcal{F}) & \longrightarrow & H_c^n(Z, G; i^* \mathcal{F}) & \longrightarrow & H_c^{n+1}(U, G; j^* \mathcal{F}) \end{array}$$

Assuming further that f is finite flat of degree d , the composition $f_ f^*$ is the multiplication by d .*

(2) *Consider morphisms in $\mathcal{C}_{G/k}$*

$$(U, G) \xrightarrow{j} (X, G) \xleftarrow{i} (Z, G)$$

where i is a strict closed immersion and j is its open complement. Take $\mathcal{F} \in \mathrm{Shv}_G(X, \Lambda)$. Let k'/k be an extension of fields and X' ,

U', Z', \mathcal{F}' be the base changes of X, U, Z, \mathcal{F} via k'/k respectively. Then the following diagram is commutative.

$$\begin{array}{ccccccc} H_c^n(U, G; j^* \mathcal{F}) & \longrightarrow & H_c^n(X, G; \mathcal{F}) & \longrightarrow & H_c^n(Z, G; i^* \mathcal{F}) & \longrightarrow & H_c^{n+1}(U, G; j^* \mathcal{F}) \\ \downarrow \iota_{k'/k} & & \downarrow \iota_{k'/k} & & \downarrow \iota_{k'/k} & & \downarrow \iota_{k'/k} \\ H_c^n(U', G; j'^* \mathcal{F}') & \longrightarrow & H_c^n(X', G; \mathcal{F}') & \longrightarrow & H_c^n(Z', G; i'^* \mathcal{F}') & \longrightarrow & H_c^{n+1}(U', G; j'^* \mathcal{F}') \end{array}$$

Proof. The corresponding fact in the nonequivariant case (i.e., the case G is trivial) are well-known and the lemma is proved by the same argument as in that case. \square

For $(X, G) \in \mathcal{C}_{\text{eq}/k}$, X is quasi-projective over k by definition, and by SGA1, V§1, the geometric quotient X/G exists. Let $\pi : X \rightarrow X/G$ be the natural projection, which is viewed as a map $(X, G) \rightarrow (X/G, e)$ in $\mathcal{C}_{\text{eq}/k}$, where e is the trivial group. A sheaf \mathcal{F} of torsion Λ -modules on X/G gives rise to $\pi^* \mathcal{F} \in \text{Shv}_G(X, \Lambda)$ and a map

$$(6.10) \quad \pi^* : H_c^n(X/G, \mathcal{F}) \rightarrow H_c^n(X, G; \pi^* \mathcal{F}).$$

by the formalism of (6.8).

Proposition 6.3. *If G acts freely on X , π^* is an isomorphism for all n .*

Proof. Take an equivariant compactification $j : X \hookrightarrow \bar{X}$ and let

$$\bar{\pi} : \bar{X} \rightarrow \bar{X}/G$$

be the projection to the geometric quotient. Then j induces a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{j} & \bar{X} \\ \downarrow \pi & & \downarrow \bar{\pi} \\ X/G & \xrightarrow{j} & \bar{X}/G \end{array}$$

As in (6.1) we have a canonical pushforward functor

$$\pi_* : \text{Shv}_G(X, \Lambda) \rightarrow \text{Shv}_G(X/G, \Lambda),$$

where we consider X/G with the trivial G -action, and similarly for $\bar{\pi}$. Putting $Y = X/G$ and $\bar{Y} = \bar{X}/G$. we have

$$\begin{aligned} H_c^n(\bar{X}, G; \pi^* \mathcal{F}) &= H^n(\bar{X}, G; j_! \pi^* \mathcal{F}) \\ &= H^n(\bar{Y}, G; \bar{\pi}_* j_! \pi^* \mathcal{F}) \\ &= H^n(\bar{Y}, G; j_! \pi_* \pi^* \mathcal{F}) \\ &= H^n(\bar{Y}, B_G(j_! \pi_* \pi^* \mathcal{F})) \\ &= H^n(\bar{Y}, j_! B_G(\pi_* \pi^* \mathcal{F})) \\ &= H_c^n(\bar{Y}, B_G(\pi_* \pi^* \mathcal{F})) \end{aligned}$$

where the second and third equality hold since π and $\bar{\pi}$ are finite. The fourth equality holds by Lemma 6.1. Finally it is well known that there is a quasi-isomorphism $B_G(\pi_*\pi^*\mathcal{F}) \simeq \mathcal{F}$, since π is an étale covering with Galois group G . Indeed, on stalks $\pi_*\pi^*\mathcal{F}$ is given by G -representations which are induced from the trivial group. This completes the proof of the proposition. \square

A similar isomorphism to that in Proposition 6.3 holds in a radicial situation:

Lemma 6.4.

- (1) Let $f : (X, G) \rightarrow (Y, G)$ be a map in $\mathcal{C}_{G/k}$ where the underlying morphism is finite, surjective and radicial. For $\mathcal{F} \in \text{Shv}_G(Y, \Lambda)$, the pullback map (6.8)

$$f^* : H_c^n(Y, G; \mathcal{F}) \rightarrow H_c^n(X, G; f^*\mathcal{F})$$

is an isomorphism.

- (2) For a purely inseparable extension k'/k of fields and for $(X, G) \in \mathcal{C}_{\text{eq}/k}$, the base change map (6.9)

$$\iota_{k'/k} : H_c^n(X, G; \mathcal{F}) \rightarrow H_c^n(X', G; f^*\mathcal{F})$$

is an isomorphism. Here $X' = X \otimes_k k'$.

Proof. This follows from [SGA4] XVIII 1.2 and XVII 5.2.6. \square

7. Equivariant étale homology

Let the assumption be as in §6. Let G_k be the absolute Galois group of k and M be a discrete $\Lambda[G_k]$ -module which is torsion as a Λ -module and continuous as a G_k module. We let M denote the corresponding étale sheaf on $\text{Spec}(k)$. When M is finite, we write $M^\vee = \text{Hom}_\Lambda(M, \Lambda_\infty)$, the dual $\Lambda[G_k]$ -module, where Λ_∞ be the torsion Λ -module \mathbb{Q}/Λ .

Definition 7.1. The equivariant étale homology groups of $(X, G) \in \mathcal{C}_{\text{eq}/k}$ with coefficient in a finite $\Lambda[G_k]$ -module M are defined as

$$H_a^{\text{ét}}(X, G; M) = H_c^a(X, G; M^\vee)^\vee \quad \text{for } a \geq 0,$$

where M^\vee on the right-hand side denotes the pullback to X of the étale sheaf M^\vee on $\text{Spec}(k)$. For a $\Lambda[G_k]$ -module M which is torsion as a Λ -module, we write

$$M = \varinjlim_n M_n$$

with M_n finite $\Lambda[G_k]$ -modules and set

$$H_a^{\text{ét}}(X, G; M) = \varinjlim_n H_a^{\text{ét}}(X, G; M_n).$$

By Lemma 6.2, the functor

$$\mathcal{C}_{\text{eq}/k} \rightarrow \text{Mod}_\Lambda ; (X, G) \rightarrow H_a^{\text{ét}}(X, G; M) \quad (a \in \mathbb{Z})$$

provides us with a homology theory on $\mathcal{C}_{\text{eq}/k}$ in the sense of Definition 1.1.

For $(X, G) \in \mathcal{C}_{\text{eq}/k}$ and an integer $a \geq 0$, let $X_{(a)}$ denotes the set of such $x \in X$ that $\dim(\overline{\{x\}}) = a$. The group G acts on the set $X_{(a)}$ and we let $X_{(a)}/G$ denotes the quotient set. For $x \in X_{(a)}/G$, let Gx be the corresponding G -orbit in X .

For a homology theory H on $\mathcal{C}_{\text{eq}/k}$ (Definition 1.1), we have a spectral sequence of homological type, called the niveau spectral sequence:

$$(7.1) \quad E_{a,b}^1(X, G; M) = \bigoplus_{x \in X_{(a)}/G} H_{a+b}(x, G; M) \Rightarrow H_{a+b}(X, G; M),$$

where

$$H_a(x, G; M) = \lim_{\substack{\longrightarrow \\ V \subseteq \overline{\{Gx\}}} H_a(V, G; M),$$

with the limit taken over all nonempty G -stable open subschemes V in the closure $\overline{\{Gx\}}$ of Gx in X . This spectral sequence was constructed by Bloch–Ogus [BO] in the nonequivariant case. The same construction works for the equivariant case and one can show the following:

Proposition 7.2. *For the spectral sequence (7.1), the following facts hold.*

- (1) *The spectral sequence is covariant with respect to morphisms in $\mathcal{C}_{\text{eq}/k}$ whose underlying morphism of schemes are proper, and contravariant with respect to strict open immersions in $\mathcal{C}_{\text{eq}/k}$.*
- (2) *Consider a commutative diagram in $\mathcal{C}_{\text{eq}/k}$*

$$\begin{array}{ccc} (V', G') & \xrightarrow{j'} & (X', G') \\ \downarrow f_V & & \downarrow f \\ (V, G) & \xrightarrow{j} & (X, G), \end{array}$$

such that the diagram of the underlying schemes is cartesian, and f and f_V are proper, and j and j' are strict open immersions. Then the following square commutes:

$$\begin{array}{ccc} (E_{a,b}^1(X', G'; M), d_{a,b}^1) & \xrightarrow{(j')^*} & (E_{a,b}^1(V', G'; M), d_{a,b}^1) \\ (f_V)_* \downarrow & & \downarrow f_* \\ (E_{a,b}^1(X, G; M), d_{a,b}^1) & \xrightarrow{j^*} & (E_{a,b}^1(V, G; M), d_{a,b}^1). \end{array}$$

For the homology theory in Definition 7.1, the above spectral sequence for $(X, G) \in \mathcal{C}_{\text{eq}/k}$ is written as

$$(7.2) \quad E_{a,b}^1(X, G; M) = \bigoplus_{x \in X_{(a)}/G} H_{a+b}^{\text{ét}}(x, G; M) \Rightarrow H_{a+b}^{\text{ét}}(X, G; M),$$

Proposition 7.3. *For the spectral sequence (7.2), the following facts hold.*

- (1) *If $\text{ch}(k)$ is invertible in Λ , the spectral sequence is contravariantly functorial for strict flat quasi-finite maps. For a commutative diagram in $\mathcal{C}_{\text{eq}/k}$*

$$\begin{array}{ccc} (V', G') & \xrightarrow{j'} & (X', G') \\ \downarrow f_V & & \downarrow f \\ (V, G) & \xrightarrow{j} & (X, G), \end{array}$$

such that the diagram of the underlying schemes is cartesian, and f and f_V are proper, and j and j' are strict and flat quasi-finite, the following square commutes:

$$\begin{array}{ccc} (E_{a,b}^1(X', G'; M), d_{a,b}^1) & \xrightarrow{(j')^*} & (E_{a,b}^1(V', G'; M), d_{a,b}^1) \\ (f_V)_* \downarrow & & \downarrow f_* \\ (E_{a,b}^1(X, G; M), d_{a,b}^1) & \xrightarrow{j^*} & (E_{a,b}^1(V, G; M), d_{a,b}^1). \end{array}$$

Assuming further that f is finite flat of degree d , the composition $f_ f^*$ is the multiplication by d .*

- (2) *If $f : (X', G) \rightarrow (X, G)$ is a map in $\mathcal{C}_{G/k}$ where the underlying morphism is finite, surjective and radicial, it induces an isomorphism of spectral sequences*

$$(E_{a,b}^1(X', G; M), d_{a,b}^1) \xrightarrow{\cong} (E_{a,b}^1(X, G; M), d_{a,b}^1).$$

- (3) *For a purely inseparable extension k'/k of fields and for $(X, G) \in \mathcal{C}_{\text{eq}/k}$, there is a natural isomorphism of spectral sequences*

$$(E_{a,b}^1(X', G; M), d_{a,b}^1) \xrightarrow{\cong} (E_{a,b}^1(X, G; M), d_{a,b}^1),$$

where $X' = X \times_{\text{Spec}(k)} \text{Spec}(k')$ with the G -action induced from that on X .

- (4) We have $E_{a,b}^1(X, G; M) = 0$ for $b < 0$. The projection $\pi : X \rightarrow X/G$ induces isomorphisms for all $a \in \mathbb{Z}$

$$\begin{aligned} E_{a,0}^1(X, G; M) &\simeq \left(\bigoplus_{x \in X_{(a)}} H_a^{\text{ét}}(x; M) \right)_G \\ &\simeq \bigoplus_{y \in (X/G)_{(a)}} H_a^{\text{ét}}(y; M) = E_{a,0}^1(X/G; M) \end{aligned}$$

and hence an isomorphism of complexes

$$(E_{\bullet,0}^1(X, G; M), d_{\bullet,0}^1) \xrightarrow{\cong} (E_{\bullet,0}^1(X/G; M), d_{\bullet,0}^1).$$

Proof. (1) through (3) follow immediately from Lemma 6.2 and Lemma 6.4. To show (4), we may assume M is finite. In the proof all homology (resp. cohomology) functors have coefficients M (resp. M^\vee). For simplicity we suppress M in the notation. Recall for $x \in X_{(a)}/G$

$$H_{a+b}^{\text{ét}}(x, G) = \lim_{\substack{\longrightarrow \\ U \subseteq \overline{\{Gx\}}} \text{Hom}_\Lambda(H_c^{a+b}(U, G), \Lambda_\infty),$$

where the limit is taken over all nonempty G -stable affine open subschemes U in the closure $\overline{\{Gx\}}$ of Gx in X . By the affine Lefschetz theorem we have $H_c^i(U) = 0$ for all $i < a$. Hence the spectral sequence (6.4) implies

$$(7.3) \quad H_c^i(U, G) = 0 \text{ for } i < a, \quad H_c^a(U, G) \simeq H_c^a(U)^G.$$

This immediately implies part (1) of the proposition and an isomorphism

$$E_{a,0}^1(X, G) = \bigoplus_{x \in X_{(a)}/G} H_a^{\text{ét}}(x, G) \simeq \left(\bigoplus_{x \in X_{(a)}} H_a^{\text{ét}}(x) \right)_G$$

noting that for $x \in X_{(a)}$ we have

$$H_a^{\text{ét}}(x) = \lim_{\substack{\longrightarrow \\ V \subseteq \overline{\{x\}}} \text{Hom}_\Lambda(H_c^a(V), \Lambda_\infty)$$

where the limit is taken over all nonempty affine open subscheme V in the closure $\overline{\{x\}}$ of x in X . To complete the proof, it remains to show

$$(7.4) \quad H_a^{\text{ét}}(x, G) \simeq H_a^{\text{ét}}(y) \quad \text{for } x \in X_{(a)}/G \text{ and } y = \pi(x),$$

noting $X_{(a)}/G \simeq (X/G)_{(a)}$, an isomorphism of sets. Let H be the image of G in $\text{Aut}(Gx)$, the scheme theoretic automorphism group, and put $H_x = \{\sigma \in H \mid \sigma(x) = x\}$. One can choose a H_x -stable affine open subschemes U in $\overline{\{x\}}$ such that H_x acts freely on U and $\sigma(U) \cap \sigma'(U) = \emptyset$ for $\sigma, \sigma' \in H$ with $\sigma^{-1}\sigma' \notin H_x$. Put

$$G \cdot U = \bigcup_{\sigma \in G} \sigma(U) = \coprod_{\sigma \in H/H_x} \sigma(U).$$

By (7.3), (6.4) implies

$$H_c^a(U, G) \simeq \left(\bigoplus_{\sigma \in H/H_x} H_c^a(\sigma(U)) \right)^H = H_c^a(U)^{H_x} \simeq H_c^a(U, H_x),$$

where the last group denotes the equivariant cohomology with compact support for $(U, H_x) \in \mathcal{C}_{\text{eq}/k}$. By Proposition 6.3 we get

$$H_c^a(U, H_x) \simeq H_c^a(U/H_x) = H_c^a(G \cdot U/G),$$

where the last equality holds since $U/H_x = G \cdot U/G$. Thus we get

$$(7.5) \quad H_a^{\text{ét}}(x, G) \simeq \varinjlim_{V \subseteq \overline{\{x\}}} \text{Hom}_\Lambda(H_c^a(G \cdot U/G), \Lambda_\infty).$$

As U runs over all H_x -stable nonempty open subsets of $\overline{\{x\}}$, $G \cdot U/G$ runs over all nonempty open subsets of T/G where $T = \overline{\{Gx\}}$. Consider the finite morphism $\pi_T : T/G \rightarrow S$ where $S \subset X/G$ is the closure of $y = \pi(x)$ in X/G .

Claim 7.4. *Letting $z \in T/G$ be the generic point (note that T/G is integral), $\kappa(z)/\kappa(y)$ is purely inseparable. In particular π_T is an isomorphism on the underlying topological spaces.*

The claim follows from [Bo, Ch.V §2.2 Thm.2(ii)].

Thanks to the above claim, we get from (7.5) and Proposition 6.4

$$H_a^{\text{ét}}(x, G) \simeq \varinjlim_{V \subseteq \overline{\{y\}}} \text{Hom}_\Lambda(H_c^a(V), \Lambda_\infty) = H_a^{\text{ét}}(y),$$

where V ranges over all nonempty open subsets of $\overline{\{y\}}$, which proves the desired claim (7.4). This completes the proof of Proposition 7.3. \square

8. Equivariant Kato homology

Let the assumption be as in §7.

Definition 8.1. For $(X, G) \in \mathcal{C}_{\text{eq}/k}$, we define the equivariant Kato complex $KC(X, G; M)$ of X with coefficient M to be

$$E_{d,0}^1(X, G; M) \cdots \rightarrow E_{a,0}^1(X, G; M) \rightarrow E_{a-1,0}^1(X, G; M) \rightarrow \cdots \rightarrow E_{0,0}^1(X, G; M).$$

Here $d = \dim(X)$, the boundary maps are d^1 -differentials and $E_{a,0}^1(X, G; M)$ is put in degree a .

In case $G = e$, $KC(X, G; M)$ is simply denoted by $KC(X; M)$. This complex has been introduced first in [JS] as a generalization of a seminal work by Kato [K1]. The complex $KC(X; \mathbb{Z}/n\mathbb{Z})$ is isomorphic to the complex (0.1) by duality, see [KeS, Sec. 1].

Lemma 8.2. *The correspondence*

$$\begin{aligned} KC(-, G; M) : \mathcal{C}_{\text{eq}/k} &\rightarrow C_+(\text{Mod}_\Lambda) \\ (X, G) &\mapsto KC(X, G; M) \end{aligned}$$

is a homological functor on $\mathcal{C}_{\text{eq}/k}$ in the sense of Definition 3.1.

Proof. The properties 3.1(i) and (ii) follow from the exact sequence of complexes

$$0 \rightarrow KC(Y, G; M) \xrightarrow{i_*} KC(X, G; M) \xrightarrow{j^*} KC(U, G; M) \rightarrow 0$$

for a closed immersion $i : Y \rightarrow X$ in $\mathcal{C}_{G/k}$ and its complement $j : U = X - Y \rightarrow X$, which is an easy consequence of the fact $X_{(a)} = Y_{(a)} \coprod U_{(a)}$. The functoriality for proper morphisms and strict open immersions follows from Proposition 7.2(1). The property 3.1(iii) follows from Proposition 7.2(2). The property 3.1(iv) is obvious. \square

Proposition 8.3. *For $(G, X) \in \mathcal{C}_{\text{eq}/k}$ with the projection $\pi : X \rightarrow X/G$, the natural map*

$$\pi_* : KC(X, G; M) \rightarrow KC(X/G; M)$$

is an isomorphism of complexes.

Proof. This follows from Proposition 7.3(4). \square

Proposition 8.4.

- (1) *If $f : (X', G) \rightarrow (X, G)$ is a map in $\mathcal{C}_{G/k}$ where the underlying morphism is finite, surjective and radicial, it induces an isomorphism of complexes*

$$f_* : KC(X', G; M) \xrightarrow{\cong} KC(X, G; M).$$

- (2) *For a purely inseparable extension k'/k of fields and for $(X, G) \in \mathcal{C}_{\text{eq}/k}$, there is a natural isomorphism of complexes*

$$KC(X', G; M) \xrightarrow{\cong} KC(X, G; M).$$

where $\overline{X}' = \overline{X} \times_{\text{Spec}(k)} \text{Spec}(k')$ with the G -action induced from that on X .

Proof. This follows from Proposition 7.3(2) and (3). \square

Proposition 8.5. *The homological functor $KC(-, G; M)$ on $\mathcal{C}_{\text{eq}/k}$ satisfies the descent condition $(\mathbf{D})_{KC}$ from Section 3.*

Proof. This follows from Proposition 7.3(1) and Theorem 4.1. \square

Definition 8.6. The equivariant Kato homology with coefficient M is the homology theory on $\mathcal{C}_{\text{eq}/k}$ associated to the homology functor $KC(-, G; M)$ (see Definition 3.2). By definition, for $(X, G) \in \mathcal{C}_{\text{eq}/k}$,

$$KH_a(X, G; M) = H_a(KC(X, G; M)) = E_{a,0}^2(X, G; M) \quad (a \in \mathbb{Z}_{\geq 0}).$$

In case $G = e$, $KH_a(X, G; M)$ is simply denoted by $KH_a(X; M)$.

By Proposition 7.3(1) we have an edge homomorphism for each integer $a \geq 0$:

$$(8.1) \quad \epsilon_X^a : H_a^{\text{ét}}(X, G; M) \rightarrow KH_a(X, G; M)$$

which is viewed as a map of homology theories (see Definition 1.1) on $\mathcal{C}_{\text{eq}/k}$.

Now we assume that G_k acts trivially on M and construct a map of homology theories on $\mathcal{C}_{\text{eq}/k}$

$$\gamma_M : \{KH_a(-; M)\}_{a \in \mathbb{Z}} \rightarrow \{H_a^W(-; M)\}_{a \in \mathbb{Z}}.$$

First note $H_0^{\text{ét}}(\text{Spec}(k); M) \cong M$, and hence

$$KC(\text{Spec}(k); M) = M[0],$$

where $\text{Spec}(k)$ is viewed as an object of $\mathcal{C}_{\text{eq}/k}$ with the trivial action of e . For $(X, G) \in \mathcal{S}_{/k}^{\text{prim}}$, the natural map $\pi : X \rightarrow \text{Spec}(k)$ induces a map of complex

$$\pi_* : KC(X, G; M) \rightarrow KC(\text{Spec}(k); M) = M[0].$$

For $(X, G) \in \mathcal{S}_{\text{eq}/k}$, we get a natural map (see (5.4))

$$(8.2) \quad \gamma_M : KC(X, G; M) \rightarrow M^{X^{(0)}/G}[0] \simeq F_M^W(X, G)$$

by taking the sum of the above maps for the G -orbits of connected components of X . This gives us a map of prehomology theories

$$(8.3) \quad \gamma_M : KC(-; M) \rightarrow F_M^W.$$

Theorem 8.7. *Assume the condition (\star) of Theorem 1.4. There exists a map of homology theories on $\mathcal{C}_{\text{eq}/k}$:*

$$\gamma_M : \{KH_a(-; M)\}_{a \in \mathbb{Z}} \rightarrow \{H_a^W(-; M)\}_{a \in \mathbb{Z}}$$

such that for $(X, G) \in \mathcal{S}_{\text{eq}/k}$, it coincides with the map induced from (8.2).

Proof. If k is perfect, this is an immediate consequence of Proposition 8.5 and Construction 3.6(4) with Lemma 3.5(1). The general case follows from the above case thanks to Theorem 1.4(vi)(a) and Proposition 8.4(2). \square

9. Proof of McKay principle for weight homology

Let the notation be as in §8 and assume that G_k acts trivially on M . Recall that Λ denotes either \mathbb{Z} or $\mathbb{Z}[\frac{1}{p}]$ where p is the exponential characteristic of k . Write $\Lambda_\infty = \mathbb{Q}/\Lambda$ and $\Lambda_n = \Lambda/n\Lambda$ for an integer $n > 0$. Concerning the map of homology theories on $\mathcal{C}_{\text{eq}/k}$:

$$\gamma_M : \{KH_a(-; M)\}_{a \in \mathbb{Z}} \rightarrow \{H_a^W(-; M)\}_{a \in \mathbb{Z}},$$

we have the following result, which is one of the main results of this paper.

Theorem 9.1. *Assume that k is a purely inseparable extension of a finitely generated field. Assume the condition (\star) of Theorem 1.4. Then γ_M is an isomorphism for $M = \Lambda_\infty$. In particular, for $(X, G) \in \mathcal{S}_{\text{eq}/k}$ we have*

$$KH_a(X, G; \Lambda_\infty) = \begin{cases} (\Lambda_\infty)^{X^{(0)}/G} & \text{for } a = 0, \\ 0 & \text{for } a \neq 0, \end{cases}$$

where $X^{(0)}$ is the set of the generic points of X . If k is finite, the same holds by replacing Λ_∞ by Λ_n for any integer $n > 0$.

Theorem 1.7 and Proposition 1.11 will be deduced from from Theorem 9.1. Theorem 9.1 will be deduced from the following result quoted from [JS] Theorem 3.8 and [KeS] Theorem 3.5.

Theorem 9.2. *Assume that k is a purely inseparable extension of a finitely generated field. Assume the condition (\star) of Theorem 1.4. Then, for a smooth projective scheme X over k , we have $KH_a(X, \Lambda_\infty) = 0$ for $a \neq 0$. If k is finite, the same holds by replacing Λ_∞ by Λ_n for any integer $n > 0$.*

Note that Theorem 9.1 and Proposition 8.3 imply the following extension of Theorem 9.2 to a singular case.

Corollary 9.3. *Let the assumption be as in Theorem 9.2. For $(X, G) \in \mathcal{S}_{\text{eq}/k}$, we have $KH_a(X/G, \Lambda_\infty) = 0$ for $a \neq 0$. If k is finite, the same holds by replacing Λ_∞ by Λ_n for any integer $n > 0$.*

For the proof of Theorem 9.1, we prepare some lemmas.

Lemma 9.4. *For $X \in \mathcal{S}_{\text{eq}/k}$, the map (8.2) induces an isomorphism*

$$KH_0(X, G; M) \xrightarrow{\sim} H_0^W(X, G; M).$$

Proof. We may suppose that X is primitive. By Proposition 7.3(4), the edge homomorphism

$$\epsilon_X^0 : H_0^{\text{ét}}(X, G; M) \rightarrow KH_0(X, G; M)$$

is an isomorphism. Thus the assertion follows from the fact that the map

$$H_0^{\text{ét}}(X, G; M) \rightarrow H_0^{\text{ét}}(\text{Spec}(k), M)$$

is an isomorphism, which is easily seen from the Definition 7.1. □

Lemma 9.5. *Fix a finite group G and assume $KH_i(Z, G; M) = 0$ for all $(Z, G) \in \mathcal{S}_{\text{eq}/k}$ and for all $i > 0$. Then*

$$(9.1) \quad \gamma_M : KH_a(X, G; M) \rightarrow H_a^W(X, G; M)$$

is an isomorphism for all $(X, G) \in \mathcal{C}_{G/k}$ and for all $a \in \mathbb{Z}$.

Proof. By Lemma 9.4, the assumption implies that (9.1) is an isomorphism for all $(X, G) \in \mathcal{S}_{G/k}$ and for all $a \in \mathbb{Z}$. Then the lemma follows from Construction 3.6(4). For convenience of the readers, we give a proof. We suppress the coefficient M from the notation. First assume that X is

proper over k . Take a Λ -admissible hyperenvelope $f : (X_\bullet, G) \rightarrow (X, G)$ in $\mathcal{C}_{G/k}$ such that $(X_a, G) \in \mathcal{S}_{G/k}$ for all $a \geq 0$. By Construction 3.6(2) and Theorems 8.5, we have spectral sequences

$$\begin{aligned} {}^K E_{a,b}^1 &= KH_b(X_a, G) \Rightarrow KH_{a+b}(X, G), \\ {}^W E_{a,b}^1 &= H_b^W(X_a, G) \Rightarrow H_{a+b}^W(X, G), \end{aligned}$$

and γ_M induces a map of spectral sequences. By the assumption the map on the E_1 terms are all isomorphism which implies the desired isomorphism.

In case X is not proper, take an open immersion $j : X \hookrightarrow \bar{X}$ in $\mathcal{C}_{G/k}$ with \bar{X} proper over k . Putting $Y = \bar{X} - X$, we have a commutative diagram with exact horizontal sequences

$$\begin{array}{ccccccccc} KH_i(Y, G) & \longrightarrow & KH_i(\bar{X}, G) & \longrightarrow & KH_i(X, G) & \longrightarrow & KH_{i-1}(Y, G) & \longrightarrow & KH_{i-1}(\bar{X}, G) \\ \gamma \downarrow & & \gamma \downarrow & & \gamma \downarrow & & \gamma \downarrow & & \gamma \downarrow \\ H_i^W(Y, G) & \longrightarrow & H_i^W(\bar{X}, G) & \longrightarrow & H_i^W(X, G) & \longrightarrow & H_{i-1}^W(Y, G) & \longrightarrow & H_{i-1}^W(\bar{X}, G). \end{array}$$

Since \bar{X} and Y are proper over k , all vertical maps except the middle are isomorphisms and so is the middle one. This completes the proof. \square

Proof of Theorem 9.1. By Theorem 1.4(vi) and Proposition 8.4(2), we may assume that k is the perfection of a finitely generated field. Let $M = \Lambda_n$ for an integer $n > 0$ if k is finite, and let $M = \Lambda_\infty$ otherwise. For $(X, G) \in \mathcal{C}_{\text{eq}/k}$, let $\pi : (X, G) \rightarrow (X/G, e)$ be the induced map in $\mathcal{C}_{\text{eq}/k}$. Consider the commutative diagram

$$\begin{array}{ccc} KH_a(X, G; M) & \xrightarrow[\pi_*]{\cong} & KH_a(X/G; M) \\ \downarrow \gamma & & \simeq \downarrow \gamma \\ H_a^W(X, G; M) & \xrightarrow[\pi_*]{} & H_a^W(X/G; M). \end{array}$$

The map π_* in the upper row is an isomorphism by Proposition 8.3. The map γ on the right-hand side is an isomorphism by Theorem 9.2 and Lemma 9.5 with G the trivial group. In case $(X, G) \in \mathcal{S}_{\text{eq}/k}$, $H_a^W(X, G; M) = 0$ for $a \neq 0$ by Theorem 1.4(iv). Thus we get $KH_a(X, G; M) = 0$ from the diagram. By Lemma 9.5, this implies that γ_X^a is an isomorphism for all $a \in \mathbb{Z}$ and for all $(X, G) \in \mathcal{C}_{\text{eq}/k}$. This completes the proof of Theorem 9.1. \square

Theorem 9.1 now allows us to deduce Theorem 1.7.

Proof of Theorem 1.7. By Theorem 1.4(vi), we may assume that k is the perfection of a finitely generated field. By the universal coefficient spectral sequence in Theorem 1.4(iv), we may assume $M = \Lambda$. Theorem 1.4(v) shows that $H_a^W(X, G; \Lambda)$ and $H_a^W(X/G; \Lambda)$ are finitely generated Λ -modules. The subcategory $\text{Modf}_\Lambda \subset \text{Mod}_\Lambda$ of finitely generated

Λ -modules is a Serre-subcategory, so we can talk about the quotient category $\text{QMod}_\Lambda = \text{Mod}_\Lambda/\text{Mod}_\Lambda$. The long exact homology sequence for weight homology associated to

$$(9.2) \quad 0 \rightarrow \Lambda \rightarrow \mathbb{Q} \rightarrow \Lambda_\infty \rightarrow 0$$

gives us isomorphisms in QMod_Λ :

$$H_a^W(X, G; \mathbb{Q}) \simeq H_a^W(X, G; \Lambda_\infty) \quad \text{and} \quad H_a^W(X/G; \mathbb{Q}) \simeq H_a^W(X/G; \Lambda_\infty).$$

By Theorem 9.1 and Proposition 8.3 we have isomorphisms in Mod_Λ :

$$H_a^W(X, G; \Lambda_\infty) \xleftarrow{\sim} KH_a(X, G; \Lambda_\infty) \xrightarrow{\sim} KH_a(X/G, \Lambda_\infty) \xleftarrow{\sim} H_a^W(X/G; \Lambda_\infty).$$

Putting this together we get that $\pi_* : (X, G) \rightarrow (X/G, e)$ induces an isomorphism in QMod_Λ :

$$(9.3) \quad H_a^W(X, G; \mathbb{Q}) \simeq H_a^W(X/G; \mathbb{Q}).$$

Lemma 9.6 below implies that (9.3) is a true isomorphism of \mathbb{Q} -modules.

Lemma 9.6. *If a morphism of \mathbb{Q} -modules $\phi : A \rightarrow B$ becomes an isomorphism in the category QMod_Λ then ϕ is already a true isomorphism of \mathbb{Q} -modules.*

We deduce Theorem 1.7 with $M = \Lambda$ using the five-lemma from the commutative diagram with exact rows associated to the sequence (9.2)

$$\begin{array}{ccccccccc} H_{a+1}^W(X, G; \mathbb{Q}) & \rightarrow & H_{a+1}^W(X, G; \Lambda_\infty) & \rightarrow & H_a^W(X, G; \Lambda) & \rightarrow & H_a^W(X, G; \mathbb{Q}) & \rightarrow & H_a^W(X, G; \Lambda_\infty) \\ \downarrow \wr & & \downarrow \wr & & \downarrow & & \downarrow \wr & & \downarrow \wr \\ H_{a+1}^W(X/G; \mathbb{Q}) & \rightarrow & H_{a+1}^W(X/G; \Lambda_\infty) & \rightarrow & H_a^W(X/G; \Lambda) & \rightarrow & H_a^W(X/G; \mathbb{Q}) & \rightarrow & H_a^W(X/G; \Lambda_\infty). \end{array}$$

This completes the proof of Theorem 1.7. □

The following theorem is an analogue of Theorem 1.7 for a radicial morphism.

Theorem 9.7. *Assume $\text{ch}(k) = p > 0$ and that the condition (\star) of Theorem 1.4 holds. Let M be a Λ -module. A finite radicial surjective morphism $\pi : X' \rightarrow X$ in $\mathcal{C}_{/k}$ induces an isomorphism*

$$\pi_* : H_a^W(X'; M) \xrightarrow{\cong} H_a^W(X; M) \quad \text{for } a \in \mathbb{Z}.$$

Proof. By the same argument as the proof of Theorem 1.7, we are reduced to the case $M = \Lambda_\infty = \mathbb{Q}/\Lambda$. Then the theorem follows from Theorem 9.1 and Proposition 8.4. □

Proof of Proposition 1.11. We may assume that the base field k is finitely generated. For $M = \Lambda_\infty$ we have by Theorem 9.1

$$\gamma_{\Lambda_\infty} : KH_a(X, G; \Lambda_\infty) \xrightarrow{\simeq} H_a^W(X, G; \Lambda_\infty).$$

Thus we get $H_a^W(X, G; \Lambda_\infty) = 0$ for $a > \dim(X)$ since $KH_a(X, G; \Lambda_\infty)$ vanishes in degrees $a > \dim(X)$ by definition. By a simple devissage, this implies

(*) $H_a^W(X, G; M) = 0$ for $a > \dim(X)$ and M torsion.

For a finitely generated Λ -module M , we consider the exact sequences

$$0 \rightarrow M/M_{\text{tor}} \rightarrow M \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow M \otimes \mathbb{Q}/\mathbb{Z} \rightarrow 0,$$

$$0 \rightarrow M_{\text{tor}} \rightarrow M \rightarrow M/M_{\text{tor}} \rightarrow 0,$$

where M_{tor} is the torsion part of M . By (*) the first sequence implies

$$H_a^W(X, G; M/M_{\text{tor}}) \simeq H_a^W(X, G; M/M_{\text{tor}}) \otimes_{\mathbb{Z}} \mathbb{Q} \quad \text{for } a > \dim(X).$$

Since $H_a^W(X, G; M/M_{\text{tor}})$ is a finitely generated Λ -module, this implies

$$H_a^W(X, G; M/M_{\text{tor}}) = 0 \quad \text{for } a > \dim(X).$$

By (*) we deduce from the second sequence that $H_a^W(X, G; M) = 0$ for $a > \dim(X)$. This completes the proof of Proposition 1.11. \square

10. Dual complexes and cs-coverings

The dual complex of a simple normal crossing divisor is a special kind of CW -complex as sketched in the introduction (called Δ -complex in [Hat, Section 2.1]). It describes the configuration of the irreducible components of the divisor (see [St], [Pa] and [ABW1]).

Let E be a locally noetherian scheme and $\{E_v\}_{v \in K}$ be the set of irreducible components of E . For a finite subset $\sigma = \{v_0, \dots, v_k\} \subset K$, put

$$E_\sigma := E_{v_0} \cap \dots \cap E_{v_k}.$$

Let $\{E_\alpha\}_{\alpha \in K_\sigma}$ be the set of connected components of E_σ . In case $E_\sigma = \emptyset$, $K_\sigma = \emptyset$ by convention.

To a locally noetherian scheme E , we associate its dual complex $\Gamma = \Gamma(E)$ as follows. Fix a linear ordering of K . The set of vertices of $\Gamma(E)$ is identified with K . For a finite subset $\sigma = \{v_0, \dots, v_k\} \subset K$ with $v_0 < \dots < v_k$, put

$$\Delta_\sigma = \left\{ \sum_{0 \leq i \leq k} t_i v_i \mid t_i \geq 0, \sum_{0 \leq i \leq k} t_i = 1 \right\} \subset \mathbb{R}^K.$$

To each $\alpha \in K_\sigma$, is associated a k -simplex $\Delta_{\sigma, \alpha}$ of $\Gamma(E)$ which is a copy of Δ_σ . The interiors of these simplices are disjoint in $\Gamma(E)$ and we have

$$\Gamma(E) = \coprod_{\sigma \subset K} \coprod_{\alpha \in K_\sigma} \Delta_{\sigma, \alpha} / \sim,$$

where for finite subsets $\tau \subset \sigma \subset K$ and for $\alpha \in K_\sigma$ and $\beta \in K_\tau$ such that $E_\alpha \subset E_\beta$, $\Delta_{\tau, \beta}$ is identified with a face of $\Delta_{\sigma, \alpha}$ by the inclusion $\Delta_\tau \hookrightarrow \Delta_\sigma$ induced by $\tau \hookrightarrow \sigma$.

Clearly, as a topological space $\Gamma(E)$ does not depend on the ordering of K .

Let $f : E \rightarrow E'$ be a morphism of locally noetherian schemes. Let $\{E'_w\}_{w \in K'}$ be the set of irreducible components of E' . Choosing $\phi(v) \in K'$ for each $v \in K$ such that $f(E_v) \subset E'_{\phi(v)}$, we get a continuous map

$$\phi : K \rightarrow K'.$$

For each $\alpha \in K_\sigma$, there is a unique $\phi(\alpha) \in K'_{\phi(\sigma)}$ such that $f(E_\alpha) \subset E'_{\phi(\alpha)}$. We define an affine map

$$\begin{aligned} \phi : \Delta_{\sigma, \alpha} &\rightarrow \Delta_{\phi(\sigma), \phi(\alpha)} \\ \sum_{0 \leq i \leq k} t_i v_i &\mapsto \sum_{0 \leq i \leq k} t_i \phi(v_i). \end{aligned}$$

(Here $t_i \geq 0$ and $\sum t_i = 1$.) These maps glue to induce a map of Δ -complexes

$$\phi : \Gamma(E) \rightarrow \Gamma(E').$$

Lemma 10.1. *Up to homotopy, ϕ does not depend on the choice of $\phi : K \rightarrow K'$.*

Proof. Pick $v_0 \in K$ and let $\phi(v_0) = w_0 \in K'$. Assume $f(E_{v_0}) \subset E'_{w'_0}$ for $w'_0 \in K'$ with $w'_0 \neq w_0$. Let $\psi : \Gamma(E) \rightarrow \Gamma(E')$ be the map induced by the map $\psi : K \rightarrow K'$ defined as $\psi(v) = \phi(v)$ if $v \neq v_0$ and $\psi(v_0) = w'_0$. It suffices to show that ψ is homotopic to ϕ . Let $\sigma = \{v_0, \dots, v_k\} \subset K$ and $\alpha \in K_\sigma$. The assumption implies

$$f(E_\alpha) \subset f(E_\sigma) \subset E'_\theta = E'_{w_0} \cap E'_{w'_0} \cap E_{\phi(v_1)} \cap \dots \cap E_{\phi(v_k)},$$

where

$$\theta = \phi(\sigma) \cup \psi(\sigma) = \{w_0, w'_0, \phi(v_1), \dots, \phi(v_k)\} \subset K'.$$

There is a unique connected component E'_γ of E'_θ such that $f(E_\alpha) \subset E'_\gamma$. We have $E'_\gamma \subset E'_{\phi(\alpha)} \cap E'_{\psi(\alpha)}$ and hence $\Delta_{\phi(\sigma), \phi(\alpha)}$ and $\Delta_{\psi(\sigma), \psi(\alpha)}$ are faces of $\Delta_{\theta, \gamma}$. We define a map

$$\begin{aligned} H_{\sigma, \alpha} : [0, 1] \times \Delta_{\sigma, \alpha} &\rightarrow \Delta_{\theta, \gamma} \\ \left(r, \sum_{0 \leq i \leq k} t_i v_i \right) &\mapsto t_0(rw_0 + (1-r)w'_0) + \sum_{1 \leq i \leq k} t_i \phi(v_i). \end{aligned}$$

which gives a homotopy from $\phi|_{\Delta_{\sigma, \alpha}}$ to $\psi|_{\Delta_{\sigma, \alpha}}$. The maps $H_{\sigma, \alpha}$ for all σ and α glue to give a desired homotopy $H : [0, 1] \times \Gamma(E) \rightarrow \Gamma(E')$. \square

Lemma 10.2. *Let E be a G -strict simple normal crossing divisor over a field as in Definition 1.5. There is a canonical homeomorphism*

$$\Gamma(E/G) \xrightarrow{\cong} \Gamma(E)/G,$$

of topological spaces.

Proof. The proof is straightforward and left to the readers. \square

To investigate fundamental groups of dual complexes, we need a more geometric interpretation. In the following we introduce the technique of covering spaces of non-path-connected topological spaces, which we apply to the Zariski topological space underlying a locally noetherian scheme.

Definition 10.3. For a connected topological space S let $\text{Cov}(S)$ be the category of topological coverings of S . For a point $s_0 \in S$, let $\pi_1^{cs}(S, s_0)$ be the automorphism group of the fiber functor

$$\text{Cov}(S) \rightarrow \text{Sets}; S' \mapsto S' \times_S s_0.$$

In what follows we omit the base point s_0 from the notation. We call $\pi_1^{cs}(S)$ the completely split fundamental group.

If S is a connected scheme we write $\pi_1^{cs}(S)$ for the cs-fundamental group of the underlying Zariski topological space of S .

Lemma 10.4. *If the connected topological space S admits a covering by open subsets $\mathcal{U} = (U_i)_{i \in I}$ such that $\text{Cov}(U_i)$ consists only of the trivial coverings for all $i \in I$, then there is a universal connected covering $S' \rightarrow S$. We have $\pi_1^{cs}(S) = \text{Aut}(S'/S)$.*

Proof. Fix a linear ordering of I . One associates to the open covering \mathcal{U} a Δ -complex $\Gamma(\mathcal{U})$ in analogy to the construction above. Its k -simplices are the connected components of $U_{v_0} \cap \cdots \cap U_{v_k}$ for $v_0 < \cdots < v_k \in I$. There is a canonical equivalence $\text{Cov}(S) \cong \text{Cov}(\Gamma(\mathcal{U}))$. In fact objects of both sides are described in terms of the following data: A family of sets $(W_i)_{i \in I}$ and a family of isomorphisms $w_\alpha : W_i \xrightarrow{\cong} W_j$ for each connected component α of $U_i \cap U_j$ ($i < j$) which satisfy cocycle conditions (one for each connected component of $U_i \cap U_j \cap U_k$ with $i < j < k$). Since $\Gamma(\mathcal{U})$ is a CW -complex, it is locally path-connected and locally simply connected and therefore by [Hat, Thm. 1.38] it has a universal covering space. \square

The assumption of Lemma 10.4 is satisfied in particular for the topological space underlying a locally noetherian scheme.

Proposition 10.5. *Let X, Y be connected locally noetherian schemes and let $f : X \rightarrow Y$ be a proper surjective morphism with connected fibers. Then the induced map*

$$f_* : \pi_1^{cs}(X) \rightarrow \pi_1^{cs}(Y)$$

is surjective.

Proof. Let $Y' \rightarrow Y$ be the universal connected covering. Clearly Y' has a canonical scheme structure. We have to show that $X' := X \times_Y Y'$ is connected. In fact the fibers of $X' \rightarrow Y'$ are connected and the map is closed and surjective. Therefore different connected components of X' map onto different connected components of Y' , but there is just one of the latter. \square

Proposition 10.6. *For a connected locally noetherian scheme E , there is a natural isomorphism*

$$\gamma_E : \pi_1^{cs}(E) \xrightarrow{\cong} \pi_1(\Gamma(E)).$$

Proof. An irreducible scheme has only trivial cs-topological coverings. This implies that there is a canonical equivalence $\text{Cov}(E) \cong \text{Cov}(\Gamma(E))$ by an argument similar to the proof of Lemma 10.4. \square

Corollary 10.7. *Let E be a G -strict simple normal crossing divisor as in Definition 1.5. Then we have a natural isomorphism*

$$\pi_1^{cs}(E/G) \xrightarrow{\cong} \pi_1(\Gamma(E)/G).$$

Proof. One combines the isomorphism $\gamma_{E/G}$ of Proposition 10.6 with the isomorphism of Lemma 10.2. \square

11. McKay principle for homotopy type of dual complexes

Let the notation be as §1. We assume $\text{ch}(k) = 0$ or canonical resolution of singularities in the sense of [BM] holds over k . Let $(X, G) \in \mathcal{C}_{G/k}$ with X smooth and $\pi : X \rightarrow X/G$ be the projection. Fix a closed reduced subscheme $S \subset X/G$ which is projective over k and contains the singular locus $(X/G)_{\text{sing}}$ of X/G . Let $T = \pi^{-1}(S)_{\text{red}}$ be the reduced part of $\pi^{-1}(S)$. Assume that we are given the following data:

- a proper birational morphism $g : \tilde{Y} \rightarrow X/G$ such that \tilde{Y} is smooth, $E_S = g^{-1}(S)_{\text{red}}$ is a simple normal crossing divisor on \tilde{Y} and g is an isomorphism over $X/G - S$;
- a proper birational G -equivariant morphism $f : \tilde{X} \rightarrow X$ in $\mathcal{C}_{G/k}$ such that \tilde{X} is smooth, $E_T = f^{-1}(T)_{\text{red}}$ is a G -strict simple normal crossing divisor on \tilde{X} (cf. Definition 1.5) and f is an isomorphism over $X - T$.

$$\begin{array}{ccccccc} \pi^{-1}(S)_{\text{red}} = T & \longrightarrow & X & \xleftarrow{f} & \tilde{X} & \longleftarrow & E_T = f^{-1}(T)_{\text{red}} \\ & & \downarrow & & \downarrow \pi & & \\ S & \longrightarrow & X/G & \xleftarrow{g} & \tilde{Y} & \longleftarrow & E_S = g^{-1}(S)_{\text{red}} \end{array}$$

Note that we do not assume that there exists a morphism $E_T \rightarrow E_S$. By definition G act on $\Gamma(E_T)$ and we can form the topological space $\Gamma(E_T)/G$.

Theorem 11.1. *In the homotopy category of CW-complexes, there exists a canonical map*

$$\phi : \Gamma(E_T)/G \rightarrow \Gamma(E_S)$$

which induces isomorphisms on the homology and fundamental groups:

$$\begin{aligned} H_a(\Gamma(E_T)/G) &\xrightarrow{\cong} H_a(\Gamma(E_S)) \quad \text{for } \forall a \in \mathbb{Z}, \\ \pi_1(\Gamma(E_T)/G) &\xrightarrow{\cong} \pi_1(\Gamma(E_S)). \end{aligned}$$

If $\Gamma(E_T)/G$ is simply connected, then ϕ is a homotopy equivalence.

The following corollaries are immediate consequences of Theorem 11.1.

Corollary 11.2. *If $\Gamma(E_T)/G$ is contractible, then $\Gamma(E_S)$ is contractible.*

Corollary 11.3. *If T is smooth (e.g., $\dim(T) = 0$, meaning $(X/G)_{\text{sing}}$ is isolated), then $\Gamma(E_S)$ is contractible.*

Indeed, if T is smooth, one can take \tilde{X} to be the blowup of X along T and $\Gamma(E_T)/G$ consists only of 0-simplices.

Corollary 11.4. *Let A be a complete regular local ring containing \mathbb{Q} and let G be a finite group acting on A . Set $X = \text{Spec}(A)$ and assume that X/G has an isolated singularity $s \in X/G$. Let $g : \tilde{Y} \rightarrow X/G$ be a proper morphism such that g is an isomorphism outside s and $E_s = g^{-1}(s)_{\text{red}}$ is a simple normal crossing divisor in the regular scheme \tilde{Y} . Then the topological space $\Gamma(E_s)$ is contractible.*

Proof. Note that $Y = X/G = \text{Spec}(A^G)$ is automatically noetherian, $X \rightarrow Y$ is finite and Y is complete, because we are in characteristic zero [Mu, Sec. 1.2]. One observes that the morphism $X \rightarrow Y$ is flat and locally complete intersection outside s , see [Ma, Thm. 46, Thm. 36(4)].

By algebraization of isolated singularities [Ar, Thm. 3.8] the complete local scheme Y results from the completion at an isolated singularity $s' \in Y'$ of a scheme Y' of finite type over the residue field k of A^G . By [El, Thm. 4] the morphism $X \rightarrow Y$ algebraizes to a morphism $X' \rightarrow Y'$ together with the group action G on X'/Y' , after possibly replacing Y' by an étale neighborhood of s' . Again after shrinking Y' around s' and using Artin approximation [Ar] one finds a proper morphism $\tilde{Y}' \rightarrow Y'$ such that the base change $\tilde{Y}' \times_{Y'} Y$ coincides with $\tilde{Y} \rightarrow Y$ to some arbitrary high order with respect to powers of the maximal ideal of A^G . By choosing the order large one can make sure that $\tilde{Y}' \rightarrow Y'$ is an isomorphism outside s' , that Y' is regular and that the reduced preimage $E_{s'}$ of s' in \tilde{Y}' is a simple normal crossing divisor.

It follows from Corollary 11.3 that $\Gamma(E_{s'}) = \Gamma(E_s)$ is contractible. \square

Proof of Theorem 11.1. The last assertion follows from the first using the relative version of the Hurewicz theorem (cf. [Hat, 4.33]). Choose a G -equivariant proper birational morphism $h : \tilde{X}' \rightarrow \tilde{X}$ such that:

- \tilde{X}' is smooth, $E'_S = h^{-1}(E_T)_{\text{red}}$ is a G -strict simple normal crossing divisor on \tilde{X}' and h is an isomorphism over $\tilde{X} - E_T$.
- There exists a G -equivariant proper morphism $\tilde{X}' \rightarrow \tilde{Y}$.

Thus we get a commutative diagram

$$\begin{array}{ccccc}
 T & \longleftarrow & E_T & \longleftarrow & E'_T \\
 \downarrow & & \downarrow & & \downarrow \\
 X & \longleftarrow & \tilde{X} & \xleftarrow{h} & \tilde{X}' \\
 \downarrow \pi & & & \swarrow & \\
 X/G & \longleftarrow & \tilde{Y} & &
 \end{array}$$

and a G -equivariant map $\rho : E'_T \rightarrow E_T$.

Claim 11.5. ρ induces a G -equivariant homotopy equivalence

$$\Gamma(E'_T) \underset{h.e.}{\simeq} \Gamma(E_T)$$

and hence a homotopy equivalence $\Gamma(E'_T)/G \underset{h.e.}{\simeq} \Gamma(E_T)/G$.

The claim is a consequence of the following general fact:

Theorem 11.6. *Let G be a group. Let $\phi : X' \rightarrow X$ be a G -equivariant proper morphism of smooth projective G -schemes over k such that ϕ is an isomorphism over a dense open subset $U \subset X$ and such that $E = X - U$ (resp. $E' = X' - \phi^{-1}(U)$) is a G -strict simple normal crossing divisor on X (resp. X'). Then the induced map $\Gamma(E') \rightarrow \Gamma(E)$ is a G -equivariant homotopy equivalence.*

The case $G = e$ is Stepanov's theorem [St] and its generalizations [Pa, Theorem 1.1] and [ABW1, Theorem 7.5]. The G -equivariant version is proved by the exactly same argument by replacing the weak factorization theorem of Wlodarczyk [Wl] with its equivariant version [AKMW, Theorem 0.3.1 and Remark (2)]. \square

By Claim 11.5 we may assume that there exist a morphism $\phi : \tilde{X}/G \rightarrow \tilde{Y}$. Take a proper birational morphism $h : \tilde{Y}' \rightarrow \tilde{X}/G$ such that \tilde{Y}' is smooth, $E'_S = h^{-1}(E_T/G)_{\text{red}}$ is a simple normal crossing divisor on \tilde{Y}' and h is an isomorphism on $\tilde{Y}' - E'_S$. Thus we get a commutative diagram:

$$\begin{array}{ccccccccc}
 \pi^{-1}(S) = T & & \longleftarrow & & E_T & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & X & \xleftarrow{f} & \tilde{X} & & & & \\
 (11.1) & & \downarrow \pi & & & \searrow \phi & & & \\
 & & X/G & \xleftarrow{g} & \tilde{Y} & \xleftarrow{\phi} & \tilde{X}/G & \xleftarrow{h} & \tilde{Y}' \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & S & \longleftarrow & E_S & \longleftarrow & E_T/G & \longleftarrow & E'_S.
 \end{array}$$

From the diagram we get maps of CW -complexes:

$$\Gamma(E'_S) \xrightarrow{h_*} \Gamma(E_T)/G \xrightarrow{\phi_*} \Gamma(E_S).$$

The composite of these maps is a homotopy equivalence by Theorem 11.6. Hence the assertion on the fundamental groups follows if one shows the surjectivity of the induced maps

$$\pi_1(\Gamma(E'_S)) \xrightarrow{h_*} \pi_1(\Gamma(E_T)/G) \xrightarrow{\phi_*} \pi_1(\Gamma(E_S)).$$

Noting $\tilde{Y}' \rightarrow \tilde{X}/G \rightarrow \tilde{Y}$ have geometrically connected fibers by Zariski's main theorem, the assertion follows from Proposition 10.6, Corollary 10.7 and Proposition 10.5.

It remains to show the assertion on the homology groups. Recall that we are given a commutative diagram

$$(11.2) \quad \begin{array}{ccccccc} \pi^{-1}(S)_{\text{red}} = T & \longrightarrow & X & \xleftarrow{f} & \tilde{X} & \longleftarrow & E_T = f^{-1}(T)_{\text{red}} \\ & & \downarrow & & \downarrow \pi & & \downarrow \tilde{\pi} \\ S & \longrightarrow & X/G & \xleftarrow{g} & \tilde{Y} & \longleftarrow & E_S = g^{-1}(S)_{\text{red}} \end{array}$$

where the assumption is the same as in the beginning of this section. Note that we are now given $\tilde{\pi} : \tilde{X} \rightarrow \tilde{Y}$ which extends $\pi : X \rightarrow X/G$. It induces a map $\phi : E_T/G \rightarrow E_S$ and we are ought to show that it induces isomorphisms on the homology groups

$$H_a(\Gamma(E_T)/G) \xrightarrow{\cong} H_a(\Gamma(E_S)) \quad (a \in \mathbb{Z}).$$

Since S (and hence T) is assumed projective over k , we may choose X, \tilde{X} and \tilde{Y} projective over k . By Example 1.6, the homology groups of the dual complexes are described as the weight homology groups:

$$H_a(\Gamma(E_T)/G, M) \simeq H_a^W(E_T, G; M), \quad H_a(\Gamma(E_S), M) \simeq H_a^W(E_S; M),$$

where M is any coefficient module. Thus the desired assertion follows from the following claim which holds over a field k of arbitrary characteristic.

Claim 11.7. *Let the assumption be as (11.2) and assume that X, \tilde{X} and \tilde{Y} are projective over k . Assume the condition (\star) of Theorem 1.4. For a Λ -module M , the map $(E_T, G) \rightarrow (E_S, e)$ in $\mathcal{C}_{\text{eq}/k}$ induces an isomorphism*

$$H_a^W(E_T, G; M) \simeq H_a^W(E_S; M).$$

Proof. We have a commutative diagram with exact rows

$$\begin{array}{ccccccc} H_{a+1}^W(\tilde{X}, G; M) & \longrightarrow & H_{a+1}^W(\tilde{X} - E_T, G; M) & \longrightarrow & H_a^W(E_T, G; M) & \longrightarrow & H_a^W(\tilde{X}, G; M) \\ \downarrow & & \downarrow \simeq & & \downarrow & & \downarrow \\ H_{a+1}^W(\tilde{Y}; M) & \longrightarrow & H_{a+1}^W(\tilde{Y} - E_S; M) & \longrightarrow & H_a^W(E_S; M) & \longrightarrow & H_a^W(\tilde{Y}; M). \end{array}$$

The second vertical isomorphism follows from Theorem 1.7 noting

$$(\tilde{X} - E_T)/G \cong \tilde{Y} - E_S.$$

By Theorem 1.4(i) this implies the desired isomorphism of the claim. \square

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