

Curved A_∞ algebras and Landau–Ginzburg models

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ABSTRACT. We study the Hochschild (co)homology of curved A_∞ algebras that arise in the study of Landau–Ginzburg (LG) models in physics. We show that the ordinary Hochschild homology and cohomology of these algebras vanish. To correct this we introduce modified versions of these theories, Borel–Moore Hochschild homology and compactly supported Hochschild cohomology. For LG models the new invariants yield the answer predicted by physics, shifts of the Jacobian ring.

We also study the relationship between graded LG models and the geometry of hypersurfaces. We prove that Orlov’s derived equivalence descends from an equivalence at the differential graded level, so in particular the CY/LG correspondence is a dg equivalence. This leads us to study the equivariant Hochschild homology of orbifold LG models. The results we get can be seen as noncommutative analogues of the Lefschetz hyperplane and Griffiths transversality theorems.

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1. Introduction

1.1. The main goal of this paper is to study the Hochschild homology and cohomology of categories of matrix factorizations. This is achieved by exploiting the relationship that exists between matrix factorizations and

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curved associative algebras. As we shall see, the usual Hochschild invariants are trivial for curved algebras. To correct this we introduce versions of these invariants that are better behaved in this context.

1.2. Categories of matrix factorizations have appeared in several areas of mathematics and physics. They were first introduced by Eisenbud in the study of hypersurface singularities [6], and further studied by Buchweitz as the stable category of Cohen–Macaulay B/W -modules [2]. Two decades later Kontsevich argued that matrix factorizations are the correct mathematical concept to formalize the Landau–Ginzburg (LG) models from physics of string theory. Subsequently Orlov [14] argued that there is a strong relationship between a graded version of the category of matrix factorizations and the geometry of projective hypersurfaces.

1.3. The main construction in these theories associates to the data of an associative algebra B and an element W (the superpotential) in the center of B a $\mathbb{Z}/2\mathbb{Z}$ - or \mathbb{Z} -graded differential graded (dg) category $\mathrm{DG}(B, W)$ called the category of matrix factorizations. Its homotopy category $\mathrm{D}^b(B, W)$ is called the derived category of matrix factorizations of the pair (B, W) .

There is a natural way to associate to the data (B, W) a $\mathbb{Z}/2\mathbb{Z}$ -graded curved A_∞ algebra B_W . As a super-vector space it consists of B , concentrated purely in the even part. The only nonzero operations are m_0 , determined by the equality $m_0(1) = W$, and m_2 , given by the original multiplication of B . We will call such an algebra a curved associative algebra.

The relationship between the algebra B_W and the category of matrix factorizations is well understood: the category $\mathrm{Tw}(B_W)$ of twisted complexes over B_W (the natural generalization to the A_∞ setting of the dg category of complexes of projective modules over an associative algebra) is precisely $\mathrm{DG}(B, W)$.

From a geometric point of view the A_∞ algebra B_W can be thought of as the algebra of “functions” on a hypothetical noncommutative space X , with $\mathrm{Tw}(B_W)$ playing the role of the dg category of complexes of coherent sheaves on X .

1.4. We are primarily interested in understanding the Hochschild homology and cohomology of the dg category of matrix factorizations

$$\mathrm{DG}(B, W) = \mathrm{Tw}(B_W).$$

When dealing with the category of complexes of projective modules over a ring A (or, more generally, over a noncurved A_∞ algebra) the situation is greatly simplified by the existence of the identities

$$\begin{aligned} HH^*(\mathrm{Tw}(A)) &= HH^*(A), \\ HH_*(\mathrm{Tw}(A)) &= HH_*(A). \end{aligned}$$

Indeed, it is often much easier to compute the invariants of A than computing those of $\mathrm{Tw}(A)$.

However, as we shall soon see, the above isomorphisms fail for curved algebras like B_W . In fact the first result we prove in this paper is the following theorem.

1.5. Theorem. *For any curved associative algebra B_W we have*

$$HH^*(B_W) = HH_*(B_W) = 0.$$

1.6. This result should be regarded in a negative way, in the sense that ordinary Hochschild homology and cohomology are the wrong invariants to use for curved algebras. To obtain interesting invariants, we introduce in Section 3 modified versions of these invariants. Based on a close analogy with algebraic topology we call these invariants Borel–Moore Hochschild homology, HH_*^{BM} , and compactly supported Hochschild cohomology, HH_c^* . They are closely related to the ordinary Hochschild invariants, but with the twist that a Borel–Moore chain is an element of the *direct product* of homogeneous Hochschild chains, as opposed to a direct sum in the usual definition. Similarly, a compactly-supported Hochschild cochain is a direct sum of homogeneous cochains, as opposed to a direct product. Collectively we refer to $HH_*^{\text{BM}}(B_W)$ and $HH_c^*(B_W)$ as the compact type invariants of the curved algebra B_W .

1.7. The main advantage of the new invariants lies in the fact that an argument of Segal [15] shows that the compact type invariants of any algebra A , curved or not, agree with the compact type invariants of $\text{Tw}(A)$. As before this greatly simplifies the computation of the compact type invariants for categories of the type $\text{Tw}(A)$. In particular for the category of matrix factorizations $\text{DG}(B_W) = \text{Tw}(B_W)$ we get the following theorem.

1.8. Theorem. *Let $Y = \text{Spec } B$ be a smooth affine scheme of dimension n and assume that $W \in B$ is a regular function on Y with isolated critical points. Then we have*

$$\begin{aligned} HH_c^*(\text{DG}(B_W)) &= HH_c^*(B_W) = \text{Jac}(W)[0], \\ HH_*^{\text{BM}}(\text{DG}(B_W)) &= HH_*^{\text{BM}}(B_W) = \omega(W)[n], \end{aligned}$$

where $\text{Jac}(W)$ denotes the Jacobi ring of W , $\omega(W)$ denotes the canonical module for $\text{Jac}(W)$, and for a vector space M the notation $M[i]$ represents the $\mathbb{Z}/2\mathbb{Z}$ graded vector space which is M in degree $i \bmod 2$ and 0 otherwise.

1.9. Remark. This result agrees precisely with computations of the closed string sector of Landau–Ginzburg models in physics.

1.10. There are two drawbacks to the new invariants. First, as in the topological situation, the compact type Hochschild homology and cohomology are not homotopy invariant. The more serious issue is that in general we do not understand the relationship between the compact type invariants and the traditional ones. However, it turns out that for many interesting A_∞

algebras or categories the compact type invariants agree with the usual ones. We call such algebras or categories of *compact type*.

The easiest examples of compact type algebras are given by the following theorem.

1.11. Theorem. *Let A be a finite-dimensional, \mathbb{Z} -graded A_∞ algebra supported in nonpositive degrees. Then A is of compact type. In other words we have*

$$HH^*(A) \cong HH_c^*(A), \quad HH_*(A) \cong HH_*^{\text{BM}}(A).$$

1.12. As we have seen from the explicit calculations in Theorems 1.5 and 1.8, B_W is not of compact type. On the other hand, for a complete regular local ring B , Dyckerhoff [5] computes directly the ordinary Hochschild (co)homology of the category $\text{DG}(B, W)$, and his answers agree with our computations of compact type invariants for the curved algebras B_W . Putting these results together gives the following corollary.

1.13. Corollary. *The dg category $\text{DG}(B, W)$ associated to a commutative, complete regular local ring B and to a nonunit $W \in B$ with isolated critical points is of compact type. We have*

$$HH_*(\text{DG}(B_W)) \cong HH_*^{\text{BM}}(B_W) \cong HH_*^{\text{BM}}(\text{DG}(B_W)).$$

1.14. The above fact is based on calculations that are quite indirect. A direct proof of Corollary 1.13 using curved Koszul duality will appear in a separate paper [18] of the second author, thus giving an alternative approach to the calculations of Dyckerhoff.

1.15. The graded case. The categories of matrix factorizations described above share many characteristics with *affine* geometry. The natural generalization to the projective setting is realized by the so-called categories of *graded* matrix factorizations. To be precise, assume that the ring B that appears in the definition of matrix factorizations is \mathbb{Z} -graded, and that the superpotential W is homogeneous. Then we shall say that we are dealing with a graded LG-model, and there is an appropriate modification $\text{DG}_{\mathbb{Z}}(B, W)$ of the definition of $\text{DG}(B, W)$ which takes the grading into account. The resulting category is called the dg category of graded matrix factorizations, with associated homotopy category $\text{D}_{\mathbb{Z}}^b(B, W)$. It can be viewed as a particular case of the graded category of singularities introduced by Orlov [14].

The main result of [loc.cit.] is the following theorem. We state it using the original notation, which will be reviewed in Section 5. The main objects it relates are two triangulated categories associated to a Gorenstein graded ring A , $\text{D}_{\text{sg}}^{\text{gr}}(A)$ and $\text{D}^b(\text{qgr-}A)$. Their relationship to $\text{D}_{\mathbb{Z}}^b(B, W)$ and $\text{D}^b(\text{Proj } B/W)$ will be discussed in (1.17) below.

1.16. Theorem (Orlov [14]). *Let A be a graded Gorenstein algebra with Gorenstein parameter a . Then for any integer $i \in \mathbb{Z}$ we have:*

(A) *If $a > 0$ there exists a semi-orthogonal decomposition*

$$D^b(\mathbf{qgr}\text{-}A) = \langle A(-i - a + 1), \dots, A(-i), D_{\text{sg}}^{\text{gr}}(A) \rangle.$$

(B) *If $a < 0$ there exists a semi-orthogonal decomposition*

$$D_{\text{sg}}^{\text{gr}}(A) = \langle k(-i), \dots, k(-i + a + 1), D^b(\mathbf{qgr}\text{-}A) \rangle.$$

(C) *If $a = 0$ there is an equivalence*

$$D_{\text{sg}}^{\text{gr}}(A) \cong D^b(\mathbf{qgr}\text{-}A).$$

1.17. To connect the above theorem to our context, assume that B is a regular graded ring, W is a homogeneous element of B , and denote by A the graded Gorenstein ring B/W . Orlov’s construction yields a graded category of singularities $D_{\text{sg}}^{\text{gr}}(A)$, and in [14] Orlov shows that this triangulated category agrees with the derived category of graded matrix factorizations $D_{\mathbb{Z}}^b(B_W)$.

Moreover, if A is commutative (so that we can talk about $\text{Proj } A$), the category $D^b(\mathbf{qgr}\text{-}A)$ agrees with the derived category of coherent sheaves on $\text{Proj } A$. Hence the above theorem ultimately relates the derived category of graded matrix factorizations $D_{\mathbb{Z}}^b(B, W)$ with the derived category of coherent sheaves on the hypersurface $\text{Proj}(B/W)$.

1.18. Since we are interested in Hochschild invariants, and these do not behave well with respect to derived equivalence, we need to extend Orlov’s result (1.16) to the dg setting. The result we get is the following theorem. (The notion of semi-orthogonal decomposition for dg categories is defined in Appendix A.)

1.19. Theorem. *Let A be a Gorenstein algebra with Gorenstein parameter a . Then for any integer $i \in \mathbb{Z}$ we have:*

(A) *If $a > 0$ there exists a semi-orthogonal decomposition*

$$D_i = \langle \pi A(-i - a + 1), \dots, \pi A(-i), T_i \rangle.$$

(B) *If $a < 0$ there exists a semi-orthogonal decomposition*

$$T_i = \langle qk(-i), \dots, qk(-i + a + 1), D_i \rangle.$$

(C) *If $a = 0$ there is an equivalence*

$$T_i \cong D_i.$$

Here D_i and T_i are dg enhancements for $D^b(\mathbf{qgr}\text{-}A)$ and $D_{\text{sg}}^{\text{gr}}(A)$ respectively, and π and q are analogues of the functors with the same name introduced by Orlov.

1.20. The main idea of the proof is to use the notion of dg quotient (due to Drinfel'd [4]) and its universal properties. (After we proved our theorem, we became aware of the main result of Lunts and Orlov's paper [12] which provides an alternative, more general way of proving the same theorem.)

1.21. The next step is to understand graded matrix factorizations from the point of view of noncommutative geometry. Our goal is to get a description of $\mathrm{DG}_{\mathbb{Z}}(B, W)$ similar to the equivalence

$$\mathrm{DG}(B, W) \cong \mathrm{Tw}(B_W)$$

we had in the ungraded case. The idea is to exploit the observation from physics [19] that graded LG-models can be realized by an orbifold construction from ungraded ones. From a mathematical standpoint this statement is best expressed as the following result.

1.22. Theorem. *Let (B, W) be a \mathbb{Z} -graded Landau–Ginzburg model with $\deg W = d$. Then there exists a \mathbb{Z} -graded, curved A_{∞} category $B_W \sharp \mathbb{Z}_d$ such that the category $\mathrm{Tw}(B_W \sharp \mathbb{Z}_d)$ of associated twisted complexes is dg equivalent to $\mathrm{DG}_{\mathbb{Z}}(B, W)$.*

1.23. We are mainly interested in understanding the Hochschild homology of the category $\mathrm{DG}_{\mathbb{Z}}(B, W)$, and as in the ungraded case we will do this by understanding instead the Borel–Moore homology of $B_W \sharp \mathbb{Z}_d$. This is justified by a graded analogue of Corollary 1.13, which will also appear in the forthcoming paper [18]. The corresponding result states that in the situation we are interested in (hypersurfaces in projective space) there is an isomorphism of graded vector spaces

$$HH_*(\mathrm{DG}_{\mathbb{Z}}(B, W)) \cong HH_*^{\mathrm{BM}}(B_W \sharp \mathbb{Z}_d).$$

1.24. We state the result of our calculation of $HH_*^{\mathrm{BM}}(B_W \sharp \mathbb{Z}_d)$ in a more general form as a localization formula. The notations in the following theorem are explained in Section 6.

1.25. Theorem. *Let G be a finite group acting on a smooth affine scheme $Y = \mathrm{Spec} B$, and let W be a G -invariant global function on Y . Denote by $[(Y, W)/G]$ the dg category of G -equivariant matrix factorizations of (B, W) . Then we have*

$$HH_*^{\mathrm{BM}}([(Y, W)/G]) \cong \left(\bigoplus_{g \in G} HH_*^{\mathrm{BM}}(Y^g, W|_{Y^g}) \right)_G$$

where for $g \in G$, Y^g is the g -invariant subspace of Y and the subscript G denotes taking coinvariants of the induced G action.

1.26. Combining the above calculation with the dg equivalences of Theorem 1.19 gives a new proof of Griffiths’ theorem on the cohomology of a hypersurface $X = \text{Proj } B/W$ of projective space $\text{Proj } B$. Indeed, since Hochschild homology is invariant under dg equivalences, we have

$$HH_*(X) \cong HH_*(\text{DG}_{\mathbb{Z}}(B, W)) \cong HH_*^{\text{BM}}(B_W \sharp \mathbb{Z}_d),$$

and these latter groups are computed by Theorem 1.25. On the other hand, the Hochschild homology groups of X are traditionally computed using the well-known Hochschild–Kostant–Rosenberg isomorphism [17], and the answer depends directly on the computation of the Hodge numbers $h^{p,q}(X)$. These are classically computed using the Lefschetz hyperplane and Griffiths transversality theorems.

We interpret the conclusion of the previous paragraph as giving an interpretation of Theorems 1.25 and 1.19 as noncommutative analogues of the Lefschetz hyperplane and Griffiths transversality theorems from classical complex geometry.

1.27. The paper is organized as follows. In Section 2, for the purpose of fixing notation, we recall the basics of A_∞ algebras along with details of the twist construction and its relationship to categories of matrix factorizations. We also prove Theorem 1.22. The usual definitions of Hochschild homology and cohomology are presented in Section 3 in a way which extends to the curved case as well. We then prove Lemma 3.9 which states that any A_∞ algebra with only m_0 term has vanishing Hochschild (co)homology. This is the basic ingredient in the proof of Theorem 1.5.

In Section 4 we specialize to the case of the curved associative algebras that appear in the study of affine LG models. We explicitly calculate both the usual Hochschild (co)homology groups and their compact type variations. The usual ones are shown to vanish and the modified ones yield the answers predicted by physics.

Section 5 is devoted to extending Orlov’s results on derived categories of graded singularities to the dg setting. We begin by reviewing Orlov’s theorem that relates the derived category of singularities to the derived category of coherent sheaves on the corresponding projective variety. We also relate the dg versions of the category of graded singularities and the category of graded matrix factorizations.

Motivated by Theorem 1.22 we calculate in Section 6 the Borel–Moore Hochschild homology for LG orbifolds of the form $B_W \sharp G$. We prove a localization formula for homology in this general situation. Unlike the rest of the paper where homology and cohomology are on equal footing, our localization result only applies to homology. We leave the calculation of compactly supported Hochschild cohomology of orbifolds for future work.

For the reader’s convenience we collect in Appendix A certain results on dg quotients of dg categories that we use in the course of this work. We

also define the notion of semi-orthogonal decomposition of dg categories and prove Lemma A.12 that is used in the proofs in Section 5.

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2. A-infinity categories and the twist construction

In this section we recall the notion of A_∞ structure and explain the twist construction of A_∞ algebras (or, more generally, A_∞ categories) following the exposition of Seidel [16]. Using the twist construction it is easy to see that the category of matrix factorizations can be identified with the twist of a curved algebra. The situation for the graded case is more complicated and we explain how to obtain the category of graded matrix factorizations by an orbifold construction. The conclusion is that only the finite group $(\mathbb{Z}/d\mathbb{Z})$ is necessary, as opposed to \mathbb{C}^* as one might have expected for a Z -grading.

We will work over a ground field \mathbf{k} and we make once and for all the choice to work with either \mathbb{Z} - or $\mathbb{Z}/2\mathbb{Z}$ -graded vector spaces. All tensor products are understood to be in the symmetric monoidal category of these graded vector spaces, with the Koszul convention for signs. If V and W are graded vector spaces, $\mathrm{Hom}(V, W)$ is the graded vector space of all linear maps $V \rightarrow W$, not necessarily homogeneous. A map $f \in \mathrm{Hom}(V, W)$ is said to have degree i if it is homogeneous of degree i , i.e., it is a homogeneous degree zero map $V \rightarrow W[i]$.

2.1. Let A be a vector space. Form the tensor coalgebra

$$B(A) := T^c(A[1]) = \bigoplus_{i=0}^{\infty} A[1]^{\otimes i}$$

endowed with the coproduct

$$\Delta(a_1 | \cdots | a_i) = \sum_{j=0}^i (a_1 | \cdots | a_j) \otimes (a_{j+1} | \cdots | a_i),$$

where if the index j is 0 or i we use the unit of the ground field as the empty tensor product.

2.2. Definition. An A_∞ algebra structure on A consists of a degree one coderivation $m : B(A) \rightarrow B(A)$ of $B(A)$ such that $m \circ m = 0$.

2.3. As the coalgebra $B(A)$ is cogenerated by $A[1]$ giving a linear map from $B(A)$ to A is equivalent to giving a coderivation of $B(A)$. In other words $C^*(A)$ can be identified with the space of coderivations on $B(A)$.

Let m be an A_∞ algebra structure on A . Under the above identification, it corresponds to a homogeneous degree one cochain in A which we shall denote by m as well. The component of m of tensor degree i is denoted by m_i . It can be regarded as a map

$$m_i : A^{\otimes i} \rightarrow A[2 - i].$$

We remind the reader the standard terminology used in the theory of A_∞ algebras. The term m_0 is usually assumed to be zero in most definitions of A_∞ algebras. Algebras for which $m_0 = 0$ will be called flat, while those for which m_0 does not vanish will be said to be curved. We emphasize that most of the existing homological algebra constructions for flat A_∞ algebras do not generalize in an obvious way to the curved case.

For flat algebras the degree one map m_1 is called the differential as $m_1^2 = 0$ in this case. The map m_2 is usually called the product of the algebra and the terms m_i for $i \geq 3$ are referred to as higher multiplications.

2.4. The equation $m \circ m = 0$ translates into an infinite system of quadratic relations between the m_i 's that are sometimes taken as the definition of A_∞ algebras. The first few relations are

$$\begin{aligned} m_1 \circ m_0(1) &= 0, \\ m_1 \circ m_1(-) \pm m_2(m_0(1), -) \pm m_2(-, m_0(1)) &= 0, \\ m_1 \circ m_2(-, -) \pm m_2(m_1(-), -) \pm m_2(-, m_1(-)) \\ &\pm m_3(m_0(1), -, -) \pm m_3(-, m_0(1), -) \pm m_3(-, -, m_0(1)) = 0, \\ &\vdots \end{aligned}$$

where the signs are determined by the Koszul sign convention.

2.5. For the purposes of this paper we shall be primarily interested in curved associative algebras or categories. These are special cases of A_∞ algebras where the only nonzero multiplications are m_0 and m_2 . To keep with Orlov's notation denote the vector space of such an algebra by B . The A_∞ relations require m_2 to define an ordinary associative algebra structure on B and the image of m_0 is a one dimensional subspace of B generated by an element W in the center of B . Conversely, the data of an associative algebra B and an element W in its center determine a curved algebra which will be denote by B_W . (Usually we shall assume that the entire algebra is concentrated in the even part of the $\mathbb{Z}/2\mathbb{Z}$ -graded vector space B .)

2.6. Let \mathcal{C} be an A_∞ category, i.e., a category with higher composition operations such that the A_∞ identities hold whenever the compositions make sense. One can think of A_∞ algebras as A_∞ categories with one object.

The twist construction of an A_∞ category \mathcal{C} produces another A_∞ category $\text{Tw}(\mathcal{C})$ which generalizes the notion of the category of perfect complexes associated to an ordinary algebra. We recall the construction as explained in [16].

2.7. By definition the morphism sets of \mathcal{C} are graded and this grading is central to the twist construction. We shall denote by H the grading group, which can be $\mathbb{Z}/2\mathbb{Z}$ or \mathbb{Z} . The construction of $\text{Tw}(\mathcal{C})$ proceeds in two steps.

Step 1. Form an additive enlargement $\Sigma\mathcal{C}$ of \mathcal{C} , which adds direct sums and grading shifts of objects of \mathcal{C} .

Objects of $\Sigma\mathcal{C}$ are formal finite sums of the form

$$X = \bigoplus_{f \in F} X_f[\sigma_f]$$

for a finite index set $F \subset \mathbb{Z}$ and objects X_f of \mathcal{C} , $\sigma_f \in H$. The morphisms in $\Sigma\mathcal{C}$ are defined by

$$\mathrm{Hom}_{\Sigma\mathcal{C}}(\bigoplus_{f \in F} X_f[\sigma_f], \bigoplus_{g \in G} Y_g[\tau_g]) = \bigoplus_{f, g} \mathrm{Hom}_{\mathcal{C}}(X_f, Y_g)[\tau_g - \sigma_f]$$

where $[-]$ denotes shifting the H -grading. Composition of maps in $\Sigma\mathcal{C}$ is defined using matrix multiplication with signs.

Step 2. Form the category $\mathrm{Tw}_H(\mathcal{C})$.

The objects in $\mathrm{Tw}_H(\mathcal{C})$ are twisted complexes, which are defined to be pairs (X, δ) consisting of an object $X \in \mathrm{Ob}(\Sigma\mathcal{C})$ and a morphism $\delta \in \mathrm{Hom}(X, X)$ of degree one such that the generalized Maurer–Cartan equation holds:

$$\bigoplus_{i \geq 0} m_i(\delta, \dots, \delta) = 0.$$

The morphisms in $\mathrm{Tw}_H(\mathcal{C})$ are the same as in $\Sigma\mathcal{C}$, but compositions involve insertions of δ .

2.8. Remark. In the second step there is usually an upper triangular property for δ which ensures that the Maurer–Cartan equation makes sense (finiteness of summation). We will not worry about this issue because the A_∞ categories we will be considering have finitely many nonzero m_k 's. Secondly, one usually takes the idempotent completion of our definition of $\mathrm{Tw}_H(\mathcal{C})$ to have better homological properties. For the applications in this paper, this completion is not important.

2.9. As an easy example we consider the twist construction applied to a curved associative algebra B_W viewed as a curved A_∞ category with one object, graded by $H = \mathbb{Z}/2\mathbb{Z}$ with all of B concentrated in the even part.

First recall the classical definition of $\mathrm{DG}(B_W)$, the dg category of matrix factorizations of W . Objects of this category are pairs (E, Q) where E is a free, $\mathbb{Z}/2\mathbb{Z}$ -graded B -module of finite rank, and Q is an odd B -linear map on E such that $Q^2 = W \cdot \mathrm{id}$. The set of morphisms between two objects (E, Q) and (F, P) is simply $\mathrm{Hom}_B(E, F)$, the space of all B -linear maps.

There is a differential d on $\mathrm{Hom}_B(E, F)$ which makes $\mathrm{DG}(B_W)$ into a differential graded category. It is defined by the formula

$$d(\varphi) := P \circ \varphi - (-1)^{|\varphi|} \varphi \circ Q.$$

The identity $d^2 = 0$ follows from the fact that W is in the center of B .

2.10. We can identify $\mathrm{DG}(B_W)$ with $\mathrm{Tw}_{\mathbb{Z}/2\mathbb{Z}}(B_W)$, as follows. The objects of the additive enlargement ΣB_W are $\mathbb{Z}/2\mathbb{Z}$ -graded free B -modules of finite rank. The twisted complexes add the odd map δ , and the Maurer–Cartan equation reduces to

$$W \cdot \mathrm{id} + \delta \circ \delta = 0.$$

As the degree of δ is odd, the above equation defines a matrix factorizations of W (up to a sign). It is easy to check that composition of morphisms is the same as in $\mathrm{DG}(B_W)$, so we conclude that

$$\mathrm{Tw}_{\mathbb{Z}/2\mathbb{Z}}(B_W) \cong \mathrm{DG}(B_W).$$

2.11. Of interest to us is also the category of graded matrix factorizations $\mathrm{DG}_{\mathbb{Z}}(B_W)$ introduced by Orlov [14]. This category is a \mathbb{Z} -graded dg category associated to a graded algebra B and a homogeneous potential W . An object of $\mathrm{DG}_{\mathbb{Z}}(B_W)$ consists of two graded free B -module E_0, E_1 along with homogeneous maps

$$\begin{aligned} E_0 &\xrightarrow{P_0} E_1 \text{ and} \\ E_1 &\xrightarrow{P_1} E_0 \end{aligned}$$

of degrees 0 and d , respectively, satisfying the matrix factorization identity

$$P_1 \circ P_0 = W \cdot \mathrm{id}, \quad P_0 \circ P_1 = W \cdot \mathrm{id}.$$

It is convenient to denote such data by (E, P) with $E = E_0 \oplus E_1$ and P the odd map which satisfies the matrix factorization identity $P^2 = W \cdot \mathrm{id}$. This notation is thus the same as that in the un-graded case, but we should keep in mind the degree requirements for E and P .

The morphism space in the graded case takes into account the \mathbb{Z} -grading of B . Explicitly the morphism space between two objects (E, P) and (F, Q) is

$$\bigoplus_k \mathrm{Hom}_{\mathrm{gr}\text{-}B}(E, F(kd))$$

where the subscript $\mathrm{gr}\text{-}B$ means that we only take homogeneous B -linear maps of degree zero. This space can be characterized as the G -invariant subspace of the space of all B -linear maps $\mathrm{Hom}_B(E, F)$. The differential on the Hom space is defined by the same commutator formula as in the un-graded case.

Our next goal is to realize $\mathrm{DG}_{\mathbb{Z}}(B_W)$ as the twist construction of a curved category.

2.12. Theorem. *Let B be a \mathbb{Z} -graded algebra, and let W be a homogeneous element in the center of B . Denote the degree of W by d . Then there exists a \mathbb{Z} -graded, curved A_∞ category $B_W \sharp \mathbb{Z}/d\mathbb{Z}$ with finitely many objects such that the twisted category $\mathrm{Tw}_{\mathbb{Z}}(B_W \sharp \mathbb{Z}/d\mathbb{Z})$ is isomorphic to the category of graded matrix factorizations $\mathrm{DG}_{\mathbb{Z}}(B_W)$.*

2.13. If G is a group acting on an associative algebra B then we can form the cross product algebra $B\sharp G$. As a B -module it is the same as $B \otimes_{\mathbb{Z}} \mathbb{Z}G$, and we shall denote an element $b \otimes g$ by $b\sharp g$. The multiplication in $B\sharp G$ is given by the formula

$$(a\sharp g)(b\sharp h) = ab^g\sharp gh,$$

where b^g is the result of the action of g on b .

If W is a central element of B which is invariant under the action of G then $W\sharp 1$ is central in $B\sharp G$, so we can form the curved algebra $(B\sharp G)_{W\sharp 1}$. For simplicity we shall denote this algebra by $B_W\sharp G$.

The category of twisted modules over $B_W\sharp G$ can be identified with the category of G -equivariant twisted modules over B_W , so we can think of the resulting theory as an orbifolding of the original Landau–Ginzburg model B_W .

2.14. We now return to the context of Theorem 2.12. As the vector space B is \mathbb{Z} -graded it carries a natural \mathbb{C}^* -action. The cyclic subgroup $\mathbb{Z}/d\mathbb{Z} = \{\bar{i} \mid 0 \leq i \leq d-1\}$ embeds into \mathbb{C}^* by $\bar{i} \mapsto \zeta^i$ where $\zeta = e^{\frac{2\pi\sqrt{-1}}{d}}$. Thus $\mathbb{Z}/d\mathbb{Z}$ also acts on B by

$$\bar{i} \rightarrow (f \mapsto \zeta^{i|f|} f).$$

In the following arguments we shall denote the group $\mathbb{Z}/d\mathbb{Z}$ by G .

Using the construction of (2.13) we produce a $\mathbb{Z}/2\mathbb{Z}$ -graded curved algebra $B_W\sharp G$. For the proof of Theorem 2.12 we shall construct a \mathbb{Z} -graded curved category whose total space of morphisms is $B_W\sharp G$. The new category shall also be denoted by $B_W\sharp G$, where no risk of confusion exists.

2.15. The idea is to consider $B\sharp G$ as a category with d objects instead of just an algebra. The objects of this category are the characters of G (one dimensional representations of G) and the morphisms between two objects χ_i and χ_j are given by the invariant part of the G action on B twisted by χ_i and χ_j . Explicitly, in our case, we denote the objects of this category by

$$\left(\frac{0}{d}\right), \left(\frac{1}{d}\right), \dots, \left(\frac{d-1}{d}\right).$$

The morphisms between two objects $\left(\frac{i}{d}\right)$ and $\left(\frac{j}{d}\right)$ consist of elements of B of degree $(j-i) \bmod d$. Composition of morphisms is multiplication in B .

2.16. This category has a \mathbb{Z} -grading induced from that of B . However, with this grading we can not add the curvature element W , since it is of degree d and not 2 as needed for a \mathbb{Z} -graded A_∞ category.

To fix this we define a new \mathbb{Z} -grading on the category $B\sharp G$. The new grading of an element $f \in \text{Hom}\left(\left(\frac{i}{d}\right), \left(\frac{j}{d}\right)\right)$ is given by

$$\hat{f} := \frac{2(|f| - j + i)}{d}.$$

The result is an integer since we required that $|f| = (j - i) \pmod d$. In fact, \hat{f} is always in $2\mathbb{Z}$ for any morphism f .

The curvature element W has degree two in the new grading, and hence we can define a \mathbb{Z} -graded A_∞ category structure $B_W\sharp G$ on $B\sharp G$ by adding the curvature term $m_0(1) = W$ in the endomorphism space of every object in $B\sharp G$. This is the desired category for Theorem 2.12.

2.17. Since the category $\mathcal{C} = B_W\sharp G$ is \mathbb{Z} -graded, we can perform the \mathbb{Z} -graded twist construction to it. Our goal is to show that $\text{Tw}_{\mathbb{Z}}(\mathcal{C})$ can be identified with the category of graded matrix factorizations $\text{DG}_{\mathbb{Z}}(B_W)$.

2.18. To identify $\text{DG}_{\mathbb{Z}}(B_W)$ with $\text{Tw}_{\mathbb{Z}}(B_W\sharp G)$ we first identify the objects. Recall that objects of $\text{Tw}_{\mathbb{Z}}(B_W\sharp G)$ are pairs of the form

$$\left(\bigoplus_i \frac{a_i}{d} [k_i], \delta \right)$$

with $a_i, k_i \in \mathbb{Z}$, and δ an endomorphism of hat-degree one satisfying the Maurer–Cartan equation. Define a map on the level of objects by the formula

$$\left(\bigoplus_i \frac{a_i}{d} [k_i], \delta \right) \mapsto (E_0 \oplus E_1, P),$$

where

$$\begin{aligned} E_0 &:= \bigoplus_i B \left(\frac{k_i d}{2} + a_i \right), \\ E_1 &:= \bigoplus_i B \left(\frac{(k_i + 1)d}{2} + a_i \right), \text{ and} \\ P &:= \text{the corresponding matrix defined by } \delta. \end{aligned}$$

Here the first summation is over those indices i such that the corresponding k_i is an even integer, while the second one is over those for which k_i is odd.

2.19. The matrix factorization identity is precisely the Maurer–Cartan equation. It remains to check that each component of P has the correct degree. Components of P arise from components of δ , which are morphisms in $B_W\sharp G$ of hat-degree 1. In other words such components are maps in $B_W\sharp G$ of the form

$$\left(\frac{a}{d} [k] \right) \rightarrow \left(\frac{b}{d} [l] \right),$$

i.e., elements $x \in B$ whose degree $|x|$ satisfy

$$l - k + 1 = \frac{2(|x| - b + a)}{d}.$$

Note that since the right hand side is necessarily even, l and k are of different parity, and hence the only nonzero components of the map P will be from

E_0 to E_1 and vice-versa (no self-maps). Moreover, the above equality can be rewritten as

$$\begin{aligned} |x| &= \left(\frac{(l+1)d}{2} + j \right) - \left(\frac{kd}{2} + i \right) \\ &= \left(\frac{ld}{2} + j \right) - \left(\frac{(k+1)d}{2} + i \right) + d. \end{aligned}$$

The first equality can be applied to the case when k is even and l is odd, and it implies that the degree of the corresponding component of the resulting map $P_0 : E_0 \rightarrow E_1$ is zero. The second equality similarly shows that if k is odd and l is even, x corresponds to an entry of degree d of the resulting homogeneous matrix P_1 from E_1 to E_0 .

2.20. To check that the morphism sets are the same between the two categories $\mathrm{Tw}_{\mathbb{Z}}(B_W \sharp G)$ and $\mathrm{DG}_{\mathbb{Z}}(B_W)$ we need to check that every component of a morphism in $\mathrm{Tw}_{\mathbb{Z}}(B_W \sharp G)$ corresponds to an element of B of (ordinary) degree divisible by d . Consider such a component, which is a morphism

$$\varphi \in \mathrm{Hom}_{\mathrm{Tw}(B_W \sharp G)} \left(\frac{i}{d}[k], \frac{j}{d}[l] \right) = \mathrm{Hom}_{B_W \sharp G} \left(\frac{i}{d}, \frac{j}{d} \right) [l - k].$$

We can express its ordinary degree as an element in B using the hat degree, the natural degree of $B_W \sharp G$:

$$\hat{\varphi} + l - k = \frac{2(|\varphi| - j + i)}{d}.$$

Solving for $|\varphi|$ yields

$$\begin{aligned} |\varphi| &= \frac{(\hat{\varphi} + l - k)d}{2} + j - i \\ &= \left(\frac{(l+1)d}{2} + j \right) - \left(\frac{kd}{2} + i \right) + \frac{\hat{\varphi} - 1}{2}d && \text{or} \\ &= \left(\frac{ld}{2} + j \right) - \left(\frac{(k+1)d}{2} + i \right) + \frac{\hat{\varphi} + 1}{2}d && \text{or} \\ &= \left(\frac{ld}{2} + j \right) - \left(\frac{kd}{2} + i \right) + \frac{\hat{\varphi}}{2}d && \text{or} \\ &= \left(\frac{(l+1)d}{2} + j \right) - \left(\frac{(k+1)d}{2} + i \right) + \frac{\hat{\varphi}}{2}d. \end{aligned}$$

On the other hand φ corresponds to a morphism x between shifts of B , as explained in (2.18). Combining these facts we see that the degree $|x|$ of x as a graded map in $\mathrm{DG}_{\mathbb{Z}}(B_W)$ is given by

$$\frac{\hat{\varphi} - 1}{2}d, \frac{\hat{\varphi} + 1}{2}d, \text{ or } \frac{\hat{\varphi}}{2}d,$$

according to the parity of k and l .

Observe also that the fractions $\frac{\hat{\varphi}-1}{2}$, $\frac{\hat{\varphi}+1}{2}$ and $\frac{\hat{\varphi}}{2}$ which multiply d are in fact all integers. The reason is that the hat grading is in $2\mathbb{Z}$ for even shifts,

and in $2\mathbb{Z} + 1$ after odd shifts. So the components of every morphism in $\text{Tw}_{\mathbb{Z}}(B_W \sharp G)$ are all elements of B of degree divisible by d , which is precisely the same as in $\text{DG}_{\mathbb{Z}}(B_W)$.

Thus we have identified the morphism spaces between the two categories $\text{DG}_{\mathbb{Z}}(B_W)$ and $\text{Tw}_{\mathbb{Z}}(B_W \sharp G)$. It is easy to see that composition in both categories is given by matrix multiplications and the differential on Hom spaces is given by commutators. This concludes the proof of our identification of the two categories. □

3. Hochschild-type invariants of curved A -infinity algebras

In this section we review the definitions of Hochschild invariants of A_∞ algebras, ensuring that these definitions include the curved case. We present a vanishing result showing that these invariants vanish for an algebra which only has nontrivial m_0 . Finally we introduce the compact type invariants discussed in the introduction and give a first example of algebras of compact type.

3.1. Let A be an A_∞ algebra. The space of cochains in A is the vector space

$$C^*(A) = \text{Hom}(B(A), A[1]) = \prod_{i=0}^{\infty} \text{Hom}(A[1]^{\otimes i}, A[1]),$$

where $B(A)$ is the free coalgebra generated by $A[1]$. As in (2.3) $C^*(A)$ can be identified with the space of coderivations of $B(A)$.

3.2. The space of cochains admits two different gradings: the internal grading arising from the grading of A , and a secondary grading by tensor degree. For example, a map $A \otimes A \rightarrow A$ of degree zero can be regarded as an element of $\text{Hom}(A[1] \otimes A[1], A[1])$ of internal degree one and of tensor degree two. In general, a map $A^{\otimes i} \rightarrow A$ of degree j will have internal degree $i + j - 1$. We shall use the internal grading as the default one.

Note that while the tensor grading is always by integers, the internal grading is by \mathbb{Z} or $\mathbb{Z}/2\mathbb{Z}$, depending on the grading of A . Therefore the Hochschild (co)homology of A only carries a $\mathbb{Z}/2\mathbb{Z}$ -grading if the algebra A is itself only $\mathbb{Z}/2\mathbb{Z}$ -graded (and it is not a strictly associative algebra, in which case the Hochschild differentials are homogeneous of degree one with respect to the tensor grading).

We emphasize that $C^*(A)$ is a direct *product* of Hom -sets, which is a direct consequence of the fact that $B(A)$ is a direct *sum*.

3.3. The space $C^*(A)$, being identified with the space of coderivations on $B(A)$, naturally carries the structure of a Lie algebra with bracket given by the commutator of coderivations. The A_∞ condition $m \circ m = 0$ is equivalent to $[m, m] = 0$ as m is of degree one.

The Jacobi identity implies that the operator $d : C^*(A) \rightarrow C^*(A)$ given by $d = [m, -]$ is a degree one differential ($d^2 = 0$). The Hochschild cohomology

$HH^*(A)$ is defined as the cohomology of the complex $(C^*(A), d)$. Note that d is homogeneous with respect to the internal grading on $C^*(A)$, but not, in general, with respect to the tensor degree. The latter happens only if m_2 is the only nonzero structure map.

3.4. Let us denote by $|a|$ the parity of a in the vector space $A[1]$, and let φ be a cochain in $\mathbf{Hom}(A[1]^{\otimes k}, A[1])$. Then the explicit formula for the component of $d\varphi$ of tensor degree l is

$$\begin{aligned} (d\varphi)(a_1 | \cdots | a_l) &:= \sum_{j, l \geq k} (-1)^{j+|a_1|+\cdots+|a_j|} m_{l-k+1}(a_1 | \cdots | a_j | \varphi(a_{j+1} | \cdots | a_{j+k}) | \cdots | a_l) \\ &\quad + \sum_i (-1)^{|\varphi|+|a_1|+\cdots+|a_i|} \varphi(a_1 | \cdots | a_i | m_{l-k+1}(a_{i+1} | \cdots | a_{i+l-k+1}) | \cdots | a_l). \end{aligned}$$

Note that the second term in the above sum includes contributions from m_0 only if $l = k - 1$. In this case the contribution is given by

$$\sum_{i=0}^{k-1} (-1)^{|\varphi|+|a_1|+\cdots+|a_i|} \varphi(a_1 | \cdots | a_i | m_0(1) | \cdots | a_{k-1}).$$

3.5. We are also interested in studying the Hochschild homology of curved algebras. The space of chains on A is

$$C_*(A) := A \otimes B(A) = \bigoplus_{i=0}^{\infty} A \otimes A^i.$$

Again, like in the case of the space of cochains, there are two gradings on $C_*(A)$, an internal one and one given by tensor degree, and we use the internal grading by default.

In order to define the differential we mimic the standard definition used in the flat case. The best way to illustrate this formula is to use trees drawn on a cylinder as explained in [10]. The formula one gets for the Hochschild differential $b : C_*(A) \rightarrow C_*(A)$ is

$$\begin{aligned} b(a_0 | \cdots | a_i) &= \sum_{j,k} (-1)^{\epsilon_k} m_j(a_{i-k+1} | \cdots | a_0 | \cdots | a_{j-k-1}) | a_{j-k} | \cdots | a_{i-k} \\ &\quad + \sum_{j,k} (-1)^{\lambda_k} a_0 | \cdots | a_k | m_j(a_{k+1} | \cdots | a_{k+j}) | \cdots | a_i \end{aligned}$$

where the signs ϵ_k and λ_k are determined by the Koszul sign convention:

$$\begin{aligned} s_i &:= \sum_{0 \leq l \leq i} |a_l|, \\ \epsilon_k &= \sum_{0 \leq l \leq k-1} (|a_{n-l}|)(s_i - |a_{n-l}|), \\ \lambda_k &= \sum_{0 \leq l \leq k} (|a_l|). \end{aligned}$$

It is straightforward to check that $b^2 = 0$. The Hochschild homology of the algebra A is defined as the cohomology of the complex $(C_*(A), b)$.

3.6. Remark. The key feature of the first summation is that a_0 has to be inserted in m_j . Thus the appearance of m_0 does not affect these terms as m_0 allows no insertion. In the second summation a_0 is not inserted in m_j and has to be placed at the first spot. Thus if m_0 is present, we should insert the term $m_0(1)$ into any spot after the first term a_0 . Explicitly, the terms involving m_0 are

$$\sum_{k=0}^i (-1)^{\lambda_k} a_0 | \cdots | a_k | m_0(1) | \cdots | a_i.$$

3.7. From a computational point of view the Hochschild (co)homology of curved A_∞ algebras is far more complicated than that of noncurved ones. The main difficulty is caused by the fact that the Hochschild (co)homology differential does not preserve the filtration on (co)chains induced by tensor degree in the curved case. (It is easy to see that in the flat case this filtration is preserved, even though the differentials are not homogeneous with respect to the tensor degree.) Thus the usual spectral sequences associated with this filtration that are used for flat algebras are unusable for curved ones. We shall not go into more details as we shall not use these spectral sequences.

3.8. We now present a vanishing result which will motivate the discussion of compact type invariants for curved algebras later.

3.9. Lemma. *Let A be an A_∞ algebra over a field k such that $m_0 \neq 0$ and $m_k = 0$ for any $k > 0$. This is equivalent to the data of a graded vector space A together with the choice of a degree two element $W = m_0(1)$ in A . Then both the Hochschild homology and cohomology of A are zero.*

Proof. To avoid sign issues we shall assume that A is concentrated in even degrees. The proof still works with corrected signs in the general case. Also, throughout this proof the degree of a (co)chain will mean the tensor degree, so that $C_k(A)$ means $A^{\otimes(k+1)}$.

We begin with the statement for homology. The Hochschild chain complex for our algebra A is

$$C_*(A) = \bigoplus_i A^{\otimes(i+1)}$$

with differential given by

$$b(a_0 | \cdots | a_i) = \sum_{k=0}^i (-1)^k a_0 | \cdots | a_k | W | \cdots | a_i.$$

We will prove that this complex is acyclic by constructing an explicit homotopy between the identity map and the zero map.

Let $L : A \rightarrow k$ be a k -linear map such that $L(W) \neq 0$. Such a map exists as k is a field and A is free as a k -module, so we can extend any nonzero

map from the one-dimensional subspace spanned by W to the whole space A . Once such a functional L is chosen we can use it to define a homotopy $h_k : C_k(A) \rightarrow C_{k-1}(A)$ by the formula:

$$\begin{aligned} h_k(a_0 | \cdots | a_k) &= (-1)^{k+1} L(a_k)(a_0 | \cdots | a_{k-1}), \\ h_0 &= 0. \end{aligned}$$

An explicit calculation shows that h_* is a homotopy between the identity map on $C_*(A)$ and the zero map, hence $C_*(A)$ is acyclic.

A similar argument yields the result for cohomology, with homotopy given by $h_k : C^k(A) \rightarrow C^{k+1}(A)$,

$$[h_k(\varphi_k)](a_1 | \cdots | a_{k+1}) = L(a_1)\varphi_k(a_2 | \cdots | a_{k+1}). \quad \square$$

Motivated by the vanishing results in the above lemma we introduce the following modifications of the Hochschild invariants.

3.10. Definition. Let A be an A_∞ algebra or category with finitely many nonzero operations m_k . Consider modified spaces of chains and cochains

$$\begin{aligned} C_*^\Pi(A) &= \prod_{i=0}^{\infty} A \otimes A^{\otimes i}, \\ C_\oplus^*(A) &= \bigoplus_{i=0}^{\infty} \text{Hom}(A^{\otimes i}, A). \end{aligned}$$

The Borel–Moore Hochschild homology $HH_*^{\text{BM}}(A)$ is defined to be the homology of the complex $(C_*^\Pi(A), b)$, and the compactly supported Hochschild cohomology $HH_c^*(A)$ is defined to be the cohomology of $(C_\oplus^*(A), d)$, where b and d are given by the same formulas as before.

3.11. Remark. Without the assumption that A has finitely many nonzero higher multiplications these invariants might not be well-defined due to the possible nonconvergence of the infinite sum. Note that the property required of A is not homotopy invariant: it is easy to construct dg algebras (A_∞ algebras that only have nonzero m_1 and m_2) whose minimal models have infinitely many nonzero m_k 's.

3.12. Let A be an A_∞ algebra or category for which the compact type invariants can be defined. There exist natural maps

$$HH_c^*(A) \rightarrow HH^*(A) \quad \text{and} \quad HH_*(A) \rightarrow HH_*^{\text{BM}}(A)$$

induced by the chain maps $(C_\oplus^*(A), d) \rightarrow (C^*(A), d)$ and $(C_*(A), b) \rightarrow (C_*^\Pi(A), b)$ given by the natural inclusion of the direct sum in the direct product.

3.13. Definition. When the above maps are isomorphisms we shall say that A is of *compact type*.

3.14. It is reasonable to expect that under certain finiteness hypotheses we get compact type algebras. The following proposition shows that this is the case in a simple situation.

3.15. Proposition. *Let A be a \mathbb{Z} -graded, finite dimensional A_∞ algebra concentrated in nonpositive degrees. Assume that A has only finitely many nonzero higher multiplications over a field k . Then A is of compact type. In other words we have*

$$HH_*(A) \cong HH_*^\pi(A), \text{ and}$$

$$HH^*(A) \cong HH_\oplus^*(A).$$

Proof. It suffices to observe that under the assumptions of this proposition, the ordinary Hochschild cochain complex $C^*(A)$ is the completion of the compactly supported Hochschild cochain complex $C_c^*(A)$ with respect to the natural internal \mathbb{Z} -grading. The case for Hochschild homology is similar. \square

4. Hochschild invariants of LG models

In this section we compute the usual and compact type Hochschild invariants of the curved A_∞ algebras that arise in the study of LG models. More precisely we show that the usual Hochschild invariants vanish for any curved algebra while the modified invariants agree with the predictions from physics.

4.1. The setup we shall work with is the following. Let $Y = \text{Spec } B$ be a smooth affine scheme of dimension n and assume that $W \in B$ is a regular function on Y . Let Z denote the set of critical points of W , i.e., the subscheme of Y cut out by the section dW of Ω_Y^1 . The ring of regular functions on Z shall be denoted by $\text{Jac}(W)$, the Jacobi ring of W . The relative dualizing sheaf $\omega_{Z/Y}$ of the closed embedding $Z \hookrightarrow Y$ is a coherent sheaf on Z , i.e., a module $\omega(W)$ over $\text{Jac}(W)$. Algebraically $\text{Jac}(W)$ and $\omega(W)$ are the cokernels of the maps $\lrcorner dW : \text{Der}(B) \rightarrow B$ and $\wedge dW : \Omega_B^{n-1} \rightarrow \Omega_B^n$, respectively.

Denote by B_W the curved algebra with m_2 given by multiplication in B and with m_0 given by W . Since B is commutative W is trivially in the center and hence B_W is an A_∞ algebra.

4.2. Theorem. *With the above notations, we have:*

- (a) $HH^*(B_W) = HH_*(B_W) = 0$.
- (b) *If $\dim Z = 0$, i.e., the critical points of W are isolated, then we have isomorphisms of $\mathbb{Z}/2\mathbb{Z}$ -graded vector spaces*

$$HH_c^*(B_W) = \text{Jac}(W)[0],$$

$$HH_*^{\text{BM}}(B_W) = \omega(W)[n].$$

(Here $M[i]$ denotes the $\mathbb{Z}/2\mathbb{Z}$ graded vector space which is M in degree $i \pmod 2$ and 0 otherwise.)

4.3. Remark. We see that the usual result that $HH^*(A) = HH^*(\mathrm{Tw}(A))$ for flat (noncurved) A_∞ algebras fails for curved ones. Indeed, for the curved algebra B_W we have $HH^*(B_W) = 0$, while

$$HH^*(\mathrm{Tw}(B_W)) = HH^*(\mathrm{DG}_{\mathbb{Z}/2\mathbb{Z}}(B_W)) = \mathrm{Jac}(W)[0] = HH_c^*(B_W).$$

Here the second equality is due to Dyckerhoff [5] (but some more assumptions are needed for B and W). See also [18]. The third equality is part (b) of the theorem above.

4.4. The differential b on Hochschild chains decomposes as

$$b = b_- + b_+$$

where

$$b_+ : C_n(B) \rightarrow C_{n+1}(B), \quad b_- : C_{n-1}(B) \rightarrow C_n(B)$$

are given by

$$b_+(a_0 | \cdots | a_n) = \sum_{i=0}^n (-1)^i a_0 | \cdots | a_i | W | a_{i+1} | \cdots | a_n,$$

$$b_-(a_0 | \cdots | a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 | \cdots | a_i a_{i+1} | \cdots | a_n + (-1)^n a_n a_0 | a_1 | \cdots | a_{n-1}.$$

Here we are using the tensor degree for the spaces of chains or cochains.

4.5. Lemma. *We have $b_+^2 = 0$, $b_-^2 = 0$, $b_+ b_- + b_- b_+ = 0$.*

Proof. The fact that $b_+^2 = b_-^2 = 0$ is a straightforward calculation. Alternatively for the second equality one could observe that b_- is the Hochschild differential for the algebra B (without m_0) for which it is well known that $b_-^2 = 0$. Finally the last equation follows from the first two and the fact that $b^2 = (b_+ + b_-)^2 = 0$ as the Hochschild differential of the A_∞ algebra B_W . \square

4.6. From the above lemma we see that the Hochschild chain complex $C_*(B_W)$ is actually a so-called mixed complex. Recall [20, 9.8] that a mixed complex is a \mathbb{Z} graded complex with two anti-commuting differentials of degrees 1 and -1 respectively. Mixed complexes appear in the theory of cyclic homology where the two differentials are the Hochschild boundary map and the Connes cyclic operator. We find this analogy quite useful and we will calculate the Hochschild (co)homology of B_W using techniques developed for cyclic homology. We refer the reader to Loday's book [11] for a detailed discussion of cyclic homology. The idea for computing the homology of a mixed complex C is to associate a double complex BC to it, which on one hand avoids the problems caused by the inhomogeneity of b and on the other hand allows us to use the powerful machinery of spectral sequences. As the double complexes we are going to study are usually unbounded, it will make a difference if we take the direct sum or the direct product total complex. To distinguish between the two we shall denote the direct sum and direct

product total complex of a double complex BC by $\text{tot}^\oplus(BC)$ and $\text{tot}^\Pi(BC)$, respectively. For a bounded double complex BC the direct sum and the direct product total complex are the same so we will simply denote any of them by $\text{tot}(BC)$.

4.7. Consider the double complex $BC_{i,j} = C_{j-i}(B_W) = B \otimes B^{\otimes(j-i)}$ associated to the mixed complex $(C_*(B_W), b_+, b_-)$. It is a bicomplex supported above the diagonal on the (i, j) -plane as illustrated below (here i and j are horizontal and vertical coordinates respectively):

$$\begin{array}{ccccccc}
 & \cdots & & \cdots & & \cdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 \leftarrow & C_2(B_W) & \leftarrow & C_1(B_W) & \leftarrow & C_0(B_W) & \leftarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 \leftarrow & C_1(B_W) & \leftarrow & C_0(B_W) & \leftarrow & 0 & \leftarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
 \leftarrow & C_0(B_W) & \leftarrow & 0 & \leftarrow & 0 & \leftarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
 \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \cdots &
 \end{array}$$

The horizontal differential of BC is given by b_+ and the vertical one is b_- .

Note that BC is periodic with respect to shifts along the main diagonal, hence the two total complexes $\text{tot}^\oplus(BC)$ and $\text{tot}^\Pi(BC)$ are 2-periodic and this periodicity descends to their homology. Therefore it makes sense to talk about the even or the odd homology of $\text{tot}^\oplus(BC)$ or $\text{tot}^\Pi(BC)$ and for $*$ = even or odd we have

$$\begin{aligned}
 H_*(\text{tot}^\oplus(BC)) &\cong HH_*(B_W), \\
 H_*(\text{tot}^\Pi(BC)) &\cong HH_*^{\text{BM}}(B_W).
 \end{aligned}$$

4.8. We will need another double complex which is the part of BC that lies in the first quadrant of the (i, j) -plane. Denote this double complex by BC^+ . Note that BC^+ is bounded, hence the direct sum and the direct product total complexes agree. The relationship between BC^+ and BC is that the former is a quotient of the latter by the subcomplex consisting of those terms whose i -coordinate is strictly negative.

4.9. Proof of part (b) of Theorem 4.2. Consider the positive even shifts $\text{tot}(BC^+)[2r]$ for $r \in \mathbb{N}$. For $r < t$ the complex $\text{tot}(BC^+)[2r]$ is a quotient of $\text{tot}(BC^+)[2t]$ due to the 2-periodicity. Hence the quotient maps

$$\text{tot}(BC^+)[2t] \rightarrow \text{tot}(BC^+)[2r]$$

form an inverse system whose inverse limit is $\text{tot}^\Pi(BC)$ (see [11, 5.1] for more details). In short we have realized $\text{tot}^\Pi(BC)$ as the inverse limit

$$\text{tot}^\Pi(BC) \cong \varprojlim \text{tot}(BC^+)[2r].$$

Moreover the tower $(\mathrm{tot}(BC^+)[2r], r \in \mathbb{N})$ satisfies the Mittag-Leffler condition [20, 3.5.6] as these maps are all onto. By [20, 3.5.8] we get a short exact sequence for $k \in \mathbb{Z}$

$$\begin{aligned} 0 \rightarrow \lim_{\leftarrow} {}^1H_{k+1}(\mathrm{tot}(BC^+)[2r]) &\rightarrow H_k(\mathrm{tot}^\Pi(BC)) \\ &\rightarrow \lim_{\leftarrow} H_k(\mathrm{tot}(BC^+)[2r]) \rightarrow 0. \end{aligned}$$

Since we have

$$H_*(\mathrm{tot}(BC^+)[2r]) \cong H_{*+2r}(\mathrm{tot}(BC^+)),$$

the problem of computing

$$HH_*^{\mathrm{BM}}(B_W) = H_*(\mathrm{tot}^\Pi(BC))$$

reduces to the problem of computing the homology of $\mathrm{tot}(BC^+)$. Since the bicomplex BC^+ is nonzero only in the first quadrant, the standard bounded convergence theorem [20, 5.6.1] applies. We will use the spectral sequence associated to the vertical filtration of the bicomplex BC^+ whose 1E page consists precisely of the Hochschild homology of the associative ring B . (The vertical differential b_- is precisely the Hochschild differential of B .) Since B is regular the classical Hochschild–Kostant–Rosenberg isomorphism applies and gives

$${}^1E_{ij} = \Omega_B^{j-i}.$$

Moreover, the horizontal differential at the 1E page is induced from the original horizontal differential b_+ on BC^+ , and the former can be calculated using the splitting map e of the HKR-isomorphism, see [20, 9.4.4]

$$e_k(a_0 | \cdots | a_k) = \frac{1}{k!} a_0 da_1 \wedge \cdots \wedge da_k.$$

Since we have

$$\begin{aligned} e_{k+1} \circ b_+(a_0 | \cdots | a_k) &= e_{k+1} \left(\sum_{i=0}^k (-1)^i a_0 | \cdots | a_i | W | \cdots | a_k \right) \\ &= \frac{1}{(k+1)!} \sum_{i=0}^k (-1)^i a_0 \wedge \cdots \wedge da_i \wedge dW \wedge \cdots \wedge da_k \\ &= \frac{1}{k!} dW \wedge a_0 \wedge da_1 \wedge \cdots \wedge da_k \end{aligned}$$

we conclude that the horizontal differential on the 1E page is the map $\Omega_B^{j-i} \rightarrow \Omega_B^{j-i+1}$ given by $\alpha \mapsto dW \wedge \alpha$.

To calculate the 2E page, we compute homology with respect to this horizontal differential. Observe that the rows in the 1E page are precisely truncations of the dual of the Koszul complex associated to the section dW of Ω_B^1 . The assumption that the critical points of W are isolated implies that this complex is exact except at the n -th (last) spot (recall that $n = \dim B$).

Thus the 2E page is everywhere zero except for the spots $(i, i+n)$ for $i > 0$ and the spots $(0, j)$ for $0 \leq j \leq n$. More precisely it is $(\wedge^j \Omega_B)/(dW)$ at the spots $(0, j), 0 \leq j \leq \dim_k B$ and $\omega_B/(dW)$ at the spots $(i, i + \dim(B)), \forall i \in \mathbb{N}^+$. So the spectral sequence degenerates at the E^2 page already. We conclude that the inverse limit of the homology groups of $\text{tot}(BC^+)[2r]$ is

$$\varprojlim H_*(\text{tot}(BC^+)[2r]) \cong \begin{cases} \omega_B/(dW) & * = \dim_k B \bmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

It also follows that the maps in the tower of these homology groups are actually isomorphisms for large r . Therefore the tower of homology groups also satisfies the Mittag-Leffler condition, which implies the vanishing of the first derived functor of the inverse limit

$$\varprojlim^1 H_k((\text{tot } BC^+)[2r]) = 0, \forall k \in \mathbb{Z}.$$

Hence we conclude that

$$HH_*^{\text{BM}}(B_W) \cong H_*(\text{tot}^\Pi(BC)) \cong \begin{cases} \omega_B/(dW) & * \equiv \dim_k B \bmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

4.10. A similar argument can be adapted to compute $HH_c^*(B_W)$. In fact, in this case, the proof is easier as we will not need the argument involving inverse limits. The conclusion is that

$$HH_{\oplus}^*(B_W) \cong H_*(\text{tot}^\oplus(BC)) \cong \begin{cases} \text{Jac}(W) & * = \text{even}, \\ 0 & * = \text{odd}. \end{cases}$$

4.11. Proof of part (a) of Theorem 4.2. The usual Hochschild homology of B_W can be computed by the direct sum total complex of BC . For this, the convergence theorem of spectral sequences shows that the spectral sequence associated to the horizontal filtration converges to $\text{tot}^\oplus(BC)$. To calculate the 1E page of this spectral sequence, we can apply Lemma 3.9, which shows that the 1E page is already all zero. Thus we have proved that

$$HH_*(B_W) \cong H_*(\text{tot}^\oplus(BC)) \cong 0.$$

4.12. For the case of the ordinary Hochschild cohomology of B_W , again by Lemma 3.9, we can deduce that the usual Hochschild cohomology vanish by an argument using the idea of taking inverse limit.

The usual Hochschild cohomology $HH^*(B_W)$ was defined to be the homology of $d = d_+ + b_-$ on the direct product $\prod_{n \geq 0} C^n(B_W)$ of Hochschild cochains. Thus we should consider $\text{tot}^\Pi(BC)$ of the double complex BC associated to the mixed complex $(C^*(B_W), d_+, d_-)$. Again the vertical filtration realizes $\text{tot}^\Pi(BC)$ as the inverse limit of the tower of complexes

$\mathrm{tot}(BC^+)[2r]$ for $r \in \mathbb{N}$. Hence we have the following short exact sequence:

$$0 \rightarrow \lim_{\leftarrow}^1 H_{k+1}(\mathrm{tot}(BC^+)[2r]) \rightarrow H_k(\mathrm{tot}^{\mathrm{II}}(BC)) \\ \rightarrow \lim_{\leftarrow} H_k(\mathrm{tot}(BC^+)[2r]) \rightarrow 0.$$

To compute the homology of $\mathrm{tot}(BC^+)$ we run the spectral sequence associated to the vertical filtration of BC^+ . Then the first page of the spectral sequence can be calculated using the vertical differential d_- . But according to Lemma 3.9 this is already zero. Using the above short exact sequence, the vanishing of the homology of $\mathrm{tot}(BC^+)$ immediately implies the vanishing of $H_k(\mathrm{tot}^{\mathrm{II}}(BC))$. We conclude that

$$HH^*(B_W) \cong H_*(\mathrm{tot}^{\mathrm{II}}(BC)) \cong 0.$$

4.13. Remark. The above proof of the vanishing of the usual Hochschild homology and cohomology actually works for any curved algebra over any field of arbitrary characteristic.

5. Orlov's results in the dg setting

In [14] Orlov relates the derived category of graded matrix factorizations of a homogeneous superpotential W to the derived category of coherent sheaves on the projective hypersurface defined by $W = 0$. In this section we argue that Orlov's results can be extended to the dg setting by exhibiting a chain of quasi-equivalences between appropriate dg enhancements of the categories he considered. The main context in which we work is that of the category Hqe (see Appendix A) in which quasi-equivalences are inverted, and thus in Hqe the resulting categories are isomorphic.

5.1. We begin by reviewing Orlov's results from [14]. Let $A = \bigoplus_{i \geq 0} A_i$ be a connected graded Gorenstein Noetherian k -algebra. Recall the Gorenstein property means that A has finite injective dimension n , and $R\mathrm{Hom}_A(k, A)$ is isomorphic to $k(a)[-n]$ for some integer a which is called the *Gorenstein parameter* of A . Here $()$ and $[]$ are the internal degree shift and homological degree shift respectively.

Denote by $\mathrm{gr}\text{-}A$ the abelian category of finitely generated graded left A -modules. Let $\mathrm{tors}\text{-}A$ be the full subcategory of $\mathrm{gr}\text{-}A$ consisting of all graded A -modules that are finite dimensional over k and form the quotient abelian category $\mathrm{qgr}\text{-}A := \mathrm{gr}\text{-}A/\mathrm{tors}\text{-}A$. When A is commutative $\mathrm{qgr}\text{-}A$ is equivalent to the category of coherent sheaves on $\mathrm{Proj} A$.

For convenience we will sometimes omit the algebra A in our notation. For example $\mathrm{gr}\text{-}A$ will simply be denoted by gr .

There is a canonical quotient functor π from gr to qgr , which in the commutative case corresponds to the operation associating a sheaf on $\mathrm{Proj} A$ to a graded A -module. Moreover π admits a right adjoint ω defined by

$$\omega(N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{qgr}\text{-}A}(A, N(i)).$$

Again, in the geometric setting, ω is the analogue of the functor Γ_* of [7, II.5].

For any integer i we also consider the abelian category $\mathbf{gr}_{\geq i}$ which is the full subcategory of \mathbf{gr} consisting of graded A -modules M such that $M_p = 0$ for $p < i$. The categories $\mathbf{tors}_{\geq i}$ and $\mathbf{qgr}_{\geq i}$ are similarly defined, and the restrictions of the functors π and ω to these subcategories shall be denoted by $\pi_{\geq i}$ and $\omega_{\geq i}$.

Deriving these abelian categories yields triangulated categories $D^b(\mathbf{gr})$, $D^b(\mathbf{gr}_{\geq i})$ and $D^b(\mathbf{qgr})$. However, when defining the derived functor of $\omega_{\geq i}$ we run into trouble as the category \mathbf{qgr} has neither enough projectives nor enough injectives. We can however bypass this difficulty by giving up the condition of our modules being finitely generated. Namely consider the category \mathbf{Gr} which consists of all graded A -modules, finitely generated or not, and form the corresponding quotient \mathbf{QGr} . Note that $D^+(\mathbf{qgr})$ is naturally a full subcategory of $D^+(\mathbf{QGr})$, and similarly for D^b , etc.

Since \mathbf{QGr} has enough injectives we can define the derived functor

$$R\omega_{\geq i} : D^+(\mathbf{QGr}) \rightarrow D^+(\mathbf{Gr}_{\geq i}).$$

It can be checked that the restriction of $R\omega_{\geq i}$ to the subcategory $D^+(\mathbf{qgr})$ lands inside $D^+(\mathbf{gr}_{\geq i})$. If we assume further that A has finite injective dimension then the restriction of $R\omega_{\geq i}$ to $D^b(\mathbf{qgr})$ lands inside $D^b(\mathbf{gr}_{\geq i})$. One can also show that the functor $\pi_{\geq i}$ is exact (and hence does not need to be derived) and it is a one-sided inverse of $R\omega_{\geq i}$: $\pi_{\geq i} \circ R\omega_{\geq i} \cong \text{id}$.

The main object of study in this section is the quotient of $D^b(\mathbf{gr})$ by the full triangulated subcategory $\mathbf{perf}(\mathbf{gr})$ consisting of perfect objects. (Recall that a complex is perfect if it is quasi-isomorphic to a bounded complex of projectives.) This category is known as the (graded) category of singularities and is denoted by $D_{\text{sg}}^{\text{gr}}$.

5.2. Theorem (Orlov [14]). *Let A be a Gorenstein algebra with Gorenstein parameter a . Then the triangulated categories $D_{\text{sg}}^{\text{gr}}$ and $D^b(\mathbf{qgr})$ are related as follows:*

- (A) *If $a > 0$ then for each integer $i \in \mathbb{Z}$ there exists a fully faithful functor $\Phi_i : D_{\text{sg}}^{\text{gr}} \rightarrow D^b(\mathbf{qgr})$ and a semi-orthogonal decomposition*

$$D^b(\mathbf{qgr}) = \langle \pi A(-i - a + 1), \dots, \pi A(-i), \Phi_i(D_{\text{sg}}^{\text{gr}}) \rangle.$$

- (B) *If $a < 0$ then for each integer $i \in \mathbb{Z}$ there exists a fully faithful functor $\Psi_i : D^b(\mathbf{qgr}) \rightarrow D_{\text{sg}}^{\text{gr}}$ and a semi-orthogonal decomposition*

$$D_{\text{sg}}^{\text{gr}} = \langle qk(-i), \dots, qk(-i + a + 1), \Psi_i(D^b(\mathbf{qgr})) \rangle$$

where $q : D^b(\mathbf{gr}) \rightarrow D_{\text{sg}}^{\text{gr}}(A)$ is the natural projection.

- (C) *If $a=0$ then $D_{\text{sg}}^{\text{gr}} \cong D^b(\mathbf{qgr})$.*

5.3. Our purpose is to generalize the above result to the dg setting so we can apply invariance of Hochschild structures under dg equivalences. For this we need appropriate dg enhancements of the triangulated categories involved. For the category $D^b(\mathbf{qgr})$ we take the dg category of complexes of injective objects in \mathbf{QGr} that are quasi-isomorphic to objects in $D^b(\mathbf{qgr})$. We denote this dg category by $\mathbf{DG}(\mathbf{qgr})$.

For the category $D_{\mathbf{sg}}^{\mathbf{gr}}$ there is no obvious candidate for a dg enhancement. However observe that $D_{\mathbf{sg}}^{\mathbf{gr}}$ is obtained as the quotient of $D^b(\mathbf{gr})$ by $\mathbf{perf}(\mathbf{gr})$, and these two categories admit natural dg enhancements by using either injective or projective resolutions of the corresponding complexes. Both dg enhancements will be used in this paper and the relationship between them will be clarified in Lemma 5.15.

We first consider the category obtained by injective resolutions. Let $\underline{\mathbf{DG}}(\mathbf{gr})$ be the dg category of bounded below complexes of injective graded A -modules that are quasi-isomorphic to objects of $D^b(\mathbf{gr})$ (the left arrow is used to indicate that we are using injective resolutions). Denote by $\underline{\mathbf{P}}(\mathbf{gr})$ the full subcategory of $\underline{\mathbf{DG}}(\mathbf{gr})$ whose objects are perfect complexes. Then the dg enhancement of $D_{\mathbf{sg}}^{\mathbf{gr}}$ that we will consider is the dg quotient $\underline{\mathbf{DG}}(\mathbf{gr})/\underline{\mathbf{P}}(\mathbf{gr})$, which we shall denote by $\underline{\mathbf{DG}}_{\mathbf{sg}}^{\mathbf{gr}}$. Similarly, we denote by $\underline{\mathbf{DG}}(\mathbf{gr}_{\geq i})$ the dg version of $D^b(\mathbf{gr}_{\geq i})$ and by $\underline{\mathbf{P}}_{\geq i}(\mathbf{gr})$ the dg version of $\mathbf{perf}_{\geq i}(\mathbf{gr})$.

5.4. Lemma. *The functor $\omega_{\geq i}$ from \mathbf{QGr} to $\mathbf{Gr}_{\geq i}$ sends injectives to injectives.*

Proof. This is immediate from the fact that the left adjoint $\pi_{\geq i}$ of $\omega_{\geq i}$ is exact. \square

5.5. Because of this lemma we have a well-defined dg functor

$$\omega_{\geq i} : \mathbf{DG}(\mathbf{qgr}) \rightarrow \underline{\mathbf{DG}}(\mathbf{gr}_{\geq i}).$$

We denote by D_i the full subcategory of $\underline{\mathbf{DG}}(\mathbf{gr}_{\geq i})$ consisting of objects in the image of $\omega_{\geq i}$. Note that the functor $\omega_{\geq i}$ induces a quasi-equivalence between $\mathbf{DG}(\mathbf{qgr})$ and D_i by the corresponding results in the triangulated setting. Moreover, the subcategory D_i is an admissible subcategory of $\underline{\mathbf{DG}}(\mathbf{gr}_{\geq i})$ as the adjoint to the inclusion functor is given by $\pi_{\geq i}$.

5.6. Lemma. *The dg functor*

$$\iota_i : \underline{\mathbf{DG}}(\mathbf{gr}_{\geq i})/\underline{\mathbf{P}}_{\geq i}(\mathbf{gr}) \rightarrow \underline{\mathbf{DG}}(\mathbf{gr})/\underline{\mathbf{P}}(\mathbf{gr})$$

induced by the natural morphism between localization pairs

$$(\underline{\mathbf{P}}_{\geq i}(\mathbf{gr}), \underline{\mathbf{DG}}(\mathbf{gr}_{\geq i})) \rightarrow (\underline{\mathbf{P}}(\mathbf{gr}), \underline{\mathbf{DG}}(\mathbf{gr}))$$

is a quasi-equivalence. (See Appendix A for details on localization pairs).

Moreover, the full subcategory $P_{\geq i}(\text{gr})$ is left admissible in $\underline{\text{DG}}(\text{gr}_{\geq i})$. Thus by Lemma A.12 in the Appendix the category $T_i := {}^\perp P_{\geq i}(\text{gr})$ in $\underline{\text{DG}}(\text{gr}_{\geq i})$ is canonically quasi-isomorphic to the quotient $\underline{\text{DG}}(\text{gr}_{\geq i})/P_{\geq i}(\text{gr})$.

Proof. We only need to show that the morphism ι induces an equivalence at the level of homotopy categories. Since all our dg categories are pre-triangulated, the problem is reduced to that of showing that the morphism

$$D^b(\text{gr}_{\geq i})/\text{perf}_{\geq i} \rightarrow D^b(\text{gr})/\text{perf}(\text{gr})$$

is an equivalence, a result which is proved by Orlov in [14].

The second statement also follows from the definition and from results of Orlov. □

5.7. We have two admissible full subcategories D_i and T_i inside $\underline{\text{DG}}(\text{gr}_{\geq i})$ such that D_i is isomorphic to $\text{DG}(\text{qgr})$ and T_i is isomorphic to $\underline{\text{DG}}_{\text{sg}}^{\text{gr}}(A)$ in Hqe . Once these categories have been put inside the same dg category Orlov’s proof of the existence of the corresponding semi-orthogonal decompositions (Theorem 5.2) also yields a proof of the following dg analogue, as the notion of orthogonality is strictly on the level of objects (see Lemma A.12 in the Appendix A). The proof of the following theorem will be omitted as it is identical to Orlov’s original proof of Theorem 5.2.

5.8. Theorem. *Let A be a Gorenstein algebra with Gorenstein parameter a . Then we have:*

(A) *If $a > 0$ there exists a semi-orthogonal decomposition*

$$D_i = \langle \pi A(-i - a + 1), \dots, \pi A(-i), T_i \rangle.$$

(B) *If $a < 0$ there exists a semi-orthogonal decomposition*

$$T_i = \langle qk(-i), \dots, qk(-i + a + 1), D_i \rangle.$$

(C) *If $a = 0$ there exists an equivalence $T_i \cong D_i$.*

The following result is a reformulation of the above theorem in the geometric setting, corresponding to a similar theorem stated by Orlov in [14] for triangulated categories (the LG/CY correspondence).

5.9. Theorem. *Let X be a Gorenstein projective variety of dimension n and let \mathcal{L} be a very ample line bundle such that $\omega_X \cong \mathcal{L}^{-r}$ for some integer r . Suppose that $H^j(X, \mathcal{L}^k) = 0$ for all $k \in \mathbb{Z}$ when $j \neq 0, n$. Set $A = \bigoplus_{i \geq 0} H^0(X, \mathcal{L}^i)$. Then we have:*

(A) *If $r > 0$ (X is Fano) then there is a semi-orthogonal decomposition of dg categories*

$$\text{DG}(X) = \langle \mathcal{L}^{-r+1}, \dots, \mathcal{O}_X, T \rangle$$

for some T that is isomorphic to $\underline{\text{DG}}_{\text{sg}}^{\text{gr}}$ in Hqe .

(B) If $r < 0$ (X is of general type) then there is a semi-orthogonal decomposition of dg categories

$$\overleftarrow{\mathrm{DG}}_{\mathrm{sg}}^{\mathrm{gr}} = \langle k(r+1), \dots, k, D \rangle$$

for some D that is isomorphic to $\mathrm{DG}(X)$ in Hqe .

(C) If $r = 0$ (X is Calabi–Yau) then there is an equivalence in Hqe

$$\overleftarrow{\mathrm{DG}}_{\mathrm{sg}}^{\mathrm{gr}} \cong \mathrm{DG}(X).$$

5.10. We also want to relate the categories considered above to the category of graded matrix factorizations from Section 2. Let $B = \bigoplus_{i \geq 0} B_i$ be a finitely generated connected graded algebra over k and let $W \in B_n$ be a central element of degree n which is not a zero-divisor. Consider the quotient graded algebra $A := B/W$, the quotient of B by the two-sided ideal generated by W . We are interested in the category of graded matrix factorizations $\mathrm{DG}_{\mathbb{Z}}(B_W)$.

5.11. To relate $\mathrm{DG}_{\mathbb{Z}}(B_W)$ to the dg category of graded singularities we will use a different model of the latter, given by projective resolutions instead of injective ones. Denote by $\underline{\mathrm{DG}}(\mathrm{gr})$ the dg category whose objects are bounded above complexes of free A -modules that are quasi-isomorphic to objects of $D^b(\mathrm{gr})$, and let $\underline{P}(\mathrm{gr})$ be the full subcategory consisting of perfect objects. As before we denote by $\underline{\mathrm{DG}}_{\mathrm{sg}}^{\mathrm{gr}}(A)$ the corresponding quotient.

5.12. For any dg category \mathcal{D} , we can associate a dg category $\mathcal{D}_{\leq 0}$ whose objects are the same as that of \mathcal{D} . The morphism space $\mathcal{D}_{\leq 0}$ between two objects E and F is the smart truncation of the complex $\mathrm{Hom}_{\mathcal{D}}(E, F)$ in nonpositive degrees. The inclusion functor

$$\mathcal{D}_{\leq 0} \rightarrow \mathcal{D}$$

is a homotopy equivalence. Next we define a dg functor

$$\mathrm{Coker} : \mathrm{DG}_{\mathbb{Z}}(B_W)_{\leq 0} \rightarrow \underline{\mathrm{DG}}(\mathrm{gr}).$$

For this purpose we think of objects in $\mathrm{DG}_{\mathbb{Z}}(B_W)$ as graded periodic complexes

$$E = \cdots E_1 \xrightarrow{p_1} E_0 \rightarrow E_1(n) \rightarrow E_0(n) \rightarrow \cdots.$$

The functor Coker is defined by

$$E \mapsto (\cdots E_1 \rightarrow E_0 \rightarrow 0 \rightarrow \cdots) \otimes_B A,$$

that is the object $\mathrm{Coker}(E)$ is the truncation of the graded periodic complex E followed by tensoring with the ring $A = B/W$. The reason this functor is called Coker is that $\mathrm{Coker}(E)$ gives a natural projective resolution over A of the Cohen–Macaulay module $E_0/\mathrm{Im} p_1$.

We can compose the functor Coker with the natural universal quotient functor $Q : \underline{\mathrm{DG}}(\mathrm{gr}) \rightarrow \underline{\mathrm{DG}}(\mathrm{gr})/\underline{P}(\mathrm{gr}) = \underline{\mathrm{DG}}_{\mathrm{sg}}^{\mathrm{gr}}(A)$ in Hqe to form a morphism in Hqe

$$Q \circ \mathrm{Coker} : \mathrm{DG}_{\mathbb{Z}}(B_W)_{\leq 0} \rightarrow \underline{\mathrm{DG}}_{\mathrm{sg}}^{\mathrm{gr}}(A).$$

Again, by passing to homotopy categories, Orlov’s results immediately imply the following theorem, which further implies that the two dg categories $\mathrm{DG}_{\mathbb{Z}}(B_W)$ and $\underline{\mathrm{DG}}_{\mathrm{sg}}^{\mathrm{gr}}(A)$ are isomorphic in Hqe .

5.13. Theorem. *The morphism $Q \circ \mathrm{Coker}$ is an isomorphism in Hqe .*

5.14. As a final part of this section we want to argue that there is an isomorphism in Hqe

$$\underline{\mathrm{DG}}_{\mathrm{sg}}^{\mathrm{gr}}(A) \cong \underline{\mathrm{DG}}_{\mathrm{sg}}^{\mathrm{gr}}(A).$$

For this consider the dg category $\mathrm{DG}_{\mathrm{gr}}(\mathrm{Mod}\text{-}A)$ of complexes of graded A -modules that are quasi-isomorphic to objects in $D^b(\mathrm{gr})$. Let P be the full subcategory of $\mathrm{DG}_{\mathrm{gr}}(\mathrm{Mod}\text{-}A)$ consisting of perfect objects. (Note that by definition P contains all the acyclic objects.) We can consider the following two morphisms of localization pairs:

$$\begin{aligned} j &: (\underline{P}(\mathrm{gr}), \underline{\mathrm{DG}}(\mathrm{gr})) \rightarrow (P, \mathrm{DG}_{\mathrm{gr}}(\mathrm{Mod}\text{-}A)), \\ k &: (\underline{P}(\mathrm{gr}), \underline{\mathrm{DG}}(\mathrm{gr})) \rightarrow (P, \mathrm{DG}_{\mathrm{gr}}(\mathrm{Mod}\text{-}A)). \end{aligned}$$

5.15. Lemma. *Both j and k induce quasi-equivalences on the corresponding quotient categories. Thus $\underline{\mathrm{DG}}_{\mathrm{sg}}^{\mathrm{gr}}(A)$ and $\underline{\mathrm{DG}}_{\mathrm{sg}}^{\mathrm{gr}}(A)$ are isomorphic in Hqe .*

Proof. Consider the induced map on the homotopy categories of the quotients. As these categories are all pre-triangulated, we have

$$\begin{aligned} H^0(\underline{\mathrm{DG}}_{\mathrm{sg}}^{\mathrm{gr}}(A)) &\cong H^0(\mathrm{Tw}(\underline{\mathrm{DG}}_{\mathrm{sg}}^{\mathrm{gr}}(A))) \\ &\cong H^0(\mathrm{Tw}(\underline{\mathrm{DG}}(\mathrm{gr}))/H^0(\mathrm{Tw}(\underline{P}(\mathrm{gr})))) \\ &\cong H^0(\underline{\mathrm{DG}}(\mathrm{gr}))/H^0(\underline{P}(\mathrm{gr})) \\ &\cong D_{\mathrm{sg}}^{\mathrm{gr}}(A). \end{aligned}$$

We also have

$$H^0(\mathrm{DG}_{\mathrm{gr}}(\mathrm{Mod}\text{-}A)/P) \cong D_{\mathrm{sg}}^{\mathrm{gr}}(A).$$

Thus we have shown that j induces a quasi-equivalence on the quotients. The proof of the corresponding statement for k is similar. \square

6. Hochschild homology of LG orbifolds

In this section we give a localization formula for Borel–Moore Hochschild homology groups of algebras of the form that arise in the study of orbifold LG models.

6.1. The main motivation for this calculation is the following. Let $X = \text{Proj } B/W$ be a smooth projective Calabi–Yau hypersurface of degree d in $\mathbb{P}^{d-1} = \text{Proj } B$, $B = \mathbf{k}[x_0, \dots, x_{d-1}]$. By the results in the previous sections there is a dg equivalence

$$\text{DG}(X) \cong \text{DG}_{\mathbb{Z}}(B_W) = \text{Tw}_{\mathbb{Z}}(B_W \sharp G),$$

where $\text{DG}(X)$ is a dg enhancement of the derived category of X and $G = \mathbb{Z}/d\mathbb{Z}$. (See Section 5 for the appropriate results in the non-CY case.) Hochschild homology is invariant under dg equivalences hence

$$HH_*(X) = HH_*(\text{Tw}_{\mathbb{Z}}(B_W \sharp G))$$

and the latter groups can be related to the Borel–Moore homology groups $HH_*^{\text{BM}}(B_W \sharp G)$ (see [18]), which are the groups we compute in this section.

6.2. Let G be a finite group acting on a smooth affine scheme $Y = \text{Spec } B$ and let W be a G -invariant global function on Y . Then we can form the curved cross product algebra $B_W \sharp G$ (2.13). We shall denote the Borel–Moore Hochschild homology of this algebra by $HH_*^{\text{BM}}([(Y, W)/G])$ to emphasize the geometric point of view.

6.3. Theorem. *In the above context there exists a graded vector space isomorphism*

$$HH_*^{\text{BM}}([(Y, W)/G]) \cong \left(\bigoplus_{g \in G} HH_*^{\text{BM}}(Y^g, W|_{Y^g}) \right)_G,$$

where Y^g is the g -invariant subspace of Y and the subscript G means taking coinvariants of the induced G action.

Proof. We first recall the following result of Baranovsky [1] for the non-curved case. Let $Y = \text{Spec } B$ be an affine scheme with a finite group G acting on it. Then we have

$$HH_*([Y/G]) \cong (\bigoplus_{g \in G} HH_*(Y^g))_G.$$

In fact this result is stronger in the sense that there is a natural chain level map that induces the above isomorphism. The map Ψ that goes from the Hochschild chain complex of the cross product algebra $B \sharp G$ to the G -coinvariants of the direct sum of the Hochschild chain complexes on the invariant subspaces is given explicitly by

$$\Psi(a_0 \sharp g_0 | \cdots | a_n \sharp g_n) = (a_0 | g_0(a_1) | g_0 g_1(a_2) | \cdots | g_0 \cdots g_{n-1}(a_n))_g$$

where the subscript $g = g_0 \cdots g_n$ means that all the functions on the right hand side are viewed as functions restricted on Y^g .

To see that Ψ indeed defines a map of chain complexes we note that the Hochschild differential consists of direct sums of face maps d_i , and Ψ commutes with all these face maps except the last one (d_n). However one can

check by a direct computation that Ψ also commutes with d_n after taking coinvariants.

Theorem 6.3 can now be proved quite easily with the quasi-isomorphism Ψ in hand. Let us denote by B^g the global regular functions on the subspace Y^g . (This notation is slightly misleading: in fact B^g is the set of *coinvariants* in B of the action of g ; we keep our notation for geometric purposes.) Denote by B_W^g the curved algebra $(B^g)_W$.

Consider the Hochschild chain complexes

$$C_*^\Pi(B_W \sharp G) = \prod_n C_n(B_W \sharp G),$$

$$\left(\bigoplus_{g \in G} C_*^\Pi(B_W^g) \right)_G = \left(\bigoplus_{g \in G} \prod_n C_n(B_W^g) \right)_G,$$

which are both mixed complexes with differentials given by the Hochschild differential b_- and the differential b_+ coming from inserting W . The map Ψ defines a linear map between the associated double complexes of these mixed complexes which commutes with b_- . A direct computation shows that Ψ commutes with b_+ as well:

$$\begin{aligned} & b_+(\Psi(a_0 \sharp g_0 | \cdots | a_n \sharp g_n)) \\ &= b_+((a_0 | g_0(a_1) | g_0 g_1(a_2) | \cdots | g_0 \cdots g_{n-1}(a_n))_g) \\ &= \sum_{i=0}^n (-1)^i (a_0 | g_0(a_1) | \cdots | g_0 \cdots g_{i-1}(a_i) | W | \cdots | g_0 \cdots g_{n-1}(a_n))_g, \end{aligned}$$

$$\begin{aligned} & \Psi(b_+(a_0 \sharp g_0 | \cdots | a_n \sharp g_n)) \\ &= \Psi \left(\sum_{i=0}^n (-1)^i (a_0 \sharp g_0 | \cdots | a_i \sharp g_i | W \sharp e | \cdots | a_n \sharp g_n) \right) \\ &= \sum_{i=0}^n (-1)^i (a_0 | g_0(a_1) | \cdots | g_0 \cdots g_i(W) | \cdots | g_0 \cdots g_{n-1}(a_n))_g \\ &= \sum_{i=0}^n (-1)^i (a_0 | g_0(a_1) | \cdots | g_0 \cdots g_{i-1}(a_i) | W | \cdots | g_0 \cdots g_{n-1}(a_n))_g. \end{aligned}$$

We conclude that Ψ is a map of double complexes. Taking the spectral sequences associated to the vertical filtrations of these double complexes as in Section 4 yields a map between spectral sequences which we shall denote by Ψ as well. Baranovsky’s isomorphism shows that the induced map on the E^1 page is everywhere an isomorphism. It follows from the comparison theorem of spectral sequences (see for example [20]) that the two spectral sequences converge to the same homology groups. Since the first one converges to $HH_*^{\text{BM}}((Y, W)/G)$ by definition, while the second

one converges to $(\bigoplus_{g \in G} HH_*^{\text{BM}}(Y^g, W|_{Y^g}))_G$ by the results in Section 4, the result is proved. \square

6.4. Example. Consider the case where $B = \mathbf{k}[x_1, \dots, x_d]$, $Y = \text{Spec } B = \mathbb{C}^d$, and $W \in B$ is a homogeneous polynomial of degree d . We take $G = \mathbb{Z}/d\mathbb{Z}$ acting diagonally on Y . Assume that $X = \text{Proj } B/W$ is smooth. Then Theorem 6.3 yields

$$HH_*^{\text{BM}}([(Y, W)/G]) = \left(HH_*^{\text{BM}}(Y, W) \right)_G \oplus \mathbb{C} \oplus \dots \oplus \mathbb{C},$$

where we have $d - 1$ copies of \mathbb{C} indexed by the nontrivial elements in the group G . (The contribution of the nontrivial elements of G to orbifold homology is traditionally called the twisted sectors of the theory.) Note that by the considerations in (6.1) the above is also a computation of $HH_*(X)$.

The above calculation can be regarded as a noncommutative version of the Lefschetz hyperplane theorem (for the part involving the twisted sectors) and of the Griffith transversality theorem (for the G -coinvariant part of the identity component, which can be computed using a Jacobian ring calculation as in Section 3).

For example for a smooth quartic surface in \mathbb{P}^3 its even Hochschild homology groups are of dimensions 1, 22, 1, respectively, obtained by adding the even verticals in the Hodge diamond below:

$$\begin{array}{ccccc} & & & & 1 \\ & & & 0 & 0 \\ & & 1 & 20 & 1 \\ & & 0 & 0 & \\ & & & & 1. \end{array}$$

The degree 0, 4, 8 components of the Jacobian ring of the defining quartic have dimensions 1, 19, 1 respectively. Compared with the above Hodge diamond what is missing is another 3 dimensional vector space down the middle vertical (which corresponds to HH_0), which are precisely the 3 one-dimensional twisted sectors. (The odd Hochschild homology is trivial on both sides.)

Appendix A. Dg categories and dg quotients

In this appendix we collect some results on dg categories and on the notion of dg quotients. Our main reference for dg categories is Keller's paper [9]. For dg quotients we refer to the original construction of Keller [8], see also Drinfel'd [4] for an alternative construction.

A.1. We shall fix a ground field k and all categories will be assumed to be k -linear categories. (All the constructions below can be generalized to any ground ring after suitable flat resolutions.)

A dg category is a category \mathcal{D} such that the Hom spaces are complexes over k . Composition maps are required to be not only k -linear but also maps

of complexes. Explicitly, for any objects $X, Y, Z \in \mathcal{D}$ the homomorphism space $\mathbf{Hom}_{\mathcal{D}}(X, Y)$ is a k -complex and the composition map

$$\mathbf{Hom}_{\mathcal{D}}(X, Y) \otimes \mathbf{Hom}_{\mathcal{D}}(Y, Z) \rightarrow \mathbf{Hom}_{\mathcal{D}}(X, Z)$$

is a map of complexes. Here the tensor product is the tensor product of complexes.

A.2. Similarly a dg functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is defined to be a k -linear functor such that the map

$$\mathbf{Hom}_{\mathcal{C}}(X, Y) \rightarrow \mathbf{Hom}_{\mathcal{D}}(FX, FY)$$

is a map of complexes for any objects X and Y of \mathcal{C} .

A.3. One can associate to a dg category \mathcal{D} its homotopy category $H^0(\mathcal{D})$ whose objects are the same as those of \mathcal{D} and the homomorphism set between two objects is defined by

$$\mathbf{Hom}_{H^0(\mathcal{D})}(X, Y) = H^0(\mathbf{Hom}_{\mathcal{D}}(X, Y)).$$

Moreover the composition in $H^0(\mathcal{D})$ is defined to be the one induced from \mathcal{D} (well-defined as the original composition maps are maps of complexes and hence induce maps on the cohomology). It is easy to see that dg functors induce maps on the corresponding homotopy categories. For a dg functor F we shall denote the induced functor on homotopy categories by $H^0(F)$.

A.4. A dg functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between dg categories \mathcal{C} and \mathcal{D} is said to be a quasi-equivalence if $H^0(F)$ is an equivalence. Two dg categories are said to be quasi-equivalent if they belong to the same equivalence class with respect to the equivalence relation generated by the above notion of quasi-equivalence.

A.5. Consider the category of small dg categories over k , denoted by $\mathbf{dgc}at_k$. Its objects are small dg categories and its morphisms are dg functors. As quasi-equivalences induce natural isomorphisms on most homological algebra constructions we would like to consider a modification of $\mathbf{dgc}at_k$ wherein quasi-equivalences are inverted. This is more or less in the same spirit as the construction of derived categories where quasi-isomorphisms between complexes are inverted. This construction can be carried out as explained in [9], and the resulting category \mathbf{Hqe} is obtained as the localization of $\mathbf{dgc}at_k$ with respect to quasi-equivalences.

One can show that various types of homological algebra invariants of dg categories factor through \mathbf{Hqe} . These include for example Hochschild homology, cyclic homology, Hochschild cohomology and more sophisticated invariants like open-closed string operations constructed by Costello [3].

In particular in Section 5 we proved the existence of a relationship between the dg category of coherent sheaves and the dg category of matrix factorization, both regarded as objects in \mathbf{Hqe} . In the Calabi–Yau case this

implies that these two categories are isomorphic in \mathbf{Hqe} , which further implies that these two dg categories carry isomorphic homological invariants as mentioned above. See Section 6 for an application.

A.6. There exists a twist construction for dg categories analogous to the one for A_∞ categories. It allows us to relate dg categories to triangulated categories. Explicitly the twist of a dg category \mathcal{D} is a new dg category $\mathbf{Tw}(\mathcal{D})$ such that the homotopy category of $\mathbf{Tw}(\mathcal{D})$ has a natural triangulated structure. We shall denote this resulting triangulated category by $\mathcal{D}^{\mathrm{tr}}$.

For example, an ordinary algebra A can be seen as a category with only one object. Then $\mathbf{Tw}(A)$ is the dg category of bounded complexes of free A -modules. Its homotopy category is in general not the derived category of A , but rather a fully faithful subcategory consisting of free objects. This example is a special case of the general statement that $\mathcal{D}^{\mathrm{tr}}$ is the fully faithful subcategory of $D(\mathcal{D})$ consisting of representable objects.

The dg category $\mathbf{Tw}(\mathcal{D})$ is also called the pre-triangulated envelope of \mathcal{D} . Intuitively speaking it is the smallest dg category containing \mathcal{D} such that its homotopy category has a triangulated structure. Thus we shall call a dg category \mathcal{D} pre-triangulated if the following two conditions hold:

- For any object $X \in \mathcal{D}$, $X[1]$ is isomorphic to an object of \mathcal{D} inside $\mathcal{D}^{\mathrm{tr}}$.
- For any closed morphism $f : X \rightarrow Y \in \mathcal{D}$, $\mathbf{Cone}(f)$ is isomorphic to an object of \mathcal{D} inside $\mathcal{D}^{\mathrm{tr}}$.

(The object $X[1]$, if it exists, is the unique object of \mathcal{D} representing the functor $\mathbf{Hom}_{\mathcal{D}}(-, X)[1]$, where the latter is the shift-by-one of the complex $\mathbf{Hom}_{\mathcal{D}}(-, X)$. A similar definition applies to the notion of the cone of a morphism.)

It follows easily from the definition that if \mathcal{D} is pre-triangulated then $H^0(\mathcal{D})$ has a triangulated structure, and the natural embedding of \mathcal{D} into $\mathbf{Tw}(\mathcal{D})$ is a quasi-equivalence. The induced functor on homotopy categories is a triangulated equivalence between triangulated categories.

A.7. If \mathcal{C} is a triangulated category and \mathcal{E} is a full triangulated subcategory of \mathcal{C} then the triangulated quotient category \mathcal{C}/\mathcal{E} is obtained by localizing the category \mathcal{C} with respect to the multiplicative system

$$S = \{f \in \mathbf{Hom}(\mathcal{C}) \mid \mathbf{Cone}(f) \in \mathcal{E}\}.$$

This is known as the triangulated quotient construction. There is a natural quotient functor from $\mathcal{C} \rightarrow \mathcal{C}/\mathcal{E}$.

There is a quotient construction in the dg context initiated by Keller [8] and later elaborated by Drinfel'd [4]. The following theorem summarizes the main results on dg quotients.

A.8. Theorem. *Let \mathcal{E} be a full subcategory of a dg category \mathcal{D} . Then there exists a dg category \mathcal{D}/\mathcal{E} together with a quotient map*

$$Q : \mathcal{D} \rightarrow \mathcal{D}/\mathcal{E}$$

in the category \mathbf{Hqe} such that Q and \mathcal{D}/\mathcal{E} have the universal property that every morphism in \mathbf{Hqe} from \mathcal{D} to some other dg category \mathcal{F} that annihilates \mathcal{E} (the image of any object in \mathcal{E} is isomorphic to zero in $H^0(\mathcal{F})$) factors through \mathcal{D}/\mathcal{E} .

Moreover, the dg quotient is the dg analogue of the triangulated quotient: we have

$$(\mathcal{D}/\mathcal{E})^{\text{tr}} \cong \mathcal{D}^{\text{tr}}/\mathcal{E}^{\text{tr}}.$$

A.9. Dg quotients have good functorial properties with respect to *localization pairs*. A localization pair \mathcal{B} is a pair of dg categories $(\mathcal{B}_1, \mathcal{B}_2)$ such that \mathcal{B}_1 is a full subcategory of \mathcal{B}_2 . A morphism F between localization pairs \mathcal{B} and \mathcal{D} is a dg functor from \mathcal{B}_2 to \mathcal{D}_2 that sends objects of \mathcal{B}_1 to objects of \mathcal{D}_1 . Keller showed that such a map F induces a map on the dg quotients

$$F : \mathcal{B}_2/\mathcal{B}_1 \rightarrow \mathcal{D}_2/\mathcal{D}_1.$$

A.10. For the remainder of this section we are interested in understanding semi-orthogonal decompositions at the dg level, and these make best sense in the case of pre-triangulated dg categories. Let \mathcal{D} be a pre-triangulated dg category, and denote by $\mathcal{C} = H^0(\mathcal{D})$ the homotopy category of \mathcal{D} . By the previous discussion \mathcal{C} is triangulated.

Recall the notion of semi-orthogonal decomposition of a triangulated category \mathcal{C} . Let \mathcal{E} be a full triangulated subcategory of \mathcal{C} . We would like to decompose \mathcal{C} as a “sum” of \mathcal{E} and its orthogonal complement. In order to do this we need the subcategory \mathcal{E} to be *admissible*: \mathcal{E} is said to be right (left) admissible if the inclusion functor from \mathcal{E} to \mathcal{C} admits a right (left) adjoint.

Assume that \mathcal{E} is a right admissible full triangulated subcategory of \mathcal{C} . Define the right orthogonal complement of \mathcal{E} to be the full subcategory \mathcal{E}^\perp of \mathcal{C} consisting of those objects $X \in \mathcal{C}$ such that

$$\text{Ext}^*(E, X) = 0 \quad \forall E \in \mathcal{E}.$$

One can show that \mathcal{E}^\perp is also triangulated and the functor defined as the composition

$$\mathcal{E}^\perp \rightarrow \mathcal{C} \rightarrow \mathcal{C}/\mathcal{E}$$

is a triangulated equivalence. One usually denotes this situation by $\mathcal{C} = \langle \mathcal{E}^\perp, \mathcal{E} \rangle$ to illustrate the fact that every object in \mathcal{C} can be obtained from \mathcal{E} and \mathcal{E}^\perp by taking cones and shifts.

We can make a similar definition of the notion of being admissible in the dg setting.

A.11. Definition. Let \mathcal{D} be a pre-triangulated dg category and let \mathcal{E} be a full subcategory of \mathcal{D} which is also pre-triangulated. Then \mathcal{E} is said to be right (left) admissible in \mathcal{D} if $H^0(\mathcal{E})$ is right (left) admissible in $H^0(\mathcal{D})$.

We shall denote by \mathcal{E}^\perp the full subcategory of \mathcal{D} consists of those objects in \mathcal{D} that are inside $H^0(\mathcal{E})^\perp$, or, equivalently, \mathcal{E}^\perp consists of those objects X such that $\text{Hom}_{\mathcal{D}}(E, X)$ is acyclic for any $E \in \mathcal{E}$.

As before we write

$$\mathcal{D} = \langle \mathcal{E}^\perp, \mathcal{E} \rangle.$$

We end this section with an easy lemma that is used in the proofs in Section 5.

A.12. Lemma. *Let \mathcal{E} be a right (left) admissible pre-triangulated subcategory of the pre-triangulated category \mathcal{D} . Then \mathcal{E}^\perp (${}^\perp\mathcal{E}$) is quasi-equivalent to the dg quotient \mathcal{D}/\mathcal{E} .*

Proof. Consider the composition of morphisms in \mathbf{Hqe}

$$\mathcal{E}^\perp \hookrightarrow \mathcal{D} \rightarrow \mathcal{D}/\mathcal{E}.$$

To show that this map is an isomorphism in \mathbf{Hqe} we only need to check the statement in the homotopy category. Consider the induced map on the corresponding homotopy categories

$$H^0(\mathcal{E}^\perp) \hookrightarrow H^0(\mathcal{D}) \rightarrow H^0(\mathcal{D}/\mathcal{E}).$$

Since both \mathcal{E} and \mathcal{D} are pre-triangulated we have

$$H^0(\mathcal{D}/\mathcal{E}) \cong (\mathcal{D}/\mathcal{E})^{\text{tr}}.$$

By Theorem A.8 we have

$$(\mathcal{D}/\mathcal{E})^{\text{tr}} \cong \mathcal{D}^{\text{tr}}/\mathcal{E}^{\text{tr}},$$

and the fact that the morphism

$$H^0(\mathcal{E}^\perp) \rightarrow \mathcal{D}^{\text{tr}}/\mathcal{E}^{\text{tr}}$$

is an isomorphism is the known result for triangulated categories. \square

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