New York Journal of Mathematics

New York J. Math. **18** (2012) 523–549.

Growth of maximal functions

S. Butler and **J. Rosenblatt**

Abstract. We consider the integrability of $\phi(f^*)$ for various maximal functions f^* and various increasing functions ϕ. We show that for some of the standard maximal functions arising in harmonic analysis and ergodic theory, there is never integrability of $\phi(f^*)$ for all Lebesgue integrable functions f except in cases where the growth of ϕ is slow enough so that the integrability follows from the standard weak maximal inequalities.

Contents

1. Introduction

Suppose that $(X, \mathcal{B}, \mu)$ is a σ-finite positive measure space and (T_n) is a sequence of bounded linear operators on $L_1(X, \mu)$. Consider the maximal function

$$f^* = \sup_{n \geq 1} |T_n f|$$

Received March 9, 2012.

2010 *Mathematics Subject Classification.* 42B25, 28D05, 37A05.

Key words and phrases. Maximal functions, maximal inequalities, Lebesgue derivatives, ergodic averages.

The second author was partially supported by NSF grant DMS 0555905.

ISSN 1076-9803/2012

for $f \in L_1(X,\mu)$. Pointwise almost everywhere convergence of $(T_n f)$ guarantees that f^* is finite almost everywhere. In many well-known cases, because of additional information that is available, it is enough to prove that f^* is finite a.e. for all $f \in L_1(X,\mu)$ in order to show that $(T_n f)$ converges a.e. for all $f \in L_1(X,\mu)$. The classical approach to this typically includes obtaining a weak $(1,1)$ maximal inequality of this form: there is a constant C such that for all $\lambda > 0$ and $f \in L_1(X,\mu)$,

$$\mu\{f^* > \lambda\} \leq \frac{C}{\lambda}\|f\|_1. \tag{1.1}$$

One way to prove this would be to have a strong maximal inequality: for some constant C, we have for all $f \in L_1(X,\mu)$,

$$\|f^*\|_1 \leq C\|f\|_1.$$

However, such a strong maximal inequality often fails to be true.

Nonetheless, it is important to characterize which $f \in L_1(X,\mu)$ have $f^* \in L_1(X,\mu)$. This is in general a difficult issue because of the following proposition. Given a subsequence $\mathbf{n} = (n_m)$ and $f \in L_1(X,\mu)$, let $f^*_{\mathbf{n}} = \sup\limits_{m\geq 1} |T_{n_m} f|$.

Proposition 1.1. *Suppose (T_n) are bounded linear operators on $L_1(X,\mu)$ and $(T_n f)$ is L_1-norm convergent for all $f \in L_1(X,\mu)$. Then for each $f \in L_1(X,\mu)$, there exists a subsequence $\mathbf{n}$ such that $f^*_{\mathbf{n}} \in L_1(X,\mu)$.*

Proof. Let $Lf \in L_1(X,\mu)$ denote the L_1-norm limit of $(T_n f)$. Take a subsequence $\mathbf{n} = (n_m)$ such that $\|T_{n_m} f - Lf\|_1 \leq \frac{1}{2^m}$. Then

$$f^*_{\mathbf{n}} \leq |Lf| + \sup_{m\geq 1} |T_{n_m} f - Lf| \leq |Lf| + \sum_{m=1}^{\infty} |T_{n_m} f - Lf|.$$

Hence

$$\|f^*_{\mathbf{n}}\|_1 \leq \|Lf\|_1 + \sum_{m=1}^{\infty} \|T_{n_m} f - Lf\|_1 \leq \|Lf\|_1 + \sum_{m=1}^{\infty} \frac{1}{2^m} < \infty. \qquad \square$$

Proposition 1.1 suggests the possibility that there is one subsequence $\mathbf{n}$ such that for all $f \in L_1(X,\mu)$ we have $f^*_{\mathbf{n}} \in L_1(X,\mu)$. However, we will see in this article that this is too much to expect because it is typically not the case. Can we reduce our expectations and obtain something nontrivial along these lines? Suppose momentarily $(X,\mathcal{B},\mu)$ is actually a probability space. Then the integrability of $f^*_{\mathbf{n}}$ would follow from having

$$\sum_{k=1}^{\infty} \mu\{f^*_{\mathbf{n}} > k\} < \infty.$$

See (2.6) and (2.7) below. What happens if we somewhat weaken our expectations here and replace (k) by a more rapidly increasing sequence of weights (w_k)? Given that there is a weak inequality as in (1.1), we would not want to take (w_k) increasing so rapidly that $\sum_{k=1}^{\infty} \frac{1}{w_k} < \infty$ because then clearly $\sum_{k=1}^{\infty} \mu\{f_{\mathbf{n}}^* > w_k\} < \infty$ for all f. So we assume that $\sum_{k=1}^{\infty} \frac{1}{w_k} = \infty$. We may assume without loss of generality that (w_k) is strictly increasing and $\lim_{k\to\infty} w_k = \infty$. Then we know that there is a strictly increasing smooth function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi^{-1}(k) = w_k$ for all $k \geq 1$. Also, in the case of a probability measure μ, it follows that $\phi(f_{\mathbf{n}}^*)$ would be integrable if and only if

$$\sum_{k=1}^{\infty} \mu\{\phi(f_{\mathbf{n}}^*) > k\} = \sum_{k=1}^{\infty} \mu\{f_{\mathbf{n}}^* > \phi^{-1}(k)\} = \sum_{k=1}^{\infty} \mu\{f_{\mathbf{n}}^* > w_k\} < \infty.$$

Remark 1.2. Here is an integral characterization of $\sum_{n=1}^{\infty} \frac{1}{\phi^{-1}(n)} = \infty$. Assume that ϕ is smooth and increasing. By the integral test, this is the same as $\int_1^{\infty} \frac{1}{\phi^{-1}(x)}\, d\mu(x) = \int_1^{\infty} \frac{\phi'(y)}{y}\, d\mu(y)$ is infinite.

This is the manner in which we arrive at the central question that is treated in various cases in this article. Take a sequence (T_n) of bounded linear operators on $L_1(X, \mathcal{B}, \mu)$. For generality, we do not restrict μ to being a probability measure, but assume that $(X, \mathcal{B}, \mu)$ is a σ-finite measure space. Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing smooth function.

Question 1.3. When do we have

$$\phi\left(\sup_{n\geq 1} |T_n f|\right) \in L_1(X, \mathcal{B}, \mu)$$

for all $f \in L_1(X, \mathcal{B}, \mu)$?

Answer. In this article, we show that for most natural settings (e.g., for differentiation, for ergodic averages, and for martingales), there is never a subsequence (T_n) of the process and a nontrivial function ϕ for which $\phi\left(\sup_{n\geq 1} |T_n f|\right)$ in $L_1(X, \mathcal{B}, \mu)$ for all $f \in L_1(X, \mathcal{B}, \mu)$.

In obtaining results as above, it certainly does matter what the stochastic process is. If (T_n) converges in operator norm to a limit operator L, then there is always of a subsequence (T_{n_m}) such that $\sup_{m\geq 1} T_{n_m}|f| \in L_1(X, \mu)$ for all $f \in L_1(X, \mu)$. Just take (T_{n_m}) such that $C = \sum_{m=1}^{\infty} \|T_{n_m} - L\|_1 < \infty$. It follows that $\sup_{m\geq 1} T_{n_m}|f| \leq \sum_{m=1}^{\infty} \|T_{n_m}|f| - L|f|\|_1 + \|L|f|\|_1 \leq (C+1)\|f\|_1$.

Example 1.4. An interesting example of this principle is the following. Let τ be an ergodic invertible measure-preserving transformation of a nonatomic probability space $(X,\mathcal{B},p)$. Let $T_nf(x) = \frac{1}{n}f(\tau^n x) + \frac{n-1}{n}f(x)$ for all $f \in L_1(X,\mu)$. It is not hard to see that T_n is a positive linear operator on $L_1(X,p)$ which has norm one, even if one restricts to the subspace of mean-zero functions in $L_1(X,p)$. Also, (T_n) converges to $L = Id$ in the operator norm limit. So there is a subsequence (T_{n_m}) such that $\sup\limits_{m\geq 1} T_{n_m}|f| \in L_1(X,p)$ for all $f \in L_1(X,p)$. On the other hand, $\sup\limits_{n\geq 1}|T_nf| \notin L_1(X,p)$ for the generic function in $L_1(X,p)$. To see this, use Proposition 2.16 in this article and the following construction that shows that for some $f \in L_1(X,p)$ one has $\sup\limits_{n\geq 1}|T_nf| \notin L_1(X,p)$. First, observe that if B_k is the base of a Rokhlin tower for τ of height N_k, and $f_k = \frac{1}{p(B_k)}1_{B_k}$, then $\|\sup_n |T_nf_k|\|_1 \geq \|\sum\limits_{n=0}^{N_k}\frac{1}{n}\frac{1}{p(B_k)}1_{\tau^{-n}B_k}\|_1 \geq C\log(N_k)$. So it is not hard see then that a series $f = \sum\limits_{k=1}^{\infty}\frac{1}{2^k}\frac{1}{p(B_k)}1_{B_k}$, where B_k is a base of a Rokhlin tower of height $N_k = \exp(k2^k)$, will give a function $f \in L_1(X,p)$ such that $\|\sup\limits_{n\geq 1}|T_nf|\|_1 = \infty$.

Example 1.5. Here is a very simple example that illustrates one can have $\sup\limits_{n\geq 1} T_n|f|$ always integrable, although no subsequence (T_{n_m}) converges in the operator norm. Let $T_nf = f1_{[0,1/n]}$ for any $f \in L_1([0,1],p)$ where p is Lebesgue measure on $[0,1]$. Then clearly $\sup\limits_{n\geq 1} T_n|f| \leq |f|$ and so $\sup\limits_{n\geq 1} T_n|f|$ is always integrable. But also (T_n) converges in the strong operator topology to $L = 0$, while $\|T_n\|_1 = 1$ for all $n \geq 1$. Hence, no subsequence (T_{n_m}) can converge in the operator norm since it would have to converge to 0 and yet $\|T_{n_m}\|_1 = 1$ for all $m \geq 1$.

2. Lebesgue derivatives

First, we consider the particular maximal functions that arise in the study of Lebesgue derivatives. It is enough to consider the one-sided Lebesgue derivatives which are defined by

$$D_\epsilon f(x) = \frac{1}{\epsilon}\int_0^\epsilon f(x+t)\,d\mu(t)$$

for functions $f \in L_1(\mathbb{R},\mu)$ where μ is the usual Lebesgue measure. We know that for all $f \in L_1(\mathbb{R},\mu)$, $(D_\epsilon f)$ converges in L_1-norm to f as $\epsilon \to 0$, and $(D_\epsilon f)$ converges in L_1-norm to 0 as $\epsilon \to \infty$. Also, it is a classical fact that the associated Hardy–Littlewood maximal functions $f^*_{HL}(x) = \sup\limits_{\epsilon>0}|D_\epsilon f(x)|$

satisfies a weak $(1,1)$ inequality

$$\mu\{f^*_{HL} > \lambda\} \le \frac{C}{\lambda}\int |f(t)|\, d\mu(t). \tag{2.1}$$

In addition, a theorem of Stein [15] shows that a positive function $f \in L_1(\mathbb{R},\mu)$ has f^*_{HL} locally integrable if and only if $f \in L\log L$, i.e.,

$$\int f(t)\log^+ f(t)\, d\mu(t) < \infty.$$

Suppose we restrict the maximal function so that we are only using a sequence $\mathbf{E} = (\epsilon_n)$ decreasing to 0 instead of all $\epsilon > 0$. We denote the maximal function in this case by $f^*_{\mathbf{E}}$, i.e., $f^*_{\mathbf{E}}(x) = \sup\limits_{n\ge 1} |D_{\epsilon_n} f(x)|$. This maximal function will of course satisfy a weak $(1,1)$ inequality as in (2.1). Also, if (ϵ_n) is not decreasing too quickly, e.g., $\epsilon_n = \frac{1}{2^n}$ for all $n \ge 1$, then $f^*_{\mathbf{E}}$ will again be integrable if and only if $f \in L\log L$. However, if we take a more quickly shrinking sequence (ϵ_n), the situation is different. Indeed, we have the following specific instance of Proposition 1.1.

Proposition 2.1. *For any $f \in L_1(\mathbb{R},\mu)$, there exists a decreasing sequence $\mathbf{E} = (\epsilon_n)$ such that $f^*_{\mathbf{E}} \in L_1(\mathbb{R},\mu)$.*

Remark 2.2. Hagelstein [7] gives a characterization of when one has an integrable maximal function for a general class of averaging operators that applies here. He gives a characterization of when $f^*_{\mathbf{E}}$ is integrable in terms of Córdoba–Fefferman collections. It is not clear how to relate his characterization to Proposition 2.1. In particular, it would be interesting to somehow link the decreasing subsequence $\mathbf{E}$ that one chooses in Proposition 2.1 to the structure of the Córdoba–Fefferman collections in [7]. One might not be able to do this directly for f but need to look at associated functions that locally overestimate f when it is highly oscillatory because if f oscillates a great deal within bounds on the range of f then it may require choosing $\mathbf{E}$ to grow faster in order to use Proposition 2.1, while the relevant Córdoba–Fefferman collections do not need to change much.

Remark 2.3. If we had taken (ϵ_n) increasing to infinity, the situation would be different because then the Lebesgue derivatives converge in operator norm to zero. To distinguish the notation in this case, we write $f^*_{\mathbf{B}}$ when $\mathbf{B} = (B_n)$ is a sequence increasing to infinity and $f^*_{\mathbf{B}} = \sup\limits_{n\ge 1} |D_{B_n} f|$. Now $\|D_{B_n} f\|_\infty \le \frac{\|f\|_1}{B_n}$. So, if we had $C = \sum\limits_{n=1}^{\infty} \frac{1}{B_n} < \infty$, then $\|f^*_{\mathbf{B}}\|_\infty \le C\|f\|_1$ for all $f \in L_1(\mathbb{R},\mu)$. On the other hand, we generally do not have integrability of $f^*_{\mathbf{B}}$ is this case. For example, take $f = 1_{[0,1]}$ and any increasing (B_n) tending to infinity; then $\|f^*_{\mathbf{B}}\|_1 = \infty$.

Proposition 2.1 says that by taking a rarer (i.e., more rapidly decreasing) sequence $\mathbf{E}$, we can control the size of $f^*_{\mathbf{E}}$. The question is whether or not the right choice of $\mathbf{E}$ would allow this to hold on all of $L_1(\mathbb{R},\mu)$. In addition, as discussed in Section 1, we can measure this size by knowing whether or not for a given increasing function $\phi:\mathbb{R}^+\to\mathbb{R}^+$, we have $\|\phi(f^*_{\mathbf{E}})\|_1<\infty$ for all functions $f\in L_1(\mathbb{R},\mu)$. Again, if ϕ increases slowly enough, then $\sum\limits_{n=1}^{\infty}\frac{1}{\phi^{-1}(n)}<\infty$. In this case, (2.1) tells us that for some constant C and for all compactly supported $f\in L_1(\mathbb{R},\mu)$,

$$\|\phi(f^*_{\mathbf{E}})\|_1\le C\sum_{k=1}^{\infty}\mu\{f^*_{\mathbf{E}}>\phi^{-1}(k)\}\le C\|f\|_1\sum_{k=1}^{\infty}\frac{1}{\phi^{-1}(k)}<\infty.$$

For example, take $\phi(x)=\sqrt{x}$. Then for every $f\in L_1(\mathbb{R},\mu)$, we have $\|\phi(f^*)\|_1\le C\|f\|_1$. It follows that for all $f\in L_1(\mathbb{R},\mu)$, we have $\|\phi(f^*)\|_1\le C\|f\|_1$.

Hence to have a nontrivial result, we would have to assume

$$\sum_{k=1}^{\infty}\frac{1}{\phi^{-1}(k)}=\infty. \tag{2.2}$$

That is, given this growth assumption on ϕ, does there exist a sequence $\mathbf{E}$ decreasing to zero such that for all $f\in L_1(\mathbb{R},\mu)$, we have $\|\phi(f^*_{\mathbf{E}})\|_1<\infty$? This question and related issues are discussed in this section for the Hardy–Littlewood maximal function. First, in Section 2.1 we look at the case of slowly decreasing sequences (ϵ_n) and next in Section 2.2 we look at the case of rapidly decreasing sequences (ϵ_n). The reason that we consider these cases separately is that different, interesting issues arise in these two cases.

2.1. The nonrare case for differentiation. Suppose we are dealing with the maximal function f^* such that there is a reverse weak $(1,1)$ inequality of the following form: there is a constant C such that for any $f\in L_1(X,\mu)$, and some lower limit λ_f depending on f, whenever $\lambda\ge\lambda_f$, we have

$$\mu\{f^*>\lambda\}\ge\frac{1}{C\lambda}\int_{\{f\ge C\lambda\}}f\,d\mu. \tag{2.3}$$

Reverse inequalities like this, or weaker forms of this type of inequality, will be the basis of much of the analysis that follows in this and later sections.

Remark 2.4. We sometimes obtain an inequality like (2.3) because a stronger fact is true: that this holds where the set being integrated over is $\{f^*\ge C_1\lambda\}$ for some constant C_1, and also there is a constant C_2 such that $f\le C_2f^*$.

In proving the $L\log L$ result in [15], Stein used the inequality in (2.3). He showed that with no restriction on $\lambda>0$, if $f\in L_1(\mathbb{R},\mu)$, then

$$\mu\{f^*_{HL}>\lambda\}\ge\frac{1}{C\lambda}\int_{\{f\ge C\lambda\}}f\,d\mu. \tag{2.4}$$

However, if one takes a smaller maximal function like $f^*_{\mathbf{E}}$, in the case that $\mathbf{E} = (\epsilon_n)$ with $\epsilon_n = \frac{1}{2^n}$, then some lower limit on λ is needed. Take for example, the larger maximal function $f^*_1 = \sup_{0<\epsilon\leq 1} D_\epsilon|f|$. Then a reverse inequality as in (2.3) cannot hold for arbitrarily small λ when the function f has bounded support. For example, suppose $f = 1_{[0,1]}$. Then $\mu\{f^*_1 > \lambda\} \leq 2$ for all λ. But if $\lambda \leq \frac{1}{C}$, then $\frac{1}{C\lambda} \int\limits_{\{f\geq C\lambda\}} f\,d\mu = \frac{1}{C\lambda}$. So (2.3) implies that $\lambda \geq \frac{1}{2C}$. What we can say for f^*_1 is that for all $f \in L_1(\mathbb{R},\mu)$ and $\lambda \geq \|f\|_1$, we have

$$\mu\{f^*_1 > \lambda\} \geq \frac{1}{C\lambda}\int_{\{f\geq C\lambda\}} f\,d\mu. \tag{2.5}$$

This equation holds because for $\lambda \geq \|f\|_1$, $f^*_{HL}(x) > \lambda$ occurs exactly when $f^*_1(x) > \lambda$ since $D_\epsilon|f|(x) \leq \|f\|_1$ when $\epsilon \geq 1$. So (2.4) gives (2.5).

We show now how reverse inequalities can be used to give a negative answer to Question 1.3.

Proposition 2.5. *Given a maximal function such that* (2.3) *holds for a universal constant C, and a smooth, increasing function ϕ such that $\sum\limits_{k=1}^{\infty} \frac{1}{\phi^{-1}(k)} = \infty$, there exists $f \in L_1(X,\mathcal{B},\mu)$, supported on a set of finite measure, such that $\|\phi(f^*)\|_1 = \infty$.*

Proof. For a positive $\mathcal{B}$-measurable function F on X, we have the inequality

$$\int F\,d\mu \geq \sum_{n=1}^{\infty} \mu\{F > n\}. \tag{2.6}$$

Now suppose $\phi(f^*)$ is integrable. Let $w_n = \phi^{-1}(n)$ for all $n \geq 1$. Then by (2.6) and (2.3), we would know that

$$\sum_{n=1}^{\infty} \frac{1}{w_n}\int_{\{f\geq Cw_n\}} f\,d\mu < \infty.$$

This holds because for some N_f, we would know that

$$\begin{aligned}\int \phi(f^*)\,d\mu &\geq \sum_{n=1}^{\infty} \mu\{\phi(f^*) > n\}\\ &= \sum_{n=1}^{\infty} \mu\{f^* > w_n\}\\ &\geq \sum_{n=N_f}^{\infty} \mu\{f^* > w_n\}\\ &\geq \sum_{n=N_f}^{\infty} \frac{1}{Cw_n}\int_{\{f\geq Cw_n\}} f\,d\mu.\end{aligned}$$

This proposition is asserting that for some $f \in L_1$ this cannot hold if $\sum\limits_{n=1}^{\infty} \frac{1}{w_n} = \infty$.

Take $f = \sum\limits_{n=1}^{\infty} C w_n 1_{E_n}$ for some sequence of sets such that $\sum\limits_{n=1}^{\infty} w_n \mu(E_n) < \infty$. Then $f \in L_1(X,\mu)$ and f is supported on $\bigcup\limits_{n=1}^{\infty} E_n$, which is necessarily of finite measure since $\lim\limits_{n\to\infty} w_n = \infty$. We are going to see how to choose (E_n) such that this function gives our result. Let $\rho_n > 0$ be such that $\sum\limits_{n=1}^{\infty} \rho_n < \infty$. Choose any (E_n) such that $\mu(E_n) = \frac{\rho_n}{w_n}$. We will see how to choose (ρ_n) decreasing to zero slowly enough so that this choice of (E_n) gives us what we want. Notice that we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{w_n} \int_{\{f \geq C w_n\}} f\, d\mu &\geq \sum_{n=1}^{\infty} \frac{1}{w_n} \int_{\bigcup\limits_{k=n}^{\infty} E_k} f\, d\mu \\
&\geq \sum_{n=1}^{\infty} \frac{1}{w_n} \int_{\bigcup\limits_{k=n}^{\infty} E_k} \sum_{j=n}^{\infty} C w_j 1_{E_j}\, d\mu \\
&= C \sum_{n=1}^{\infty} \frac{1}{w_n} \int \sum_{j=n}^{\infty} w_j 1_{E_j}\, d\mu \\
&= C \sum_{n=1}^{\infty} \frac{1}{w_n} \sum_{j=n}^{\infty} \rho_j.
\end{aligned}$$

But since $\sum\limits_{n=1}^{\infty} \frac{1}{w_n} = \infty$, we can choose (ρ_n) such that $\sum\limits_{j=n}^{\infty} \rho_j$ decreases so slowly that

$$\sum_{n=1}^{\infty} \frac{1}{w_n} \sum_{j=n}^{\infty} \rho_j = \infty.$$

Therefore, $f \in L_1(X,\mu)$, has support of finite measure, and

$$\sum_{n=1}^{\infty} \mu\{f^* > w_n\} = \infty.$$

Hence, $\|\phi(f^*)\|_1 = \infty$ by (2.6). □

Remark 2.6.

(a) If μ is finite, then (2.6) can be reversed in the sense that

$$\int F\, d\mu \leq \mu(X) + \sum_{n=1}^{\infty} \mu\{F > n\}. \tag{2.7}$$

This and (2.6) are what give the standard fact that on a finite measure space $(X, \mathcal{B}, \mu)$, a positive, $\mathcal{B}$-measurable function F is integrable if and only if $\sum_{n=1}^{\infty} \mu\{F > n\} < \infty$.

(b) The argument above may seem odd because it may not be clear why it works so easily. After all, we could use $Cf^* \geq f$ directly and try to show that $\sum_{n=1}^{\infty} \mu\{f^* > w_n\} = \infty$ because $\sum_{n=1}^{\infty} m\{f > Cw_n\} = \infty$. However, with the examples where we do not know yet if there is $f \in L_1(\mathbb{R}, \mu)$ such that $\sum_{n=1}^{\infty} \mu\{f > Cw_n\} = \infty$, we have (w_n) increasing faster than n and so generally $\sum_{n=1}^{\infty} \mu\{f > Cw_n\} < \infty$. So this approach does not work.

(c) We can also get some insights into this argument by asking what is happening when $w_n = n$, in which case we are trying to show that f^* is not integrable. This is equivalent to having f not in $L \log L$ for positive functions f. So our required condition on (ρ_n) should not be possible when $f \in L \log L$. But

$$\int f \log f \, d\mu = \sum_{n=1}^{\infty} (n \log n) \frac{\rho_n}{n} = \sum_{n=1}^{\infty} (\log n) \rho_n.$$

Hence, $f \in L \log L$ implies by summation by parts that $\sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=n}^{\infty} \rho_k$ converges, which is the opposite of the condition that we need to show that f^* is not integrable.

Remark 2.7. Proposition 2.5 shows why it is an important issue to decide when a maximal function f^* satisfies (2.3). For example, in Theorem 1 in Hare and Stokolos [8], they consider the case of the Hardy–Littlewood maximal function, $f^*_{\mathbf{E}}$, with $\mathbf{E} = (\frac{1}{2^{m_n}})$ for some $m_n \to \infty$ as $n \to \infty$. They give an argument that (2.3) holds only when $m_{n+1} - m_n$ is bounded. However, their argument needs some clarification because the inequality they are considering never holds for functions of bounded support as λ goes to zero. Perhaps what was intended in [8] was to prove their result with a restriction on λ such as $\lambda \geq \|f\|1$. But in any case, one can see that Hagelstein [6] completely clarifies this result.

Proposition 2.5 gives this specific result for the maximal function f_1^*, the Hardy–Littlewood maximal function for ϵ with $0 < \epsilon \leq 1$.

Corollary 2.8. *Let ϕ be a smooth, increasing function such that $\sum_{k=1}^{\infty} \frac{1}{\phi^{-1}(k)} = \infty$. Then there exists $f \in L_1(\mathbb{R}, \mu)$, supported on a set of finite measure, such that $\|\phi(f_1^*)\|_1 = \infty$.*

Remark 2.9. Kita [10] proves a result like Corollary 2.8 without an explicit use of reverse inequalities. Using Kita's notation, just take $b(s) = 1$ and so $\Psi(t) = t$. Assume that Φ satisfies the doubling condition so that f_1^* is in the Orlicz space L^Φ if and only if $\|a(f_1^*)\|_1$ is finite. We take our function ϕ to be a. Then the condition from Remark 1.2 is what Kita assumes to prove the main result in [10]: that $\int\limits_0^\infty \frac{a(s)}{s}\,ds = \infty$. But then if $\phi(f_1^*)$ is integrable for all $f \in L_1([0,1],\mu)$, by the choice of b it follows that $\int\limits_0^s \frac{a(t)}{t}\,dst$ is bounded in s. This contradiction shows that there must exist $f \in L_1([0,1],\mu)$ such that $\|\phi(f_1^*)\|_1 = \infty$.

2.2. The rare case for differentiation. We now want to consider maximal functions $f_{\mathbf{E}}^*$ for which the results of Section 2.1 do not apply because the sequence $\mathbf{E} = (\epsilon_n)$ is decreasing very rapidly. We call these *rare maximal functions*. For example, if $\epsilon_n = \frac{1}{2^n}$ for all $n \geq 1$, then the same result that holds for f_1^* proved in Corollary 2.8 holds for $f_{\mathbf{E}}^*$ because $f_1^* \leq 2f_{\mathbf{E}}^*$. However, if $\epsilon_n = \frac{1}{2^{n^2}}$ for all $n \geq 1$, then this proportionality no longer holds and so the results in Section 2.1 do not apply. Here we will see that there is a restricted reverse inequality that we can use that will allow us to prove results in general for maximal functions $f_{\mathbf{E}}^*$.

First, there is a special case of the type of result we are discussing in this section which gives some insight into the computational issues. Take our function $\phi(x) = x$ for all $x \geq 0$. Then we are asking if there is always an $f \in L_1(\mathbb{R},\mu)$ such that $f_{\mathbf{E}}^* \notin L_1(\mathbb{R},\mu)$?

Proposition 2.10. *Suppose* $\mathbf{E}$ *is a sequence decreasing to zero. Then there exists* $f \in L_1(\mathbb{R},\mu)$ *with support in* $[0,1]$ *such that* $f_{\mathbf{E}}^*$ *is not in* $L_1(\mathbb{R},\mu)$.

Proof. Consider $f^*(N,\delta) = \sup\limits_{1\leq n\leq N} D_{\epsilon_n} f_\delta$ where $f_\delta = \frac{1}{\delta}1_{[0,\delta]}$, for some $\delta > 0$. As $\delta \to 0$, $f^*(N,\delta)$ converges in L_1-norm to $\sup\limits_{1\leq n\leq N} \frac{1}{\epsilon_n}1_{[-\epsilon_n,0]}$. Hence, we have $\|f^*(N,\delta)\|_1 \to 1 + \sum\limits_{n=1}^{N-1} \frac{\epsilon_n-\epsilon_{n+1}}{\epsilon_n}$. It is elementary to show that $\sum\limits_{n=1}^{\infty} \frac{\epsilon_n-\epsilon_{n+1}}{\epsilon_n}$ diverges. Indeed, this series only decreases by taking a subsequence and clearly diverges if (ϵ_n) is lacunary. See the argument in the proof of Proposition 5.1 in Butler, Pavlov, and Rosenblatt [3]. So we can choose δ_k and (N_k) such that $\|f^*(N_k,\delta_k)\|_1 \geq k2^k$. But then consider $f = \sum\limits_{k=1}^{\infty} \frac{1}{2^k} f_{\delta_k}$. We have f supported in $[0,1]$ and $\|f\|_1 = 1$. But also, $f_{\mathbf{E}}^* \geq \frac{1}{2^k} f^*(N_k,\delta_k)$ and so $\|f_{\mathbf{E}}^*\|_1 \geq \frac{1}{2^k}\|f^*(N_k,\delta_k)\|_1 \geq k$ for all k. Hence, $\|f_{\mathbf{E}}^*\|_1 = \infty$. □

Remark 2.11. In proving Proposition 2.10, by passing to a subsequence, we could have assumed at the outset that (ϵ_n) is lacunary. In this case, the divergence of $\sum\limits_{n=1}^{\infty} \frac{\epsilon_n-\epsilon_{n+1}}{\epsilon_n}$ is immediately clear.

We could try this same approach to the integrability of $\phi(f^*_{\mathbf{E}})$. We would need to know that $\sum_{n=1}^{\infty} \phi(\frac{1}{\epsilon_n})(\epsilon_n - \epsilon_{n+1})$ always diverges. But take a function ϕ like $\phi(x) = x/\log^+ x$. Then $\phi^{-1}(n) \le Cn \log n$ and so we do have $\sum_{n=1}^{\infty} \frac{1}{\phi^{-1}(n)} = \infty$. However, if $\epsilon_n = 1/2^{n^2}$, then $\sum_{n=1}^{\infty} \phi(\frac{1}{\epsilon_n})\epsilon_n$ converges, and of course then so does $\sum_{n=1}^{\infty} \phi(\frac{1}{\epsilon_n})(\epsilon_n - \epsilon_{n+1})$. So the simple method that we used above does not work this time.

Therefore, we need another approach with rare maximal functions. The idea for the necessary step may be provided by this theorem in Stokolos [17], in the case that $n = 1$. This result says the following.

Proposition 2.12 (Stokolos)**.** *For any E, there is a constant C such that for each λ, $0 < \lambda \le 1$, there is a bounded measurable set Q such that*

$$\mu\{(1_Q)^*_{\mathbf{E}} > \lambda\} \ge \frac{1}{C\lambda}\mu(Q).$$

We need to extend this so that we can use more than one λ for a given set Q. Notice that for a fixed C, if λ is too small related to the size of $\mu(Q)$, then the right hand side would be larger than 1 and the left hand side is not. So if we are going to use smaller values of λ, then we have to shrink $m(Q)$ too, and this creates an issue of the balance between these two factors.

Proposition 2.13. *There exists a constant C such that for all sequences $\mathbf{E}$ decreasing to 0 and ϵ, $0 < \epsilon < 1$, there exist a measurable set $Q \subset [0,1]$ such that $\mu(Q) < \epsilon$ and for any $\lambda \in [\mu(Q), 1]$*

$$\mu\left\{(1_Q)^*_{\mathbf{E}} > \frac{\lambda}{C}\right\} \ge \frac{1}{C\lambda}\mu(Q).$$

Proof. The proof uses ideas of the proof of the basic theorem in Stokolos [17]. First, without loss of generality, if necessary we can replace $\mathbf{E}$ by a subsequence so that if we write for each $n, 1/2^{k_n-1} \ge \epsilon_n > 1/2^{k_n}$, for a whole number k_n, then (k_n) is strictly increasing, and even $k_{n+1} \ge k_n + 2$. Let $\mathbf{E_0} = (1/2^{k_n} : n \ge 1)$. Now for any n,

$$D_{\epsilon_n}|f|(x) \ge \frac{1}{1/2^{k_n-1}} \int_0^{1/2^{k_n}} |f(x+t)|\, d\mu(t).$$

So $f^*_{\mathbf{E}} \ge \frac{1}{2} f^*_{\mathbf{E_0}}$.

Now we first work with the Hardy–Littlewood maximal function $f^*_{\mathbf{E_0}}$. We extend the index sequence $(k_n : n \ge 1)$ by letting $k_0 = 0$. Let

$$r_n(x) = \text{ sign } \sin(2^n \pi x), \quad n = 0, 1, 2, \ldots$$

be the Rademacher functions on $[0,1]$. Choose a whole number J such that $2^{-J} < \epsilon$. For $j = 0, \dots, J$ consider sets

$$V_j = \{x \in [0,1] : r_{k_n}(x) = 1 \text{ for } n = 0, \dots, j\}.$$

Each set V_j consists of disjoint dyadic intervals of length 2^{-k_j}. It is easy to see that $V_0 = [0,1]$, $V_j \subset V_{j-1}$ and $\mu(V_j) = \frac{1}{2}\mu(V_{j-1})$ for $j = 1, \dots, J$. Thus, for $j = 0, \dots, J$

$$\mu(V_j) = \frac{1}{2^j}.$$

Let $Q = V_J$. Note that $\mu(Q) = 2^{-J} < \epsilon$, and $Q \subset V_j$ for $j = 0, \dots, J$. Let I be any of the constituent intervals of length 2^{-k_j} that make up V_j. Observe that

$$\frac{\mu(I \cap Q)}{\mu(I)} = \frac{\frac{1}{2^{J-j}}\mu(I)}{\mu(I)} = \frac{1}{2^{J-j}}.$$

Because of the periodic structure of the constituent intervals in the sets V_j, one can see that for any x in the left hand half of I, $x + [0, 1/2^{k_j}]$ contains the right hand half of I. Hence, since $k_{n+1} \geq k_n + 2$ for all n, we have

$$\frac{\mu((x + [0, 2^{-k_j}]) \cap Q)}{2^{-k_j}} \geq \frac{1}{2}\frac{\mu(I \cap Q)}{\mu(I)} = \frac{1}{2}\frac{1}{2^{J-j}}.$$

So we have

$$(1_Q)^*_{\mathbf{E_0}}(x) \geq \frac{1}{2^{-k_j}} \int_{x+[0,2^{-k_j}]} 1_Q \, d\mu = \frac{\mu((x + [0, 2^{-k_j}]) \cap Q)}{2^{-k_j}} \geq \frac{1}{2}\frac{1}{2^{J-j}}.$$

It follows that if V_j^0 is the union of the left hand halves of all of the constituent intervals I in V_j, then

$$V_j^0 \subset \left\{x : (1_Q)^*_{\mathbf{E_0}}(x) \geq \frac{1}{2^{J-j+1}}\right\}$$

for $j = 0, \dots, J$.

Now if $\lambda \in [\mu(Q), 1]$ choose j, $1 \leq j \leq J$, such that

$$\frac{1}{2^{J-j+1}} \leq \lambda \leq \frac{1}{2^{J-j}}.$$

Then

$$\begin{aligned}
\mu\left\{x : (1_Q)^*_{\mathbf{E_0}}(x) \geq \frac{\lambda}{2}\right\} &\geq \mu\left\{x : (1_Q)^*_{\mathbf{E_0}}(x) \geq \frac{1}{2^{J-j+1}}\right\} \\
&\geq \mu(V_j^0) \\
&= \frac{1}{2}\mu(V_j) = \frac{1}{2^{j+1}} \\
&= \mu(Q)2^{J-j-1} = \mu(Q)\frac{1}{4}2^{J-j+1} \\
&\geq \mu(Q)\frac{1}{4\lambda}.
\end{aligned}$$

Using this inequality, we see that for $\lambda \in [\mu(Q), 1]$, we have

$$\mu\left\{x : (1_Q)^*_{\mathbf{E}} \geq \frac{\lambda}{4}\right\} \geq \mu(Q)\frac{1}{4\lambda}.$$

So let $C = 5$ and we get the inequality that we wanted. □

Remark 2.14.

(a) The set Q in Proposition 2.13 depends on the choice of $\mathbf{E}$.

(b) The inequality in Proposition 2.13 does not hold, even in a suitably altered form, for all measurable sets in place of Q. So also there can not be a reverse maximal inequality for a rare (enough) maximal function in this context. See Hare and Stokolos [8]. This is what Hare and Stokolos [8] and Hagelstein [6] are characterizing, as discussed in Remark 2.7.

Proposition 2.13 gives this result.

Proposition 2.15. *Suppose ϕ is a smooth, increasing function such that $\sum\limits_{n=1}^{\infty} \frac{1}{\phi^{-1}(n)} = \infty$. Let $\mathbf{E}$ be a sequence decreasing to 0. Then there exists $f \in L_1(\mathbb{R}, \mu)$, supported in $[0, 1]$, such that $\|\phi(f^*_{\mathbf{E}})\|_1 = \infty$.*

Proof. Let $\phi^{-1}(n) = w_n$. This result follows if we can show that there exists a positive function $f \in L_1(\mathbb{R}, \mu)$, supported in $[0, 1]$, such that

$$\sum_{n=1}^{\infty} \mu\{f^*_{\mathbf{E}} > w_n\} = \infty.$$

We may assume that $w_n \geq 1$ for each n. Choose a convergent series $\sum\limits_{k=1}^{\infty} t_k < \infty$, $0 < t_k \leq 1$. For each k, choose $Q = Q_k \subset [0, 1]$ satisfying the conditions in Proposition 2.13 but with $\mu(Q_k)$ so small that if

$$N_k = \left\{n \in \mathbb{N} : \ w_n \leq \frac{t_k}{\mu(Q_k)}\right\},$$

then

$$\sum_{n \in N_k} \frac{1}{w_n} \geq \frac{k}{t_k}. \tag{2.8}$$

Then for $n \in N_k$

$$\mu(Q_k) \leq \frac{w_n \mu(Q_k)}{t_k} \leq 1,$$

and by Proposition 2.13

$$\mu\left\{\left(\frac{Ct_k}{\mu(Q_k)} 1_{Q_k}\right)^*_{\mathbf{E}} > w_n\right\} = \mu\left\{(1_{Q_k})^*_{\mathbf{E}} > \frac{w_n \mu(Q_k)}{Ct_k}\right\} \geq \frac{t_k}{Cw_n}. \tag{2.9}$$

Define the positive function $f \in L_1(\mathbb{R}, \mu)$ supported in $[0,1]$ by:

$$f = \sum_{k=1}^{\infty} \frac{Ct_k}{\mu(Q_k)} 1_{Q_k}.$$

Then using (2.9) and (2.8) we have for each k:

$$\begin{aligned}
\sum_{n=1}^{\infty} \mu\{f^*_{\mathbf{E}} > w_n\} &\geq \sum_{n=1}^{\infty} \mu \left\{ \left(\frac{Ct_k}{\mu(Q_k)} 1_{Q_k} \right)^*_{\mathbf{E}} > w_n \right\} \\
&\geq \sum_{n \in N_k} \mu \left\{ \left(\frac{Ct_k}{\mu(Q_k)} 1_{Q_k} \right)^*_{\mathbf{E}} > w_n \right\} \\
&\geq \sum_{n \in N_k} \frac{t_k}{Cw_n} \\
&\geq t_k \, \frac{k}{Ct_k} = \frac{k}{C}.
\end{aligned}$$

Since k is arbitrary, it follows that

$$\sum_{n=1}^{\infty} m\{f^*_{\mathbf{E}} > w_n\} = \infty. \qquad \square$$

2.3. Generic counterexamples. We can usually also obtain a result that says that our counterexamples to integrability are generic. Here is one approach to such a result.

Proposition 2.16. *Suppose $(\mathcal{T}_n)$ is a sequence of positive, continuous linear operators on $L_1(X, \mathcal{B}, \mu)$. Suppose ϕ is a increasing smooth function. Assume that for all K, there is a positive function $f \in L_1(X, \mu)$, $\|f\|_1 = 1$ such that $\left\| \phi \left(\sup_{n \geq 1} \mathcal{T}_n |f| \right) \right\|_1 \geq K$. Then there is a dense G_δ set $\mathcal{L}$ in $L_1(X, \mu)$ such that for any $f \in \mathcal{L}$, we have $\left\| \phi \left(\sup_{n \geq 1} \mathcal{T}_n |f| \right) \right\|_1 = \infty$.*

Proof. Consider the set

$$\mathcal{G}_K = \left\{ f \in L_1(X, \mu) : \left\| \phi \left(\sup_{n \geq 1} \mathcal{T}_n |f| \right) \right\|_1 \leq K \right\}.$$

We claim that this is closed in L_1-norm and has no interior. It follows that $\mathcal{L} = L_1(X, \mu) \backslash \bigcup_{K \geq 1} \mathcal{G}_K$ is a dense G_δ set with the property that we wanted.

To see that $\mathcal{G}_K$ is closed, we observe that with

$$\mathcal{G}_{K,N} = \left\{ f \in L_1(X, \mu) : \left\| \phi \left(\sup_{n \leq N} \mathcal{T}_n |f| \right) \right\|_1 \leq K \right\},$$

we have $\mathcal{G}_K = \bigcap_{N\geq 1} \mathcal{G}_{K,N}$. So it suffices to show each $\mathcal{G}_{K,N}$ is L_1-norm closed. But if (f_k) is a sequence in $\mathcal{G}_{K,N}$ converging in L_1-norm to f, then without loss of generality we may assume the convergence is also pointwise a.e. since we can pass to a subsequence which converges almost everywhere. Then $\phi\left(\sup_{n\leq N} \mathcal{T}_n|f_k|\right)$ converges pointwise a.e. to $\phi\left(\sup_{n\leq N} \mathcal{T}_n|f|\right)$, and so by Fatou's Lemma,

$$\left\|\phi\left(\sup_{n\leq N} \mathcal{T}_n|f|\right)\right\|_1 \leq \liminf_{k\to\infty} \left\|\phi\left(\sup_{n\leq N} \mathcal{T}_n|f_k|\right)\right\|_1 \leq K.$$

Assume now that $\mathcal{G}_K$ contains interior. Then there is some $f_0 \in L_1(X,\mu)$ and $\delta > 0$ such that for all $f \in L_1(X,\mu), \|f\|_1 \leq 1$, we have $f_0 + \delta f \in \mathcal{G}_K$. Let σ be a measurable function with $|\sigma| = 1$ a.e. such that $|f_0| = \sigma f_0$. We have $||f_0| + \delta f| = |\sigma f_0 + \delta f| = |f_0 + \overline{\sigma}\delta f|$. So for any $f \in L_1(X,\mu)$ with $\|f\|_1 \leq 1$, we have $|f_0| + \delta f \in \mathcal{G}_K$ also. Now take a positive function $f \in L_1(X,\mu)$ with $\|f\|_1 = 1$ and $\left\|\phi\left(\sup_{n\geq 1} \mathcal{T}_n f\right)\right\|_1 \geq \frac{2K}{\delta}$. Then, because the operators $\mathcal{T}_n$ are positive and ϕ is nondecreasing,

$$2K \leq \left\|\phi\left(\sup_{n\geq 1} \mathcal{T}_n(\delta f)\right)\right\|_1 \leq \left\|\phi\left(\sup_{n\geq 1} \mathcal{T}_n(|f_0| + \delta f)\right)\right\|_1 \leq K.$$

This is impossible. □

Remark 2.17. It should be just a technical adjustment to prove the same result with the maximal function $\sup_{n\geq 1} |\mathcal{T}_n f|$, but at this time we do not have a proof of this stronger fact.

This general result gives the following.

Proposition 2.18. *Suppose ϕ is a smooth, increasing function such that $\sum_{n=1}^{\infty} \frac{1}{\phi^{-1}(n)} = \infty$. Let $\mathbf{E}$ be a sequence decreasing to 0. Then there exists a dense G_δ set $\mathcal{L}$, in the closed subspace of functions in $L_1(\mathbb{R},\mu)$. which are supported on $[0,1]$, such that for all $f \in \mathcal{L}$ we have $\|\phi(f^*_{\mathbf{E}})\|_1 = \infty$.*

Proof. This follows immediately from Proposition 2.16 and Proposition 2.8. □

3. Ergodic maximal functions

We now consider the behavior of the maximal function f^*_τ for ergodic averages on a standard Lebesgue probability space $(X,\mathcal{B},p)$. We have an invertible ergodic measure-preserving transformation τ on (X,p) and let $A_N f = \frac{1}{N}\sum_{n=0}^{N-1} f\circ\tau^n$. Then $f^*_\tau = \sup_{N\geq 1} |A_N f|$ for $f \in L_1(X,p)$. Again, we could consider the size of the full maximal function f_τ^* or more generally the maximal function $f_{\mathbf{N}}^* = \sup_{m\geq 1} |A_{N_m} f|$ for a subsequence $\mathbf{N} = (N_m)$ of $\mathbb{N}$. In this section, we will just consider these simultaneously instead of breaking the discussion into two separate subsections.

Ornstein's Theorem [14] says that for ergodic transformations τ and positive functions $f \in L_1(X,p)$, f_τ^* is integrable if and only if $f \in L\log L$. This was proved using this reverse inequality for the ergodic maximal function: for any positive function $f \in L_1(X,p)$, if $p\{f_\tau^* > \lambda\} < 1$, then

$$p\{f_\tau^* > \lambda\} \geq \frac{1}{2\lambda}\int_{\{f_\tau^*>\lambda\}} f\,dp. \tag{3.1}$$

Remark 3.1. Since $f_\tau^* \geq f$, we see that $\{f_\tau^* > \lambda\}$ can be replaced by $\{f > \lambda\}$ in (3.1). So (2.3) holds with $C = 2$.

Remark 3.2. See Ornstein [14], and Jones [9], for the reverse weak inequality (3.1). In this inequality, λ must be restricted from being too small. The criterion used in Ornstein [14], is that $E_f = \{f_\tau^* > \lambda\}$ has $p(X\backslash E_f) > 0$. In particular, this requires that at least $\lambda \geq \int |f|\,dp$. Some restriction on λ should have also been included in Jones [9] because without that, one might have $p(X\backslash E_f) = 0$ and then the stopping time τ that is used would not be defined.

Despite Ornstein's Theorem, by passing to subsequences we have this improvement.

Proposition 3.3. *For every $f \in L_1(X,p)$, there exists a sequence $\mathbf{N} = (N_m)$ such that $f_{\mathbf{N}}^* \in L_1(X,\mu)$.*

Proof. This is an immediate consequence of Proposition 1.1 because the ergodic averages converge in L_1-norm. □

However, as with the Hardy–Littlewood maximal functions, we expect that this result requires adjusting the sequence to the function, even if we modulate the maximal function by considering $\phi(f_{\mathbf{n}}^*)$ for functions ϕ growing more slowly than $\phi_0(x) = x$. But first, there is a direct analog of Proposition 2.10 where we now consider the issue of integrability of $f_{\mathbf{N}}^*$ for all $f \in L_1(X,\mu)$ simultaneously.

Proposition 3.4. *Suppose τ is ergodic and $\mathbf{N}$ is a subsequence of $\mathbb{N}$. Then there exists a positive function $f \in L_1(X,p)$ such that $f_{\mathbf{N}}^*$ is not integrable.*

Proof. It is an elementary argument to show that $\sum_{m=1}^{\infty} \frac{N_{m+1}-N_m}{N_{m+1}} = \infty$. Indeed, with $\epsilon_m = 1/N_m$, we have $\sum_{m=1}^{\infty} \frac{N_{m+1}-N_m}{N_{m+1}} = \sum_{m=1}^{\infty} \frac{\epsilon_m-\epsilon_{m+1}}{\epsilon_m}$. So $\sum_{m=1}^{\infty} \frac{N_{m+1}-N_m}{N_{m+1}}$ is an analogue of the divergent series in Proposition 2.10. We now use the Rokhlin Lemma. Suppose we take a stack $(B, TB, \ldots, T^L B)$ of pairwise disjoint sets of positive measure. For a fixed M, if L is sufficiently large, one can see that

$$\sup_{1\le m\le M} A_{N_m}\left(\frac{1}{p(B)}1_B\right) \ge \frac{1}{p(B)}\sum_{m=1}^{M-1}\frac{1}{N_{m+1}}\sum_{l=N_m}^{N_{m+1}-1} 1_{T^l B}.$$

Hence,

$$\left\|\sup_{1\le m\le M} A_{N_m}\left(\frac{1}{p(B)}1_B\right)\right\|_1 \ge \sum_{m=1}^{M-1}\frac{N_{m+1}-N_m}{N_{m+1}}.$$

So, we can choose (M_k) and stacks $(B_k, TB_k, \ldots, T^{L_k}B_k)$ such that

$$\left\|\sup_{1\le m\le M_k} A_{N_m}\left(\frac{1}{p(B_k)}1_{B_k}\right)\right\|_1 \ge k2^k$$

for all k. Let $f = \sum_{k=1}^{\infty}\frac{1}{2^k}\frac{1_{B_k}}{p(B_k)}$. Then for all k,

$$\begin{aligned}\|f_{\mathbf{N}}^*\|_1 &\ge \left\|\sup_{1\le m\le M_k} A_{N_m} f\right\|_1\\ &\ge \frac{1}{2^k}\left\|\sup_{1\le m\le M_k} A_{N_m}\left(\frac{1}{p(B_k)}1_{B_k}\right)\right\|_1\\ &\ge k.\end{aligned}$$

Hence, $\|f_{\mathbf{N}}^*\|_1 = \infty$. □

Remark 3.5. In proving Proposition 3.4, by passing to a subsequence we could have assumed that (N_m) is lacunary. Then it is immediate, just as in Remark 2.11, that $\sum_{m=1}^{\infty}\frac{N_{m+1}-N_m}{N_{m+1}} = \infty$.

It follows that if we want to have a nontrivial result concerning the growth of $f_{\mathbf{N}}^*$, we will have to consider again $\phi(f_{\mathbf{N}}^*)$ where ϕ is a smooth, increasing function. Of course, we do have again a weak $(1,1)$ maximal inequality for the ergodic maximal function.

There is a constant C such that for all $\lambda > 0$ and $f \in L_1(X,p)$, we have

$$p\{f_\tau^* > \lambda\} \le \frac{C}{\lambda}\int |f|\,dp. \tag{3.2}$$

Consequently, if we want to entertain the idea that $\phi(f_{\mathbf{n}}^*)$ is always integrable, and that this is a nontrivial fact, then we have to assume that $\sum_{n=1}\frac{1}{\phi^{-1}(n)} = \infty$. The basic result that we are going to prove here is that there is no such result. By comparison with Section 2.2, it will be good to have a restricted reverse weak inequality for the rare maximal function.

Proposition 3.6. *There is a constant C such that the following holds. Assume that τ is ergodic. Given a sequence $\mathbf{N}$ increasing to ∞, and ϵ, $0 < \epsilon < 1$, there exist a measurable set $Q \subset X$ such that $p(Q) < \epsilon$ and for any $\lambda \in [p(Q), \frac{99}{100}]$*

$$p\left\{(1_Q)^*_{\mathbf{N}} > \frac{\lambda}{C}\right\} \geq \frac{1}{C\lambda}p(Q).$$

Proof. This proof is a direct analogue of the proof of Proposition 2.13. We first may assume that we can take whole numbers k_m such that $2^{k_m} \leq N_m \leq 2^{k_m+1}$ and $k_{m+1} \geq k_m + 2$ for all m. Then $f^*_{\mathbf{N}} \leq 2 \sup\limits_{m \geq 1} A_{N_m} f$ for all positive functions $f \in L_1(X,p)$. So without loss of generality we replace (N_m) by (2^{k_m}).

Now we use the Rokhlin Lemma to choose a large stack

$$\mathcal{S} = (B, TB, \ldots, T^{2^L-1}B)$$

of pairwise disjoint subsets of X with $p(B) > 0$. The sets $T^l B$ are the levels of $\mathcal{S}$. We let $X_{\mathcal{S}} = \bigcup\limits_{l=0}^{2^L-1} T^l B$. We assume that the stack has been chosen so that $s = p(X \backslash X_{\mathcal{S}}) < \frac{1}{100}$. Choose integers M and L so that $1/2^M < \epsilon$ and $L \geq k_M$. For $n \leq L$ we can define an analogue of the Rademacher function, the function $R_n(l)$ on the indices $l = 0, \ldots, 2^L - 1$, by periodically extending to $\{0, \ldots, 2^L - 1\}$ the function that is 1 for $l = 0, \ldots, 2^n - 1$ and 0 for $l = 2^n, \ldots, 2^{n+1} - 1$. Let $E_j = \bigcup\{T^l B : \ R_{k_i}(l) = 1, \ i = j, \ldots, M\}$. Each E_j consists of a union of the levels $T^l B$ with l in particular dyadic blocks $\mathcal{L}$ of length 2^{k_j}. Call these blocks $\mathcal{L}$ the defining blocks for E_j. It is easy to see that $E_j \subset E_{j+1}$ and $p(E_{j+1}) = 2p(E_j)$ for $j = 1, \ldots, M$. Also, $p(E_M) = \frac{1}{2}(1-s)$. So $p(E_{M-j}) = \frac{1}{2^{j+1}}(1-s)$ and $p(E_1) < \epsilon$. Let $Q = E_1$. Then $p(Q) < \epsilon$ and $Q \subset E_j$ for all $j = 1, \ldots, M$. Consider a constituent block $I = I_{\mathcal{L}}$, i.e., the union of all the levels $T^l B$ with l in some defining block $\mathcal{L}$. Observe that

$$\frac{p(I \cap Q)}{p(I)} = \frac{\frac{1}{2^{j-1}}p(I)}{p(I)} = \frac{2}{2^j}.$$

Take any x in the lower half of $I = I_{\mathcal{L}}$, i.e., x is in a level whose index is in the left hand half of the corresponding dyadic block $\mathcal{L}$. Since $k_{m+1} \geq k_m + 2$, it follows that we have $A_{2^{k_j}} 1_Q(x) \geq \frac{1}{2^j}$. So, with E^0_j denoting the union of the lower halves of the constituent blocks I of E_j, we have for any $x \in E^0_j$, $(1_Q)^*_{\mathbf{N}}(x) \geq \frac{1}{2^j}$.

Now if $\lambda \in [p(Q), 1-s]$ choose j, $0 \leq j \leq M-1$, such that

$$\frac{1}{2^{j+1}}(1-s) \leq \lambda \leq \frac{1}{2^j}(1-s).$$

Then

$$\begin{aligned}
p\left\{x : (1_Q)^*_{\mathbf{N}}(x) \geq \frac{\lambda}{1-s}\right\} &\geq p\left\{x : (1_Q)^*_{\mathbf{N}}(x) \geq \frac{1}{2^j}\right\} \\
&\geq p(E_j^0) \\
&= \frac{1}{2}p(E_j) = \frac{1}{2}\frac{1-s}{2^{M-j+1}} \\
&= p(Q)2^{j-2} \\
&\geq p(Q)\frac{1-s}{8\lambda}.
\end{aligned}$$

That is, for $\lambda \in [p(Q), 1-s]$, we have

$$p\left\{x : (1_Q)^*_{\mathbf{N}}(x) \geq \frac{\lambda}{1-s}\right\} \geq p(Q)\frac{1-s}{8\lambda}.$$

So with $C = 9$, we have the inequality that we wanted. □

In the same manner we obtained Proposition 2.15 from Proposition 2.13, we obtain this result from Proposition 3.6.

Proposition 3.7. *Suppose ϕ is a smooth, increasing function such that $\sum\limits_{n=1}^{\infty} \frac{1}{\phi^{-1}(n)} = \infty$. Let $\mathbf{N}$ be a sequence increasing to ∞. Then there exists $f \in L_1(X,p)$ such that $\|\phi(f^*_{\mathbf{N}})\|_1 = \infty$.*

In addition, we again get a generic result using Proposition 3.7 and Proposition 2.16.

Proposition 3.8. *Suppose ϕ is a smooth, increasing function such that $\sum\limits_{n=1}^{\infty} \frac{1}{\phi^{-1}(n)} = \infty$. Let $\mathbf{N}$ be a sequence increasing to ∞. Then there exists a dense G_δ set $\mathcal{L}$ in $L_1(X,p)$ such that for all $f \in \mathcal{L}$ we have $\|\phi(f^*_{\mathbf{N}})\|_1 = \infty$.*

4. Other possibilities

4.1. Approximate identities. Here is a natural harmonic analysis context for trying to generalize Proposition 2.15. Suppose (ψ_n) is a normalized, positive *approximate identity* on $L_1(\mathbb{R},\mu)$. That is, for each n, $\|\psi_n\|_1 = 1$, $\psi_n \geq 0$, and for all $f \in L_1(\mathbb{R},\mu)$, $\lim\limits_{n\to\infty} \|\psi_n * f - f\|_1 = 0$. By Proposition 1.1, we have this general fact.

Proposition 4.1. *If (ψ_n) is a normalized, positive approximate identity on $L_1(\mathbb{R},\mu)$ and $f \in L_1(\mathbb{R},\mu)$, then there is a subsequence (ψ_{n_m}) such that $\sup\limits_{m\geq 1} \psi_{n_m} * |f| \in L_1(\mathbb{R},\mu)$.*

We do generally have the following contrasting result. This result makes it clear that our question is about general functions ϕ, not just $\phi = \mathrm{Id}$.

Proposition 4.2. *Given any approximate identity* (ψ_n)*, there exists a positive compactly supported function* $f \in L_1(\mathbb{R}, \mu)$ *such that*

$$\left\| \sup_{n\geq 1} \psi_n * f \right\|_1 = \infty.$$

Proof. The proof is a straightforward modification of the proof of Proposition 2.10 using the fact that $\left\| \sup_{n\geq 1} \psi_n \right\|_1 = \infty$. One way to see this is to choose a rapidly increasing subsequence (ψ_{n_m}) such that the functions ψ_{n_m} are approximately singular to each other. Then one can see that

$$\left\| \sup_{n\geq 1} \psi_n \right\|_1 \geq \left\| \sup_{m\geq 1} \psi_{n_m} \right\|_1 \geq \frac{1}{2} \sum_{m=1}^{\infty} \|\psi_{n_m}\|_1 = \infty. \qquad \square$$

Remark 4.3. There is a result analogous to this for averages in ergodic theory. Suppose (μ_n) is a sequence of probability measures on $\mathbb{Z}$ and (X, β, p) is a probability space. Given an ergodic transformation τ of (X, β, p), consider the averaging operators $\mu_n^\tau f = \sum_{k\in\mathbb{Z}} \mu(k) f \circ \tau^k$. In order to have a nontrivial averaging process, we would usually assume that the measure are uniformly dissipative, i.e., $\sup_{k\geq 1} \mu_n(k) \to 0$ as $n \to \infty$. But it is enough here to assume they are dissipative, i.e., for all k, $\mu_n(k) \to 0$ as $n \to \infty$. Then in a manner analogous to the proof of Proposition 4.2 one can show that $\|\sup_{n\geq 1} \mu_n\|_1 = \infty$. It follows from a simple argument using the Rokhlin Lemma that there exists a positive $f \in L_1(X, p)$ such that $\|\sup_{n\geq 1} \mu_n^\tau f\|_1 = \infty$. So in this context, as above, the unresolved issue is if we can also have $\|\phi(\sup_{n\geq 1} \mu_n^\tau f)\|_1 = \infty$ for some positive $f \in L_1(X, p)$ as long as we knew that $\sum_{n=1}^{\infty} \frac{1}{\phi^{-1}(n)} = \infty$.

Sometimes there is a pointwise a.e. convergence result on $L_1(\mathbb{R}, \mu)$ for the operators $T_n f = \psi_n * f$, and sometimes there is not. If there is no pointwise convergence result, then it is a standard fact that there would be a positive function $f \in L_1(\mathbb{R}, \mu)$ such that $\sup_{n\geq 1} \psi_n * f = \infty$ a.e. and so certainly $\sup_{n\geq 1} \psi_n * f$ is not in $L_1(\mathbb{R}, \mu)$. But in any case, there is always a pointwise good subsequence of (ψ_n); see Kostyukovsky and Olevskii [11]. So assume that we know for all $f \in L_1(\mathbb{R}, \mu)$, that $f_{\mathbf{P}}^* = \sup_{n\geq 1} |\psi_n * f| < \infty$ almost everywhere. Then it makes sense to ask for a characterization of when $f_{\mathbf{P}}^* \in L_1(\mathbb{R}, \mu)$.

Question 4.4. Given any approximate identity (ψ_n) and a strictly increasing smooth function ϕ such that $\sum_{n=1}^{\infty} \frac{1}{\phi^{-1}(n)} = \infty$, is there a compactly supported function $f \in L_1(\mathbb{R}, \mu)$ such that $\|\phi(f_{\mathbf{P}}^*)\|_1 = \infty$?

We know the answer to this question for certain classical approximate identities like the Fejér kernels, giving the averages of the partial sums of the Fourier series of a function in $L_1[0,1]$, and the Poisson kernels, giving the harmonic extension of a function into the disc (or upper half plane). First, taking a function ψ on $\mathbb{R}$, the dilations ψ_t are given by $\psi_t(x) = t\psi(tx)$. If ψ is positive and $\int \psi \, d\mu = 1$, then $(\psi_t : t = 1, 2, 3, \dots)$ is a positive normalized approximate identity on $L_1(\mathbb{R}, \mu)$. For example, consider the function $\psi(x) = \frac{1}{\pi}\frac{1}{x^2+1}$. Then with $y = \frac{1}{t}$, the Poisson kernels $\psi(x,y) = \frac{1}{\pi}\frac{y}{x^2+y^2}$ are just the dilations $\psi_t(x)$. So the following general results apply in particular to the Poisson kernels.

Proposition 4.5. *Suppose ψ is a positive even function on $\mathbb{R}$ which is decreasing on $\mathbb{R}^+$. Assume that $\int \psi \, d\mu = 1$. Then the approximate identity $(\psi_t : t \geq 1)$ satisfies the following: for some constants c and C,*

$$c\, f_1^* \leq \sup_{t \geq 1} \psi_t * f \leq C\, f_1^*$$

for all positive $f \in L_1(\mathbb{R}, \mu)$.

Proof. The upper estimate is obtained by approximating ψ_t by a convex combination of functions of the form $\frac{1}{2\epsilon}1_{[-\epsilon,\epsilon]}$. The lower estimate is obtained from the fact that for $x_0 > 0$, with $\psi(x_0) > 0$, if $x \in [0, \frac{x_0}{t}]$, then $\psi_t(x) \geq t\psi(t\frac{x_0}{t}) = c\frac{t}{x_0}$ for some constant c. □

Corollary 4.6. *Suppose ψ is a positive even function on $\mathbb{R}$ which is decreasing on $\mathbb{R}^+$. Assume that $\int \psi \, d\mu = 1$. The for a positive compactly supported function $f \in L_1(\mathbb{R}, \mu)$, $\sup\limits_{t \geq 1} \psi_t * f$ is integrable if and only if $f \in L \log L$.*

Proof. This follows immediately from Proposition 4.5 and Stein [15]. □

Corollary 4.7. *Let ϕ be a smooth, increasing function such that $\sum\limits_{k=1}^{\infty} \frac{1}{\phi^{-1}(k)} = \infty$. For any increasing sequence (t_m), there exists a positive function $f \in L_1(\mathbb{R}, \mu)$, supported on $[0,1]$, such that $\left\| \sup\limits_{m \geq 1} \psi_{t_m} * f \right\|_1 = \infty$.*

Proof. This follows from Proposition 4.5 and Proposition 2.15. □

Remark 4.8. Another example of the idea used above is the Fejér kernels given by $\sigma_n(x) = \frac{1}{n}\left(\frac{\sin(nx/2)}{\sin(x/2)}\right)^2$ on $[-\pi, \pi]$. Take our measure space to be $([-\pi,\pi], \beta, \mu)$ where μ is Lebesgue measure. One feature of the Fejér kernels that causes issues in calculations is that they are not decreasing on $[0,\pi)$. However, there is a constant c such that $\sigma_n(1/n) \geq cn$. Since σ_n is an even function and decreasing on $[0, 1/n)$, this says that for come constant c, $\sigma_n \star f \geq cn1_{[0,1/n]} \star f$ for positive functions f. Hence, certainly there is a constant c such that for positive functions f, we have $c \sup\limits_{n \geq 1} n1_{[0,1/n]} * f \leq$

$\sup_{n\geq 1} \sigma_n * f$. It follows from Stein [15] that for a positive function, if the Fejér maximal function $\sup_{n\geq 1} \sigma_n * f$ is integrable, then $f \in L \log L$. Moreover, for any rare maximal function $\sup_m \sigma_{n_m} * f$, we have the same result as in Corollary 4.7.

Since σ_n is not decreasing on $[0, \pi)$, we cannot argue as with the dilation approximate identities above, that for positive functions the maximal Fejér function is dominated by the maximal Hardy–Littlewood function. However, one of the results in Móricz [12], see also Brown, Dai, and Móricz [2], shows that for functions in $L \log L$ the maximal Fejér function is integrable. Combining that with the above, we see for positive functions the maximal Fejér function is integrable if and only if $f \in L \log L$.

Remark 4.9. The comparisons above between the maximal functions for the Poisson kernels, the maximal function for the Fejér kernels, and the Hardy–Littlewood maximal function, show that for positive functions the maximal functions for the Poisson kernels and the maximal function for the Fejér kernels satisfy reverse weak inequalities (with suitable restrictions on λ) as in (2.3). Also, the corresponding rare maximal functions have restricted reverse weak inequalities as in Proposition 2.13.

4.2. Moving averages. One particular class of approximate identities that we can handle as above with only a slight modification is the case of moving averages. See Nagel and Stein [13], Sueiro [16], and Bellow, Jones, and Rosenblatt [1].

First we prove this proposition.

Proposition 4.10. *Suppose we have a sequence (k_n) in $\mathbb{N}$ with $\lim_{n\to\infty} k_n = \infty$, and a sequence (x_n) with $\lim_{n\to\infty} x_n = 0$. Suppose $\theta_n = 2^{k_n} 1_{[x_n, x_n + 2^{-k_n}]}$. Then for every $\epsilon \in (0,1)$ there exists a measurable set $Q \subset [0,1]$ such that $\mu(Q) < \epsilon$ and for any $\lambda \in [\mu(Q), 1]$*

$$\mu\left\{\sup_{n\geq 1} \theta_n * 1_Q > \frac{\lambda}{4}\right\} \geq \frac{1}{4\lambda}\mu(Q).$$

Proof. The proof proceeds just as the proof of Proposition 2.13 We extend the index sequence $(k_n : n \geq 1)$ by letting $k_0 = 0$. Let

$$r_n(x) = \text{ sign } \sin(2^n \pi x), \quad n = 0, 1, 2, \dots$$

be the Rademacher functions on $[0,1]$. Choose a whole number J such that $2^{-J} < \epsilon$. For $j = 0, \dots, J$ consider sets

$$V_j = \{x \in [0,1] : \ r_{k_n}(x) = 1 \text{ for } n = 0, \dots, j\}.$$

Each set V_j consists of disjoint dyadic intervals of length 2^{-k_j}. It is easy to see that $V_0 = [0,1]$, $V_j \subset V_{j-1}$ and $\mu(V_j) = \frac{1}{2}\mu(V_{j-1})$ for $j = 1, \dots, J$. Thus, for $j = 0, \dots, J$

$$\mu(V_j) = \frac{1}{2^j}.$$

Let $Q = V_J$. Note that $\mu(Q) = 2^{-J} < \epsilon$, and $Q \subset V_j$ for $j = 0, \dots, J$. Let I be any of the constituent intervals of length 2^{-k_j} that make up V_j. Observe that

$$\frac{\mu(I \cap Q)}{\mu(I)} = \frac{\mu(V_j \cap Q)}{\mu(V_j)} = \frac{\mu(Q)}{\mu(V_j)} = \frac{1}{2^{J-j}}. \tag{4.1}$$

Let L be the left half of I and let R be the right one. For any $y \in R + x_j$

$$[y - x_j - 2^{-k_j}, y - x_j] \supseteq L,$$

so

$$\mu(Q \cap [y - x_j - 2^{-k_j}, y - x_j]) \geq \frac{1}{2}\mu(I \cap Q).$$

Since

$$\theta_j * 1_Q(y) = 2^{k_j}\mu(Q \cap [y - x_j - 2^{-k_j}, y - x_j])$$

we see that for any $y \in R + x_j$

$$\begin{aligned}\sup_{n\geq 1} \theta_n * 1_Q(y) &\geq \theta_j * 1_Q(y)\\ &\geq \frac{1}{2}2^{k_j}\mu(I \cap Q)\\ &= \frac{1}{2}\frac{\mu(I \cap Q)}{\mu(I)} = \frac{1}{2^{J-j+1}}\end{aligned}$$

by (4.1). Let V_j^0 be the union of the right halves of all of the constituent intervals I in V_j. Then $\mu(V_j^0 + x_j) = \mu(V_j^0) = \frac{1}{2}\mu(V_j) = \frac{1}{2^{j+1}}$ and

$$V_j^0 + x_j \subset \left\{ y : \sup_{n\geq 1} \theta_n * 1_Q(y) \geq \frac{1}{2^{J-j+1}} \right\}$$

for $j = 0, \dots, J$.

Now if $\lambda \in [\mu(Q), 1]$ choose j, $1 \leq j \leq J$, such that

$$\frac{1}{2^{J-j+1}} \leq \lambda \leq \frac{1}{2^{J-j}}.$$

Then

$$\begin{aligned}
\mu\left\{y : \sup_{n\geq 1} \theta_n * 1_Q(y) \geq \frac{\lambda}{4}\right\} &\geq \mu\left\{y : \sup_{n\geq 1} \theta_n * 1_Q(y) \geq \frac{\lambda}{2}\right\} \\
&\geq \mu\left\{y : \sup_{n\geq 1} \theta_n * 1_Q(y) \geq \frac{1}{2^{J-j+1}}\right\} \\
&\geq \mu\left\{y : \theta_j * 1_Q(y) \geq \frac{1}{2^{J-j+1}}\right\} \\
&\geq \mu(V_j^0 + x_j) = \frac{1}{2^{j+1}} \\
&= \frac{1}{2}\mu(Q)2^{J-j} \\
&\geq \frac{1}{2}\mu(Q)\frac{1}{2\lambda} \\
&= \frac{1}{4\lambda}\mu(Q). \qquad \square
\end{aligned}$$

This proposition gives immediately the following result, just as Proposition 2.13 gave Proposition 2.15.

Proposition 4.11. *Suppose ϕ is a smooth, increasing function such that $\sum_{n=1}^{\infty} \frac{1}{\phi^{-1}(n)} = \infty$. Suppose that we have sequences (ϵ_n) with all $\epsilon_n > 0$ and $\lim_{n\to\infty} \epsilon_n = 0$, and (x_n) such that $\lim_{n\to\infty} x_n = 0$. Let $\theta_n = \frac{1}{\epsilon_n} 1_{[x_n, x_n+\epsilon_n]}$. Then there exists $f \in L_1(\mathbb{R}, \mu)$, supported in $[0,1]$, such that*

$$\left\| \phi\left(\sup_{n\geq 1} \theta_n * |f|\right) \right\|_1 = \infty.$$

Remark 4.12.

(a) Of course, it follows from Proposition 2.16 that generically the result in Proposition 4.11 holds.

(b) In both moving averages for differentiation and the ergodic averages, when the moving averages yield an L_1 pointwise result, then the maximal function of the moving average is dominated in distribution by the maximal function of the corresponding nonmoving averages. Hence, if the function f is in $L \log L$, then the maximal function is locally integrable (integrable) respectively in the real case (ergodic theory case). However, since the moving averages are inherently defined along some subsequence, there can be no reverse result as in Stein [15] and Ornstein [14].

The following analog of Proposition 4.11 holds for moving averages in ergodic theory. The argument for this proceeds using Proposition 4.10 and Proposition 4.11 as models, just as the results in Section 3 were proved.

Proposition 4.13. *Suppose ϕ is a smooth, increasing function such that $\sum_{n=1}^{\infty} \frac{1}{\phi^{-1}(n)} = \infty$. Let $\mathbf{N}$ be a sequence increasing to ∞. Let $\mathbf{v} = (v_m)$ be any sequence of whole numbers. Let $f^*_{\mathbf{N},\mathbf{v}} = \sup_{m\geq 1} |A_{N_m} f \circ T^{v_m}|$ Then there exists $f \in L_1(X,p)$ such that $\|\phi(f^*_{\mathbf{N},\mathbf{v}})\|_1 = \infty$.*

4.3. Dyadic martingales. Let $\mathcal{E}_n f$ denote the usual *classical dyadic martingale* defined on $L_1([0,1],p)$ where p is Lebesgue measure on $[0,1]$. The maximal functions $\sup_{m\geq 1} \mathcal{E}(|f||\beta_{k_m})$ and $\sup_{m\geq 1} D_{1/2^{k_m}}|f|$ are distributionally equivalent. This means that for some constant C, one has for all sequences (k_m), for all $\lambda > 0$, and for all positive functions $f \in L_1([0,1],p)$,

$$p\left\{\sup_{m\geq 1} \mathcal{E}(f|\beta_{k_m}) > \lambda\right\} \leq Cp\left\{\sup_{m\geq 1} D_{1/2^{k_m}} f > \frac{\lambda}{C}\right\} \tag{4.2}$$

and

$$p\left\{\sup_{m\geq 1} D_{1/2^{k_m}} f > \lambda\right\} \leq Cp\left\{\sup_{m\geq 1} \mathcal{E}(f|\beta_{k_m}) > \frac{\lambda}{C}\right\}. \tag{4.3}$$

See Goubran [4], Lemma 2.1.11, where this idea appears in a comparison of ergodic averages on $\ell_1(\mathbb{Z})$ and the usual dyadic reverse martingales on $\ell_1(\mathbb{Z})$. The same technique can be used to prove the inequalities in (4.2) and (4.3). The argument requires a cover lemma step and the estimate that Lemma 2.4 in [3] provides. See also Goubran [5] for a discussion of pointwise versions of (4.2) and (4.3).

This equivalence in distribution shows that Stein [15] proves that for a positive function $f \in L_1[0,1]$, the full dyadic martingale $(E(f|\beta_k))$ has $\sup_{k\geq 1} E(f|\beta_k)$ integrable if and only if $f \in L\log L$. Moreover, the following also is immediate from our earlier result Proposition 2.8 by a simple comparison of level sets.

Proposition 4.14. *Suppose ϕ is a smooth, increasing function such that $\sum_{n=1}^{\infty} \frac{1}{\phi^{-1}(n)} = \infty$. Let (k_m) be any increasing sequence of whole numbers. Then there exists a positive function $f \in L_1[0,1]$ such that*

$$\left\|\phi\left(\sup_{m\geq 1} E(f|\beta_{k_m})\right)\right\|_1 = \infty.$$

Remark 4.15. Because the martingale is L_1-norm convergent, Proposition 1.1 applies. So as in the above discussions about differentiation and ergodic averages, Proposition 4.14 provides us a limitation on what we can do by dropping to a subsequence of the dyadic martingale.

Remark 4.16. From (4.2), we can see there is there is a reverse weak inequality for the full dyadic martingale maximal function $\sup_{k\geq 1} E(f|\beta_k)$ analogous to the results of Stein [15] and Ornstein [14]. Moreover, there would be restricted reverse maximal inequalities as in Proposition 2.13, and these could also be used to prove Proposition 4.14 directly.

5. Additional issues

Here are some particular questions that we have not yet been able to answer.

(1) For which classical maximal functions is there a result analogous to Stein's result in [15]? Alternatively, for which ones are there reverse weak inequalities for a useful range of λ?

(2) For what other maximal functions can we obtain divergence results as above? In particular, for which ones are there restricted reverse weak inequalities?

There are a number of other issues raised by the results in this paper. For example, generally we cannot characterize the subspaces where rare maximal functions are integrable. Also, if we have two rare maximal functions, we do not know how to determine when the functions integrable with respect to one of them are the same as the functions that are integrable with respect to the other one. These questions and related ones will be discussed more in a later paper.

References

[1] Bellow, Alexandra; Jones, Roger; Rosenblatt, Joseph. Convergence for moving averages. *Ergodic Theory and Dynamical Systems* **10** (1990), 43–62. MR1053798 (91g:28021), Zbl 0674.60035, doi: 10.1017/S0143385700005381.

[2] Brown, Gavin; Dai, Feng; Móricz, Ferenc. The maximal Riesz, Fejér, and Cesàro operators on real Hardy spaces. *J. Fourier Anal. Appl.* **10** (2004), no. 1, 27–50. MR2045523 (2005b:42019), Zbl 1062.42003, doi: 10.1007/s00041-004-8002-6.

[3] Butler, S.; Pavlov, S.; Rosenblatt, J. Large deviations for martingales and derivatives. *J. Math. Anal. Appl.* **366** (2010), no. 1, 67–80. MR2593634 (2011b:60090), Zbl 1214.60014, doi: 10.1016/j.jmaa.2009.12.054.

[4] Goubran, Nader. Pointwise relations between ergodic averages and martingales. Thesis (Ph.D.)- University of Illinois at Urbana-Champaign. 2002. 51 pp. ISBN: 978-0493-90037-7. MR2704006.

[5] Goubran, N. Pointwise inequalities for ergodic averages and reversed martingales. *J. Math. Anal. Appl.* **290** (2004), no. 1, 10–20. MR2032221 (2005j:37005), Zbl 1041.37003, doi: 10.1016/j.jmaa.2003.08.002.

[6] Hagelstein, Paul Alton. A note on rare maximal functions. *Colloq. Math.* **95** (2003), no. 1, 49–51. MR1967553 (2004b:42039), Zbl 1047.42014, doi: 10.4064/cm95-1-4.

[7] Hagelstein, Paul Alton. Córdoba-Fefferman collections in harmonic analysis. *Pacific J. Math.* **216** (2004), no. 1, 95–109. MR2094583 (2005h:42047), Zbl 1085.42010.

[8] Hare, K.; Stokolos, A. On weak type inequalities for rare maximal functions. *Colloq. Math.* **83** (2000), no. 2, 173–182. MR1758313 (2002e:42019), Zbl 1030.42017.

[9] Jones, Roger. Ornstein's $L\log^+ L$ theorem. *Proc. Amer. Math. Soc.* **91** (1984), no. 2, 262–264. MR0740182 (85k:28023), Zbl 0553.42007, doi: 10.1090/S0002-9939-1984-0740182-6 .

[10] Kita, Hiro-o. On maximal functions in Orlicz spaces. *Proc. Amer. Math. Soc.* **124** (1996), no. 10, 3019–3025. MR1376993 (97b:42031), Zbl 0862.42014, doi: 10.1090/S0002-9939-96-03807-5.

[11] Kostyukovsky, Sergey; Olevskii, Alexander. Compactness of families of convolution operators with respect to convergence almost everywhere. *Real Anal. Exchange* **30** (2004/05), no. 2, 755–765. MR2177432 (2006e:42001), Zbl 1105.42002.

[12] Móricz, Ferenc. The maximal Fejér operator is bounded from $H^1(\mathbb{T})$ to $L^1(\mathbb{T})$. *Analysis* **16** (1996), no. 2, 125–135. MR1397575 (97f:42014), Zbl 0864.47016.

[13] Nagel, Alexander; Stein, Elias M. On certain maximal functions and approach regions. *Adv. in Math.* **54** (1984), no. 1, 83–106. MR0761764 (86a:42026), Zbl 0546.42017, doi: 10.1016/0001-8708(84)90038-0.

[14] Ornstein, Donald. A remark on the Birkhoff ergodic theorem. *Illinois J. Math.* **15** (1971), 77–79. MR0274719 (43 #479), Zbl 0212.40102.

[15] Stein, E. M. Note on the class $L\log L$. *Studia Math.* **32** (1969) 305–310. MR0247534 (40 #799), Zbl 182.47803.

[16] Sueiro, Juan. A note on maximal operators of Hardy-Littlewood type. *Math. Proc. Cambridge Philos. Soc.* **102** (1987), no. 1, 131–134. MR0886442 (88e:42037), Zbl 0638.42018, doi: 10.1017/S0305004100067116.

[17] Stokolos, A. M. On weak type inequalities for rare maximal functions in $\mathbb{R}^n$. *Colloq. Math.* **104** (2006), no. 2, 311–315. MR2197080 (2006m:42032), Zbl 1085.42013, doi: 10.4064/cm104-2-7.

Department of Mathematics, University of Illinois at Urbana-Champaign, 273 Altgeld Hall, 1409 West Green Street, Urbana, IL 61801, USA
svbutler@illinois.edu
rosnbltt@illinois.edu

This paper is available via http://nyjm.albany.edu/j/2012/18-29.html.