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A sharp norm estimate for weighted Bergman projections on the minimal ball

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Abstract. We show that, for $1 < p < \infty$, the norm of the weighted Bergman projection $\mathbf{P}_{s,\mathbb{B}_*}$ on $L^p(\mathbb{B}_*, |z \bullet z|^{\frac{p-2}{2}} dv_s)$ is comparable to $\csc(\pi/p)$, where $\mathbb{B}_*$ is the minimal unit ball in $\mathbb{C}^n$.

Contents

1. Introduction

We consider the domain $\mathbb{B}_*$ in $\mathbb{C}^n$, $n \geq 2$, defined by

$$\mathbb{B}_* = \{z \in \mathbb{C}^n : |z|^2 + |z \bullet z| < 1\},$$

where

$$z \bullet w = \sum_{j=1}^{n} z_j w_j$$

for z and w in $\mathbb{C}^n$. This is the unit ball of $\mathbb{C}^n$ with respect to the norm

$$N_*(z) := \sqrt{|z|^2 + |z \bullet z|}, \qquad z \in \mathbb{C}^n.$$

The norm $N := N_*/\sqrt{2}$ was introduced by Hahn and Pflug in [1], where it was shown to be the smallest norm in $\mathbb{C}^n$ that extends the euclidean norm in $\mathbb{R}^n$ under certain restrictions.

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The domain $\mathbb{B}_*$ has since been studied by Hahn, Mengotti, Oeljiklaus, Pflug, Youssfi, and others. In particular, it is well known that the automorphism group of $\mathbb{B}_*$ is compact and its identity component is

$$\mathrm{Aut}^0_{\mathcal{O}}(\mathbb{B}_*) = S^1 \cdot \mathrm{SO}(n, \mathbb{R}),$$

where the S^1-action is diagonal and the $\mathrm{SO}(n,\mathbb{R})$-action is by matrix multiplication; see [5] for example. It is also well known that $\mathbb{B}_*$ is a nonhomogeneous domain. Its singular boundary consists of all points z with $z \bullet z = 0$, and the regular part of the boundary of $\mathbb{B}_*$ consists of strictly pseudoconvex points.

As a nonhomogeneous domain, it is not surprising that $\mathbb{B}_*$ exhibits certain exotic behavior. For example, it was recently used in [6] to construct counterexamples to the Lu Qi-Keng conjecture. What is a bit unexpected is that the norm of the Bergman projection on some L^p spaces on $\mathbb{B}_*$ can be estimated in such a way that resembles the situation on the Euclidean ball in $\mathbb{C}^n$. This constitutes the main result of the paper.

Theorem 1. *For any $s > -1$ there exists a constant $C > 0$, depending only on s and n but not on p, such that the norm $\|\boldsymbol{P}_{s,\mathbb{B}_*}\|_p$ of the linear operator*

$$\boldsymbol{P}_{s,\mathbb{B}_*} : L^p\left(\mathbb{B}_*, |z \bullet z|^{\frac{p-2}{2}} dv_s\right) \to L^p\left(\mathbb{B}_*, |z \bullet z|^{\frac{p-2}{2}} dv_s\right)$$

satisfies the estimates

$$C^{-1} \csc(\pi/p) \le \|\boldsymbol{P}_{s,\mathbb{B}_*}\| \le C \csc(\pi/p)$$

for all $1 < p < \infty$.

A similar, optimal estimate for the Bergman projection on L^p spaces of the Euclidean ball in $\mathbb{C}^n$ was obtained in [10]. So this paper can be considered a sequel to [10]. On the other hand, it was shown in [3] that the operator $\mathbf{P}_{s,\mathbb{B}_*}$ is bounded on $L^p\big(\mathbb{B}_*, |z \bullet z|^{\frac{p-2}{2}} dv_s\big)$ for $1 < p < \infty$. Thus our main result here is a complement to [3].

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2. Preliminaries

For each $s > -1$ we let v_s denote the measure on $\mathbb{B}_*$ defined by

$$dv_s(z) := (1 - N_*^2(z))^s dv(z),$$

where v denotes the normalized Lebesgue measure on $\mathbb{B}_*$. For all $0 < p < \infty$ we define

$$\mathcal{A}^p_s(\mathbb{B}_*) := \mathcal{H}(\mathbb{B}_*) \cap L^p\left(\mathbb{B}_*, |z \bullet z|^{\frac{p-2}{2}} dv_s\right),$$

where $\mathcal{H}(\mathbb{B}_*)$ is the space of all holomorphic functions on $\mathbb{B}_*$. Naturally, the spaces $\mathcal{A}^p_s(\mathbb{B}_*)$ are called weighted Bergman spaces of $\mathbb{B}_*$.

When $p = 2$, there exists an orthogonal projection from $L^2(\mathbb{B}_*, dv_s)$ onto $\mathcal{A}^2_s(\mathbb{B}_*)$. This will be called the weighted Bergman projection and is denoted by $\mathbf{P}_{s,\mathbb{B}_*}$.

It is well known that $\mathbf{P}_{s,\mathbb{B}_*}$ is an integral operator on $L^2(\mathbb{B}_*, dv_s)$, namely,

$$\mathbf{P}_{s,\mathbb{B}_*} f(z) = \int_{\mathbb{B}_*} \mathbf{K}_{s,\mathbb{B}_*}(z,w) f(w)\, dv_s(w),$$

where

$$\mathbf{K}_{s,\mathbb{B}_*}(z,w) = \frac{\mathbf{A}(X,Y)}{(n^2+n-s)v_s(\mathbb{B}_*)(X^2-Y)^{n+1+s}}$$

is the weighted Bergman kernel. Here

$$X = 1 - z \bullet \bar{w}, \quad Y = (z \bullet z)\, \overline{w \bullet w},$$

and $\mathbf{A}(X,Y)$ is the sum

$$\sum_{j=0}^{\infty} c_{n,s,j} X^{n+s-1-2j} Y^j \left[2(n+s)X - \frac{(n-2j+s)(n+1+2s)}{n+s+1}(X^2-Y) \right].$$

As usual,

$$c_{n,s,j} = \binom{n+s+1}{2j+1} = \frac{(n+s+1)(n+s)\cdots(n+s-2j+1)}{(2j+1)!}$$

is the binomial coefficient and $v_s(\mathbb{B}_*)$ is the weighted Lebesgue volume of $\mathbb{B}_*$. See [3] and [5] for these formulas and more information about the weighted Bergman kernels.

Because of the infinite sum $\mathbf{A}(X,Y)$, the formula for $\mathbf{K}_{s,\mathbb{B}_*}$ in the previous paragraph is not really a closed form. As such, it is inconvenient for us to do estimates for $\mathbf{P}_{s,\mathbb{B}_*}$ directly. Thus we employ a technique that was used in [3]. More specifically, we relate the domain $\mathbb{B}_*$ to the hypersurface $\mathbb{M}$ of the Euclidean unit ball in $\mathbb{C}^{n+1}$ defined by

$$\mathbb{M} = \{z \in \mathbb{C}^{n+1} \setminus \{0\} : z \bullet z = 0, |z| < 1\}.$$

If $\mathrm{Pr} : \mathbb{C}^{n+1} \to \mathbb{C}^n$ is defined by

$$\mathrm{Pr}(z_1, \dots, z_n, z_{n+1}) = (z_1, \dots, z_n),$$

and $\mathbf{F} = \mathrm{Pr}_{|\mathbb{M}}$, then $\mathbf{F} : \mathbb{M} \to \mathbb{B}_* - \{0\}$.

Let

$$\mathbb{H} = \{z \in \mathbb{C}^{n+1} \setminus \{0\} : z \bullet z = 0\}.$$

It was proved in [5] that there is an $\mathrm{SO}(n+1,\mathbb{C})$-invariant holomorphic form α on $\mathbb{H}$. Moreover, this form is unique up to a multiplicative constant. In fact, after appropriate normalization, the restriction to $\mathbb{H} \cap (\mathbb{C} \setminus \{0\})^{n+1}$ of this form is given by

$$\alpha(z) = \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{z_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1}.$$

Our norm estimates for the weighted Bergman projection $\mathbf{P}_{s,\mathbb{B}_*}$ on L^p spaces will be based on the corresponding estimates on $\mathbb{M}$. Therefore, we will consider the spaces $L^p_s(\mathbb{M})$ consisting of measurable complex-valued functions f on $\mathbb{M}$ such that

$$\|f\|^p_{L^p_s(\mathbb{M})} = \int_{\mathbb{M}} |f(z)|^p (1-|z|^2)^s \frac{\alpha(z)\wedge\bar{\alpha}(z)}{\tilde{C}} < \infty,$$

where

$$\tilde{C} := (-1)^{\frac{n(n+1)}{2}}(2i)^n.$$

We use $\mathcal{A}^p_s(\mathbb{M})$ to denote the subspace of all holomorphic functions in $L^p_s(\mathbb{M})$. The weighted Bergman projection $\mathbf{P}_{s,\mathbb{M}}$ is then the orthogonal projection from $L^2_s(\mathbb{M})$ onto $\mathcal{A}^2_s(\mathbb{M})$. Again, it is well known that $\mathbf{P}_{s,\mathbb{M}}$ is an integral operator on $L^2_s(\mathbb{M})$ given by the formula

$$\mathbf{P}_{s,\mathbb{M}} f(z) = \int_{\mathbb{M}} \mathbf{K}_{s,\mathbb{M}}(z,w) f(w)(1-|w|^2)^s \frac{\alpha(w)\wedge\bar{\alpha}(w)}{\tilde{C}},$$

where $\mathbf{K}_{s,\mathbb{M}}$ is the corresponding Bergman kernel.

The starting point for our analysis is the following closed form of $\mathbf{K}_{s,\mathbb{M}}$, which was obtained as Theorem 3.2 in [3].

Lemma 2. *The weighted Bergman kernel $\boldsymbol{K}_{s,\mathbb{M}}$ of $\mathcal{A}^2_s(\mathbb{M})$ is given by*

$$\boldsymbol{K}_{s,\mathbb{M}}(z,w) = \frac{C\big(n-1+(n+1+2s)z\bullet\bar{w}\big)}{(1-z\bullet\bar{w})^{n+s+1}},$$

where C is a certain constant that depends on n and s.

Let $f : \mathbb{B}_* \to \mathbb{C}$ be a measurable function. We define a function $\mathbf{T}f$ on $\mathbb{M}$ by

$$(\mathbf{T}f)(z) := \frac{z_{n+1}}{(2(n+1)^2)^{1/p}}(f\circ\mathbf{F})(z) = \frac{z_{n+1} f(z_1,\ldots,z_n)}{(2(n+1)^2)^{1/p}}.$$

The operator $\mathbf{T}$ will also play a key role in our analysis. In particular, we need the following result which was obtained as Lemma 4.1 in [3].

Lemma 3. *For each $p \geq 1$ and $s > -1$ the linear operator $\boldsymbol{T}$ is an isometry from $L^p\big(\mathbb{B}_*,\ |z\bullet z|^{\frac{p-2}{2}}dv_s\big)$ into $L^p_s(\mathbb{M})$. Moreover, we have $\boldsymbol{P}_{s,\mathbb{M}}\boldsymbol{T} = \boldsymbol{T}\boldsymbol{P}_{s,\mathbb{B}_*}$ on $L^p\big(\mathbb{B}_*, |z\bullet z|^{\frac{p-2}{2}}dv_s\big)$.*

Theorem B of [3] states that the operator $\mathbf{P}_{s,\mathbb{B}_*}$ maps $L^p\big(\mathbb{B}_*, |z\bullet z|^{\frac{p-2}{2}}dv_s\big)$ boundedly onto $\mathcal{A}^p_s(\mathbb{B}_*)$ for all $p > 1$. Our goal is to obtain a sharp norm estimate of $\mathbf{P}_{s,\mathbb{B}_*}$ on $L^p\big(\mathbb{B}_*, |z\bullet z|^{\frac{p-2}{2}}dv_s\big)$. To this end, we need to derive an improved version of the classical Forelli–Rudin integral estimates in the case of the minimal ball.

3. Refined Forelli–Rudin type estimates

We use μ to denote the unique $\mathcal{O}(n+1,\mathbb{R})$-invariant measure on the boundary $\partial\mathbb{M}$ of $\mathbb{M}$ that satisfies $\mu(\partial\mathbb{M}) = 1$. For any integers n and k we will need the following constant,

$$N(k,\, n) = \frac{(2k+n-1)(k+n-2)!}{k!(n-1)!}.$$

Lemma 4. *Let d be any nonnegative integer. For any $T > 0$ there exists a constant $C > 0$, depending on n, T and d but not on t, such that*

$$\int_{\partial\mathbb{M}} \frac{|z \bullet \bar{\xi}|^{2d}}{|1 - z \bullet \bar{\xi}|^{n+t}}\, d\mu(\xi) \le \frac{C\Gamma(t)}{(1-|z|^2)^t}$$

for all $z \in \mathbb{M}$ and $0 < t < T$.

Proof. Let I denote the integral above and let $\lambda = (n+t)/2$. By the proof of Lemma 5.1 in [3], we have

$$I = \frac{|z|^{2d}}{\Gamma^2(\lambda)} \sum_{k=0}^{\infty} \left(\frac{\Gamma(k+\lambda)}{\Gamma(k+1)}\right)^2 \frac{|z|^{2k}}{N(k+d,n)}. \tag{1}$$

Since

$$\frac{1}{(1-|z|^2)^t} = \frac{1}{\Gamma(t)} \sum_{k=0}^{\infty} \frac{\Gamma(k+t)}{\Gamma(k+1)} |z|^{2k}. \tag{2}$$

Lemma 4 will be proved if we can show that there exists a constant $C > 0$, depending only on n, T, and d, such that

$$\frac{\Gamma^2(k+\lambda)}{\Gamma^2(k+1)N(k+d,\; n)} \le \frac{C\Gamma(k+t)}{\Gamma(k+1)} \tag{3}$$

for $k \in \mathbb{N}$.

Let

$$A_k(t) = \frac{\Gamma^2(k+\lambda)}{\Gamma(k+1)N(k+d,\; n)\Gamma(k+t)}.$$

By Stirling's formula there exist two positive constants C_1 and M such that

$$C_1^{-1} \le \frac{\Gamma(x)}{x^{x-\frac{1}{2}} e^{-x}} \le C_1$$

for all $x \ge M$. It follows that the constant $N(k+d,\; n)$ is comparable to k^{n-1}. Moreover, there exist positive constants C_2 and C_3, depending only on n, T, and d, such that

$$\begin{aligned} A_k(t) &\le C_2 \frac{\left((k+\lambda)^{k+\lambda-\frac{1}{2}} e^{-k-\lambda}\right)^2}{(k+1)^{k+1-\frac{1}{2}} e^{-k-1} k^{n-1} (k+t)^{k+t-\frac{1}{2}} e^{-k-t}} \\ &\le C_3 \left[1 + \frac{\lambda}{k}\right]^{n-1} \left[1 + \frac{\lambda - t}{k+t}\right]^{k+t} \left[1 + \frac{\lambda - 1}{k+1}\right]^{k+1} \left[\frac{k+1}{k+\lambda}\right]^{\frac{3}{2}}. \end{aligned}$$

By virtue of uniform convergence of the limit

$$\lim_{\theta\to\infty}\left(1+\frac{x}{\theta}\right)^{\theta}=e^x \tag{4}$$

for x in any bounded interval, we have $A_k(t)\le C$, where C is a positive constant independent of t and k. This proves the desired estimate (3). $\square$

Lemma 5. *Suppose $T>0$, $A>-1$, and d is a nonnegative integer. Then there exists a constant $C>0$ (depending on d, T, and A, but not on t and s) such that*

$$\int_{\mathbb{M}}\frac{|z\bullet\bar{w}|^{2d}}{|1-z\bullet\bar{w}|^{n+t+s+1}}(1-|w|^2)^s\alpha(w)\wedge\bar{\alpha}(w)\le\frac{C\Gamma(s+1)\Gamma(t)}{(1-|z|^2)^t}$$

for all $-1<s<A$, $0<t<T$, and $z\in\mathbb{M}$.

Proof. Let J denote the integral above. According to Lemma 2.1 on page 506 in [3], we have

$$J=\int_0^1(1-r^2)^s r^{2n-3}\left(\int_{\partial\mathbb{M}}\frac{|(rz)\bullet\bar{\xi}|^{2d}}{|1-(rz)\bullet\bar{\xi}|^{n+t+s+1}}d\mu(\xi)\right)dr.$$

Using the binomial series and the orthogonality of the sequence of functions $\xi\mapsto(z\bullet\bar{\xi})^k$, $k\in\mathbb{N}$, in $L^2(\partial\mathbb{M},\ \mu)$, we obtain

$$J=\frac{\Gamma(s+1)|z|^{2d}}{2\Gamma^2(\lambda)}\sum_{k=0}^{\infty}\frac{\Gamma^2(k+\lambda)\Gamma(n+k+d-1)|z|^{2k}}{\Gamma^2(k+1)\Gamma(s+n+k+d)N(k+d,n)},$$

where $\lambda=(n+t+s+1)/2$.

As t goes from 0 to T, and s goes from -1 to A, the parameter λ goes from $n/2$ to $(n+T+A+1)/2$, so $\Gamma^2(\lambda)$ is bounded below away from 0 and bounded above away from infinity.

Therefore, just like in the proof of Lemma 4, we only need to show that there is a positive constant C, independent of k and t, such that $B_k(t,s)\le C$ for all $k\ge 0$, $0<t<T$, and $-1<s<A$, where

$$B_k(t,s)=\frac{\Gamma(n+k+d-1)}{\Gamma(s+n+k+d)}A_k(t).$$

A little computing shows that the factor $\frac{\Gamma(n+k+d-1)}{\Gamma(s+n+k+d)}$ is uniformly bounded for $s\in(-1,A)$. So, from estimate for A_k, there exists a positive constant $C=C(A,T,d)$ such that $B_k(t,s)\le C_2$ for all $k\ge 0$, $t\in(0,T)$, and $s\in(-1,A)$. This proves the desired estimate. $\square$

4. An optimal pointwise estimate

The proof of our main result depends on two estimates. One is the refined version of the Forelli–Rudin estimates obtained in the previous section. The other is an optimal pointwise estimate for functions in weighted Bergman spaces in our context. We refer the interested reader to [8] and [11] for similar estimates about functions in weighted Bergman spaces of the Euclidean ball in $\mathbb{C}^n$.

More specifically, we will obtain an optimal pointwise estimate for the functions in $\mathcal{A}_s^p(\mathbb{M})$. To this end, we consider the following commutative diagram

$$\begin{array}{ccc} \mathbb{M}\cup\{0\} & \xrightarrow{\ \phi\ } & \mathbb{M}\cup\{0\} \\ i\big\downarrow & & \big\downarrow i \\ \mathbb{B}_{n+1} & \xrightarrow{\ \varphi_z\ } & \mathbb{B}_{n+1} \end{array}$$

where $\mathbb{B}_{n+1}$ is the unit ball in $\mathbb{C}^{n+1}$, $z \in \mathbb{M}$, φ_z is the involutive automorphism determined by z (see [11], for example, for more information about these automorphisms), i is the identity map, and ϕ is given by $i \circ \phi = \varphi_z \circ i$. Thus $\mathbb{M}$ is invariant by the mapping φ_z so that one indeed can define the inverse mapping i^{-1}.

For any $s > -1$ we consider the measure

$$d\lambda_s(z) = \frac{\Gamma(n+s)}{2\omega(\partial\mathbb{M})\Gamma(n-1)\Gamma(s+1)}(1-|z|^2)^s\,\alpha(z)\wedge\bar{\alpha}(z)$$

on $\mathbb{M}$, where ω is the $(2n-1)$-form on $\partial\mathbb{M}$ defined by

$$\omega(z)(V_1,\dots,V_{2n-1}) = \alpha(z)\wedge\bar{\alpha}(z)(z, V_1,\dots,V_{2n-1}).$$

Proposition 6. *Suppose* $1 \le p < \infty$ *and* $-1 < s < \infty$. *Then*

$$|g(z)|^p \le \frac{1}{(1-|z|^2)^{n+1+s}}\int_{\mathbb{M}} |g(w)|^p\,d\lambda_s(w)$$

for all $g \in \mathcal{A}_s^p(\mathbb{M})$ *and* $z \in \mathbb{M}$.

Proof. Let $g \in \mathcal{A}_s^p(\mathbb{M})$. By the mean value property for holomorphic functions,

$$g(z) = \int_{\partial\mathbb{M}} g(z+r\xi)d\mu(\xi)$$

for all $z \in \mathbb{M}$ and $0 \le r < 1-|z|$. By Lemma 2.1 in [3] and Hölder's inequality,

$$|g(z)|^p \le \int_{\mathbb{M}} |g(z+rw)|^p\,d\lambda_s(w).$$

Moreover, from Lemma 3.2 in [2], g can be uniquely extented to the complex hypersurface $\mathbb{M}\cup\{0\}$. Let $z \to 0$ and $r \to 1^-$ in the above inequality. Then

$$|g(0)|^p \le \int_{\mathbb{M}} |g(w)|^p\,d\lambda_s(w). \tag{5}$$

More generally, for $g \in \mathcal{A}_s^p(\mathbb{M})$ and $z \in \mathbb{M}$, we consider the function defined on $\mathbb{M}$ by

$$F(w) = g\circ i^{-1}\circ\varphi_z\circ i(w)\frac{(1-|z|^2)^{(n+1+s)/p}}{(1-w\bullet\bar{z})^{2(n+1+s)/p}}.$$

It can be checked that

$$\int_{\mathbb{M}} |F(w)|^p \, d\lambda_s(w) = \int_{\mathbb{M}} |g(w)|^p \, d\lambda_s(w). \tag{6}$$

In fact, if we let

$$\mathcal{X} = \{(x,y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : \quad x \bullet x = y \bullet y = 1, \ x \bullet y = 0\},$$

then it is clear that the mapping

$$(x,y) \mapsto z = \frac{x+iy}{\sqrt{2}}$$

injects $\mathcal{X}$ into $\mathbb{C}^{n+1}$ and the image is $\partial\mathbb{M}$. It follows from the $O(n+1)$-invariance of $\mathcal{X}$ (see [7] for example) and Lemma 1.7 in [11] that

$$\int_{\partial\mathbb{M}} |F(r\xi)|^p d\mu(\xi) = \int_{\mathcal{X}} |g(r\zeta)|^p dv(\zeta),$$

where dv is the normalized Lebesgue measure on $\mathbb{C}^{n+1}$. Using Lemma 2.1 in [3] again, we obtain (6).

The proposition is proved if we combine the estimates in (5) and (6). □

5. Proof of the main result

The following result is a standard boundedness criterion for integral operators on L^p-spaces and is usually referred to as Schur's test.

Lemma 7. *Suppose $H(x,y)$ is a positive kernel and*

$$Tf(x) = \int_X H(x,y) f(y) \, d\nu(y)$$

is the associated integral operator. Let $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If there exists a positive function $h(x)$ and positive constants C_1 and C_2 such that

$$\int_X H(x,y)(h(y))^q \, d\nu(y) \le C_1 (h(x))^q, \quad x \in X$$

and

$$\int_X H(x,y)(h(x))^p \, d\nu(x) \le C_1 (h(y))^p, \quad y \in X.$$

Then the operator T is bounded on $L^p(X, d\nu)$. Moreover, the norm of T on $L^p(X, d\nu)$ does not exceed $C_1^{\frac{1}{q}} C_2^{\frac{1}{p}}$.

Proof. See [11]. □

We can now prove the main result of the paper.

Theorem 8. *For any $s > -1$ there exists a constant $C > 0$ (depending only on s and n but not on p) such that the norm $\|\boldsymbol{P}_{s,\mathbb{M}}\|_p$ of the linear operator*

$$\boldsymbol{P}_{s,\mathbb{M}} : L^p_s(\mathbb{M}) \to \mathcal{A}^p_s(\mathbb{M})$$

satisfies the estimates

$$C\csc(\pi/p) \le \|\boldsymbol{P}_{s,\mathbb{M}}\|_p \le C\csc(\pi/p)$$

for all $1<p<\infty$, *where* $\boldsymbol{P}_{s,\mathbb{M}}$ *denotes the orthogonal projection from* $L^2_s(\mathbb{M})$ *onto* $\mathcal{A}^2_s(\mathbb{M})$.

Proof. Fix $1 < p < \infty$ and let q be the conjuguate exponent, namely, $1/p+1/q=1$. Consider the function

$$h(z) = (1-|z|^2)^{-(s+1)/(pq)}, \qquad z\in\mathbb{M}.$$

By Lemmas 2 and 5, the integral

$$I = \int_{\mathbb{M}} |\mathbf{K}_{s,\mathbb{M}}(z,w)| h^q(w)(1-|w|^2)^s \frac{\alpha(w)\wedge\bar{\alpha}(w)}{\tilde{C}}$$

satisfies the following estimates,

$$\begin{aligned} I &\le C_1 \int_{\mathbb{M}} \frac{(1-|w|^2)^{-\frac{s+1}{p}+s}}{|1-z\bullet w|^{n+s+1}} \alpha(w)\wedge\bar{\alpha}(w) \\ &= C_1 \int_{\mathbb{M}} \frac{(1-|w|^2)^{-\frac{s+1}{q}-1}}{|1-z\bullet w|^{n+\frac{s+1}{q}-1+\frac{s+1}{p}+1}} \alpha(w)\wedge\bar{\alpha}(w) \\ &\le \frac{C_2\,\Gamma(\frac{s+1}{p})\Gamma(\frac{s+1}{q})}{(1-|z|^2)^{\frac{s+1}{p}}} \\ &= C_2\,\Gamma\left(\frac{s+1}{p}\right)\Gamma\left(\frac{s+1}{q}\right) h^q(z), \end{aligned}$$

where C_1 and C_2 are positive constants independent of p.

Similary, the integral

$$J = \int_{\mathbb{M}} |\mathbf{K}_{s,\mathbb{M}}(z,w)| h^p(z)(1-|z|^2)^s \frac{\alpha(z)\wedge\bar{\alpha}(z)}{\tilde{C}}$$

satisfies

$$J \le C_3\,\Gamma\left(\frac{s+1}{p}\right)\Gamma\left(\frac{s+1}{q}\right) h^p(w),$$

where C_3 is a positive constant independent of p. It follows from Lemma 7 that the norm of the operator $\mathbf{P}_{s,\mathbb{M}}$ on $L^p_s(\mathbb{M})$ does not exceed

$$C_4\,\Gamma\left(\frac{s+1}{p}\right)\Gamma\left(\frac{s+1}{q}\right),$$

where C_4 is a positive constant independent of p. So the norm estimate $\|\mathbf{P}_{s,\mathbb{M}}\|_p \le C\csc(\pi/p)$ follows from the following well-known property of the gamma function:

$$\Gamma\left(\frac{s+1}{p}\right)\Gamma\left(\frac{s+1}{q}\right) \le \frac{C}{\sin(\pi/p)},$$

where C is a positive constant independent of p; see [10].

Observe that $\csc(\pi/p)$ is comparable to p when p is away from 0. Therefore, to prove that the above estimate for $\|\mathbf{P}_{s,\mathbb{M}}\|_p$ is sharp, we only need to establish the norm estimate $\|\mathbf{P}_{s,\mathbb{M}}\|_p \geq pC^{-1}$ for all $p > 2$. The case $1 < p < 2$ will follow from duality and the symmetry of the sine function.

So we assume $p > 2$ and consider the function

$$f(z) = \log\left(\frac{1}{\sqrt{2}} - z_1\right) - \overline{\log\left(\frac{1}{\sqrt{2}} - z_1\right)}, \quad z = (z_1, z_2, \ldots, z_n) \in \mathbb{B}_*.$$

Alternatively,

$$f(z) = 2iArg\left(\frac{1}{\sqrt{2}} - z_1\right),$$

with

$$-\pi < Arg\left(\frac{1}{\sqrt{2}} - z_1\right) < \pi.$$

Thus the norm of f on $L^p(\mathbb{B}_*, |z \bullet z|^{\frac{p-2}{2}} dv_s)$ does not exceed $2\pi C_s^{-1/p}$, where C_s is a positive constant that only depends on s and n.

By Proposition 6, we have

$$|g(z)| \leq \frac{C_s^{-1/p} \|g\|_{\mathcal{A}_s^p(\mathbb{M})}}{(1 - |z|^2)^{\frac{n+s+1}{p}}} \tag{7}$$

for all $g \in \mathcal{A}_s^p(\mathbb{M})$ and $z \in \mathbb{M}$. We now take $g = \mathbf{P}_{s,\mathbb{M}}\mathbf{T}f = \mathbf{T}\mathbf{P}_{s,\mathbb{B}_*}f$ and

$$z = \left(\frac{r}{\sqrt{2}}, 0, \ldots, 0, i\frac{r}{\sqrt{2}}\right)$$

in (7) with $0 < r < 1$. Using the definition of $\mathbf{T}$, the fact that $\mathbf{T}$ is an isometry, and the formula

$$\mathbf{P}_{s,\mathbb{B}_*}f(z) = \log\left(\frac{1}{\sqrt{2}} - z_1\right),$$

we obtain

$$\|\mathbf{P}_{s,\,\mathbb{B}_*}f\|_{\mathcal{A}_s^p(\mathbb{B}_*)} \geq \frac{C_s^{1/p}\, r(1-r)^{\frac{n+s+1}{p}}}{(2(n+1)^2)^{1/p}\sqrt{2}} \log\frac{\sqrt{2}}{1-r}.$$

In particular, if $r = 1 - e^{-p}$, then

$$\|\mathbf{P}_{s,\,\mathbb{B}_*}f\|_{\mathcal{A}_s^p(\mathbb{B}_*)} \geq \frac{(\log\sqrt{2} + p)(1 - e^{-p})e^{-(n+s+1)}C_s^{1/p}}{(2(n+1)^2)^{1/p}\sqrt{2}}.$$

This shows that there exists a positive constant C, independent of p, such that

$$\|\mathbf{P}_{s,\mathbb{B}_*}\|_p \geq Cp, \qquad 2 < p < \infty. \tag{8}$$

Since $\mathbf{P}_{s,\mathbb{M}}\mathbf{T} = \mathbf{T}\mathbf{P}_{s,\mathbb{B}_*}$ and since $\mathbf{T}$ is an isometry, we have $\|\mathbf{P}_{s,\mathbb{M}}\| \geq \|\mathbf{P}_{s,\mathbb{B}_*}\|$. Combining this with (8) we obtain the desired lower estimate for $\|\mathbf{P}_{s,\mathbb{M}}\|$. □

Our main result, Theorem 1, is now a consequence of Theorem 8 above. In fact, it follows from Lemma 3 that

$$\|\mathbf{P}_{s,\mathbb{B}_*}f\|_p = \|\mathbf{T}\mathbf{P}_{s,\mathbb{B}_*}f\|_p = \|\mathbf{P}_{s,\mathbb{M}}\mathbf{T}f\|_p \tag{9}$$

for all $f \in L^p(\mathbb{B}_*, |z \bullet z|^{\frac{p-2}{2}}\, dv_s)$. Using the upper bound for the operator $\mathbf{P}_{s,\mathbb{M}}$ from Theorem 8 and Lemma 3 again, we obtain

$$\|\mathbf{P}_{s,\mathbb{B}_*}f\|_p \le C \csc(\pi/p)\|\mathbf{T}f\|_p = C \csc(\pi/p)\|f\|_p$$

for all $f \in L^p(\mathbb{B}_*, |z \bullet z|^{\frac{p-2}{2}}\, dv_s)$, where C is a constant indepndent of p. This shows that $\|\mathbf{P}_{s,\mathbb{B}_*}\|_p \le C \csc(\pi/p)$ for all $1 < p < \infty$.

On the other hand, it follows from (8) that $\|\mathbf{P}_{s,\mathbb{B}_*}\|_p \ge C \csc(\pi/p)$ for all $p \ge 2$. By duality, this holds for $1 < p \le 2$ as well, which completes the proof of Theorem 1.

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