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Jordan type of a $k[C_p \times C_p]$-module

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Abstract. Let E be the elementary abelian group $C_p \times C_p$, k a field of characteristic p, M a finite dimensional module over the group algebra $k[E]$ and J the Jacobson radical J of $k[E]$. We prove that the decomposition of M when considered as a $k[\langle 1+x\rangle]$-module for a p-point x in J is well defined modulo J^p.

Contents

1. Introduction

Throughout this note k denotes a field of characteristic $p > 0$, unless it is stated otherwise, E denotes the elementary abelian p-group of rank 2, generated by a and b, i.e., $E = C_p \times C_p = \langle a, b\rangle$, and M denotes a finite dimensional $k[E]$-module, $M{\downarrow_H}$ denotes M as a $k[H]$-module for a subgroup H of units of $k[E]$.

The set of indecomposable $k[C_{p^t}]$-modules (up to isomorphism) consists of the ideals of $k[C_{p^t}]$, namely,

$$k[C_{p^t}],\ J,\ J^2,\ \ldots,\ J^{p^t-1} \tag{1}$$

where J is the Jacobson radical of $k[C_{p^t}]$. However, when a finite group G contains E as a subgroup by Higman's theorem there are infinitely many indecomposable $k[G]$-modules (up to isomorphism) [Hi]. When $p = 2$, the infinite set of indecomposable $k[E]$-modules is determined in [Ba], and a cohomological characterization is given in [Ca]. However, when $p \geq 3$, there is no classification for indecomposable $k[E]$-modules. Thus, alternative means are used in the study of $k[E]$-modules so that new subcategories of modules

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are defined and characterized. For instance, information, namely the number of i-dimensional indecomposable $k[E]$-modules, for $i = 1, \dots, p$, in the restriction of a $k[E]$-module at p-points led to the definition of the subcategory of modules of constant Jordan type [CFP].

The *p-points* of $k[E]$ are elements $x = \alpha(a-1) + \beta(b-1) + w$ of J with α, β in k, not both zero, and $w \in J^2$ so that $\langle 1+x \rangle$ is cyclic group of order p. For such an x, $k[\langle 1+x \rangle] \cong k[C_p] \cong k[x]/(x^p)$ is a subalgebra of $k[E]$ for which $k[E]{\downarrow}_{\langle 1+x \rangle}$ is free. This property distinguishes p-points form arbitrary points of J_E. For a p-point x, the subgroup $\langle 1+x \rangle$ of the group of units of $k[E]$ is called a *shifted cyclic subgroup* (following [Ca]). For a $k[E]$-module M and a p-point x, by (1) the decomposition of $M{\downarrow}_{\langle 1+x \rangle}$ into indecomposable $k[\langle 1+x \rangle]$-modules is as follows;

$$M{\downarrow}_{\langle 1+x \rangle} \cong (k[\langle 1+x \rangle])^{a_p} \oplus (J)^{a_{p-1}} \oplus (J^2)^{a_{p-2}} \oplus \cdots \oplus (J^{p-1})^{a_1}.$$

Thus $M{\downarrow}_{\langle 1+x \rangle}$ is determined by the p-tuple $\underline{\mathrm{a(x)}} = (a_1, \dots, a_p)$ where a_i denotes the number of the i-dimensional indecomposable $k[\langle 1+x \rangle]$-module J^{p-i} for $J = \mathrm{rad}(k[\langle 1+x \rangle])$. Hence a $k[E]$-module M can be studied through such p-tuples $\underline{\mathrm{a(x)}}$ where x is a p-point.

Dade's [Da] criterion was the first significant result which used p-points, namely, for an arbitrary elementary abelian p-group E, a $k[E]$-module M is free if and only if $M{\downarrow}_{\langle 1+x \rangle}$ is free for all shifted cyclic subgroups of $k[E]$. In [CFS] modules for $C_p \times C_p$, especially *modules of constant Jordan type*, i.e., modules having the same $\underline{\mathrm{a(x)}}$ for all p-points x, are studied thoroughly.

In fact, p-points are defined and studied in the much more general context of finite group schemes in [FP]. Later, in [FPS] generic and maximal Jordan types for modules are introduced and studied; this is followed by [CFP] where modules of constant Jordan type are introduced. Recently, in [Ka] this type of study has been generalized to include the restrictions of modules to subalgebras of $k[G]$ that are of the form $k[\langle 1+x \rangle] \cong k[C_{p^t}]$, for $t \geq 1$, and $x \in J$ is a p^t-point. A *p^t-point* of $k[G]$ is an element of J defined analogous to a p-point, yet they are much more intricate to characterize. The p^t-points led to the definition of modules of constant p^t-Jordan type and modules of constant p^t-power Jordan type for an abelian p-group G. Also, a filtration of modules of constant Jordan type by modules of constant p^t-power Jordan type is obtained. Studying modules by means of p-points is an active research area, see also [Fr], [BP], et al. The main result of this article is a variation on that theme for $k[E]$-modules:

Theorem 1. *If M is a finite dimensional $k[E]$-module, and x, y are elements of $J - J^2$ with $x \equiv y \pmod{J^p}$, then the kernels of x^i and y^i on M are the same for all $i \geq 1$. In particular, $M{\downarrow}_{\langle 1+x \rangle}$ and $M{\downarrow}_{\langle 1+y \rangle}$ have the same decomposition.*

This theorem is a generalization of Lemma 6.4 in [Ca] which states that $M{\downarrow}_{\langle 1+x \rangle}$ is free if and only if $M{\downarrow}_{\langle 1+y \rangle}$ is free which makes the rank variety,

$V_E^r(M)$, of M well defined. The rank variety is defined as points $\bar{x}$ in J/J^2 at which $M{\downarrow}_{\langle 1+x\rangle}$ is not free. Likewise, Theorem 1 makes the following subset of $J/J^p \times (\mathbb{N} \cup \{0\})^p$, denoted by $Jt_E(M)$, called *the Jordan set* of M, well defined.

$$Jt_E(M) = \{(\bar{x}, \underline{\mathrm{a(x)}}) \mid \bar{x} \in J/J^p\}.$$

The Jordan set of M is an invariant of the module finer than its rank variety. Although it is possible for two nonisomorphic $k[E]$-modules to have the same Jordan set, the Jordan set may distinguish two nonisomorphic modules. The Jordan set of M was first defined in [Öz] and used in [Ka1], under the name *multiplicities set of* M, to distinguish some types of $k[C_2 \times C_4]$-modules.

The significance of J/J^2 in the modular representation theory of elementary abelian p-groups, especially when the freeness of a module is concerned, is manifested in Dade's Theorem, in the definition of the rank variety, etc. By our theorem it becomes clear that J/J^p has a significance as well for the Jordan decompostion of a module at a p-point x in J, for instance in the study of modules of constant Jordan type. At this point there is a need for a "geometric" interpretation for J/J^p similar to that of J/J^2.

When stated in terms of matrices our theorem takes the following form.

Corollary 2. *Let* A, B *be commuting nilpotent nonzero matrices over* k *with* $A^p = 0$, $B^p = 0$. *If* $X = f(A,B)$, $Y = g(A,B)$ *for polynomials* $f, g \in k[z_1, z_2]$ *with no constant term, having at least one linear term and* $f - g$ *in the ideal* $(z_1, z_2)^p$, *then* $\mathrm{null}(X^i) = \mathrm{null}(Y^i)$ *for all* i. *In particular,* A *and* B *have the same Jordan canonical form.*

2. A lemma

The formula in Lemma 3(i) below for counting the Jordan blocks of a given size in the Jordan canonical form of a nilpotent matrix is used in the proof of Theorem 1.

Lemma 3. *Let* X *be a* $d \times d$ *matrix over a field* $\mathbb{F}$ *and* a_t *denote the number of* $t \times t$ *Jordan blocks in the Jordan form of* X. *Suppose that* $X^s = 0$. *Then*

(i) $a_t = \mathrm{rank}(X^{t-1}) - 2\,\mathrm{rank}(X^t) + \mathrm{rank}(X^{t+1})$ *for* $1 \le t \le s$,

(ii) $\displaystyle\sum_{i=1}^{s} a_i = \#\{\textit{Jordan blocks in } X\} = \mathrm{rank}(X^0) - \mathrm{rank}(X) = \mathrm{null}(X)$,

(iii) $$\mathrm{rank}(X^r) = \sum_{r+1 \le t \le s} (t-r)\, a_t.$$

Proof. In the course of the proof of this lemma we will use the notation $a(i)$ to denote a_i in order not to use too small indices. Note also that (ii) follows from (i). To prove (i) and (iii), without loss of generality, assume that X is in Jordan canonical form. Since $X^s = 0$, X consists of Jordan blocks of sizes less than or equal to s. Thus

$$X = \begin{bmatrix} [j_s]^{\oplus a(s)} & & \\ & \ddots & \\ & & [j_1]^{\oplus a(1)} \end{bmatrix}$$

where $[j_t]$ denotes the $t{\times}t$ upper triangular Jordan block with zero eigenvalue, and $\oplus\, a(t)$ in the exponent denotes the multiplicity of $[j_t]$ in X. Hence $d = \sum_{t=1}^{s} t\, a(t)$ and $\operatorname{rank}(X) = \sum_{1\le t\le s} \operatorname{rank}([j_t])\, a(t)$. Note that

$$\operatorname{rank}([j_t]^r) = \begin{cases} t-r, & \text{if } r < t; \\ 0, & \text{if } r \ge t. \end{cases}$$

Thus $\operatorname{rank}([j_t]^r) \neq 0$ if and only if $t \ge r+1$ and

$$\begin{aligned} \operatorname{rank}([j_{r+1}]^r) &= 1, \\ \operatorname{rank}([j_{r+2}]^r) &= 2, \\ &\vdots \\ \operatorname{rank}([j_s]^r) &= s-r. \end{aligned}$$

Therefore

$$\begin{aligned} \operatorname{rank}(X^r) &= \sum_{1\le t\le s} a(t)(\operatorname{rank}([j_t]^r)) \\ &= 0 + \cdots + 0 + a(r+1) + 2\,a(r+2) + \ldots + (s-r)\,a(s) \\ &= \sum_{r+1\le t\le s} (t-r)\,a(t). \end{aligned}$$

In particular, for $r = s-1$, when computing $\operatorname{rank}(X^{s-1})$ the only possibly nonzero rank in the summation is $\operatorname{rank}([j_s]^{s-1}) = 1$. Hence one obtains

$$\operatorname{rank}(X^{s-1}) = a(s).$$

For $r = s-2$, (since $\operatorname{rank}([j_s]^{s-2}) = 2$) there are only two possibly nonzero terms in the summation, hence

$$\operatorname{rank}(X^{s-2}) = a(s-1) + 2a(s),$$

By substituting the resulting formulas for $a(s-1)$ and $a(s)$ in the formula for $\operatorname{rank}(X^{s-3})$, one obtains

$$\begin{aligned} a(s-2) &= \operatorname{rank}(X^{s-3}) - 2a(s-1) - 3a(s) \\ &= \operatorname{rank}(X^{s-3}) - 2\operatorname{rank}(X^{s-2}) + \operatorname{rank}(X^{s-1}). \end{aligned}$$

This suggests the formula

$$a(s-i) = \operatorname{rank}(X^{s-(i+1)}) - 2\operatorname{rank}(X^{s-i}) + \operatorname{rank}(X^{s-(i-1)}), \quad \text{for } 0 \le i \le s.$$

The proof is by induction on i in the above formula. Having seen that it is true for $i = 1, 2$ and 3, suppose the above equality holds for all $1 \leq i \leq r$. To prove it for $r + 1$, recall that

$$\begin{aligned}\operatorname{rank}(X^{s-(r+1)-1}) = \operatorname{rank}(X^{s-r-2}) &= \sum_{s-(r+1)\leq t\leq s} \operatorname{rank}([j_t]^{s-(r+1)-1})\, a(t)\\ &= \sum_{s-(r+1)\leq t\leq s} (t-(s-(r+1)-1))\, a(t)\\ &= a(s-(r+1)) + 2a(s-r) + 3a(s-(r-1))\\ &\quad + \cdots + (s-(s-(r+1)-1))\, a(s).\end{aligned}$$

Therefore

$$\begin{aligned}a(s-(r+1)) = \operatorname{rank}(X^{s-(r+1)-1}) &- 2\, a(s-r) - 3\, a(s-(r-1))\\ &- 4\, a(s-(r-2)) - \cdots - (r+2)\, a(s).\end{aligned}$$

By the induction hypothesis, one obtains

$$\begin{aligned}&a(s-(r+1))\\ &= \operatorname{rank}(X^{s-(r+1)-1})\\ &\quad - 2\Big(\operatorname{rank}(X^{s-(r+1)}) - 2\operatorname{rank}(X^{s-r}) + \operatorname{rank}(X^{s-(r-1)})\Big)\\ &\quad - 3\Big(\operatorname{rank}(X^{s-r}) - 2\operatorname{rank}(X^{s-(r-1)}) + \operatorname{rank}(X^{s-(r-2)})\Big)\\ &\quad - 4\Big(\operatorname{rank}(X^{s-(r-1)}) - 2\operatorname{rank}(X^{s-(r-2)}) + \operatorname{rank}(X^{s-(r-3)})\Big)\\ &\qquad \vdots\\ &\quad - r\Big(\operatorname{rank}(X^{s-(r-(r-3))}) - 2\operatorname{rank}(X^{s-(r-(r-2))})\\ &\qquad\quad + \operatorname{rank}(X^{s-(r-(r-1))})\Big)\\ &\quad - (r+1)\Big(\operatorname{rank}(X^{s-(r-(r-2))}) - 2\operatorname{rank}(X^{s-(r-(r-1))})\Big)\\ &\quad - (r+2)\operatorname{rank}(X^{s-(r-(r-1))}).\end{aligned}$$

Thus one obtains

$$\begin{aligned}&a(s-(r+1))\\ &= \operatorname{rank}(X^{s-(r+1)-1}) - 2\operatorname{rank}(X^{s-(r+1)}) + (4-3)\operatorname{rank}(X^{s-r})\\ &\quad + \Big(-2 + (-3)(-2) - 4\Big) \operatorname{rank}(X^{s-(r-1)}) + \cdots +\\ &\quad + \Big(-r + (-(r+1))(-2) - (r+2)\Big) \operatorname{rank}(X^{s-1})\\ &= \operatorname{rank}(X^{s-(r+2)}) - 2\operatorname{rank}(X^{s-(r+1)}) + \operatorname{rank}(X^{s-r}). \qquad \square\end{aligned}$$

3. Proof of Theorem 1

In the following discussion to simplify the notation we use J for $\mathrm{rad}(k[E])$. Note that $J^{2p-1}=0$.

Let X, Y be the matrices which represent the action of x, y, and also let A, B denote the matrices repesenting the actions of $a-1$ and $b-1$, on M respectively. Note that $J=\langle A,B\rangle$ and $J^{2p-1}=0$. Since X and Y commute, if $\mathrm{null}(X)=\mathrm{null}(Y)$, then $\mathrm{null}(X^i)=\mathrm{null}(X^i)$ for every $i\geq 1$.

Claim. $\mathrm{null}(X)=\mathrm{null}(Y)$.

Proof. Since the situation is symmetric with respect to X and Y, it is sufficient to show that $\mathrm{null}(X)\subseteq\mathrm{null}(Y)$. By the hypothesis on x and y we can write $Y=X+w(A,B)$ with $X=\alpha A+\beta B+c(A,B)$ for some $\gamma\in k$, for α, β in k not both 0, $c(A,B)$ in J^2 but not containing any terms with more than $p-1$ factors, i.e., can only contain γA^iB^j with $i+j$ in $\{2,\dots,p-1\}$ as a term, and $w(A,B)$ containing only terms with at least p factors. Since the situation is symmetric with respect to A and B, without loss of generality assume that $\alpha\neq 0$.

Suppose $Xm=0$ for some nonzero m in M. Then

$$-(\alpha A+\beta B)m=c(A,B)m, \tag{2}$$

$$Ym=w(A,B)m. \tag{3}$$

Multiplying (2) with $A^{p-2}B^{p-1}$ we get

$$-A^{p-2}B^{p-1}(\alpha A+\beta B)m=A^{p-2}B^{p-1}c(A,B)m\in J^{2p-1}m=0.$$

Since $\alpha\neq 0$, we have $A^{p-1}B^{p-1}m=0$, and hence, $J^{2p-2}m=0$. Multiplying (2) with $A^{p-3}B^{p-1}$ gives

$$-A^{p-3}B^{p-1}(\alpha A+\beta B)m=A^{p-3}B^{p-1}c(A,B)m\in J^{2p-2}m=0.$$

Hence $A^{p-2}B^{p-1}m=0$ as $\alpha\neq 0$. Similarly, multiplying (2) with $A^{p-2}B^{p-2}$ gives that $A^{p-1}B^{p-2}m=0$. Thus $J^{2p-3}m=0$. Using $J^{2p-3}m=0$, and multiplying (2) with the terms $A^{p-2}B^{p-3}$, $A^{p-3}B^{p-2}$, $A^{p-4}B^{p-1}$ we obtain that $J^{2p-4}m=0$. Then by induction on l in J^{2p-l}, for $2\leq l\leq p$, we obtain that $J^pm=0$. Hence by (3) $Ym\in J^pm=0$ proving the claim. □

Thus by the above remarks we have $\mathrm{null}(X^i)=\mathrm{null}(Y^i)$.

The second statement of the theorem follows from the formula

$$a_i=\mathrm{rank}(X^{i-1})-2\,\mathrm{rank}(X^i)+\mathrm{rank}(X^{i+1})$$

given in Lemma 3(i). Since $\mathrm{null}(X^i)=\mathrm{null}(Y^i)$, we have $\mathrm{rank}(X^i)=\mathrm{rank}(Y^i)$ for all i. Hence each Jordan block occurs with the same multiplicity in the Jordan form of X and Y. That is, $M{\downarrow_{\langle 1+x\rangle}}$ and $M{\downarrow_{\langle 1+y\rangle}}$ have the same decomposition. □

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