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Rotation methods in operator ergodic theory

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Abstract. Let $E(\cdot) : \mathbb{R} \to \mathfrak{B}(\mathfrak{X})$ be the spectral decomposition of a trigonometrically well-bounded operator U acting on the arbitrary Banach space $\mathfrak{X}$, and suppose that the bounded function $\phi : \mathbb{T} \to \mathbb{C}$ has the property that for each $z \in \mathbb{T}$, the spectral integral $\int_{[0,2\pi]} \phi(e^{it}) dE_z(t)$ exists, where $E_z(\cdot)$ denotes the spectral decomposition of the (necessarily) trigonometrically well-bounded operator (zU). We show this implies that for each $z \in \mathbb{T}$, the spectral integral with respect to $E(\cdot)$ of the rotated function $\phi_z(\cdot) \equiv \phi((\cdot)z)$ exists. In particular, these considerations furnish the preservation under rotation of spectral integration for the Marcinkiewicz r-classes of multipliers $\mathfrak{M}_r(\mathbb{T})$, which are not themselves rotation-invariant. In the setting of an arbitrary super-reflexive space, we pursue a different aspect of the impact of rotations on the operator ergodic theory framework by applying the rotation group to the spectral integration of functions of higher variation so as to obtain strongly convergent Fourier series expansions for the operator theory counterparts of such functions. This vector-valued Fourier series convergence can be viewed as an extension of classical Calderón–Coifman–Weiss transference without being tied to the need of the latter for power-boundedness assumptions.

Contents

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1. Introduction and notation

We begin by recalling the requisite operator-theoretic ingredients, fixing some notation and terminology in the process. In what follows the symbol "K" with a (possibly empty) set of subscripts will signify a constant which depends only on those subscripts, and which may change in value from one occurrence to another. The Banach algebra of all continuous linear operators mapping a Banach space $\mathfrak{X}$ into itself will be denoted by $\mathfrak{B}(\mathfrak{X})$, and the identity operator of $\mathfrak{X}$ will be designated I. Given a function $\phi : \mathbb{T} \to \mathbb{C}$ and $z \in \mathbb{T}$, we shall denote by ϕ_z the corresponding rotate of ϕ, specified for all $w \in \mathbb{T}$ by writing $\phi_z(w) \equiv \phi(zw)$. A trigonometric polynomial will be a complex-valued function on $\mathbb{T}$ expressible as a finite linear combination of the functions z^n $(n \in \mathbb{Z})$. The notion of a trigonometrically well-bounded operator $U \in \mathfrak{B}(\mathfrak{X})$ (introduced in [5] and [6]) and its characterization by a "unitary-like" spectral representation rest on the following vehicle for spectral decomposability.

Definition 1.1. A *spectral family of projections* in a Banach space $\mathfrak{X}$ is an idempotent-valued function $E(\cdot) : \mathbb{R} \to \mathfrak{B}(\mathfrak{X})$ with the following properties:

(a) $E(\lambda)E(\tau) = E(\tau)E(\lambda) = E(\lambda)$ if $\lambda \le \tau$;
(b) $\|E\|_u \equiv \sup\{\| E(\lambda) \| : \lambda \in \mathbb{R}\} < \infty$;
(c) with respect to the strong operator topology, $E(\cdot)$ is right-continuous and has a left-hand limit $E(\lambda^-)$ at each point $\lambda \in \mathbb{R}$;
(d) $E(\lambda) \to I$ as $\lambda \to \infty$ and $E(\lambda) \to 0$ as $\lambda \to -\infty$, each limit being with respect to the strong operator topology.

If, in addition, there exist $a, b \in \mathbb{R}$ with $a \le b$ such that $E(\lambda) = 0$ for $\lambda < a$ and $E(\lambda) = I$ for $\lambda \ge b$, $E(\cdot)$ is said to be concentrated on $[a, b]$.

Their definition endows spectral families of projections with properties reminiscent of, but weaker than, those that would be inherited from a countably additive Borel spectral measure on $\mathbb{R}$. Given a spectral family $E(\cdot)$ in the Banach space $\mathfrak{X}$ concentrated on a compact interval $J = [a, b]$, an associated notion of spectral integration can be developed as follows (for these and further basic details regarding spectral integration, see [23]). For each bounded function $\psi : J \to \mathbb{C}$ and each partition $\mathcal{P} = (\lambda_0, \lambda_1, ..., \lambda_n)$ of J, where we take $\lambda_0 = a$ and $\lambda_n = b$, set

$$\mathcal{S}(\mathcal{P}; \psi, E) = \sum_{k=1}^{n} \psi(\lambda_k)\{E(\lambda_k) - E(\lambda_{k-1})\}. \tag{1.1}$$

If the net $\{\mathcal{S}(\mathcal{P}; \psi, E)\}$ converges in the strong operator topology of $\mathfrak{B}(\mathfrak{X})$ as $\mathcal{P}$ runs through the set of partitions of J directed to increase by refinement, then the strong limit is called the *spectral integral of* ψ with respect to $E(\cdot)$, and is denoted by $\int_J \psi(\lambda) dE(\lambda)$. In this case, we define $\int_J^{\oplus} \psi(\lambda) dE(\lambda)$ by writing $\int_J^{\oplus} \psi(\lambda) dE(\lambda) \equiv \psi(a)E(a) + \int_J \psi(\lambda) dE(\lambda)$. It can be shown that the spectral integral $\int_J \psi(\lambda) dE(\lambda)$ exists for each complex-valued function ψ

having bounded variation on J (in symbols, $\psi \in BV(J)$), and that the mapping $\psi \mapsto \int_J^{\oplus} \psi(\lambda) dE(\lambda)$ is an identity-preserving algebra homomorphism of $BV(J)$ into $\mathfrak{B}(\mathfrak{X})$ satisfying

$$\left\| \int_J^{\oplus} \psi(t) dE(t) \right\| \leq \|\psi\|_{BV(J)} \|E\|_u , \tag{1.2}$$

where $\|\cdot\|_{BV(J)}$ denotes the usual Banach algebra norm expressed by

$$\|\psi\|_{BV(J)} \equiv \sup_{x \in J} |\psi(x)| + \operatorname{var}(\psi, J).$$

(Similarly, $BV(\mathbb{T})$ denotes the Banach algebra of all $f \in BV([0, 2\pi])$ such that $f(0) = f(2\pi)$. More generally, when there is no danger of confusion, we shall, as convenient, tacitly adopt the conventional practice of identifying a function Ψ defined on $\mathbb{T}$ with its (2π)-periodic counterpart $\Psi\left(e^{i(\cdot)}\right)$ defined on $\mathbb{R}$.) In connection with (1.2), we recall here the following handy tool for gauging the oscillation of a spectral family $E(\cdot)$ in the arbitrary Banach space $\mathfrak{X}$ concentrated on a compact interval $J = [a, b]$. For each $x \in \mathfrak{X}$, and each partition of $[a, b]$, $\mathcal{P} = (a = a_0 < a_1 < \cdots < a_N = b)$, we put:

$$\omega(\mathcal{P}, E, x) = \max_{1 \leq j \leq N} \sup \{\|E(t)x - E(a_{j-1})x\| : a_{j-1} \leq t < a_j\}.$$

Now, as $\mathcal{P}$ increases through the set of all partitions of $[a, b]$ directed to increase by refinement, we have (see Lemma 4 of [23]):

$$\lim_{\mathcal{P}} \omega(\mathcal{P}, E, x) = 0. \tag{1.3}$$

We can now state the following "spectral theorem" characterization of trigonometrically well-bounded operators (Definition 2.18 and Proposition 3.1 of [5]).

Definition 1.2. An operator $U \in \mathfrak{B}(\mathfrak{X})$ is said to be *trigonometrically well-bounded* if there is a spectral family $E(\cdot)$ in $\mathfrak{X}$ concentrated on $[0, 2\pi]$ such that $U = \int_{[0,2\pi]}^{\oplus} e^{i\lambda} dE(\lambda)$. In this case, it is possible to arrange that $E(2\pi^-) = I$, and with this additional property the spectral family $E(\cdot)$ is uniquely determined by U, and is called *the spectral decomposition* of U.

The circle of ideas in Definition 1.2 entails the following convenient tool for our considerations. Let $\mathfrak{X}$ be a Banach space, and $U \in \mathfrak{B}(\mathfrak{X})$ be trigonometrically well-bounded, with spectral decomposition $E(\cdot)$. By, for instance, Theorem 2.1 and Corollary 2.2 in [11], we see that zU is trigonometrically well-bounded for each $z \in \mathbb{T}$, and, upon denoting its spectral decomposition by $E_z(\cdot)$, we have

$$\eta(U) \equiv \sup\{\|E_z\|_u : z \in \mathbb{T}\} < \infty. \tag{1.4}$$

We remark that if $\mathfrak{X}$ is a UMD Banach space, and $U \in \mathfrak{B}(\mathfrak{X})$ is merely assumed to be invertible, then by Theorem (4.5) of [15] U is automatically trigonometrically well-bounded provided U is power-bounded:

$$c(U) \equiv \sup\{\|U^n\| : n \in \mathbb{Z}\} < \infty. \tag{1.5}$$

For the proof of Theorem 2.1 below, we shall require the following proposition, which is readily deducible by using the beginning of the proof of Proposition (3.11) in [2] as starting point — alternatively, it can be shown by using direct calculations to verify (1.6) below.

Proposition 1.3. *Let $\mathfrak{X}$ be a Banach space, and $U \in \mathfrak{B}(\mathfrak{X})$ be trigonometrically well-bounded, with spectral decomposition $E(\cdot)$. Let $z = e^{i\theta}$, where $0 \leq \theta < 2\pi$, denote by $E_z(\cdot)$ the spectral decomposition of (zU), and decompose $\mathfrak{X}$ by writing $\mathfrak{X} = \mathfrak{X}_0 \oplus \mathfrak{X}_1 \oplus \mathfrak{X}_2$, where $\mathfrak{X}_0 = \left\{E(2\pi - \theta) - E\left((2\pi - \theta)^-\right)\right\}\mathfrak{X}$, $\mathfrak{X}_1 = E\left((2\pi - \theta)^-\right)\mathfrak{X}$, $\mathfrak{X}_2 = \{I - E(2\pi - \theta)\}\mathfrak{X}$. Then we have*

$$(1.6)\quad zU = I\,|\mathfrak{X}_0 \oplus \int_{[0,2\pi]}^{\oplus} e^{it}\,dE(t-\theta)\,|\mathfrak{X}_1 \oplus \int_{[0,2\pi]}^{\oplus} e^{it}\,dE(t+2\pi-\theta)\,|\mathfrak{X}_2\ ,$$

and consequently, upon symbolizing by χ the characteristic function, defined on $\mathbb{R}$, of $[0,\infty)$, we infer that for each $\lambda \in \mathbb{R}$,

$$(1.7)\qquad \begin{aligned} E_z(\lambda) = {} & \chi(\lambda)\left\{E(2\pi-\theta) - E\left((2\pi-\theta)^-\right)\right\} \\ & + E(\lambda-\theta)\,E\left((2\pi-\theta)^-\right) \\ & + E(\lambda+2\pi-\theta)\,\{I - E(2\pi-\theta)\}\,. \end{aligned}$$

2. Rotation of integrands in spectral integration

The stage is now set for deriving the following result in the general Banach space setting.

Theorem 2.1. *Assume the hypotheses and notation of Proposition* 1.3*, and suppose that the bounded function $\phi : \mathbb{T} \to \mathbb{C}$ has the property that for each $z \in \mathbb{T}$, the spectral integral $\int_{[0,2\pi]} \phi\left(e^{it}\right) dE_z(t)$ exists. Then for each $z \in \mathbb{T}$, the spectral integral $\int_{[0,2\pi]} \phi_z\left(e^{it}\right)\, dE(t)$ exists, where ϕ_z denotes the rotate of ϕ defined by writing $\phi_z(w) \equiv \phi(zw)$ for all $w \in \mathbb{T}$, and we have*

$$(2.1)\qquad \int_{[0,2\pi]}^{\oplus} \phi_z\left(e^{it}\right)\, dE(t) = \int_{[0,2\pi]}^{\oplus} \phi\left(e^{it}\right) dE_z(t)\,.$$

Proof. We shall tacitly take into account the aforementioned basic properties possessed by the spectral decomposition of a trigonometrically well-bounded operator, such as concentration on $[0, 2\pi]$. Using an appropriate change of variable while treating the right side of (2.1) on the basis of (1.7), we find upon simplifying the outcome that for each $z = e^{i\theta} \in \mathbb{T}$,

$$\begin{aligned} \int_{[0,2\pi]}^{\oplus} \phi\left(e^{it}\right)\, dE_z(t) = {} & \phi\left(e^{i\theta}\right) E(0) + \int_{[0,2\pi-\theta]} \phi\left(e^{i\theta}e^{it}\right)\, dE(t) \\ & + \int_{[0,2\pi]} \phi\left(e^{it}\right)\, dE(t+2\pi-\theta)\,\{I - E(2\pi-\theta)\}\,, \end{aligned}$$

with all the spectral integrals in sight existing. Hence a change of variable in the second spectral integral on the right completes the proof of Theorem 2.1.by yielding the equality

$$\int_{[0,2\pi]}^{\oplus} \phi\left(e^{it}\right) dE_z(t) = \phi\left(e^{i\theta}\right) E(0) + \int_{[0,2\pi-\theta]} \phi\left(e^{i\theta}e^{it}\right) dE(t) + \int_{[2\pi-\theta,2\pi]} \phi\left(e^{i\theta}e^{it}\right) dE(t). \qquad \square$$

Corollary 2.2. *Assume the hypotheses and notation of Theorem* 2.1*. Then for each* $w \in \mathbb{T}$*, the spectral integral* $\int_{[0,2\pi]} \phi_w\left(e^{it}\right) dE_z(t)$ *exists for each* $z \in \mathbb{T}$*.*

Proof. Fix $\zeta \in \mathbb{T}$, and notice that for all $z \in \mathbb{T}$, $\left(E_\zeta\right)_z(\cdot) = E_{\zeta z}(\cdot)$. By hypothesis the spectral integral $\int_{[0,2\pi]} \phi\left(e^{it}\right) dE_{\zeta z}(t)$ exists for all $z \in \mathbb{T}$. Hence by Theorem 2.1, the spectral integral $\int_{[0,2\pi]} \phi_w\left(e^{it}\right) dE_\zeta(t)$ exists for each $w \in \mathbb{T}$. $\square$

3. Applications to Marcinkiewicz multiplier classes defined by r-variation

This section is devoted to applications of Theorem 2.1 to the spectral integration of the Marcinkiewicz r-classes of Fourier multipliers, $\mathfrak{M}_r(\mathbb{T})$, $1 \le r < \infty$, which are defined in (3.8) below. For our present and our later purposes, we first describe the algebras (under pointwise operations) $V_r(\mathbb{T})$, consisting of all $f : \mathbb{T} \to \mathbb{C}$ whose r-variation

$$\text{var}_r(f, \mathbb{T}) \equiv \sup \left\{ \sum_{k=1}^{N} \left| f\left(e^{ix_k}\right) - f\left(e^{ix_{k-1}}\right) \right|^r \right\}^{1/r} < \infty, \tag{3.1}$$

where the supremum is extended over all partitions $0 = x_0 < x_1 < \cdots < x_N = 2\pi$ of $[0, 2\pi]$. When endowed with the following norm, $V_r(\mathbb{T})$ becomes a Banach algebra:

$$\|f\|_{V_r(\mathbb{T})} \equiv \sup\{|f(z)| : z \in \mathbb{T}\} + \text{var}_r(f, \mathbb{T}).$$

For key fundamental features of the r-variation, see, e.g., [9], [18], [24]. In particular, it is elementary that if $f \in V_r(\mathbb{T})$, then $\lim_{x \to y^+} f\left(e^{ix}\right)$ and $\lim_{x \to y^-} f\left(e^{ix}\right)$exist for each $y \in \mathbb{R}$, and $f\left(e^{i(\cdot)}\right)$ has only countably many discontinuities on $\mathbb{R}$. The analogous formulation to (3.1) is employed to define the r-variation of a function on an arbitrary compact interval of $\mathbb{R}$. It is clear that the Banach algebras $V_1(\mathbb{T})$ and $BV(\mathbb{T})$ coincide, and also clear that $V_q(\mathbb{T}) \subseteq V_r(\mathbb{T})$, when $1 \le q \le r < \infty$, since $\|\cdot\|_{V_p(\mathbb{T})}$ is a decreasing function of p. There is also a rotation-invariant equivalent notion for the r-variation of f on $\mathbb{T}$, which serves as an alternative to $\text{var}_r(f, \mathbb{T})$ defined above. Specifically, we can define

$$\mathfrak{v}_r(f,\mathbb{T}) = \sup \left\{ \sum_{k=1}^{N} \left| f\left(e^{it_k}\right) - f\left(e^{it_{k-1}}\right) \right|^r \right\}^{1/r},$$

where the supremum is taken over all finite sequences $-\infty < t_0 < t_1 < \cdots < t_N = t_0 + 2\pi < \infty$. It is evident that for $1 \le r < \infty$,

$$\mathrm{var}_r(f,\mathbb{T}) \le \mathfrak{v}_r(f,\mathbb{T}) \le 2\mathrm{var}_r(f,\mathbb{T}),$$

and, moreover, $\mathfrak{v}_1(f,\mathbb{T}) = \mathrm{var}_1(f,\mathbb{T})$. For $1 \le r < \infty$, and $f \in V_r(\mathbb{T})$, the partial sums $S_n(f,z) \equiv \sum_{k=-n}^{n} \widehat{f}(n)\, z^n$ of the Fourier series of f satisfy

$$\sup_{n\ge 0} \|S_n(f,\cdot)\|_{L^\infty(\mathbb{T})} \le K_r \|f\|_{V_r(\mathbb{T})}. \tag{3.2}$$

(In the case $p = 1$, this is shown by the reasoning in Theorem III(3.7) of [26]. The case $p > 1$ is covered by the reasoning in §§10, 12 of [24].)

Straightforward application of the generalized Minkowski inequality shows that if $F \in L^1(\mathbb{T})$ and $\psi \in V_r(T)$, then the convolution $F * \psi$ belongs to $V_r(T)$, with

$$\|F * \psi\|_{V_r(\mathbb{T})} \le 2\|F\|_{L^1(\mathbb{T})} \|\psi\|_{V_r(\mathbb{T})}.$$

This fact comes in handy when the need arises for regularization by an $L^1(\mathbb{T})$-summability kernel such as $\{\kappa_n\}_{n=0}^{\infty}$, Fejér's kernel for $\mathbb{T}$, specified by $\kappa_n(z) \equiv \sum_{k=-n}^{n} (1 - |k|/(n+1))\, z^n$.

The following notation will come in handy–particularly whenever Fejér's Theorem is invoked. Given any function $f : \mathbb{R} \to \mathbb{C}$ which has a right-hand limit and a left-hand limit at each point of $\mathbb{R}$, we shall denote by $f^\# : \mathbb{R} \to \mathbb{C}$ the function defined for every $t \in \mathbb{R}$ by putting

$$f^\#(t) = \frac{1}{2}\left\{ \lim_{s\to t^+} f(s) + \lim_{s\to t^-} f(s) \right\}.$$

In the case of a function $\phi : \mathbb{T} \to \mathbb{C}$ such that $\phi\left(e^{i(\cdot)}\right) : \mathbb{R} \to \mathbb{C}$ has everywhere a right-hand and a left-hand limit, we shall, by a slight abuse of notation, write

$$\phi^\#(t) = \frac{1}{2}\left\{ \lim_{s\to t^+} \phi\left(e^{is}\right) + \lim_{s\to t^-} \phi\left(e^{is}\right) \right\}, \quad \text{for all } t \in \mathbb{R}.$$

In accordance with our convention, we may regard the (2π)-periodic function $\phi^\#$ as also being defined on $\mathbb{T}$. In this case it is clear that for each $z \in \mathbb{T}$, $(\phi_z)^\#$ likewise exists, and

$$(\phi_z)^\# = \left(\phi^\#\right)_z \text{ on } \mathbb{T}. \tag{3.3}$$

Note that for each $\phi \in V_r(\mathbb{T})$, $\phi^\# \in V_r(\mathbb{T})$, with $\left\|\phi^\#\right\|_{V_r(\mathbb{T})} \le 2\|\phi\|_{V_r(\mathbb{T})}$. By virtue of Fejér's Theorem, the Fourier series of any function $\phi \in V_r(\mathbb{T})$ is $(C,1)$ summable to $\phi^\#$ pointwise on $\mathbb{T}$, $1 \le r < \infty$. Moreover, Theorem 1 of [21] is a Tauberian result which, in particular, converts this $(C,1)$

summability of the Fourier series corresponding to arbitrary $\phi \in V_r(\mathbb{T})$ into pointwise convergence on $\mathbb{T}$.

Theorem 1 of [21] applies here, because $V_r(\mathbb{T})$ is a subset of the integrated Lipschitz class Λ_r (also denoted $\mathrm{Lip}\left(r^{-1}, r\right)$), consisting of all $f \in L^r(\mathbb{T})$ such that as $h \to 0^+$,

$$\int_0^{2\pi} |f(t+h) - f(t)|^r \, dt = 0(h).$$

This standard inclusion

$$V_r(\mathbb{T}) \subseteq \Lambda_r \tag{3.4}$$

was established for $r = 1$ in Lemma 9 of [21], and for $1 < r < \infty$ on pages 259, 260 of [24]. In Theorem 4.1 below, we shall see that the above-noted Fourier series convergence for functions of higher variation has an abstract vector-valued counterpart. The inclusion in (3.4) also combines with Lemma 11 of [21] to furnish the following rate of decay for the Fourier coefficients of arbitrary $f \in V_r(\mathbb{T})$:

$$\sup_{k\in\mathbb{Z}} |k|^{1/r} \left|\widehat{f}(k)\right| < \infty. \tag{3.5}$$

For each $n \in \mathbb{Z}$, if we denote by $\epsilon_n \in BV(\mathbb{T})$ the function defined by writing $\epsilon_n(z) \equiv z^n$, then the reasoning on pages 275, 276 of [24] shows that for $1 \le r < \infty$,

$$\sup_{n\in\mathbb{Z}\setminus\{0\}} \frac{\mathrm{var}_r(\epsilon_n, \mathbb{T})}{|n|^{1/r}} < \infty. \tag{3.6}$$

We conclude this review of higher variation by recalling the following notion, which will play a central role in §4. Given an arbitrary spectral family of projections $F(\cdot)$ in a Banach space $\mathfrak{X}$ and an index $q \in [1, \infty)$, $\mathrm{var}_q(F)$, the (possibly infinite) q-variation of $F(\cdot)$, is defined to be:

$$\sup \left\{ \sum_{k=1}^{N} \|(F(t_k) - F(t_{k-1}))\, x\|^q \right\}^{1/q},$$

where the supremum is extended over all $x \in \mathfrak{X}$ such that $\|x\| = 1$, all $N \in \mathbb{N}$, and all $\{t_k\}_{k=0}^N \subseteq \mathbb{R}$ such that $-\infty < t_0 < t_1 < \cdots < t_{N-1} < t_N < \infty$.

The dyadic points relevant to the study of (2π)-periodic functions are the terms of the sequence $\{t_k\}_{k=-\infty}^{\infty} \subseteq (0, 2\pi)$ given by

$$t_k = \begin{cases} 2^{k-1}\pi, & \text{if } k \le 0; \\ 2\pi - 2^{-k}\pi, & \text{if } k > 0. \end{cases} \tag{3.7}$$

The dyadic arcs Δ_k, $k \in \mathbb{Z}$, are specified by $\Delta_k = \left\{e^{ix} : x \in [t_k, t_{k+1}]\right\}$, and for $\phi : \mathbb{T} \to \mathbb{C}$, we shall write $\mathrm{var}_r(\phi, \Delta_k)$ to stand for $\mathrm{var}_r\left(\phi\left(e^{i(\cdot)}\right), [t_k, t_{k+1}]\right)$.

With this notation, the *Marcinkiewicz* r-class for $\mathbb{T}$, $\mathfrak{M}_r(\mathbb{T})$, $1 \leq r < \infty$, is defined as the class of all functions $\phi : \mathbb{T} \to \mathbb{C}$ such that

$$\|\phi\|_{\mathfrak{M}_r(\mathbb{T})} \equiv \sup_{z\in\mathbb{T}} |\phi(z)| + \sup_{k\in\mathbb{Z}} \operatorname{var}_r(\phi, \Delta_k) < \infty. \tag{3.8}$$

The Fourier multiplier properties of Marcinkiewicz r-classes were originally established in [18] (for the unweighted framework). For key background details regarding Marcinkiewicz r-classes, including their Fourier multiplier properties in various settings, see, e.g., [9], [12] (for weighted settings), [18]. Clearly, each space $\mathfrak{M}_r(\mathbb{T})$ is not rotation-invariant.

We now apply Theorem 2.1 to various settings in which, for appropriate values of r, the classes $\mathfrak{M}_r(\mathbb{T})$ admit spectral integration. Throughout this discussion we continue with our umbrella hypotheses that $\mathfrak{X}$ is a Banach space, and $U \in \mathfrak{B}(\mathfrak{X})$ is trigonometrically well-bounded, with spectral decomposition $E(\cdot)$.

(a) If $\mathfrak{X}$ is a UMD space, and U is power-bounded, then by Theorem (1.1)-(ii) of [8], for each $\phi \in \mathfrak{M}_1(\mathbb{T})$ the spectral integral

$$\int_{[0,2\pi]} \phi\left(e^{it}\right) dE(t)$$

exists, and we have, in the notation of (1.5),

$$\left\| \int_{[0,2\pi]}^{\oplus} \phi\left(e^{it}\right) dE(t) \right\| \leq (c(U))^2 K_{\mathfrak{X}} \|\phi\|_{\mathfrak{M}_1(\mathbb{T})}.$$

Applying Theorem 2.1 to this state of affairs in (a), we see that for each $\phi \in \mathfrak{M}_1(\mathbb{T})$ and each $z \in \mathbb{T}$, the spectral integral

$$\int_{[0,2\pi]} \phi_z\left(e^{it}\right) dE(t)$$

exists, and

$$\left\| \int_{[0,2\pi]}^{\oplus} \phi_z\left(e^{it}\right) dE(t) \right\| \leq (c(U))^2 K_{\mathfrak{X}} \|\phi\|_{\mathfrak{M}_1(\mathbb{T})}.$$

We remark that although (1.2) and (1.4) would allow us to apply Theorem 2.1 to $BV(\mathbb{T})$ in analogous fashion when U is a trigonometrically well-bounded operator on an arbitrary Banach space $\mathfrak{X}$, the conclusion thereby reached loses its significance because of the rotation invariance of $BV(\mathbb{T})$.

(b) Suppose that $\mathfrak{X}$ belongs to the broad class $\mathcal{I}$ of UMD spaces which was considered in [14], and includes, in particular: all closed subspaces of $L^p(\sigma)$, where $1 < p < \infty$ and σ is an arbitrary measure; any UMD space with an unconditional basis; all the closed subspaces of the von Neumann–Schatten classes $\mathcal{C}_p$, $1 < p < \infty$. Suppose further that $U \in \mathfrak{B}(\mathfrak{X})$ is power-bounded. Then Theorem 3.8 of [14] furnishes a number $q_0(\mathfrak{X})$, depending only on $\mathfrak{X}$ and belonging to

$(1, \infty)$, such that whenever $1 \le r < q_0(\mathfrak{X})$, and $\phi \in \mathfrak{M}_r(\mathbb{T})$, the spectral integral $\int_{[0,2\pi]} \phi\left(e^{it}\right) dE(t)$ exists, and satisfies

$$\left\| \int_{[0,2\pi]}^{\oplus} \phi\left(e^{it}\right) dE(t) \right\| \le (c(U))^2 K_{\mathfrak{X},r} \|\phi\|_{\mathfrak{M}_r(\mathbb{T})}.$$

Upon application of Theorem 2.1 to this setting in (b), we find that for $1 \le r < q_0(\mathfrak{X})$, $\phi \in \mathfrak{M}_r(\mathbb{T})$, and $z \in \mathbb{T}$, the spectral integral $\int_{[0,2\pi]} \phi_z\left(e^{it}\right) dE(t)$ exists, and

$$\left\| \int_{[0,2\pi]}^{\oplus} \phi_z\left(e^{it}\right) dE(t) \right\| \le (c(U))^2 K_{\mathfrak{X},r} \|\phi\|_{\mathfrak{M}_r(\mathbb{T})}.$$

(c) In the subcase of (b) wherein $\mathfrak{X}$ is given as a closed subspace of $L^p(\sigma)$ ($1 < p < \infty$, σ an arbitrary measure), and $U \in \mathfrak{B}(\mathfrak{X})$ is power-bounded, then by appeal to Theorem 4.10 of [9] one can specify the outcome much more precisely than was provided by the discussion in case (b). Specifically, Theorem 4.10 of [9] provides that if $1 \le r < \infty$ and $\left|p^{-1} - (1/2)\right| < r^{-1}$, then for each $\phi \in \mathfrak{M}_r(\mathbb{T})$, the spectral integral $\int_{[0,2\pi]} \phi\left(e^{it}\right) dE(t)$ exists, and satisfies

$$\left\| \int_{[0,2\pi]}^{\oplus} \phi\left(e^{it}\right) dE(t) \right\| \le (c(U))^2 K_{p,r} \|\phi\|_{\mathfrak{M}_r(\mathbb{T})}.$$

Consequently, by Theorem 2.1, for any such value of r, if $\phi \in \mathfrak{M}_r(\mathbb{T})$, and $z \in \mathbb{T}$, the spectral integral $\int_{[0,2\pi]} \phi_z\left(e^{it}\right) dE(t)$ exists, and we have

$$\left\| \int_{[0,2\pi]}^{\oplus} \phi_z\left(e^{it}\right) dE(t) \right\| \le (c(U))^2 K_{p,r} \|\phi\|_{\mathfrak{M}_r(\mathbb{T})}.$$

(d) We now depart from the power-boundedness assumption for U that was imposed in cases (a), (b), and (c) by passing to the treatment of modulus mean-bounded operators in [12], to which we refer the reader for background details omitted here for expository reasons. Suppose that (Ω, μ) is a sigma-finite measure space, and $1 < p < \infty$. Let $\mathfrak{X} = L^p(\mu)$, and suppose that $U \in \mathfrak{B}(L^p(\mu))$ is invertible, separation-preserving, and modulus mean-bounded — this last condition means that the linear modulus of U, denoted $|U|$, satisfies

$$\gamma(U) \equiv \sup\left\{ \left\| (2N+1)^{-1} \sum_{k=-N}^{N} |U|^k \right\| : N = 0, 1, \cdots \right\} < \infty.$$

(These assumptions guarantee, in particular, that U is trigonometrically well-bounded, and we continue to denote its spectral decomposition by $E(\cdot)$. For some natural examples wherein the operator U in this context is not power-bounded, see §4 of [10].) Theorem 10 of [12], in conjunction with the condition in Theorem 6-(iii) of [12], shows that there is a constant $q(p, \gamma(U)) \in (1, \infty)$, depending only on p and $\gamma(U)$, such that if $1 \le r < q(p, \gamma(U))$ and $\phi \in \mathfrak{M}_r(\mathbb{T})$, then the spectral integral $\int_{[0,2\pi]} \phi\left(e^{it}\right) dE(t)$ exists, and satisfies

$$\left\| \int_{[0,2\pi]}^{\oplus} \phi\left(e^{it}\right) dE(t) \right\| \le K_{p,\gamma(U),r} \|\phi\|_{\mathfrak{M}_r(\mathbb{T})}.$$

Once again we invoke Theorem 2.1 — this time to infer that if $1 \le r < q(p, \gamma(U))$, $\phi \in \mathfrak{M}_r(\mathbb{T})$, and $z \in \mathbb{T}$, then the spectral integral $\int_{[0,2\pi]} \phi_z\left(e^{it}\right) dE(t)$ exists, and we have

$$\left\| \int_{[0,2\pi]}^{\oplus} \phi_z\left(e^{it}\right) dE(t) \right\| \le K_{p,\gamma(U),r} \|\phi\|_{\mathfrak{M}_r(\mathbb{T})}.$$

Remark 3.1.

(i) One sees from Theorem 3.3 of [13] that a special case of (d) arises by taking $\mathfrak{X}$ to be the sequence space $\ell^p(w)$ ($1 < p < \infty$, w a bilateral weight sequence satisfying the discrete Muckenhoupt A_p weight condition), and then taking U to be the left bilateral shift on $\ell^p(w)$. As an added bonus in this specialized scenario, the spectral decomposition $E(\cdot)$ of the left bilateral shift U has an explicit "concrete" description (see Scholium (5.13) of [10]).

(ii) If $\mathfrak{X}$ is a super-reflexive Banach space, and $U \in \mathfrak{B}(\mathfrak{X})$ is a trigonometrically well-bounded operator with spectral decomposition $E(\cdot)$, then an R.C. James inequality for normalized basic sequences in [22] can be combined with the Young–Stieltjes integration of [24] and with careful use of delicate spectral integration techniques to furnish a number $q \in (1, \infty)$ such that

$$\operatorname{var}_q(E) < \infty, \tag{3.9}$$

and to show for any such number q that whenever $r \in (1, q')$ and $\psi \in V_r(\mathbb{T})$, then $\int_{[0,2\pi]} \psi\left(e^{it}\right) dE(t)$ exists, and

$$\left\| \int_{[0,2\pi]}^{\oplus} \psi\left(e^{it}\right) dE(t) \right\| \le K_{r,q}\left(\operatorname{var}_q(E)\right) \|\psi\|_{V_r(\mathbb{T})}. \tag{3.10}$$

(See Theorem 4.5 of [4], or, alternatively, Theorem 3.7 of [3].) It is not difficult to see from its method of proof that Proposition 4.2 of [4], followed by application of (3.10), furnishes the following slight generalization of this situation.

Theorem 3.2. *If $\mathcal{F}$ is a collection of spectral families of projections acting in a super-reflexive Banach space $\mathfrak{X}$, and* $\sup\{\|F\|_u : F \in \mathcal{F}\} < \infty$*, then there is $q \in (1, \infty)$ such that*

$$\tau(\mathcal{F}) \equiv \sup\left\{\operatorname{var}_q(F) : F \in \mathcal{F}\right\} < \infty.$$

Hence by Theorem 4.5 of [4] in combination with (1.4) above, we deduce that if $\mathfrak{X}$ is a super-reflexive Banach space, and $U \in \mathfrak{B}(\mathfrak{X})$ is a trigonometrically well-bounded operator with spectral decomposition $E(\cdot)$, then there is $q \in (1, \infty)$ such that whenever $r \in (1, q')$ (where $q' = q(q-1)^{-1}$ is the conjugate index of q), $\psi \in V_r(\mathbb{T})$, and $z \in \mathbb{T}$, we have $\int_{[0,2\pi]} \psi(e^{it})\, dE_z(t)$ exists, and satisfies

$$\left\| \int_{[0,2\pi]}^{\oplus} \psi\left(e^{it}\right) dE_z(t) \right\| \leq K_{r,q}\, \tau(\mathcal{E})\, \|\psi\|_{V_r(\mathbb{T})},$$

where $\mathcal{E} \equiv \{E_w(\cdot) : w \in \mathbb{T}\}$.

While we could apply Theorem 2.1 to this state of affairs, the outcome would not provide any new information because of the rotation invariance properties of $V_r(\mathbb{T})$, which could have been invoked to begin with. However, in the next section, we shall apply this circle of ideas to obtain in the super-reflexive space framework strongly convergent Fourier series expansions for the operator theory counterparts of functions of finite r-variation.

4. Operator-valued Fourier series expansions in the strong operator topology

We continue to denote by ϕ_z the rotate by $z \in \mathbb{T}$ of a function $\phi : \mathbb{T} \to \mathbb{C}$. This section is devoted to establishing the following theorem.

Theorem 4.1. *Let $\mathfrak{X}$ be a super-reflexive Banach space, and let $U \in \mathfrak{B}(X)$ be a trigonometrically well-bounded operator. Denote by $E(\cdot)$ the spectral decomposition of U, choose $q \in (1, \infty)$ so that (3.9) holds, let $r \in (1, q')$ (where $q' = q(q-1)^{-1}$ is the conjugate index of q), and let $\psi \in V_r(\mathbb{T})$. Denote by $\mathfrak{S}_\psi : \mathbb{T} \to \mathfrak{B}(\mathfrak{X})$ the bounded function specified by*

$$\mathfrak{S}_\psi(z) = \int_{[0,2\pi]}^{\oplus} \psi_z(e^{it})\, dE(t).$$

and let $\mathcal{T}_\psi : \mathbb{T} \to \mathfrak{B}(\mathfrak{X})$ be the bounded function given by

$$\mathcal{T}_\psi(z) = \int_{[0,2\pi]}^{\oplus} (\psi_z)^{\#}(e^{it})\, dE(t).$$

Then the following conclusions are valid.

(a) *For each $x \in \mathfrak{X}$, the vector-valued series*

$$\sum_{\nu=-\infty}^{\infty} \widehat{\psi}(\nu) z^\nu U^\nu x \quad (z \in \mathbb{T}) \tag{4.1}$$

is the Fourier series of the $\mathfrak{X}$-valued function $\mathfrak{S}_\psi(\cdot)\, x : \mathbb{T} \to \mathfrak{X}$.

(b) *For each $z \in \mathbb{T}$, this Fourier series specified in (4.1) converges in the norm topology of $\mathfrak{X}$ to $\mathcal{T}_\psi(z)\, x$.*

Remark 4.2.

(i) Theorem 5.5 of [4] shows that under the hypotheses of Theorem 4.1, the series in (4.1) is also the Fourier series of $\mathcal{T}_\psi(\cdot)\,x$, and further shows the $(C,1)$ summability of this series to $\mathcal{T}_\psi(z)\,x$, at each $z \in \mathbb{T}$.

(ii) Since $BV(\mathbb{T}) = V_1(\mathbb{T}) \subseteq V_r(\mathbb{T})$, Theorem 4.1 automatically holds for $r = 1$. However, the analogue of Theorem 4.1 for the case $r = 1$ has been established earlier (see Theorems 2.6 and 4.4 of [3]). In fact, if $\mathfrak{X}$is an arbitrary Banach space, and $U \in \mathfrak{B}(X)$ is trigonometrically well-bounded with spectral decomposition $E(\cdot)$, then the conclusion of Theorem 4.1(a) continues to hold for each $\psi \in BV(\mathbb{T})$. (See [7].)

(iii) Theorem 4.1 can be viewed as a transference to the vector-valued setting of the key Fourier series convergence properties for scalar-valued functions of higher variation (see, e.g., item (ii) of the theorem on p. 275 of [24]). However, since our treatment is not confined to power-boundedness of U, it goes beyond the scope of traditional Calderón–Coifman–Weiss transference methods ([17],[19]). In the special case where the trigonometrically well-bounded operator U acting on the super-reflexive Banach space $\mathfrak{X}$ is power-bounded, the conclusion of Theorem 4.1(b) has previously been shown to hold for a suitable range of values $r > 1$, (Theorem 5.8 of [4]).

The following lemma, which is of independent interest, will readily yield Theorem 4.1(a).

Lemma 4.3. Let $\mathfrak{X}$ be a super-reflexive Banach space, and let $U \in \mathfrak{B}(X)$ be a trigonometrically well-bounded operator. Denote by $E(\cdot)$ the spectral decomposition of U, choose $q \in (1,\infty)$ so that (3.9) holds, let $r \in (1, q')$ (where $q' = q(q-1)^{-1}$ is the conjugate index of q), and let $\psi \in V_r(\mathbb{T})$. Then given $x \in \mathfrak{X}$ and $\varepsilon > 0$, there is a partition $\mathcal{P}_0$ of $[0, 2\pi]$ such that, in the notation of (1.1), we have for every refinement $\mathcal{P}$ of $\mathcal{P}_0$, and for all $z \in \mathbb{T}$,

$$\left\| \mathcal{S}(\mathcal{P}; \psi_z, E)x - \int_{[0,2\pi]} \psi_z\left(e^{it}\right)\, dE(t)\, x \right\| < \varepsilon. \tag{4.2}$$

Proof. For expository reasons, we shall confine ourselves to a sketch of the details. We choose and fix u so that $r < u < q'$. Our approach is based on the space $R_u([0, 2\pi])$ (see the discussion surrounding (2.8) in [9] for its definition), because this space provides a description of ψ in terms of convexification operations performed on functions constant on arcs. More specifically, because $r < u$, the function $\psi\left(e^{i(\cdot)}\right) |[0, 2\pi]$ belongs to $R_u([0, 2\pi])$ (Lemma 2 of [18]), and this fact permits us to obtain the following representation for ψ by performing suitable manipulations. For all $t \in \mathbb{R}$, we have

$$\psi\left(e^{it}\right) = \beta\chi\left(e^{it}; \{1\}\right) + \sum_{j=1}^{\infty} \alpha_j \sum_{k=1}^{\infty} \lambda_{j,k}\chi\left(e^{it}; J_{j,k}\right), \tag{4.3}$$

where: $\chi(\cdot, A)$ denotes the characteristic function, defined on $\mathbb{T}$, of a subset A of $\mathbb{T}$; $\{\alpha_j\}_{j=1}^{\infty} \subseteq \mathbb{C}$, with $\sum_{j=1}^{\infty} |\alpha_j| < \infty$; $\beta \in \mathbb{C}$, with $|\beta| \leq \sum_{j=1}^{\infty} |\alpha_j|$; for each $j \in \mathbb{N}$, the sequence of complex numbers $\{\lambda_{j,k}\}_{k=1}^{\infty}$ satisfies $\sum_{k=1}^{\infty} |\lambda_{j,k}|^u \leq 1$, and the sequence $\{J_{j,k}\}_{k=1}^{\infty}$ is comprised of disjoint subarcs of $\mathbb{T} \setminus \{1\}$. It follows from this that for $0 \leq \theta < 2\pi$, and all $t \in \mathbb{R}$, we have

$$\begin{aligned} \psi_{e^{i\theta}}\left(e^{it}\right) = \beta\chi\left(e^{it}; \left\{e^{-i\theta}\right\}\right) + \gamma(\theta)\,\chi\left(e^{it}; \{1\}\right) \\ + \sum_{j=1}^{\infty} \delta_j \sum_{k=1}^{\infty} \eta_{j,k}(\theta)\,\chi\left(e^{it}; \widetilde{J}_{j,k.\theta}\right), \end{aligned} \tag{4.4}$$

where: $\{\delta_j\}_{j=1}^{\infty} \subseteq \mathbb{C}$, with $\sum_{j=1}^{\infty} |\delta_j| < \infty$; $\beta \in \mathbb{C}$, $\gamma(\theta) \in \mathbb{C}$, with

$$|\beta|, |\gamma(\theta)| \leq \sum_{j=1}^{\infty} |\delta_j|;$$

for each $j \in \mathbb{N}$, the sequence $\left\{\widetilde{J}_{j,k,\theta}\right\}_{k=1}^{\infty}$ (which depends on θ) is comprised of disjoint subarcs of $\mathbb{T} \setminus \{1\}$, $\eta_{j,k}(\theta) \in \mathbb{C}$, for all $k \in \mathbb{N}$, with $\sum_{k=1}^{\infty} |\eta_{j,k}(\theta)|^u \leq 1$, and $\{2\eta_{j,k+2}(\theta)\}_{k=1}^{\infty}$ is the sequence of constants $\{\lambda_{j,k}\}_{k=1}^{\infty}$ appearing in (4.3).

We now suppose given $x \in \mathfrak{X}$ and $\varepsilon > 0$, and we shall proceed from (4.4) by adaptation of the reasoning that extends from Proposition 2.3 until just before the statement of Lemma 2.15 in [9]. For this purpose, we shall rely on (3.10) above, in place of the estimates in [9] based on domination by Fourier multiplier norms. This procedure shows that there is $N \in \mathbb{N}$ such that for each θ satisfying $0 \leq \theta < 2\pi$, and for any partition $\mathcal{P}$ of $[0, 2\pi]$, we have

$$\begin{aligned} &\left\| \mathcal{S}(\mathcal{P}; \psi_{e^{i\theta}}, E)x - \int_{[0,2\pi]} \psi_{e^{i\theta}}\left(e^{it}\right)\, dE(t)\, x \right\| \\ &\leq 2\varepsilon \|x\| + \Bigg\| \mathcal{S}(\mathcal{P}; F^{(\theta)}, E)x + \mathcal{S}(\mathcal{P}; G^{(\theta)}, E)x \\ &\quad + \sum_{j=1}^{N} \delta_j \mathcal{S}(\mathcal{P}; m_j^{(\theta)}, E)x - \int_{[0,2\pi]} F^{(\theta)}\left(e^{it}\right)\, dE(t)\, x \\ &\quad - \int_{[0,2\pi]} G^{(\theta)}\left(e^{it}\right)\, dE(t)\, x - \sum_{j=1}^{N} \delta_j \int_{[0,2\pi]} m_j^{(\theta)}\left(e^{it}\right)\, dE(t)\, x \Bigg\|, \end{aligned} \tag{4.5}$$

where we have adopted the following shorthand:

$$F^{(\theta)}\left(e^{it}\right) \equiv \beta\chi\left(e^{it}; \left\{e^{-i\theta}\right\}\right);$$
$$G^{(\theta)}\left(e^{it}\right) \equiv \gamma(\theta)\,\chi\left(e^{it}; \{1\}\right);$$

and, for each $j \in \mathbb{N}$,

$$m_j^{(\theta)}\left(e^{it}\right) \equiv \sum_{k=1}^{\infty} \eta_{j,k}\left(\theta\right)\chi\left(e^{it};\widetilde{J}_{j,k.\theta}\right).$$

At this juncture, we make use of the fact that for each $j \in \mathbb{N}$, each $\theta \in [0, 2\pi)$, and each $M \in \mathbb{N}$ such that $M \geq 3$, the following holds.

$$\left\|\mathcal{S}\left(\mathcal{P};\sum_{k=M}^{\infty}\eta_{j,k}\left(\theta\right)\chi\left(e^{i(\cdot)};\widetilde{J}_{j,k.\theta}\right),E\right)\right\| \leq K_{u,q}\left(\sum_{k=M}^{\infty}\left|\lambda_{j,k}\right|^u\right)^{1/u},$$

$$\left\|\int_{[0,2\pi]}\sum_{k=M}^{\infty}\eta_{j,k}\left(\theta\right)\chi\left(e^{it};\widetilde{J}_{j,k.\theta}\right)\,dE\left(t\right)\right\| \leq K_{u,q}\left(\sum_{k=M}^{\infty}\left|\lambda_{j,k}\right|^u\right)^{1/u}.$$

Applying this to (4.5) in order to truncate the infinite series

$$\sum_{k=1}^{\infty}\eta_{j,k}\left(\theta\right)\chi\left(e^{it};\widetilde{J}_{j,k.\theta}\right)$$

expressing the functions $m_j^{(\theta)}$, we find, after simplifying the notation, that for $x \in \mathfrak{X}$, $\varepsilon > 0$, there corresponds to each θ satisfying $0 \leq \theta < 2\pi$, a step function $\Phi^{(\theta)} \in BV\left(\mathbb{T}\right)$ such that for any partition $\mathcal{P}$ of $[0, 2\pi]$,

$$\left\|\mathcal{S}(\mathcal{P};\psi_{e^{i\theta}},E)x - \int_{[0,2\pi]}\psi_{e^{i\theta}}\left(e^{it}\right)\,dE\left(t\right)x\right\| \leq 4\varepsilon\left\|x\right\| + \left\|\mathcal{S}(\mathcal{P};\Phi^{(\theta)},E)x - \int_{[0,2\pi]}\Phi^{(\theta)}\left(e^{it}\right)\,dE\left(t\right)x\right\|. \tag{4.6}$$

Moreover, these functions $\Phi^{(\theta)}$ can be chosen so that

$$\sup\left\{\left\|\Phi^{(\theta)}\right\|_{BV(\mathbb{T})} : 0 \leq \theta < 2\pi\right\} < \infty.$$

This reduction to functions uniformly bounded with respect to the norm in $BV\left(\mathbb{T}\right)$ allows us to apply (2.8) of [7] on the right of (4.6) to get, in the notation of (1.3),

$$\left\|\mathcal{S}(\mathcal{P};\psi_{e^{i\theta}},E)x - \int_{[0,2\pi]}\psi_{e^{i\theta}}\left(e^{it}\right)\,dE\left(t\right)x\right\| \leq 4\varepsilon\left\|x\right\| + \omega\left(\mathcal{P},E,x\right)\sup_{0\leq\theta<2\pi}\left\|\Phi^{(\theta)}\right\|_{BV(\mathbb{T})},$$

valid for any $\mathcal{P}$ of $[0, 2\pi]$. The desired conclusion is evident from this by virtue of (1.3). □

Proof of Theorem 4.1(a). Let $x \in \mathfrak{X}$, and then invoke (4.2) to see that

$$\left\| \psi_z (1) E (0) x + \mathcal{S}(\mathcal{P}; \psi_z, E)x - \int_{[0,2\pi]}^{\oplus} \psi_z \left(e^{it}\right) \, dE (t) x \right\| \to 0,$$

uniformly in $z \in \mathbb{T}$, as $\mathcal{P}$ runs through the directed set of all partitions of $[0, 2\pi]$, directed to increase by refinement. It is now a simple matter to calculate that the $n^{\underline{th}}$ Fourier coefficient of $\mathfrak{S}_\psi (\cdot) x$ can be expressed, relative to the norm topology of $\mathfrak{X}$, in the form

$$\lim_{\mathcal{P}} \left\{ \widehat{\psi} (n) E (0) x + \widehat{\psi} (n) \mathcal{S}(\mathcal{P}; e^{in(\cdot)}, E)x \right\} = \widehat{\psi} (n) U^n x. \qquad \square$$

Remark 4.4. It is clear from Theorem 4.1(a) and Remark 4.2(i) that under the hypotheses and notation of Theorem 4.1 we have for each $x \in \mathfrak{X}$ the equality a.e. on $\mathbb{T}$ of $\mathfrak{S}_\psi (\cdot) x$ and $\mathcal{T}_\psi (\cdot) x$. In fact, we can apply Theorem 4.1(a) to each of the (2π)-periodic functions

$$\psi^{(+)} \left(e^{it}\right) \equiv \lim_{s \to t^+} \psi \left(e^{is}\right),$$
$$\psi^{(-)} \left(e^{it}\right) \equiv \lim_{s \to t^-} \psi \left(e^{is}\right),$$

in place of ψ, and thereby infer that for each $x \in \mathfrak{X}$, we have for almost all $z \in \mathbb{T}$,

$$\mathfrak{S}_\psi (z) x = \mathcal{T}_\psi (z) x = \mathfrak{S}_{\psi^{(+)}} (z) x = \mathfrak{S}_{\psi^{(-)}} (z) x.$$

Our starting point for the demonstration of Theorem 4.1(b) will be the following theorem.

Theorem 4.5. *Let $\mathfrak{X}$ be a super-reflexive Banach space with dual space $\mathfrak{X}^*$, and let $E (\cdot)$ be the spectral decomposition of a trigonometrically well-bounded operator $U \in \mathfrak{B} (X)$. Choose $q \in (1, \infty)$ so that (3.9) holds, let $r \in (1, q')$ (equivalently, $\min \left\{r, r^{-1} + q^{-1}\right\} > 1$), and let $\psi \in V_r(\mathbb{T})$. If $u \in \mathbb{R}$ satisfies*

$$0 < u^{-1} < r^{-1} + q^{-1} - 1 \tag{4.7}$$

(in which case $r < u$), then for each pair $x \in \mathfrak{X}$, $x^* \in \mathfrak{X}^*$, the complex-valued function $x^*\mathfrak{S}_\psi (\cdot) x$ belongs to $V_u(\mathbb{T})$, and satisfies

$$\|x^*\mathfrak{S}_\psi (\cdot) x\|_{V_u(\mathbb{T})} \le K_{u,r,q} \|\psi\|_{V_r(\mathbb{T})} (\mathrm{var}_q (E)) \|x\| \|x^*\| . \tag{4.8}$$

Proof. A derivation of (4.8) can be carried out by adapting the method of proof for Theorem (6.1) of [25] to the context of (3.10), and then simplifying the form of the outcome. In fact, the statement of (4.8) above fits in with the formulation of a result attributed to Littlewood as item (1.1) in [25]. $\square$

Proof of Theorem 4.1(b). Choose and fix u so that (4.7) is satisfied, and let $x \in \mathfrak{X}$, $x^* \in \mathfrak{X}^*$. We need only show the strong convergence at each $z \in \mathbb{T}$ of the series in (4.1), since, as mentioned in Remark 4.2(i), that series is known to be strongly $(C,1)$ summable to the desired value. In view of this observation and Theorem 4.1(a), the Fourier series of $x^*\mathfrak{S}_\psi(\cdot)\,x$, specifically

$$\sum_{\nu=-\infty}^{\infty} \widehat{\psi}(\nu) z^\nu x^* U^\nu x$$

is $(C,1)$ summable. This observation, together with the fact that

$$x^*\mathfrak{S}_\psi(\cdot)\,x \in V_u(\mathbb{T}) \subseteq \Lambda_u,$$

permits us to infer via Theorem 1 of [21] that for each $z \in \mathbb{T}$ the Fourier series of $x^*\mathfrak{S}_\psi(\cdot)\,x$ is (C,α) summable for all $\alpha > -u^{-1}$, and so, for each fixed value of $z \in \mathbb{T}$ the Fourier series of $x^*\mathfrak{S}_\psi(\cdot)\,x$ is *a fortiori* (C,α) bounded for $\alpha > -u^{-1}$. Hence by Banach–Steinhaus, at each $z \in \mathbb{T}$ the vector-valued series in (4.1) is (C,α) bounded for all $\alpha > -u^{-1}$. It now follows by the vector-valued counterpart of Lemma 2 in [21] that at each $z \in \mathbb{T}$ the strongly $(C,1)$ summable series in (4.1) is in fact (C,α) summable for all $\alpha > -u^{-1}$ in the norm topology of $\mathfrak{X}$ (see the reasoning leading to Sætning VII on page 56 of [1] for a proof that effectively includes the vector-valued case of Lemma 2 in [21]). Specializing to the value $\alpha = 0$ now completes the proof. □

5. A few closing thoughts regarding vector-valued Fourier coefficients

In this brief section, we take up a few estimates of independent interest regarding the behavior of the coefficients occurring in the Fourier series (4.1). An application of the Riemann–Lebesgue Lemma to this context shows that under the hypotheses of Theorem 4.1, the operator-valued sequence $\left\{\widehat{\psi}(\nu)\,U^\nu\right\}_{\nu=-\infty}^{\infty}$ converges to zero in the strong operator topology as $|\nu| \to \infty$ (Corollary 5.6 of [4]). In fact, this assertion can be improved to convergence in the uniform operator topology.

Theorem 5.1. *Under the hypotheses of Theorem* 4.1, $\left\{\widehat{\psi}(\nu)\,U^\nu\right\}_{\nu=-\infty}^{\infty}$ *converges to zero in the uniform operator topology as* $|\nu| \to \infty$.

Proof. In the light of (3.5) we need only show that $\left\{|\nu|^{-1/r}\,U^\nu\right\}_{\nu\neq 0}$ converges to zero in the uniform operator topology as $|\nu| \to \infty$. Upon putting $p = 2^{-1}(r+q')$ we apply (3.10) and (3.6) to the index p, and find that for $\nu \neq 0$,

$$\begin{aligned} \|U^\nu\| &= \left\| \int_{[0,2\pi]}^{\oplus} \epsilon_\nu\left(e^{it}\right) \, dE\,(t) \right\| \\ &\leq K_{p,q}\left(\operatorname{var}_q(E)\right) \|\epsilon_\nu\|_{V_p(\mathbb{T})} \\ &\leq K_{r,q}\left(\operatorname{var}_q(E)\right) |\nu|^{1/p}. \end{aligned}$$

Consequently, for $\nu \neq 0$,

$$\left\| |\nu|^{-1/r} U^{\nu} \right\| \leq K_{r,q} \left(\mathrm{var}_q \left(E \right) \right) |\nu|^{(1/p)-(1/r)} .$$

Since $(1/p) - (1/r) < 0$, the desired conclusion is evident. □

This method of proof has the following corollary.

Corollary 5.2. *Let $\mathfrak{X}$ be a super-reflexive Banach space, and let $U \in \mathfrak{B}(X)$ be a trigonometrically well-bounded operator. Then $\dfrac{U^{\nu}}{|\nu|} \to 0$ in the uniform operator topology as $|\nu| \to \infty$.*

Proof. The proof of Theorem 5.1 shows that for some $\gamma \in (0,1)$, $\dfrac{U^{\nu}}{|\nu|^{\gamma}} \to 0$ in the uniform operator topology as $|\nu| \to \infty$. □

Remark 5.3. The outcome in Corollary 5.2 improves on the corresponding general Banach space situation for trigonometrically well-bounded operators. If $\mathfrak{X}$ is an arbitrary Banach space, and $U \in \mathfrak{B}(X)$ is trigonometrically well-bounded, then $\dfrac{U^{\nu}}{|\nu|} \to 0$ in the strong operator topology as $|\nu| \to \infty$ (Theorem (2.11) of [6]). However, Example (3.1) in [6] exhibits a trigonometrically well-bounded operator U_0 defined on a certain reflexive, but not super-reflexive, Banach space $\mathfrak{X}_0$ (due to M.M. Day [20]) such that for each trigonometric polynomial Q,

$$\|Q(U_0)\| = |Q(1)| + \mathrm{var}_1(Q, \mathbb{T}) .$$

Hence for each $\nu \in \mathbb{Z}$, $\|U_0^{\nu}\| = 1 + 2\pi |\nu|$, and so the conclusion of Corollary 5.2 fails to hold here.

A tighter measure of the decay rate (relative to the strong operator topology) for the coefficients occurring in the Fourier series (4.1) can be obtained by invoking a "generalized Hausdorff–Young Inequality" for super-reflexive spaces due to Bourgain (Theorem 11 of [16]). To see this, we first take note of the following lemma.

Lemma 5.4. Under the hypotheses of Theorem 4.1,

$$\sup \left\{ \left\| \sum_{k=-n}^{n} \widehat{\psi}(k) U^k z^k \right\|_{\mathfrak{B}(\mathfrak{X})} : n \geq 0, z \in \mathbb{T} \right\} < \infty.$$

Proof. Temporarily fix $x \in \mathfrak{X}$, $x^* \in \mathfrak{X}^*$ arbitrarily, and then apply (3.2) to the function $x^* \mathfrak{S}_{\psi}(\cdot) x$, which, in accordance with Theorem 4.5, belongs to $V_u(\mathbb{T})$, for an appropriate value of u exceeding r. The desired conclusion of the lemma is now evident by Banach–Steinhaus. □

Theorem 5.5. *Under the hypotheses of Theorem* 4.1*, there is* $\xi \in (0, \infty)$ *such that for each* $x \in \mathfrak{X}$,

$$\sum_{k=-\infty}^{\infty} \left\| \widehat{\psi}(k)\, U^k x \right\|^{\xi} < \infty.$$

Proof. Use Theorem 11 of [16] in conjunction with Lemma 5.4 above. □

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