

## Further notes on a family of continuous, nondifferentiable functions

James McCollum

ABSTRACT. We examine a parametrized family of functions  $F_a$ , all of which are continuous and some of which are nowhere or almost nowhere differentiable, we explore the behavior of  $F'_a$  and  $F''_a$  almost everywhere for different values of  $a$ , focusing on specific questions regarding  $F_a$ 's differentiability for certain  $a$ , and we calculate the Hausdorff dimension of the graphs of all  $F_a$ .

### CONTENTS

1. Introduction	569
2. Remarks on differentiability	571
3. Remarks on dimension	573
4. Conclusions	577
References	577

### 1. Introduction

In this paper, we will examine a family of functions  $F_a$  that are continuous for all  $a \in (0, 1)$ , but nowhere or almost nowhere differentiable for certain  $a$ . Their construction, described by Okamoto [7] for all  $a$ , is based largely on the one given by Bourbaki [2] for  $a = 2/3$ .

Let  $F_a$  be defined inductively over  $[0, 1]$  by iterations  $f_i$  for  $i \geq 0$  as follows:  $f_0(x) = x$  for all  $x \in [0, 1]$ , every  $f_i$  is continuous on  $[0, 1]$ , every  $f_i$  is affine in each subinterval  $[k/3^i, (k+1)/3^i]$  where  $k \in \{0, 1, 2, \dots, 3^i - 1\}$ ,

---

Received April 2, 2011.

2010 *Mathematics Subject Classification*. 26A30, 37C45.

*Key words and phrases*. continuous, nowhere differentiable, fractal, Hausdorff dimension, Lebesgue measure, normal number.

This work was partially supported by a Michael D. Wilson Scholarship from the University of Maine at Farmington.

and

$$(1) \quad f_{i+1} \left( \frac{k}{3^i} \right) = f_i \left( \frac{k}{3^i} \right),$$

$$(2) \quad f_{i+1} \left( \frac{3k+1}{3^{i+1}} \right) = f_i \left( \frac{k}{3^i} \right) + a \left[ f_i \left( \frac{k+1}{3^i} \right) - f_i \left( \frac{k}{3^i} \right) \right],$$

$$(3) \quad f_{i+1} \left( \frac{3k+2}{3^{i+1}} \right) = f_i \left( \frac{k}{3^i} \right) + (1-a) \left[ f_i \left( \frac{k+1}{3^i} \right) - f_i \left( \frac{k}{3^i} \right) \right],$$

$$(4) \quad f_{i+1} \left( \frac{k+1}{3^i} \right) = f_i \left( \frac{k+1}{3^i} \right).$$

Given this construction, we say that

$$F_a = \lim_{i \rightarrow \infty} f_i.$$

We also can construct the graph of any  $F_a$  with contraction mappings, using as a basis Katsuura's construction [4] of the graph of  $F_{2/3}$ . Define contraction mappings  $w_n : X \mapsto X$ , where  $n \in \{1, 2, 3\}$  and  $X = [0, 1] \times [0, 1]$ , as follows: for all  $(x, y) \in X$ ,

$$(5) \quad w_1(x, y) = \left( \frac{x}{3}, ay \right)$$

$$(6) \quad w_2(x, y) = \left( \frac{2-x}{3}, (2a-1)y + (1-a) \right)$$

$$(7) \quad w_3(x, y) = \left( \frac{2+x}{3}, ay + (1-a) \right).$$

In this case, the graph of  $F_a$  is the unique invariant set for the iterated function system defined by (5), (6), and (7).

This paper builds primarily on a paper by Okamoto [7] regarding the same family of functions. That paper showed, among other things, that if  $a_0$  is the unique root of  $27a^2 - 54a^3 = -1$  in  $(1/2, 2/3)$ , then  $F_a$  is differentiable almost everywhere when  $a \in (0, 1/3) \cup (1/3, a_0)$ ;  $F_{1/3}(x) \equiv x$  for  $x \in [0, 1]$  and thus is differentiable everywhere on that interval;  $F_a$  is differentiable almost nowhere when  $a \in (a_0, 2/3)$ ; and  $F_a$  is differentiable absolutely nowhere when  $a \in [2/3, 1)$ . (Throughout this paper, "almost everywhere" and "almost nowhere" will refer specifically to sets of Lebesgues measure 1 and 0, respectively.) Certain issues, however, were left unresolved in Okamoto's paper. One such issue was whether  $F_{a_0}$  is differentiable almost everywhere, almost nowhere, or neither. Kobayashi [5] recently proved that  $F_{a_0}$  is in fact differentiable almost nowhere using the law of the iterated logarithm.

In this paper, we will give another proof of Kobayashi's result using only the properties of  $F_a$  and the measure of the set of normal numbers in  $[0, 1]$ . Furthermore, using the same methods, we will describe how  $F'_a$  behaves almost everywhere in  $[0, 1]$  as  $a$  varies. From here, we will prove that  $F''_a$  does not exist anywhere for any  $a \in (0, 1)$  except  $a = 1/3$  (in which case  $F_a(x) \equiv x$ , and consequently,  $F''_a(x) = 0$  for all  $x$ ) and  $a = 1/2$  (in which

case  $F'_a(x) = 0$  for infinitely many disjoint subintervals of  $[0, 1]$ . Finally, we will show that if  $a \in (0, 1/2]$ , then its graph has Hausdorff dimension 1, whereas if  $a \in (1/2, 1)$ , then the Hausdorff dimension of its graph is equal to  $\log_3(12a - 3)$ .

**Acknowledgments.** The author would like to thank Professor Daniel Jackson for direction and assistance with this project.

## 2. Remarks on differentiability

**Theorem 1.** *If  $S_a$  is the set of all points  $x$  for which  $F'_a(x)$  does not exist and  $a_0$  is the unique solution of  $27a^2 - 54a^3 = -1$  in  $(1/2, 2/3)$ , then the Lebesgue measure  $|S_{a_0}| = 1$ .*

**Proof.** Let  $S_a$  and  $a_0$  be defined as they are above. Now, we can write the ternary expansion of any  $x \in [0, 1]$  as

$$x = 0.\xi_1\xi_2\dots = \sum_{i=1}^{\infty} \frac{\xi_i}{3^i}$$

where each  $\xi_i \in \{0, 1, 2\}$ . (If an expansion terminates, we say that it ends with a infinite string of zeroes. If  $x = 1$ , we say that  $\xi_i = 2$  for all  $i$ .) For the first  $n$  digits of the ternary expansion of  $x$ , we define  $i(n)$  as the number of times  $\xi_i = 1$  in these first  $n$  digits. Finally, we let  $\gamma = \liminf_{n \rightarrow \infty} \frac{i(n)}{n}$ .

According to Okamoto [7],

$$\begin{aligned} F'_a(x) &= \lim_{n \rightarrow \infty} (3 - 6a)^{i(n)} (3a)^{n-i(n)} \\ &= \lim_{n \rightarrow \infty} [3(1 - 2a)^\gamma a^{1-\gamma}]^n \end{aligned}$$

and thus  $F'_a(x) = 0$  if  $|3(1 - 2a)^\gamma a^{1-\gamma}| < 1$ . Setting  $\gamma = 1/3$ , we can rewrite the second expression as  $-1 < 3(1 - 2a)^{1/3} a^{2/3} < 1$ . Raising each side to the third power, we get  $F'_a(x) = 0$  if  $-1 < 27a^2 - 54a^3 < 1$ ; this quantity is equal to 1 when  $a = 1/3$ , and it is equal to  $-1$  when  $a = a_0$ .

As a result, when  $a = a_0$  and  $\gamma = 1/3$ , the sequence  $\{f'_i(x)\}_{i=0}^{\infty}$  visits 1 and  $-1$  infinitely often, and therefore does not converge.<sup>1</sup> Thus,  $F'_{a_0}(x)$  does not exist for all  $x$  satisfying  $\gamma = 1/3$ . To complete the proof, we must show that this set of values for  $x$  has Lebesgue measure 1.

To do this, we turn to the *normal numbers*—a set of irrational real numbers whose digits are uniformly distributed, regardless of the base in which the numbers are written. Given any base  $b \geq 2$  and an alphabet of digits  $\{0, 1, \dots, b - 1\}$ , a normal number  $x$  must satisfy

$$\lim_{n \rightarrow \infty} \frac{i(n)}{n} = \frac{1}{b}$$

<sup>1</sup>It should be noted, however, that  $F'_{a_0}(x)$  does not converge absolutely to 1, as we might expect;  $\{f'_i(x)\}_{i=0}^{\infty}$  does not consist solely of '1's and '-1's, and as Kobayashi [5] has pointed out, it contains subsequences that diverge absolutely to infinity.

where  $n$  represents the same value it did for  $\gamma$  and  $i(n)$  can be taken as the number of any individual digit in the first  $n$  digits of  $x$ . (While this is not a sufficient property of normal numbers, it is a necessary one, and for the purposes of this proof, we only need to know that all normal numbers have this property.)

This means that the set of all  $x \in [0, 1]$  that satisfy  $\gamma = 1/3$  includes all the normal numbers in  $[0, 1]$ . Borel [1] proved that the set of all nonnormal numbers has Lebesgue measure 0, so the set of all normal numbers in  $[0, 1]$  must have Lebesgue measure 1. Because the normal numbers in  $[0, 1]$  are a subset of the numbers satisfying  $\gamma = 1/3$  in  $[0, 1]$ , the latter set must have Lebesgue measure 1. So  $S_{a_0}$ , the set of all  $x \in [0, 1]$  for which  $F'_{a_0}(x)$  does not exist, must have Lebesgue measure  $|S_{a_0}| = 1$ , as well.  $\square$

The fact that the normal numbers have Lebesgue measure 1 in  $[0, 1]$  reveals even more about  $F'_a$  for all  $a \in (0, 1)$ . Because all normal numbers satisfy  $\gamma = 1/3$  in base 3, we know that the value of  $F'_a(x)$  will depend on whether  $|27a^2 - 54a^3|$  is less than, greater than, or equal to 1 for all  $x$  in a set with Lebesgue measure 1. This information allows us to give a general description of the behavior of  $F'_a$  almost everywhere:

**Corollary.** *Let  $a \in (0, 1)$ .*

- *If  $a \in [a_0, 1)$ , then  $F'_a(x)$  diverges for almost all  $x \in [0, 1]$ .*
- *If  $a \in (0, 1/3) \cup (1/3, a_0)$ , then  $F'_a(x) = 0$  for almost all  $x \in [0, 1]$ .*
- *If  $a = 1/3$ , then  $F'_a(x) = 1$  for all  $x \in [0, 1]$ .*

We obtain the last result because  $F'_{1/3}(x) \equiv x$  and thus has the same derivative everywhere it is defined. Additionally, if we take into account the fact that  $F_a(x)$  is absolutely nowhere differentiable in  $[0, 1]$  when  $a \in [2/3, 1)$ , we can replace *almost all* with *all* above when  $a \in [2/3, 1)$ .

This leads us to our last question on the differentiability of  $F_a$ : What can we say about the second derivative  $F''_a(x)$  for different values of  $a$ ? We know that since  $F_{1/3}(x) \equiv x$ ,  $F''_{1/3}(x) = 0$  for all  $x$ . We also know that since  $F'_{1/2}(x) = 0$  for all  $x$  except when  $x$  belongs to the Cantor Set—a set with measure 0— $F'_{1/2}(x)$  is continuous on a set of points with measure 1. Thus,  $F'_{1/2}(x)$  is differentiable on this set of points, and its derivative  $F''_{1/2}(x) = 0$  at all of them. But clearly  $F''_a(x)$  does not exist at any  $x$  when  $a \in [a_0, 1)$ , since even  $F'_a(x)$  fails to converge for almost all  $x$  in this case. The following theorem answers our question for the remaining values of  $a$ .

**Theorem 2.** *For all  $a \in (0, 1/3) \cup (1/3, 1/2) \cup (1/2, a_0)$  and all  $x \in [0, 1]$ ,  $F''_a(x)$  does not exist.*

**Proof.** Let  $x \in [0, 1]$ . We recall from Theorem 1 that almost all  $x$  satisfy  $\gamma = 1/3$ , in that the set of all such values has Lebesgue measure 1. We also recall that for all  $x$  belonging to this set of numbers and for all  $a \in (0, 1/3) \cup (1/3, a_0)$ ,  $F'_a(x) = 0$  if  $|27a^2 - 54a^3| < 1$ . So for all  $a \in (0, 1/3) \cup (1/3, a_0)$  and almost all  $x \in [0, 1]$ ,  $F'_a(x) = 0$ .

The graph of  $y = F'_a(x)$  appears to be the line  $y = 0$  over  $[0, 1]$ , but there are infinitely many points at which  $F'_a(x)$  is undefined. Okamoto [7] has shown that for  $a \in (0, 1/3)$ , this set consists of all points of the form  $x = (2k + 1)/(2 \cdot 3^i)$ , for  $i \geq 0$  and  $k \in \{0, 1, \dots, 3^i - 1\}$  and for  $a \in (1/3, 1/2) \cup (1/2, a_0)$ , it consists of all points of the form  $x = k/3^i$ . In either case, the set of points where  $F'_a(x)$  does not exist is dense in  $[0, 1]$ . So  $F'_a(x)$  is discontinuous everywhere in  $[0, 1]$ , and therefore its derivative,  $F''_a(x)$ , does not exist anywhere in that interval.  $\square$

So unless  $a = 1/3$  or  $a = 1/2$ —in which case  $F''_a(x) = 0$  for all  $x$  at which it exists—the graph of  $F_a$  is neither concave up nor concave down anywhere. Also,  $F_a$  belongs to differentiability class  $C^0$  for all  $a \neq 1/3$ .

Due to its somewhat pathological nature in terms of differentiability, this family of functions may call to mind another set of counterintuitive functions—that of the everywhere differentiable but nowhere monotone functions. It is clear that these two sets have no common element; indeed, the only everywhere differentiable function in our set,  $F_{1/3}$ , is strictly increasing. Nevertheless, some similarities exist in the ways these sets of functions behave on countable, dense sets. For instance, if  $a \in (0, 1/3)$ , then on the set of all points of the form  $x = k/3^i$ , where  $i \geq 0$  and  $k \in \{0, 1, \dots, 3^i\}$ ,  $F'_a(x) = 0$ , and at all points of the form  $x = (2k + 1)/(2 \cdot 3^i)$ ,  $F'_a(x)$  is undefined; similarly, an everywhere differentiable but nowhere monotone function  $G$  can be constructed over  $[0, 1]$  using the same two disjoint dense sets under the conditions that for all  $x$  in one set,  $G'(x) < 0$  and that for all  $x$  in the other set,  $G'(x) > 0$ . Beyond this, however, the author has found no connections between the constructions for these sets of functions. This is primarily because with the dubious exceptions of  $F_{1/3}$  and  $F_{1/2}$ , Okamoto's construction assures that if  $F'_a(x) = 0$  on one set dense in  $[0, 1]$ , then  $F'_a(x)$  must be undefined on another dense set.

### 3. Remarks on dimension

We first will show that because the graph of  $F_a$  has finite arc length when  $a \in (0, 1/2]$ , it has Hausdorff dimension 1.

**Theorem 3.** *If  $\Gamma_a$  is the graph of  $F_a$ , then its Hausdorff dimension*

$$\dim_H(\Gamma_a) = 1$$

for all  $a \in (0, 1/2]$ .

**Proof.** Suppose  $\Gamma_a$  is the graph of  $F_a$ , and suppose  $a \in (0, 1/2]$ . Okamoto [7] has proven that for all  $a \leq 1/2$ ,  $F_a$  is nondecreasing. Now, if  $F_a$  is nondecreasing, then clearly the affine pieces of each of its iteration  $f_i$  must be nondecreasing as well, since  $f_i(k/3^i) = F_a(k/3^i)$  for all  $i \geq 0$  and all  $k \in \{0, 1, \dots, 3^i\}$ .

Using the Triangle Inequality, we can determine the maximum arc length of  $\Gamma_a$ . We do this by applying the inequality to each affine segment of  $f_i$

with respect to the segment's horizontal and vertical components and letting  $i \rightarrow \infty$ . More precisely, if  $l_k$  is the length of the affine segment of  $f_i$  over  $[k/3^i, (k+1)/3^i]$ , then  $l_k \leq |f_i([k+1]/3^i) - f_i(k/3^i)| + |(k+1)/3^i - k/3^i|$ .

Because this inequality holds for every affine segment of  $f_i$ , we can obtain a bound for the arc length

$$L_i = \sum_{k=0}^{3^i-1} l_k$$

of  $f_i$ . And by taking the limit as  $i \rightarrow \infty$ , we can express the maximum arc length  $L$  of  $\Gamma_a$  as

$$L \leq \lim_{i \rightarrow \infty} \sum_{k=0}^{3^i-1} \left| f_i \left( \frac{k+1}{3^i} \right) - f_i \left( \frac{k}{3^i} \right) \right| + \sum_{k=0}^{3^i-1} \left| \frac{k+1}{3^i} - \frac{k}{3^i} \right|.$$

Obviously,  $(k+1)/3^i - k/3^i = 1/3^i$  always, so the second sum becomes 1. As for the first sum, we recall that the affine pieces of every  $f_i$  are nondecreasing, so  $|f_i([k+1]/3^i) - f_i(k/3^i)| = f_i([k+1]/3^i) - f_i(k/3^i)$ . Thus, for all  $i$ ,

$$\begin{aligned} & \sum_{k=0}^{3^i-1} \left[ f_i \left( \frac{k+1}{3^i} \right) - f_i \left( \frac{k}{3^i} \right) \right] \\ &= f_i \left( \frac{1}{3^i} \right) - f_i \left( \frac{0}{3^i} \right) + f_i \left( \frac{2}{3^i} \right) - f_i \left( \frac{1}{3^i} \right) + \cdots + f_i \left( \frac{3^i}{3^i} \right) - f_i \left( \frac{3^i-1}{3^i} \right) \\ &= f_i(1) - f_i(0) \\ &= 1. \end{aligned}$$

So for  $a \in (0, 1/2]$ , the maximum arc length of  $\Gamma_a$  is 2. To find a lower bound for the arc length, we first note that every  $\Gamma_a$  has its endpoints at  $(0,0)$  and  $(1,1)$ . Because the shortest possible distance between these two points is a straight line, the minimum arc length of any  $\Gamma_a$  must be  $\sqrt{2}$ .

Hence, the graph's 1-dimensional Hausdorff measure  $\mathcal{H}^1(\Gamma_a)$  satisfies  $0 < \mathcal{H}^1(\Gamma_a) < \infty$ , which implies that  $\dim_H(\Gamma_a) = 1$  for all  $a \in (0, 1/2]$ .  $\square$

For  $a \in (1/2, 1)$ , on the other hand,  $F_a(x)$  is not nondecreasing, so we cannot find an upper bound for  $L$  using the Triangle Inequality. In fact, no such upper bound exists when  $a \in (1/2, 1)$ ; to prove this, we must show that the graph's Hausdorff dimension is greater than 1, but in order to do this, we first must calculate the box-counting dimension of the graph.

**Theorem 4.** *If  $\Gamma_a$  is the graph of  $F_a$ , then its box-counting dimension*

$$\dim_B(\Gamma_a) = \log_3(12a - 3)$$

*for all  $a \in (1/2, 1)$ .*

**Proof.** Suppose  $\Gamma_a$  is the graph of  $F_a$ , and suppose  $a \in (1/2, 1)$ .

To determine the box-counting dimension of  $\Gamma_a$ , we will use a variant of the more familiar method of box-counting. Instead of counting squares of side length  $\delta$ , we will determine the smallest area  $A(f_i)$  needed to cover the graph of iteration  $f_i$  using rectangles that share a horizontal side length of  $\delta$ . Then the “number” of squares with side length  $\delta$  needed to cover the graph of  $f_i$  can be expressed as  $N_\delta(f_i) = A(f_i)/\delta^2$ . This variant works by the same principle as box-counting dimension with squares does, but it gives more precise answers for each  $\delta$  we use.

Letting  $\delta = 1/3^i$ , we see that for all  $\Gamma_a$ ,  $A(f_0) = 1$ . Using the horizontal and vertical ratios of contraction for (5), (6), and (7), we see that

$$\begin{aligned} A(f_{i+1}) &= \frac{1}{3}aA(f_i) + \frac{1}{3}(2a-1)A(f_i) + \frac{1}{3}aA(f_i) \\ &= A(f_i) \left( \frac{4a-1}{3} \right). \end{aligned}$$

So by induction,

$$A(f_i) = \left( \frac{4a-1}{3} \right)^i$$

and thus,

$$\begin{aligned} N_\delta(f_i) &= \frac{([4a-1]/3)^i}{(1/3)^{2i}} \\ &= (12a-3)^i. \end{aligned}$$

So for  $a \in (1/2, 1)$ , the box-counting dimension of  $\Gamma_a$  is

$$\begin{aligned} \dim_B(\Gamma_a) &= \lim_{i \rightarrow \infty} \frac{\log([12a-3]^i)}{-\log(1/3^i)} \\ &= \log_3(12a-3). \end{aligned} \quad \square$$

Next, we will show that the Hausdorff dimension of  $\Gamma_a$  is equal to its box-counting dimension by the Mass Distribution Principle (see, e.g., [3]):

**Theorem 5.** *If  $\Gamma_a$  is the graph of  $F_a$ , then its Hausdorff dimension*

$$\dim_H(\Gamma_a) = \log_3(12a-3)$$

*for all  $a \in (1/2, 1)$ .*

**Proof.** We consider a small variation on Katsuura’s construction of  $\Gamma_a$  using contraction mappings. Let  $E_0 = [0, 1] \times [0, 1]$  and define further levels of the construction by  $E_{i+1} = w_1(E_i) \cup w_2(E_i) \cup w_3(E_i)$  where  $i > 0$  and  $w_1$ ,  $w_2$ , and  $w_3$  are mappings (5)–(7). Clearly  $E_{i+1} \subset E_i$  for all  $i \geq 0$ , and

$$\bigcap_{i=0}^{\infty} E_i = \Gamma_a.$$

This last relationship between the  $E_i$ s can be understood as follows: While in Okamoto’s construction, linear segments are constructed “upwards” to

a graph with infinite length, in this construction, rectangular regions are constructed “downwards” to the same graph which has zero area. Moreover, we see that each  $E_i$  can be covered by  $3^i$  rectangles of length  $(1/3)^i$ .

Using methods related to the box-counting process in Theorem 4, it can be shown that if  $a \in (1/2, 1)$ , then the area of  $E_{i+1}$  can be expressed as

$$\begin{aligned} A(E_{i+1}) &= a \left(\frac{1}{3}\right) A(E_i) + (2a - 1) \left(\frac{1}{3}\right) A(E_i) + a \left(\frac{1}{3}\right) A(E_i) \\ &= \frac{4a - 1}{3} A(E_i), \end{aligned}$$

and since  $A(E_0) = 1$ , we have  $A(E_i) = \left(\frac{4a - 1}{3}\right)^i$  for all  $i \geq 0$ .

Now, let  $\mu$  be the natural mass distribution on  $\Gamma_a$ ; we start with unit mass on  $E_0$  and repeatedly “spread” this mass over the total area of each  $E_i$ . Also, let  $U$  be any set whose diameter  $|U| < 1$ . Then there exists some  $i \geq 0$  such that

$$\left(\frac{1}{3}\right)^{i+1} \leq |U| < \left(\frac{1}{3}\right)^i,$$

an inequality that applies to any  $U$  satisfying  $0 < |U| < 1$ . Given these conditions on the diameter of  $U$ , it is clear that for every  $U$ , there is some  $i$  such that  $U$  is contained in an open square of side length  $(1/3)^i$  and  $U$  contains points in at most two level- $i$  “sub-rectangles.”

Hence, the area of  $U$  is bounded above by the area of the open square containing it; that is,  $A(U) \leq (1/9)^i$ . In terms of measure, we know that the entire area of  $U$  can be contained in  $E_i$ , so

$$\begin{aligned} \mu(U) &\leq \frac{A(U \cap E_i)}{A(E_i)} \\ &\leq \frac{(1/9)^i}{([4a - 1]/3)^i} \\ &\leq \left(\frac{1}{12a - 3}\right)^i. \end{aligned}$$

And since  $\left(\frac{1}{3}\right)^{i+1} \leq |U|$  implies that  $\left(\frac{1}{3}\right)^i \leq 3|U|$ , we have

$$\begin{aligned} \mu(U) &\leq \left(\frac{1}{12a - 3}\right)^i = \left(\frac{1}{3^i}\right)^{\log_3(12a - 3)} \leq (3|U|)^{\log_3(12a - 3)} \\ &= (12a - 3)|U|^{\log_3(12a - 3)}, \end{aligned}$$

and therefore, by the Mass Distribution Principle,

$$\log_3(12a - 3) \leq \dim_H(\Gamma_a) \leq \dim_B(\Gamma_a),$$

and given the upper bound obtained in Theorem 4, we have

$$\dim_H(\Gamma_a) = \log_3(12a - 3). \quad \square$$



This means that if  $a > 1/2$ , then  $\dim_H(\Gamma_a) > 1$ . And if the Hausdorff dimension of  $\Gamma_a$  exceeds 1, then its one-dimensional Hausdorff measure—that is, its length—must be infinite. So for  $a > 1/2$ , the graph's arc length  $L$  indeed has no upper bound.

Given the result of Theorem 5, it is clear that  $\dim_H(\Gamma_a)$  ranges continuously between 1 and 2 for all  $a \in (1/2, 1)$ . This means that for  $a \in (1/2, 1)$ ,  $\Gamma_a$  meets the criterion of a fractal, being a self-similar set whose Hausdorff dimension exceeds its topological dimension.

#### 4. Conclusions

In his paper, Okamoto [7] posed another question related to  $F_a$ 's differentiability: If  $a \in (0, 1/3)$ , there are infinitely many  $x$  for which  $F_a'(x)$  is finite but nonzero, but how do we classify them? In the corollary to Theorem 1, we have provided a partial answer to this question. Since  $F_a'(x) = 0$  if  $a \in (0, 1/3)$  and  $x$  satisfies  $\gamma = 1/3$ , we know that the points in question at least must be nonnormal numbers in  $[0, 1]$ . It follows that the set of such points must occupy a subset of  $[0, 1]$  with Lebesgue measure 0.

#### References

- [1] BOREL, ÉMILE. Les probabilités dénombrables et leurs applications arithmétiques. *Rendiconti del Circolo Matematico di Palermo* **27** (1909), no. 1, 247–71.
- [2] BOURBAKI, NICOLAS. Functions of a real variable. Elementary theory. Trans. from the 1976 French original [MR 0580296] by Philip Spain. *Springer-Verlag, Berlin*, 2004. xiv+338 pp. ISBN: 3-540-65340-6. [MR2013000](#), [Zbl 1085.26001](#).
- [3] FALCONER, KENNETH. Fractal geometry. Mathematical foundations and applications. Second edition. *John Wiley & Sons, Inc., Hoboken, NJ*, 2003. xxviii+337 pp. ISBN: 0-470-84861-8. [MR2118797](#) (2006b:28001), [Zbl 1060.28005](#).
- [4] KATSUURA, HIDEFUMI. Continuous nowhere-differentiable functions—an application of contraction mappings. *American Mathematical Monthly* **98** (1991), no.5, 411–416. [MR1104304](#) (92c:26005), [Zbl 0752.26005](#).
- [5] KOBAYASHI, KENTA. On the critical case of Okamoto's continuous non-differentiable functions. *Proc. Japan Acad. Ser. A Math. Sci.* **85** (2009), no. 8, 101–104. [MR2561897](#) (2010m:26006), [Zbl 1184.26003](#).
- [6] MCCOLLUM, JAMES. Properties of Bourbaki's function. Preprint [arXiv:1009.2817](#).
- [7] OKAMOTO, HISASHI. A remark on continuous, nowhere differentiable functions. *Proc. Japan Acad. Ser. A Math. Sci.* **81** (2005), no. 3, 47–50. [MR2128931](#) (2006a:26005), [Zbl 1083.26004](#).

DIVISION OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF MAINE AT FARMINGTON, FARMINGTON, ME 04938  
[james.mccollum@maine.edu](mailto:james.mccollum@maine.edu)

This paper is available via <http://nyjm.albany.edu/j/2011/17-24.html>.