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Further notes on a family of continuous, nondifferentiable functions

James McCollum

Abstract. We examine a parametrized family of functions F_a, all of which are continuous and some of which are nowhere or almost nowhere differentiable, we explore the behavior of F_a' and F_a'' almost everywhere for different values of a, focusing on specific questions regarding F_a's differentiability for certain a, and we calculate the Hausdorff dimension of the graphs of all F_a.

Contents

1. Introduction

In this paper, we will examine a family of functions F_a that are continuous for all $a \in (0,1)$, but nowhere or almost nowhere differentiable for certain a. Their construction, described by Okamoto [7] for all a, is based largely on the one given by Bourbaki [2] for $a = 2/3$.

Let F_a be defined inductively over $[0,1]$ by iterations f_i for $i \geq 0$ as follows: $f_0(x) = x$ for all $x \in [0,1]$, every f_i is continuous on $[0,1]$, every f_i is affine in each subinterval $[k/3^i, (k+1)/3^i]$ where $k \in \{0,1,2,\ldots,3^i-1\}$,

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and

$$f_{i+1}\left(\frac{k}{3^i}\right) = f_i\left(\frac{k}{3^i}\right), \tag{1}$$

$$f_{i+1}\left(\frac{3k+1}{3^{i+1}}\right) = f_i\left(\frac{k}{3^i}\right) + a\left[f_i\left(\frac{k+1}{3^i}\right) - f_i\left(\frac{k}{3^i}\right)\right], \tag{2}$$

$$f_{i+1}\left(\frac{3k+2}{3^{i+1}}\right) = f_i\left(\frac{k}{3^i}\right) + (1-a)\left[f_i\left(\frac{k+1}{3^i}\right) - f_i\left(\frac{k}{3^i}\right)\right], \tag{3}$$

$$f_{i+1}\left(\frac{k+1}{3^i}\right) = f_i\left(\frac{k+1}{3^i}\right). \tag{4}$$

Given this construction, we say that

$$F_a = \lim_{i\to\infty} f_i.$$

We also can construct the graph of any F_a with contraction mappings, using as a basis Katsuura's construction [4] of the graph of $F_{2/3}$. Define contraction mappings $w_n : X \mapsto X$, where $n \in \{1,2,3\}$ and $X = [0,1]\times[0,1]$, as follows: for all $(x,y) \in X$,

$$w_1(x,y) = \left(\frac{x}{3}, ay\right) \tag{5}$$

$$w_2(x,y) = \left(\frac{2-x}{3}, (2a-1)y + (1-a)\right) \tag{6}$$

$$w_3(x,y) = \left(\frac{2+x}{3}, ay + (1-a)\right). \tag{7}$$

In this case, the graph of F_a is the unique invariant set for the iterated function system defined by (5), (6), and (7).

This paper builds primarily on a paper by Okamoto [7] regarding the same family of functions. That paper showed, among other things, that if a_0 is the unique root of $27a^2 - 54a^3 = -1$ in $(1/2, 2/3)$, then F_a is differentiable almost everwhere when $a \in (0,1/3)\cup(1/3,a_0)$; $F_{1/3}(x) \equiv x$ for $x \in [0,1]$ and thus is differentiable everywhere on that interval; F_a is differentiable almost nowhere when $a \in (a_0, 2/3)$; and F_a is differentiable absolutely nowhere when $a \in [2/3, 1)$. (Throughout this paper, "almost everywhere" and "almost nowhere" will refer specifically to sets of Lebesgues measure 1 and 0, respectively.) Certain issues, however, were left unresolved in Okamoto's paper. One such issue was whether F_{a_0} is differentiable almost everywhere, almost nowhere, or neither. Kobayashi [5] recently proved that F_{a_0} is in fact differentiable almost nowhere using the law of the iterated logarithm.

In this paper, we will give another proof of Kobayshi's result using only the properties of F_a and the measure of the set of normal numbers in $[0,1]$. Furthermore, using the same methods, we will describe how F'_a behaves almost everywhere in $[0,1]$ as a varies. From here, we will prove that F''_a does not exist anywhere for any $a \in (0,1)$ except $a = 1/3$ (in which case $F_a(x) \equiv x$, and consequently, $F''_a(x) = 0$ for all x) and $a = 1/2$ (in which case $F'_a(x) = 0$ for infinitely many disjoint subintervals of $[0,1]$). Finally, we will show that if $a \in (0, 1/2]$, then its graph has Hausdorff dimension 1, whereas if $a \in (1/2, 1)$, then the Hausdorff dimension of its graph is equal to $\log_3(12a - 3)$.

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2. Remarks on differentiability

Theorem 1. *If S_a is the set of all points x for which $F'_a(x)$ does not exist and a_0 is the unique solution of $27a^2 - 54a^3 = -1$ in $(1/2, 2/3)$, then the Lebesgue measure $|S_{a_0}| = 1$.*

Proof. Let S_a and a_0 be defined as they are above. Now, we can write the ternary expansion of any $x \in [0,1]$ as

$$x = 0.\xi_1\xi_2 \ldots = \sum_{i=1}^{\infty} \frac{\xi_i}{3^i}$$

where each $\xi_i \in \{0,1,2\}$. (If an expansion terminates, we say that it ends with a infinite string of zeroes. If $x = 1$, we say that $\xi_i = 2$ for all i.) For the first n digits of the ternary expansion of x, we define $i(n)$ as the number of times $\xi_i = 1$ in these first n digits. Finally, we let $\gamma = \lim_{n\to\infty} \inf \dfrac{i(n)}{n}$.

According to Okamoto [7],

$$\begin{aligned} F'_a(x) &= \lim_{n\to\infty} (3-6a)^{i(n)}(3a)^{n-i(n)} \\ &= \lim_{n\to\infty} [3(1-2a)^{\gamma} a^{1-\gamma}]^n \end{aligned}$$

and thus $F'_a(x) = 0$ if $|3(1-2a)^{\gamma}a^{1-\gamma}| < 1$. Setting $\gamma = 1/3$, we can rewrite the second expression as $-1 < 3(1-2a)^{1/3}a^{2/3} < 1$. Raising each side to the third power, we get $F'_a(x) = 0$ if $-1 < 27a^2 - 54a^3 < 1$; this quantity is equal to 1 when $a = 1/3$, and it is equal to -1 when $a = a_0$.

As a result, when $a = a_0$ and $\gamma = 1/3$, the sequence $\{f'_i(x)\}_{i=0}^{\infty}$ visits 1 and -1 infinitely often, and therefore does not converge.[1] Thus, $F'_{a_0}(x)$ does not exist for all x satisfying $\gamma = 1/3$. To complete the proof, we must show that this set of values for x has Lebesgue measure 1.

To do this, we turn to the *normal numbers*—a set of irrational real numbers whose digits are uniformly distributed, regardless of the base in which the numbers are written. Given any base $b \geq 2$ and an alphabet of digits $\{0, 1, \ldots, b-1\}$, a normal number x must satisfy

$$\lim_{n\to\infty} \frac{i(n)}{n} = \frac{1}{b}$$

[1] It should be noted, however, that $F'_{a_0}(x)$ does not converge absolutely to 1, as we might expect; $\{f'_i(x)\}_{i=0}^{\infty}$ does not consist solely of '1's and '-1's, and as Kobayashi [5] has pointed out, it contains subsequences that diverge absolutely to infinity.

where n represents the same value it did for γ and $i(n)$ can be taken as the number of any individual digit in the first n digits of x. (While this is not a sufficient property of normal numbers, it is a necessary one, and for the purposes of this proof, we only need to know that all normal numbers have this property.)

This means that the set of all $x \in [0,1]$ that satisfy $\gamma = 1/3$ includes all the normal numbers in $[0,1]$. Borel [1] proved that the set of all nonnormal numbers has Lebesgue measure 0, so the set of all normal numbers in $[0,1]$ must have Lebesgue measure 1. Because the normal numbers in $[0,1]$ are a subset of the numbers satisfying $\gamma = 1/3$ in $[0,1]$, the latter set must have Lebesgue measure 1. So S_{a_0}, the set of all $x \in [0,1]$ for which $F'_{a_0}(x)$ does not exist, must have Lebesgue measure $|S_{a_0}| = 1$, as well. □

The fact that the normal numbers have Lebesgue measure 1 in $[0,1]$ reveals even more about F'_a for all $a \in (0,1)$. Because all normal numbers satisfy $\gamma = 1/3$ in base 3, we know that the value of $F'_a(x)$ will depend on whether $|27a^2 - 54a^3|$ is less than, greater than, or equal to 1 for all x in a set with Lebesgue measure 1. This information allows us to give a general description of the behavior of F'_a almost everywhere:

Corollary. *Let $a \in (0,1)$.*

- *If $a \in [a_0, 1)$, then $F'_a(x)$ diverges for almost all $x \in [0,1]$.*
- *If $a \in (0,1/3) \cup (1/3, a_0)$, then $F'_a(x) = 0$ for almost all $x \in [0,1]$.*
- *If $a = 1/3$, then $F'_a(x) = 1$ for all $x \in [0,1]$.*

We obtain the last result because $F_{1/3}(x) \equiv x$ and thus has the same derivative everywhere it is defined. Additionally, if we take into account the fact that $F_a(x)$ is absolutely nowhere differentiable in $[0,1]$ when $a \in [2/3,1)$, we can replace *almost all* with *all* above when $a \in [2/3,1)$.

This leads us to our last question on the differentiability of F_a: What can we say about the second derivative $F''_a(x)$ for different values of a? We know that since $F_{1/3}(x) \equiv x$, $F''_{1/3}(x) = 0$ for all x. We also know that since $F'_{1/2}(x) = 0$ for all x except when x belongs to the Cantor Set—a set with measure 0—$F'_{1/2}(x)$ is continuous on a set of points with measure 1. Thus, $F'_{1/2}(x)$ is differentiable on this set of points, and its derivative $F''_{1/2}(x) = 0$ at all of them. But clearly $F''_a(x)$ does not exist at any x when $a \in [a_0, 1)$, since even $F'_a(x)$ fails to converge for almost all x in this case. The following theorem answers our question for the remaining values of a.

Theorem 2. *For all $a \in (0,1/3) \cup (1/3,1/2) \cup (1/2, a_0)$ and all $x \in [0,1]$, $F''_a(x)$ does not exist.*

Proof. Let $x \in [0,1]$. We recall from Theorem 1 that almost all x satisfy $\gamma = 1/3$, in that the set of all such values has Lebesgue measure 1. We also recall that for all x belonging to this set of numbers and for all $a \in (0,1/3) \cup (1/3, a_0)$, $F'_a(x) = 0$ if $|27a^2 - 54a^3| < 1$. So for all $a \in (0,1/3) \cup (1/3, a_0)$ and almost all $x \in [0,1]$, $F'_a(x) = 0$.

The graph of $y = F_a'(x)$ appears to be the line $y = 0$ over $[0,1]$, but there are infinitely many points at which $F_a'(x)$ is undefined. Okamoto [7] has shown that for $a \in (0,1/3)$, this set consists of all points of the form $x = (2k+1)/(2 \cdot 3^i)$, for $i \geq 0$ and $k \in \{0,1,\ldots,3^i-1\}$ and for $a \in (1/3,1/2) \cup (1/2,a_0)$, it consists of all points of the form $x = k/3^i$. In either case, the set of points where $F_a'(x)$ does not exist is dense in $[0,1]$. So $F_a'(x)$ is discontinuous everywhere in $[0,1]$, and therefore its derivative, $F_a''(x)$, does not exist anywhere in that interval. □

So unless $a = 1/3$ or $a = 1/2$—in which case $F_a''(x) = 0$ for all x at which it exists—the graph of F_a is neither concave up nor concave down anywhere. Also, F_a belongs to differentiability class C^0 for all $a \neq 1/3$.

Due to its somewhat pathological nature in terms of differentiability, this family of functions may call to mind another set of counterintuitive functions—that of the everywhere differentiable but nowhere monotone functions. It is clear that these two sets have no common element; indeed, the only everywhere differentiable function in our set, $F_{1/3}$, is strictly increasing. Nevertheless, some similarities exist in the ways these sets of functions behave on countable, dense sets. For instance, if $a \in (0,1/3)$, then on the set of all points of the form $x = k/3^i$, where $i \geq 0$ and $k \in \{0,1,\ldots,3^i\}$, $F_a'(x) = 0$, and at all points of the form $x = (2k+1)/(2 \cdot 3^i)$, $F_a'(x)$ is undefined; similarly, an everywhere differentiable but nowhere monotone function G can be constructed over $[0,1]$ using the same two disjoint dense sets under the conditions that for all x in one set, $G'(x) < 0$ and that for all x in the other set, $G'(x) > 0$. Beyond this, however, the author has found no connections between the constructions for these sets of functions. This is primarily because with the dubious exceptions of $F_{1/3}$ and $F_{1/2}$, Okamoto's construction assures that if $F_a'(x) = 0$ on one set dense in $[0,1]$, then $F_a'(x)$ must be undefined on another dense set.

3. Remarks on dimension

We first will show that because the graph of F_a has finite arc length when $a \in (0,1/2]$, it has Hausdorff dimension 1.

Theorem 3. *If Γ_a is the graph of F_a, then its Hausdorff dimension*

$$\dim_H(\Gamma_a) = 1$$

for all $a \in (0,1/2]$.

Proof. Suppose Γ_a is the graph of F_a, and suppose $a \in (0,1/2]$. Okamoto [7] has proven that for all $a \leq 1/2$, F_a is nondecreasing. Now, if F_a is nondecreasing, then clearly the affine pieces of each of its iteration f_i must be nondecreasing as well, since $f_i(k/3^i) = F_a(k/3^i)$ for all $i \geq 0$ and all $k \in \{0,1,\ldots,3^i\}$.

Using the Triangle Inequality, we can determine the maximum arc length of Γ_a. We do this by applying the inequality to each affine segment of f_i

with respect to the segment's horizontal and vertical components and letting $i \to \infty$. More precisely, if l_k is the length of the affine segment of f_i over $[k/3^i, (k+1)/3^i]$, then $l_k \leq |f_i([k+1]/3^i) - f_i(k/3^i)| + |(k+1)/3^i - k/3^i|$.

Because this inequality holds for every affine segment of f_i, we can obtain a bound for the arc length

$$L_i = \sum_{k=0}^{3^i-1} l_k$$

of f_i. And by taking the limit as $i \to \infty$, we can express the maximum arc length L of Γ_a as

$$L \leq \lim_{i\to\infty} \sum_{k=0}^{3^i-1} \left| f_i\left(\frac{k+1}{3^i}\right) - f_i\left(\frac{k}{3^i}\right)\right| + \sum_{k=0}^{3^i-1}\left|\frac{k+1}{3^i} - \frac{k}{3^i}\right|.$$

Obviously, $(k+1)/3^i - k/3^i = 1/3^i$ always, so the second sum becomes 1. As for the first sum, we recall that the affine pieces of every f_i are nondecreasing, so $|f_i([k+1]/3^i) - f_i(k/3^i)| = f_i([k+1]/3^i) - f_i(k/3^i)$. Thus, for all i,

$$\begin{aligned}
&\sum_{k=0}^{3^i-1} \left[f_i\left(\frac{k+1}{3^i}\right) - f_i\left(\frac{k}{3^i}\right)\right] \\
&= f_i\left(\frac{1}{3^i}\right) - f_i\left(\frac{0}{3^i}\right) + f_i\left(\frac{2}{3^i}\right) - f_i\left(\frac{1}{3^i}\right) + \cdots + f_i\left(\frac{3^i}{3^i}\right) - f_i\left(\frac{3^i-1}{3^i}\right) \\
&= f_i(1) - f_i(0) \\
&= 1.
\end{aligned}$$

So for $a \in (0, 1/2]$, the maximum arc length of Γ_a is 2. To find a lower bound for the arc length, we first note that every Γ_a has its endpoints at (0,0) and (1,1). Because the shortest possible distance between these two points is a straight line, the minimum arc length of any Γ_a must be $\sqrt{2}$.

Hence, the graph's 1-dimensional Hausdorff measure $\mathcal{H}^1(\Gamma_a)$ satisfies $0 < \mathcal{H}^1(\Gamma_a) < \infty$, which implies that $\dim_H(\Gamma_a) = 1$ for all $a \in (0, 1/2]$. □

For $a \in (1/2, 1)$, on the other hand, $F_a(x)$ is not nondecreasing, so we cannot find an upper bound for L using the Triangle Inequality. In fact, no such upper bound exists when $a \in (1/2, 1)$; to prove this, we must show that the graph's Hausdorff dimension is greater than 1, but in order to do this, we first must calculate the box-counting dimension of the graph.

Theorem 4. *If Γ_a is the graph of F_a, then its box-counting dimension*

$$\dim_B(\Gamma_a) = \log_3(12a - 3)$$

for all $a \in (1/2, 1)$.

Proof. Suppose Γ_a is the graph of F_a, and suppose $a \in (1/2, 1)$.

To determine the box-counting dimension of Γ_a, we will use a variant of the more familiar method of box-counting. Instead of counting squares of side length δ, we will determine the smallest area $A(f_i)$ needed to cover the graph of iteration f_i using rectangles that share a horizontal side length of δ. Then the "number" of squares with side length δ needed to cover the graph of f_i can be expressed as $N_\delta(f_i) = A(f_i)/\delta^2$. This variant works by the same principle as box-counting dimension with squares does, but it gives more precise answers for each δ we use.

Letting $\delta = 1/3^i$, we see that for all Γ_a, $A(f_0) = 1$. Using the horizontal and vertical ratios of contraction for (5), (6), and (7), we see that

$$\begin{aligned} A(f_{i+1}) &= \frac{1}{3}aA(f_i) + \frac{1}{3}(2a-1)A(f_i) + \frac{1}{3}aA(f_i) \\ &= A(f_i)\left(\frac{4a-1}{3}\right). \end{aligned}$$

So by induction,

$$A(f_i) = \left(\frac{4a-1}{3}\right)^i$$

and thus,

$$\begin{aligned} N_\delta(f_i) &= \frac{([4a-1]/3)^i}{(1/3)^{2i}} \\ &= (12a-3)^i. \end{aligned}$$

So for $a \in (1/2, 1)$, the box-counting dimension of Γ_a is

$$\begin{aligned} \dim_B(\Gamma_a) &= \lim_{i\to\infty} \frac{\log([12a-3]^i)}{-\log(1/3^i)} \\ &= \log_3(12a-3). \end{aligned}$$ □

Next, we will show that the Hausdorff dimension of Γ_a is equal to its box-counting dimension by the Mass Distribution Principle (see, e.g., [3]):

Theorem 5. *If Γ_a is the graph of F_a, then its Hausdorff dimension*

$$\dim_H(\Gamma_a) = \log_3(12a-3)$$

for all $a \in (1/2, 1)$.

Proof. We consider a small variation on Katsuura's construction of Γ_a using contraction mappings. Let $E_0 = [0,1] \times [0,1]$ and define further levels of the construction by $E_{i+1} = w_1(E_i) \cup w_2(E_i) \cup w_3(E_i)$ where $i > 0$ and w_1, w_2, and w_3 are mappings (5)–(7). Clearly $E_{i+1} \subset E_i$ for all $i \geq 0$, and

$$\bigcap_{i=0}^{\infty} E_i = \Gamma_a.$$

This last relationship between the E_is can be understood as follows: While in Okamoto's construction, linear segments are constructed "upwards" to a graph with infinite length, in this construction, rectangular regions are constructed "downwards" to the same graph which has zero area. Moreover, we see that each E_i can be covered by 3^i rectangles of length $(1/3)^i$.

Using methods related to the box-counting process in Theorem 4, it can be shown that if $a \in (1/2, 1)$, then the area of E_{i+1} can be expressed as

$$\begin{aligned} A(E_{i+1}) &= a\left(\frac{1}{3}\right)A(E_i) + (2a-1)\left(\frac{1}{3}\right)A(E_i) + a\left(\frac{1}{3}\right)A(E_i) \\ &= \frac{4a-1}{3}A(E_i), \end{aligned}$$

and since $A(E_0) = 1$, we have $A(E_i) = \left(\dfrac{4a-1}{3}\right)^i$ for all $i \geq 0$.

Now, let μ be the natural mass distribution on Γ_a; we start with unit mass on E_0 and repeatedly "spread" this mass over the total area of each E_i. Also, let U be any set whose diameter $|U| < 1$. Then there exists some $i \geq 0$ such that

$$\left(\frac{1}{3}\right)^{i+1} \leq |U| < \left(\frac{1}{3}\right)^i,$$

an inequality that applies to any U satisfying $0 < |U| < 1$. Given these conditions on the diameter of U, it is clear that for every U, there is some i such that U is contained in an open square of side length $(1/3)^i$ and U contains points in at most two level-i "sub-rectangles."

Hence, the area of U is bounded above by the area of the open square containing it; that is, $A(U) \leq (1/9)^i$. In terms of measure, we know that the entire area of U can be contained in E_i, so

$$\begin{aligned} \mu(U) &\leq \frac{A(U \cap E_i)}{A(E_i)} \\ &\leq \frac{(1/9)^i}{([4a-1]/3)^i} \\ &\leq \left(\frac{1}{12a-3}\right)^i. \end{aligned}$$

And since $\left(\dfrac{1}{3}\right)^{i+1} \leq |U|$ implies that $\left(\dfrac{1}{3}\right)^i \leq 3|U|$, we have

$$\begin{aligned} \mu(U) &\leq \left(\frac{1}{12a-3}\right)^i = \left(\frac{1}{3^i}\right)^{\log_3(12a-3)} \leq (3|U|)^{\log_3(12a-3)} \\ &= (12a-3)|U|^{\log_3(12a-3)}, \end{aligned}$$

and therefore, by the Mass Distribution Principle,

$$\log_3(12a-3) \leq \dim_H(\Gamma_a) \leq \dim_B(\Gamma_a),$$

and given the upper bound obtained in Theorem 4, we have

$$\dim_H(\Gamma_a) = \log_3(12a-3). \qquad \square$$

This means that if $a > 1/2$, then $\dim_H(\Gamma_a) > 1$. And if the Hausdorff dimension of Γ_a exceeds 1, then its one-dimensional Hausdorff measure—that is, its length—must be infinite. So for $a > 1/2$, the graph's arc length L indeed has no upper bound.

Given the result of Theorem 5, it is clear that $\dim_H(\Gamma_a)$ ranges continuously between 1 and 2 for all $a \in (1/2, 1)$. This means that for $a \in (1/2, 1)$, Γ_a meets the criterion of a fractal, being a self-similar set whose Hausdorff dimension exceeds its topological dimension.

4. Conclusions

In his paper, Okamoto [7] posed another question related to F_a's differentiability: If $a \in (0, 1/3)$, there are infinitely many x for which $F_a'(x)$ is finite but nonzero, but how do we classify them? In the corollary to Theorem 1, we have provided a partial answer to this question. Since $F_a'(x) = 0$ if $a \in (0, 1/3)$ and x satisfies $\gamma = 1/3$, we know that the points in question at least must be nonnormal numbers in $[0, 1]$. It follows that the set of such points must occupy a subset of $[0, 1]$ with Lebesgue measure 0.

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Division of Mathematics and Computer Science, University of Maine at Farmington, Farmington, ME 04938
james.mccollum@maine.edu

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