New York Journal of Mathematics

New York J. Math. **16** (2010) 31–51.

Isoperimetric regions in the plane with density r^p

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Abstract. We consider the isoperimetric problem in the plane with density r^p, $p > 0$, and prove that the solution is a circle through the origin. We use the stability of this isoperimetric curve to prove an apparently new generalization of Wirtinger's Inequality.

Contents

1. Introduction

A *density* on a surface is a positive function weighting perimeter and area. In this paper, we study the isoperimetric problem on planes with radial density. The isoperimetric problem seeks the least-perimeter way to enclose given area. The solution is known only for a relatively small number of surfaces (see [HHM]) and for just a few densities on the plane. For

Received November 2008. Revised May 11, 2009.

2000 *Mathematics Subject Classification.* 53A10, 49Q20.

Key words and phrases. Isoperimetric, plane with density, radial densities, symmetrization.

This paper represents work of the 2008 SMALL research Geometry Group. We would like to thank Williams College and the National Science Foundation REU for funding SMALL.

ISSN 1076-9803/2010

Gaussian density $ce^{-a^2r^2}$, minimizers are straight lines; for density $ce^{a^2r^b}$, with $b \geq 2$, minimizers are circles about the origin (see [RCBM, Thm. 3.10] and [MM, Cor. 2.2]). For a few special discontinuous densities, see the paper by Cañete *et al.* [CMV]. For the plane with density $|y|^p$, $(p > 0)$, Engelstein *et al.* [EMMP, Sect. 4] prove that minimizers are semicircles closed by a segment of the x-axis.

1.1. The plane with density r^p. The plane with density r^p is especially interesting because it has vanishing generalized Gauss curvature and because it has a singularity at the origin where the density vanishes. Carroll *et al.* [CJQW, Sect. 4] prove that for $p < -2$, minimizers are circles about the origin (with the enclosed area on the exterior), prove that for $-2 \leq p < 0$ minimizers do not exist, and conjecture that for $p > 0$ minimizers are non-circular convex ovals with the origin in their interiors as in Figure 1b. Our Theorem 3.16 proves that the minimizer is a circle passing through the origin as in Figure 1c. For $p > 0$ it was already known that an isoperimetric region exists and must contain the origin (see Propositions 3.1, 3.5). Our Proposition 2.11 shows that an isoperimetric curve passing through the origin must be a circle. An isoperimetric curve not passing through the origin must have only one maximum and one minimum of r (Proposition 3.12), hence only two extrema of curvature (Lemma 3.15) contradicting the 4-Vertex Theorem [O].

Section 5 solves the isoperimetric problem on lines in the plane with density r^p, $p > 0$.

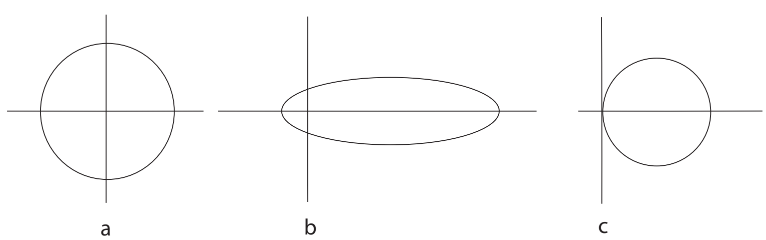

FIGURE 1. Three possibilities for an isoperimetric region in the plane with density r^p, $p > 0$. The first circle (a) is unstable, the last circle (c) is the minimizer.

1.2. Stability. Section 4 considers the second variation of the circle through the origin in the plane with density r^p, $p > 0$. Since the circle is a minimizer, its second variation must be nonnegative. Theorem 4.4 shows that nonnegative second variation is equivalent to an apparently new generalization of Wirtinger's Inequality. For $p = 2$ Theorem 4.5 proves that the circle has positive second variation.

1.3. Brakke's Evolver. In Section 6 we use Ken Brakke's Evolver program to provide computational reinforcement for our results. Figure 2 illustrates how Evolver suggests an isoperimetric region resembling the circle through the origin. We also discuss anomalies created by tiny density near the origin for high values of p, as seen in Figure 3.

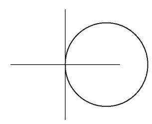

FIGURE 2. For low values of p, Brakke's Evolver produces results very close to the circle through the origin.

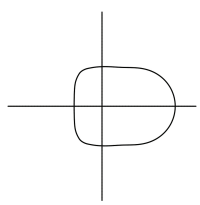

FIGURE 3. For higher values of p, the singularity at the origin gives Brakke's Evolver trouble.

1.4. Acknowledgements. We thank Ken Brakke and Richard McDowell for help with Evolver, and thank Professor Frank Morgan, whose guidance and patience have been invaluable to us in writing this paper. We also would like to thank Sean Howe and David Thompson for helpful comments. Their new paper on "Isoperimetric problems in sectors with density" with Alex Díaz and Nate Harman generalizes some of our results to planar sectors and higher dimensions.

2. Constant curvature curves in planes with density

We consider the plane with density e^ψ used to weight both perimeter and area. In terms of Riemannian perimeter and area (dP_0 and dA_0), weighted perimeter and area satisfy:

$$dP = e^\psi dP_0,$$
$$dA = e^\psi dA_0.$$

For a normal variation u, the first variation satisfies

$$\delta^1(A) = \frac{dA}{dt} = -\int u ds_\psi$$

$$\delta^1(P) = \frac{dP}{dt} = -\int u\kappa_\psi ds_\psi$$

where in terms of the Riemannian curvature κ,

$$\kappa_\psi = \kappa - \frac{\partial \psi}{\partial \mathbf{n}}. \tag{1}$$

We call κ_ψ the *generalized curvature*. It follows that an isoperimetric curve has constant generalized curvature.

Lemma 2.1. *Consider $R^2 - \{0\}$ with smooth radial density $e^{\psi(r)}$. A constant-generalized-curvature curve is symmetric under reflection across every line through the origin and a critical point of r.*

Proof. A curve with constant generalized curvature satisfies the differential equation:

$$\kappa_\psi = c.$$

At any critical point, by uniqueness of solutions of ordinary differential equations the curve must behave the same way whether going clockwise or counterclockwise. □

Lemma 2.2. *If a differentiable planar curve is symmetric across infinitely many lines through the origin, it is a circle around the origin.*

Proof. The group of symmetries of the curve includes reflections across arbitrarily close lines through the origin, hence their compositions, hence arbitrarily small rotations. Therefore for every θ, $r'(\theta) = 0$, and r is constant. □

Definition 2.3. *Define continuous functions σ and θ along a curve C as the angles counter-clockwise from the position vector to the tangent vector and from a fixed vector to the position vector. Then $\alpha = \sigma + \theta$ is the angle from the fixed vector to the tangent vector, so $\kappa = d\alpha/ds$. For a curve in polar coordinates $(r(s), \theta(s))$ parameterized by arc length, $\sin \sigma = r\theta'$ and $\cos \sigma = r'$.*

Lemma 2.4. *In $R^2 - \{0\}$ with smooth radial density $e^{\psi(r)}$, a positive generalized curvature curve that encircles the origin once and has rotation index 1 is a polar graph.*

Proof. Suppose there is such a curve C that is not a polar graph, parameterized by arc length.

Because C encircles the origin once and has rotation index 1, both θ and α change the same amount when we travel around the curve once. Therefore, for any initial point, σ returns to its original value.

Since C is not a polar graph and encircles the origin once, there exist extrema of θ, characterized by σ a multiple of π. At an extremum, $\kappa = \kappa_\psi + \partial\psi/\partial\mathbf{n} = \kappa_\psi$ is positive. Since

$$\kappa = \frac{d\alpha}{ds} = \frac{d\sigma}{ds} + \frac{d\theta}{ds}$$

and $d\theta/ds = 0$, $d\sigma/ds$ is positive whenever σ is a multiple of π. But σ must return to its original value, a contradiction. □

Remarks. If $\psi'(r) > 0$, as holds for $\psi = \log(r^p)$, for $p > 0$, a curve with constant generalized curvature must have $\kappa_\psi > 0$. Indeed, at the point farthest from the origin, κ and $-\partial\psi/\partial\mathbf{n}$ are both positive, so $\kappa_\psi > 0$. (For $p < 0$, the opposite orientation yields $\kappa_\psi > 0$.)

Any regular Jordan curve suitably oriented has rotation index 1.

In the plane with density r^p, any Jordan curve with positive generalized curvature, even if it passes through the origin, which has undefined generalized curvature, is a polar graph. To see this, introduce new coordinates $w = (x+iy)^{p+1}/(p+1)$ as in [CJQW, Prop. 4.3]. Since $|dw| = r^p|d(x+iy)|$, in these new coordinates, length is just the Euclidean $|dw|$ although area is weighted. By [CJQW, Sect. 3], generalized curvature at each point is a positive multiple of classical curvature in the w-plane, which must therefore be positive. Thus the curve is locally convex in the w-plane, implying θ a monotonic function of arc-length. Hence the curve is a polar graph.

Corollary 3.10 provides an alternative proof of Lemma 2.4 under stronger hypotheses.

Proposition 2.5. *A constant-generalized-curvature curve in a planar domain with smooth radial density has finitely many critical points unless it is a circle about the origin.*

Proof. If the curve has infinitely many critical points, it must have infinitely many lines of symmetry by Lemma 2.1 because a Jordan curve intersects a line of symmetry at most twice. Then by Lemma 2.2 it must be a circle about the origin. □

Corollary 2.6. *For a constant-generalized-curvature curve in a planar domain with smooth radial density, critical points are strict extrema unless the curve is a circle about the origin.*

Proof. Suppose the curve is not a circle about the origin. By Proposition 2.5 critical points are isolated. By symmetry (Lemma 2.1), isolated critical points are strict extrema. □

Lemma 2.7. *A constant-generalized-curvature polar graph $r(\theta) > 0$ in a planar domain with smooth radial density is symmetric under the full dihedral group acting on maxima and on minima unless it is a circle about the origin.*

Proof. Suppose the curve is not a circle about the origin. By Proposition 2.5, there are finitely many extrema. By Lemma 2.1, the curve is symmetric under reflection across a line through a minimum, which identifies adjacent maxima. Such symmetries generate the dihedral group of symmetries of maxima. A similar argument applies to minima. ☐

Proposition 2.8. *A polar graph $r(\theta)$ in the plane with density r^p, p real, has vanishing first variation for given enclosed area if and only if it is critical for the Lagrange multiplier functional*

$$P - \lambda A = \int F,$$

where the "Lagrangian" function F is given by

$$F = r^p\sqrt{r^2 + r'^2} - \lambda \frac{r^{p+2}}{p+2},$$

and λ is equal to the (constant) generalized curvature. (When $p = -2$, the second term is $-\lambda \log r$.)

Proof. This is the standard Lagrange multiplier formulation. The formula for F comes from

$$P = \int_{\theta_1}^{\theta_2} r^p ds$$
$$A = \int_{\theta_1}^{\theta_2} \int r^p r dr d\theta = \int_{\theta_1}^{\theta_2} \frac{r^{p+2}}{p+2} d\theta.$$

Note that for $p < -2$ the region of finite area A is the unbounded region. When $p = -2$, we mean the algebraic area of the region between the graph and the unit circle, and the integral of r^{p+1} is $\log r$. Finally, λ is equal to dP/dA, which equals κ_ψ. ☐

Proposition 2.9. *In a planar domain with smooth radial density $\Psi(r) = e^{\psi(r)}$, the generalized curvature of a curve parametrized by arc-length is given by*

$$\kappa_\psi = \kappa + \psi'(r)\sin\sigma. \tag{2}$$

Furthermore, if κ_ψ is constant then the function

$$f(s) = r\Psi(r)\sin\sigma - \kappa_\psi \int r\Psi(r)dr \tag{3}$$

is constant along the curve.

Proof. Equation (2) follows directly from the definition (1) of generalized curvature. To prove (3), just note that

$$\begin{aligned} f'(s) &= r'\Psi(r)\sin\sigma + \Psi'(r)rr'\sin\sigma + r\Psi(r)\sigma'\cos\sigma - \kappa_\psi\Psi(r)rr' \\ &= r'\Psi(r)\sin\sigma + \Psi'(r)rr'\sin\sigma + r\Psi(r)\sigma'\cos\sigma \\ &\quad - (\kappa + \psi'(r)\sin\sigma)\Psi(r)rr' \\ &= -rr'\Psi(r)(\kappa - \sigma' - \theta') \\ &= 0. \end{aligned}$$

□

Remark. For a polar graph $r(\theta)$, (3) is the standard first integral $F - r'\partial F/\partial r'$ constant, where F is the Lagrangian function. For the metric $ds^2 = dr^2 + \Psi^2 r^2 d\theta^2$, this is exactly the Clairaut relation [R, Prop 1.1], but with a different meaning, since Ritoré's angle is measured in the new metric.

Corollary 2.10. *Along a constant-generalized-curvature curve C in a planar domain with density r^p, $p \neq -2$,*

$$f(s) = r^{p+2}\left(\frac{\kappa_\psi}{p+2} - \frac{1}{r}\sin\sigma\right)$$

is constant, and if the constant is 0, *C is either a circle about the origin or a circle through the origin.*

Proof. By Proposition 2.9(3) for density function $\Psi(r) = r^p$, f is constant. If the constant is 0, then $r^{-1}\sin\sigma = d\theta/ds$ is constant. Hence the curve is a polar graph. For a polar graph $r(\theta)$,

$$\frac{d\theta}{ds} = \frac{1}{\sqrt{r^2 + r'^2}}.$$

Therefore, $r^2 + r'^2$ is constant, which implies that $(r^2 + r'^2)' = 2r'(r + r'') = 0$. If $r + r''$ is always 0, then the curve is a circle through the origin. Otherwise, $r + r''$ is not zero somewhere and by continuity there is a piece of the curve where $r' = 0$. By Proposition 2.5, the curve is a circle about the origin. □

Remark. For the case $p = -2$, all circles through the origin are geodesics, that is $\kappa_\psi = 0$. For them, $f(s) = \kappa_\psi \log r - r^{-1}\sin\sigma$ is a negative constant.

Proposition 2.11. *In a planar domain with density r^p, $p > -1$, if a constant-generalized-curvature closed curve passes through the origin, it must be a circle.*

Proof. Since r^{p+1} approaches 0 for $p > -1$, the constant from Corollary 2.10 must be 0. Hence, C is a circle. □

Proposition 2.12. *In $R^2 - 0$ with smooth radial density $\Psi(r)$, every circle through the origin has constant generalized curvature if and only if $\Psi(r) = cr^p$ for real p and positive c.*

Proof. By Proposition 2.9, a circle through the origin has constant generalized curvature if and only if $\psi'(r)\sin\sigma$ is constant. But

$$\psi'(r)\sin\sigma = \psi'(r)r\frac{d\theta}{ds}.$$

Since $d\theta/ds$ is constant along every circle through the origin, constant generalized curvature is equivalent to $r\psi'(r)$ constant, which is also equivalent to $\Psi(r) = cr^p$ by explicitly solving the equation

$$\psi'(r)r = c_1. \qquad \square$$

Remark. This proposition shows that the plane with density r^p is very special. For general radial density, it might be impossible explicitly to solve for constant-generalized-curvature curves, even those passing through the origin.

3. Isoperimetric problem in the plane with density r^p

Our main theorem, Theorem 3.16, proves that the solution to the isoperimetric problem in the plane with density r^p, $p > 0$, is a circle through the origin. First, by Proposition 2.11, if the isoperimetric curve C passes through the origin, it must be a circle. Otherwise, Proposition 3.12 proves that C has only one maximum and one minimum of r. Lemma 3.15 proves that if C has at most two extrema of radius it has at most two extrema of curvature, contradicting the 4-vertex Theorem [O].

Proposition 3.1 ([CJQW, Prop. 4.4], after Rosales et al. [RCBM, Thm. 2.5])**.** *In the plane with density r^p, $p > 0$, there exists an isoperimetric region for any prescribed area.*

Proposition 3.2 ([Mo, Sect. 3.10])**.** *An isoperimetric curve in a smooth surface with density is smooth.*

Definition 3.3. *A curve is* stable *if it has nonnegative second variation.*

Proposition 3.4. *In the plane with density r^p, $p > 0$, a circle centered at the origin is unstable.*

Proof. The density r^p, $p > 0$, is strictly log-concave, so circles about the origin are unstable by [RCBM, Thm. 3.10]. $\square$

Proposition 3.5 ([CJQW, Prop. 4.5])**.** *In a plane with density r^p, $p > 0$, an isoperimetric region must contain the origin in its interior or its boundary.*

Lemma 3.6. *In the plane with density r^p, homothetic expansion by a factor μ increases weighted perimeter by μ^{p+1} and increases area by μ^{p+2}.*

Proof. This follows immediately from

$$dP = r^p dP_0$$
$$dA = r^p dA_0$$

because homothetic expansion multiplies dP_0 by μ and dA_0 by μ^2. □

Lemma 3.7. *In the plane with density r^p, $p > 0$, the least-perimeter 'isoperimetric' function $I(A)$ satisfies*

$$I(A) = cA^{\frac{p+1}{p+2}}. \tag{4}$$

In particular, I is smooth, $I' > 0$, and $I'' < 0$.

Proof. Let c be the perimeter of an isoperimetric region R of area 1. Every homothetic region μR must be isoperimetric, because if another region R_1 had less perimeter, by Lemma 3.6 $\mu^{-1}R_1$ would have less perimeter than R. Hence by Lemma 3.6,

$$I(A) = cA^{\frac{p+1}{p+2}}.$$ □

Proposition 3.8. *In the plane with density r^p, or in any planar domain with smooth density and strictly concave isoperimetric function $I(A)$, the open region bounded by an isoperimetric curve is connected.*

Proof. Suppose the open region R has several connected components R_j of area A_j. By regularity (Proposition 3.2) and strict concavity of I,

$$P(R) = \sum P(R_j) \geq \sum I(A_j) > I\left(\sum A_j\right),$$

contradicting the fact that R is isoperimetric. Note that I is strictly concave in the plane with density r^p by Lemma 3.7. □

The following results give an alternative proof to that of Lemma 2.4 that an isoperimetric curve is a polar graph.

Proposition 3.9. *In the plane with density r^p, $p > 0$, an isoperimetric region is star shaped.*

Proof. By Proposition 3.5, an isoperimetric region is connected and contains the origin in its boundary or its interior. Now suppose there is an isoperimetric region which is not star shaped. Then there must be some line segment OA, where O is the origin and A is a point in the region, which lies partly outside the region. Let BC be a segment of OA which lies outside of the region, intersecting the minimizing curve at B and C. Since the region is connected, part of the minimizing curve must go from B to C. Replace that part of the curve with the line segment BC. Since we add the region between the minimizing curve and BC, this change increases area. Since the minimizing curve has to cover all the same values of r as BC, the change does not increase perimeter. So we have a region enclosing more area than our original isoperimetric region with equal or less perimeter, a contradiction since by Lemma 3.7 the isoperimetric function is strictly increasing. □

Corollary 3.10. *In the plane with density r^p, $p > 0$, an isoperimetric region is bounded by a polar graph.*

Proof. By Proposition 3.9, the region is star shaped. Suppose that the boundary is not a polar graph. Then part of the boundary curve must be a radial line segment. Along that line segment, $\kappa = 0$ and $\partial\psi/\partial\mathbf{n} = 0$ because ψ is radial. Hence $\kappa_\psi = 0$ along part of the boundary curve. But at the point farthest from the origin $\kappa_\psi > 0$, contradicting the fact that the isoperimetric region has constant generalized curvature. □

Remark. Proposition 3.9 and its proof hold for a connected isoperimetric region containing the origin for any radial density with increasing isoperimetric function $I(A)$. Corollary 3.10 holds under a slightly stronger condition, namely $I'(A) > 0$, which implies that $\kappa_\psi > 0$.

Proposition 3.11. *The isoperimetric region in the plane with density r^p is a topological disk.*

Proof. Since the open region is connected by Corollary 3.8, we just need to show that the region has no holes. But a hole could be filled in to increase area while decreasing perimeter, which contradicts I strictly increasing (Lemma 3.6). □

Proposition 3.12. *An isoperimetric curve C in the plane with density r^p, $p > 0$, has one maximum and one minimum of radius.*

Proof. By Propositions 3.5 and 2.11, we may assume that the origin lies in its interior. By Lemma 2.4 and the subsequent remarks, we may assume that the curve is a polar graph. By Proposition 3.4, it is not a circle.

By Proposition 2.5 and Corollary 2.6, C has finitely many critical points, all strict extrema. By Lemma 2.7, it is symmetric under the dihedral group on its maxima and minima, as in Figure 4a. Draw a circle of radius between the maximum and minimum. If the curve has more than one maximum or minimum, rearrange a portion of the curve as in Figure 4b, maintaining area and perimeter, to create singularities, a contradiction of regularity (Proposition 3.2).

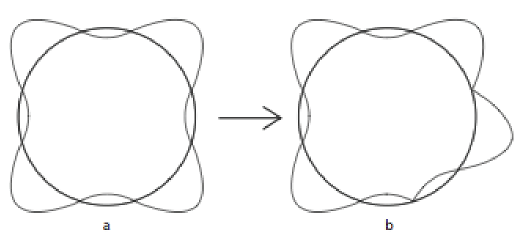

FIGURE 4. A symmetric polar graph with more than one maximum cannot be isoperimetric.

Alternative proof. We describe a transformation that preserves area and decreases perimeter if C has more than two extrema. As above, we assume

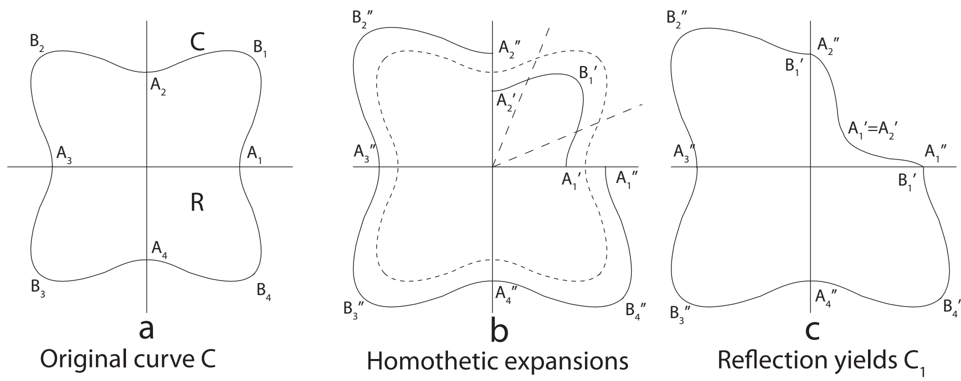

FIGURE 5. Decreasing perimeter while preserving area.

that C is a polar graph and has finitely many critical points, all strict extrema. Since between two minima A_1, A_2 there must be a maximum B_1, we can enumerate all extrema $A_1, B_1 ... A_n, B_n$.

By scaling (Lemma 3.6) and symmetry (Lemma 2.7), homothetic expansions of $A_1B_1A_2$ by $\mu_1 < 1$ and the rest of the curve by $\mu_2 > 1$ yield $A_1'B_1'A_2'$ and $A_2''B_2''...A_1''$ as in Figure 5b with perimeter and area satisfying

$$P(A_1'B_1'A_2') = \mu_1^{p+1}P(A_1B_1A_2) = \frac{1}{n}\mu_1^{p+1}P(R),$$

$$A(OA_1'B_1'A_2') = \mu_1^{p+2}A(A_1B_1A_2) = \frac{1}{n}\mu_1^{p+2}A(R),$$

$$P(A_2''...A_1'') = \mu_2^{p+1}P(A_2...A_1) = \frac{n-1}{n}\mu_2^{p+1}P(R),$$

$$A(OA_2''...A_1'') = \mu_2^{p+2}A(OA_2...A_1) = \frac{n-1}{n}\mu_1^{p+2}A(R),$$

where R is the region bounded by C. Since $n > 1$ we can choose $0 < \mu_1 < 1 < \mu_2$ such that

$$\frac{\mu_2}{\mu_1} = \frac{r_{\max}}{r_{\min}},$$

so reflections of $A_1'B_1'$ and $B_1'A_2'$ across the rays that bisect angles $A_1'OB_1'$ and $B_1'OA_2'$ yield a closed curve as in Figure 5c, and also such that

$$\frac{n-1}{n}\mu_2^{p+2} + \frac{1}{n}\mu_1^{p+2} = 1,$$

so the transformation preserves total area. We claim that the new region R' has less perimeter than R.

$$P(R') = P(A_1'B_1'B_2') + P(A_2''B_2''...A_1'') = \left(\frac{n-1}{n}\mu_2^{p+1} + \frac{1}{n}\mu_1^{p+1}\right)P(R)$$
$$< \left(\frac{n-1}{n}\mu_2^{p+2} + \frac{1}{n}\mu_1^{p+2}\right)P(R) = P(R),$$

the desired contradiction. □

Lemma 3.13. *In the plane with density r^p, the generalized curvature is*

$$\kappa_\psi = \kappa + \frac{p}{r}\sin\sigma. \tag{5}$$

For a polar graph, generalized curvature is equal to

$$\kappa_\psi = \kappa + \frac{p}{\sqrt{r^2 + r'^2}}. \tag{6}$$

Proof. These formulas follow immediately from the definition of generalized curvature, $\kappa_\psi = \kappa - \partial\psi/\partial\mathbf{n}$. □

Proposition 3.14. *In $R^2 - 0$ with density r^p, p real, a constant-generalized-curvature curve C which encircles the origin once and has rotation index 1, which by Lemma 2.4 is a polar graph $r(\theta)$, satisfies $r(\theta) + r''(\theta) > 0$ everywhere on C.*

Proof. First, we assume $p \neq -2$. By Corollary 2.10 and by Lemma 3.13,

$$\begin{aligned} c &= \frac{r^{p+2}}{p+2}\left(\kappa + \frac{p}{\sqrt{r^2+r'^2}} - \frac{p+2}{\sqrt{r^2+r'^2}}\right) \\ &= \frac{r^{p+2}}{p+2}\left(\frac{r^2 - rr'' + 2r'^2}{(r^2+r'^2)^{3/2}} - \frac{2}{\sqrt{r^2+r'^2}}\right) \\ &= -\frac{r^{p+3}}{p+2}\frac{r + r''}{(r^2+r'^2)^{3/2}}, \end{aligned}$$

where we have used the formula for curvature in polar coordinates. If $c = 0$ then by Corollary 2.10 the curve must be either a circle about the origin or a circle through the origin. Since C does not pass through the origin, it must be a circle about the origin where $r + r'' = r > 0$. If $c \neq 0$, then since r'' is nonnegative where r reaches its local minimum, $r + r'' > 0$ there and so $c < 0$. Consequently, $r + r'' > 0$ everywhere on the curve.

If $p = -2$, then by equation (5),

$$\begin{aligned} \kappa_\psi &= \kappa + \frac{p}{r}\sin\sigma \\ &= \kappa - 2\theta'. \end{aligned}$$

By the Gauss–Bonnet theorem, $\int_0^L \kappa_\psi ds = 2\pi - 4\pi = -2\pi < 0$. Therefore, $\kappa_\psi < 0$. By equation (6),

$$\begin{aligned}\kappa_\psi &= \kappa + \frac{p}{\sqrt{r^2 + r'^2}}\\ &= \frac{r^2 - rr'' + 2r'^2}{(r^2 + r'^2)^{\frac{3}{2}}} + \frac{-2r^2 - 2r'^2}{(r^2 + r'^2)^{\frac{3}{2}}}\\ &= \frac{-r(r + r'')}{(r^2 + r'^2)^{\frac{3}{2}}}.\end{aligned}$$

Therefore, $r + r'' > 0$ everywhere on the curve. □

Lemma 3.15. *In the plane with density r^p, for a constant-generalized-curvature curve which encircles the origin once and has rotation index 1, wherever κ has an extremum, r has an extremum.*

Proof. By Lemma 2.4, C is a polar graph. By Lemma 3.13,

$$\kappa_\psi = \kappa + \frac{p}{\sqrt{r^2 + r'^2}}$$

is constant. So if κ has an extremum, then so does $r^2 + r'^2$, so $2r'(r + r'') = 0$. By Proposition 3.14, $r' = 0$. By Corollary 2.6, r has an extremum. □

Theorem 3.16. *In the plane with density r^p, $p > 0$, an isoperimetric curve is a circle through the origin.*

Proof. By Proposition 3.1, a minimizer exists and has constant generalized curvature. If it passes through the origin, it is a single circle by Propositions 2.11 and 3.8. If not, the isoperimetric region contains the origin by Proposition 3.5. By Proposition 3.12, the curve has just two extrema of r. By Lemma 3.15 the curve has just two extrema of curvature, a contradiction of the 4-vertex theorem [O]. □

Conjecture 3.17. *Consider a plane with smooth radial density $\Psi(r) = e^{\psi(r)}$ except that $\Psi(0) = 0$. If $\psi'(r) > 0$ and $\psi''(r) < 0$, an isoperimetric curve must pass through the origin.*

Remark. Without the singularity at the origin, Engelstein *et al.* [EMMP, Conj. 6.9] conjecture that an isoperimetric curve exists and is a circle centered at the origin.

4. Stability

Theorem 4.4 deduces from the stability of our proven isoperimetric circle in the plane with density r^p an apparently new generalization of Wirtinger's Inequality. Theorems 4.5 and 4.6 prove that for $p = 2$, the isoperimetric circle has positive second variation.

Definition 4.1. Let H be the set of all continuous and piecewise C^2 functions on $[-\pi/2, \pi/2]$ with period π.

Lemma 4.2. *Consider the circle through the origin $r(\theta) = \alpha \cos\theta$ in the plane with density r^p where $p \geq 0$. For a variation $u \in H$ preserving weighted area to first order,*

$$0 = \delta(A) = -\alpha^p \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u \cos^p \theta d\theta,$$

the second variation of weighted perimeter is given by

$$\delta^2(P) = \alpha^{p-1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^p \theta (u'^2 - 4u^2 - pu^2(2 - \sec^2\theta))d\theta.$$

Proof. This follows from [RCBM, Prop. 3.6]. □

Lemma 4.3 (Wirtinger's Inequality [Mit, p. 127])**.** *Let $u \in H$ have integral 0 on $[-\pi/2, \pi/2]$. Then*

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u'(\theta)^2 d\theta \geq 4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u(\theta)^2 d\theta.$$

Proof. Since $u \in H$, the associated even-frequency Fourier series converge to u and u'. The result follows from plugging them in. □

Theorem 4.4 (Generalization of Wirtinger's Inequality)**.** *For $u \in H$ satisfying*

$$0 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u \cos^p \theta d\theta,$$

we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^p \theta (u'^2 - 4u^2 - pu^2(2 - \sec^2\theta))d\theta \geq 0.$$

Proof. By Theorem 3.16, the circle through the origin is the global minimizer for curves of fixed area, so its second variation must be nonnegative. The result now follows by Lemma 4.2. □

Remark. We do not know whether the generalization of Wirtinger's Inequality directly implies circles through the origin are isoperimetric, although the standard Wirtinger's Inequality implies that all circles are isoperimetric when $p = 0$ [T, Sect. 6].

Theorem 4.5. *For $p = 2$, the equality condition for Theorem 4.4 is*

$$u(\theta) = a_1 \sin 2x.$$

Proof. By plugging in the even-frequency Fourier series for $u(\theta)$ and integrating we get

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta u(\theta)^2 d\theta = -a_0^2 + \frac{\pi}{8}(a_1^2+b_1^2) + \frac{\pi}{8}\sum_{n=1}^{\infty}(a_n+a_{n+1})^2 + \frac{\pi}{8}\sum_{n=1}^{\infty}(b_n+b_{n+1})^2,$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta u'(\theta)^2 d\theta = \frac{\pi}{2}(a_1^2+b_1^2) + \frac{\pi}{2}\sum_{n=1}^{\infty}(na_n+(n+1)a_{n+1})^2 + \frac{\pi}{2}\sum_{n=1}^{\infty}(nb_n+(n+1)b_{n+1})^2,$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u(\theta)^2 d\theta = \pi a_0^2 + \frac{\pi}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2).$$

This makes the left-hand side of the inequality stated by the lemma equal to

$$6\pi a_0^2 - \frac{\pi}{2}(a_1^2+b_1^2) + \frac{\pi}{2}\sum_{n=1}^{\infty}[(na_n+(n+1)a_{n+1})^2 - 2(a_n+a_{n+1})^2 + 2a_n^2] + \cdots + \frac{\pi}{2}\sum_{n=1}^{\infty}[(nb_n+(n+1)b_{n+1})^2 - 2(b_n+b_{n+1})^2 + 2b_n^2].$$

By the helpful identity

$$(na_n+(n+1)a_{n+1})^2 - 2(a_n+a_{n+1})^2 + 2a_n^2 = (n^2+n-2)(a_n+a_{n+1})^2 + (n+1)a_{n+1}^2 + (2-n)a_n^2$$

the sum telescopes to a sum of squares, which is nonnegative. It is zero only when $f(x) = a_1 \sin 2x + b_1 \cos 2x$, but the constraint forces $b_1 = 0$, therefore $f(x) = a_1 \sin 2x$. □

Theorem 4.6. *For $p = 2$, the circle through the origin has positive second variation for all variations except rotations about the origin.*

Proof. By Theorem 4.4 and Lemma 4.5, the second variation from Lemma 4.2 is nonnegative, and zero only when $u(\theta) = a_1 \sin 2\theta$, which corresponds to rotation. □

5. Lines in the plane

We consider the isoperimetric problem on lines in the plane with density r^p for $p > 0$. Since density is unchanged by rotation about the origin, we only need to consider the lines given by $y = h$, where $h \geq 0$, with density $(x^2+h^2)^{p/2}$. Theorem 5.3 shows that the solution is an interval $[b, c]$ determined by $bc = -h^2$ and a prescribed weighted length.

Proposition 5.1. *On any line in the plane with density r^p, $p > 0$, for any given weighted length, the isoperimetric region exists as a bounded interval containing the point on the line closest to the origin.*

Proof. The density is decreasing for $x < 0$ and increasing for $x > 0$. The result follows directly by [RCBM, Thm. 4.7]. □

Proposition 5.2. *An interval $\{b \leq x \leq c\}$ on the line $y = h$ in the plane with density r^p, $p > 0$, has vanishing first variation if and only if $b = -c$ or $bc = -h^2$.*

Proof. Vanishing first variation means that dP/dL must be the same at each endpoint:

$$\frac{-pb}{b^2 + h^2} = \frac{pc}{c^2 + h^2},$$

which is equivalent to $(b + c)(h^2 + bc) = 0$, i.e., $b = -c$ or $bc = -h^2$. □

Theorem 5.3. *On the line $y = h$, $h \geq 0$, in the plane with density r^p, $p > 0$, isoperimetric regions are segments $[-b, b]$ for $0 < b \leq h$ and then segments $[-h^2/b, b]$, $[-b, h^2/b]$ with $b > h$.*

Proof. By Proposition 5.1, isoperimetric regions exist and are bounded intervals. Consider moving a segment $[a, b]$ of fixed weighted length from left to right. For fixed length we may assume that

$$\frac{da}{dt} = \frac{1}{\Psi(a)},$$
$$\frac{db}{dt} = \frac{1}{\Psi(b)}.$$

Then the rate of change of perimeter is given by

$$\frac{dP}{dt} = \frac{\Psi'(a)}{\Psi(a)} + \frac{\Psi'(b)}{\Psi(b)} = \psi'(a) + \psi'(b),$$

which is a positive quantity times $(a + b)(h^2 + ab)$.

If the length is less than or equal to the length of $[-h, h]$, then P decreases until $a = -b$, after which it increases, so that the isoperimetric region is of the form $[-b, b]$. If the length is greater than the length of $[-h, h]$, then P decreases until $a = -h^2/b$, increases until $a = -b$, decreases until once again $a = -h^2/b$ (the reflection arcoss $x = 0$ of the first such point), and then increases. So the isoperimetric regions are the two segments $[-h^2/b, b]$, $[-b, h^2/b]$ with $b > h$. □

Remarks. As a special case, the corollary says that the isoperimetric solution on a line through the origin is an interval with an endpoint at the origin.

All of the minimizers have positive second variation except for $[-h, h]$, which has vanishing second variation.

6. Brakke's Evolver for the plane with density r^p

The Evolver program, written by Ken Brakke [B1, B2], takes a given initial shape and evolves it towards the least-energy shape satisfying certain conditions. By setting energy to be weighted perimeter, we can use Evolver to make conjectures about the isoperimetric region in the plane with density r^p for various values of $p > 0$.

The program calculates the area of a region in the plane with density r^p by taking the line integral of the vector field $(r^p(-y), r^p x)$ along the boundary of the region. We thank Ken Brakke for help with this formula.

For $p = 0$, any initial condition will rapidly flow to a circle, which is the isoperimetric region in this case. Similarly, for small p, the program consistently flows toward a circle through the origin (see Figure 6).

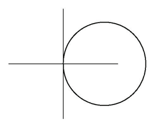

FIGURE 6. For low values of p, Brakke's Evolver produces results very close to the circle through the origin.

Unfortunately, Evolver struggles with regions of very high or very low density. This means that as p increases and density near the origin goes to zero, the program makes smaller and smaller changes near the origin and converges very slowly, as in Figures 7 and 8.

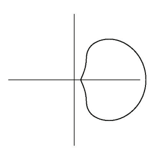

FIGURE 7. With $p = 3$, the density near the origin is so small that Evolver takes a very long time to extend the shape to the origin. This figure is evolved from a convex pentagon to the right of the origin.

The program often drastically slows its rate of movement when close to going through the origin, resulting in a situation where the program gets stuck near a particular suboptimal shape. As it converges, the shape sometimes retains vestiges of its initial condition (see Figures 8–10).

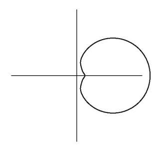

FIGURE 8. With $p = 3$ Evolver has difficulty extending the initial shape to go through the origin. This figure started out as a nonconvex pentagon, with its nonconvex vertex closest to the origin.

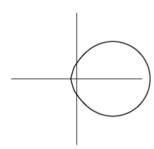

FIGURE 9. With $p = 3$, Evolver also has difficulty bringing the shape back to go through the origin, if it starts out with the origin in the interior. Here most of the shape is approximately circular, but the part near the origin is stretched out.

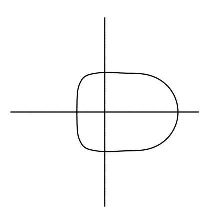

FIGURE 10. With $p = 10$, the initial square is still clearly visible.

The program can also get stuck on clearly suboptimal intermediate stages. If an edge overshoots the origin, the low density can make it very difficult for the shape to recover.

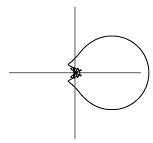

FIGURE 11. With $p = 3$, Evolver also has difficulty correcting after overshooting. The density is too small for it to quickly correct the knot at the origin.

Figures 7–9 and 11 show the results of different starting conditions for $p = 3$, after hundreds of iterations (enough to reach a circle through the origin for smaller p). The results do show a certain degree of similarity, but they behave very differently near the origin. All these shapes have perimeter within a third of a percent of one another, so we can see that they differ only in regions of extremely low density. The shapes will gradually converge with more iterations, but we do not have a practical solution to the issue of low-density regions for higher values of p.

Fortunately, we can still get useful information from the program even if its behavior is inconsistent near the origin. Lemma 2.1 tells us that a constant-generalized-curvature shape containing a circular arc must be a circle, and by Proposition 2.11 an isoperimetric circle must go through the origin. We cannot directly tell whether a shape in Evolver contains a circular arc, as the shape is numerically represented by a polygon, so this fact is not immediately useful. Still, it does show how we can guess a shape's behavior near the origin by looking at what it does further away.

There is one further caveat regarding our use of Evolver. The outcome of the simulation is sensitive not just to the starting shape, but to how much that starting shape is refined before the program starts to iterate: the more vertices the program has for each edge, the less it changes the overall shape of the edge. Although different initial conditions do converge to similar solutions for any given value of p, the shape rarely changes too much after a certain degree of refinement.

With the help of Richard McDowell, we have also calculated generalized curvature in Evolver. In calculating $\partial\psi/\partial\mathbf{n}$, we need to keep track of successive vertices.

An initial trial run of the above procedure for $p = 3$ as in Figure 11. resulted in the data in Table 1. The sign of the curvature is clearly wrong, as the version of the code used calculated the wrong unit normal.

The outliers come from the vertices and edges closest to the origin. This is to be expected since Evolver's curves do not go exactly through the origin and $\partial\psi/\partial\mathbf{n}$ blows up near the origin.

Table 1. Computed Generalized Curvature

Number	Radial Norm	Generalized Curvature
1.	0.00332	-903
2.	0.01254	-63.3
3.	0.01254	-63.3
4.	0.02454	-16.4
5.	0.02453	-16.4
6.	0.03668	-7.07
7.	0.03668	-7.07
8.	0.04886	-3.92
9.	0.04886	-3.92
10.	0.06106	-3.14

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This paper is available via http://nyjm.albany.edu/j/2010/16-4.html.