

## Unions of arcs from Fourier partial sums

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ABSTRACT. Elementary complex analysis and Hilbert space methods show that a union of at most  $n$  arcs on the circle is uniquely determined by the  $n$ th Fourier partial sum of its characteristic function. The endpoints of the arcs can be recovered from the coefficients appearing in the partial sum by solving two polynomial equations.

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We let  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$  and  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ , and for any subset  $E$  of  $\mathbb{T}$  and integer  $k$  we write

$$\widehat{E}(k) = \frac{1}{2\pi} \int_E e^{-ikt} dt$$

for the  $k$ th Fourier coefficient of the characteristic function  $\chi_E$  of  $E$ . As bounded functions with the same sequence of Fourier coefficients agree almost everywhere, any subset  $E$  of  $\mathbb{T}$  is determined up to a set of measure zero by the sequence  $\widehat{E}(k)$ . If  $E$  is known to have additional structure, the entire sequence may not be needed to recover  $E$ . Our present subject is a simple yet nontrivial illustration of this principle.

An *arc* is by definition a closed, connected, proper and nonempty subset of  $\mathbb{T}$ . We declare  $\mathbb{T}$  along with the empty set to be a “union of 0 arcs.”

**Theorem 1.** *If  $n$  is a nonnegative integer and  $E_1$  and  $E_2$  are unions of at most  $n$  arcs satisfying*

$$(1) \quad \widehat{E}_1(k) = \widehat{E}_2(k), \quad 0 \leq k \leq n,$$

*then  $E_1 = E_2$ .*

Received February 3, 2010.

1991 *Mathematics Subject Classification.* Primary: 42A16, 46N99.

*Key words and phrases.* Fourier coefficients, trigonometric moment problems, finite Blaschke products.

The author was partially supported by the University of Iowa Department of Mathematics NSF VIGRE grant DMS-0602242.

Thus a set  $E$  that is known to be a union of at most  $n$  arcs can be recovered *completely* from the  $n$ th Fourier partial sum of  $\chi_E$ , regardless of any quantitative sense in which this partial sum fails to approximate  $\chi_E$ . This stands in slight contrast to the well-known defects of Fourier partial sum approximation of functions with jump discontinuities, such as the Gibbs phenomenon (see, e.g., [4, Chapter 17]). Significantly, the property of the Fourier basis expressed by Theorem 1 is not shared by other orthonormal systems of functions on  $\mathbb{T}$  (see §3).

Our proof of Theorem 1 exploits a connection between unions of arcs and certain rational functions — the *Blaschke products*, whose properties we recall in §1. Each Blaschke product has a nonnegative integer *order*. In §2 we construct an injection  $E \mapsto b_E$  from the set of finite unions of arcs to the set of Blaschke products with the property that if  $E$  is a union of at most  $n$  arcs, then  $b_E$  has order at most  $n$ . This map has the property that if  $E_1$  and  $E_2$  satisfy (1), then  $b_{E_1}$  and  $b_{E_2}$  have the same  $n$ th order Taylor polynomial at 0. To prove Theorem 1 it then suffices to observe, as we do in §3, that a Blaschke product of order at most  $n$  is determined by its  $n$ th order Taylor polynomial.

With Theorem 1 in hand, one may ask how to recover  $E$  from a partial list of Fourier coefficients in an explicit fashion. This is the subject of §4, where we present an algorithm for testing whether or not a given tuple of complex numbers takes the form  $(\widehat{E}(k))_{k=0}^n$  for a union  $E$  of at most  $n$  arcs, and for finding the endpoints of these arcs in terms of the Fourier coefficients in this case.

Perhaps because of its elementary nature, we have not found Theorem 1 explicitly stated in the literature, though it is known, and the literature abounds with theorems on the reconstruction of a function from partial knowledge of its Fourier transform. In [6], for example, it is shown that a function on  $\mathbb{T}$  that is piecewise constant on a partition of  $\mathbb{T}$  into  $m$  arcs may be recovered from its  $m$ th Fourier partial sum. (Note that Theorem 1 concludes slightly more from a stronger hypothesis.) And our method is by no means the only route to Theorem 1. It is possible to give a purely algebraic proof exploiting the fact that for  $1 \leq k \leq n$  the numbers  $\widehat{E}(k)$  are polynomials in the endpoints of the arcs of  $E$ .

We are indebted to Donald Sarason for many valuable discussions, and to Mihalis Kolountzakis for drawing our attention to [6].

## 1. Blaschke products

**Definition.** A (finite) Blaschke product is a function of the form

$$(2) \quad b(z) = \lambda \prod_{j=1}^n \frac{z - a_j}{1 - \overline{a_j}z}$$

for some nonnegative integer  $n$ , some  $\lambda \in \mathbb{T}$ , and some  $a_1, \dots, a_n \in \mathbb{D}$ . The nonnegative integer  $n$  is called the *order* of the Blaschke product.

If  $n = 0$  we interpret the empty product as 1. The domain of a Blaschke product is either  $\mathbb{T}$ ,  $\mathbb{D}$ , or the closure  $\overline{\mathbb{D}}$  of  $\mathbb{D}$ , depending on context. A Blaschke product is evidently a rational function that maps  $\mathbb{T}$  to itself and has no poles in  $\mathbb{D}$  (it suffices to check the case  $n = 1$ ). It is well known that these properties characterize the Blaschke products.

**Proposition 1.** *If a rational function  $r$  maps  $\mathbb{T}$  to itself and has no poles in  $\mathbb{D}$ , then it is a Blaschke product of order equal to the number  $n$  of zeros of  $r$  in  $\mathbb{D}$ , counted according to multiplicity.*

**Proof.** We induct on  $n$ . If  $n = 0$ , then  $r = q^{-1}$  for some polynomial  $q$ ; write  $q(z) = \sum_{k=0}^m q_k z^k$  with  $q_m \neq 0$ . As  $q(\mathbb{T}) \subseteq \mathbb{T}$  we have

$$q(z)^{-1} = \overline{q(z)} = \overline{q(\overline{z}^{-1})} = \sum_{k=0}^m \overline{q_k} z^{-k} = \frac{\sum_{k=0}^m \overline{q_k} z^{m-k}}{z^m}, \quad z \in \mathbb{T},$$

so this holds for all nonzero  $z \in \mathbb{D}$ . As  $q$  has no zeros in  $\mathbb{D}$ , the extreme right hand side has no pole at 0; thus  $m = 0$  and  $q$  is constant as desired.

If  $r$  has  $n + 1$  zeros in  $\mathbb{D}$ , choose one,  $a$ , and note that  $r(z) \cdot (\frac{z-a}{1-\overline{a}z})^{-1}$  has  $n$  zeros in  $\mathbb{D}$  and maps  $\mathbb{T}$  to itself.  $\square$

**Definition.** If  $b$  is a Blaschke product, we let  $U_b = \{z \in \mathbb{T} : \operatorname{Im} b(z) \geq 0\}$ .

If the zeros of a Blaschke product are  $a_1, \dots, a_n$ , we calculate from (2)

$$\frac{z b'(z)}{b(z)} = \sum_{j=1}^n \frac{1 - |a_j|^2}{|z - a_j|^2} > 0, \quad z \in \mathbb{T},$$

so the argument of  $b(e^{it})$  is strictly increasing in  $t$ . The argument principle implies that  $b(e^{it})$  travels  $n$  times counterclockwise around  $\mathbb{T}$  as  $t$  runs from 0 to  $2\pi$ .

**Corollary 1.** *A Blaschke product  $b$  has order  $n$  if and only if  $U_b$  is a disjoint union of  $n$  arcs.*

This is the main reason we include  $\mathbb{T}$  as a “union of 0 arcs.”

## 2. Blaschke products from unions of arcs

Let  $S = \{z \in \mathbb{C} : 0 \leq 2 \operatorname{Re} z \leq 1\}$  and let  $\phi$  denote the function

$$\phi(z) = \frac{\exp(2\pi i(z - 1/4)) - 1}{\exp(2\pi i(z - 1/4)) + 1}.$$

It is easy to show (see, e.g., [2, §III.3]) that  $\phi$  maps  $S$  bijectively onto  $\overline{\mathbb{D}} \setminus \{\pm 1\}$ , that  $\phi$  restricts to an analytic bijection of the interior of  $S$  with  $\mathbb{D}$ , that  $\phi$  maps the right boundary line of  $S$  onto  $\{z \in \mathbb{T} : \operatorname{Im} z > 0\}$ , and that  $\phi$  maps the left boundary line of  $S$  onto  $\{z \in \mathbb{T} : \operatorname{Im} z < 0\}$ .

**Proposition 2.** *If  $E$  is a disjoint union of  $n \geq 0$  arcs and  $h_E$  is given by*

$$(3) \quad h_E(z) = \frac{1}{2}\widehat{E}(0) + \sum_{k=1}^{\infty} \widehat{E}(k)z^k, \quad z \in \mathbb{D},$$

then  $h_E$  is an analytic map of  $\mathbb{D}$  into  $S$ , and the function  $\mathbb{D} \rightarrow \overline{\mathbb{D}}$  given by

$$b_E = \phi \circ h_E$$

extends uniquely to a Blaschke product  $\overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$  of order  $n$  satisfying  $U_{b_E} = E$ .

Using the formulas for  $\phi$  and  $h_E$  one can show without much work that  $b_E$  is a rational function; the work in proving Proposition 2 is to establish that  $b_E$  has the mapping properties of Proposition 1, and hence is a Blaschke product, and to prove that  $U_{b_E} = E$ .

To motivate the argument, let us work nonrigorously for a moment. Formally we have the series expansion

$$(4) \quad \chi_E(z) = \sum_{k \in \mathbb{Z}} \widehat{E}(k)z^k, \quad z \in \mathbb{T},$$

and formal manipulation of the series (3) with  $z \in \mathbb{T}$  then shows that

$$\chi_E(z) = h_E(z) + \overline{h_E(z)} = 2 \operatorname{Re} h_E(z), \quad z \in \mathbb{T}.$$

As  $\chi_E$  is  $\{0, 1\}$  valued on  $\mathbb{T}$ , the maximum principle for harmonic functions then implies that  $h_E$  maps  $\mathbb{D}$  into  $S$ , so  $b_E = \phi \circ h_E$  maps  $\overline{\mathbb{D}}$  into  $\overline{\mathbb{D}}$  and sends the circle to itself. By Proposition 1 it follows that  $b_E$  is a Blaschke product; the equality  $U_{b_E} = E$  comes from the mapping properties of  $\phi$  on the boundary of  $S$ .

What makes this argument nonrigorous is that the series (4) does not converge for all  $z \in \mathbb{T}$ , and to equate  $\chi_E$  with  $2 \operatorname{Re} h_E$  is to ignore the distinction between a discontinuous real valued function on  $\mathbb{T}$  and a harmonic function on  $\mathbb{D}$ . To fill in these gaps, we need to use the actual connection between  $2 \operatorname{Re} h_E$  and  $\chi_E$  — the former is the Poisson integral of the latter.

**Proof.** It is easily checked that (3) does define an analytic function on  $\mathbb{D}$ , e.g., because  $\sum_{k=1}^{\infty} |\widehat{E}(k)|^2$  is convergent. One can then verify the identity

$$2h_E(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + ze^{-is}}{1 - ze^{-is}} \chi_E(e^{is}) ds, \quad z \in \mathbb{D}.$$

(Fix  $z$ , expand  $\frac{1}{1 - ze^{-is}}$  as a power series in  $z$  and interchange the sum and the integral.) Taking real parts it follows that for any  $r \in [0, 1)$  and any  $t$

$$(5) \quad 2 \operatorname{Re} h_E(re^{it}) = \frac{1}{2\pi} \int_0^{2\pi} P_r(t - s) \chi_E(e^{is}) ds,$$

where

$$P_r(t) = \operatorname{Re} \left( \frac{1 + re^{it}}{1 - re^{it}} \right)$$

is the *Poisson kernel*. It is elementary (see, e.g., [2, §X.2]) that for  $r \in [0, 1)$  the function  $P_r$  is nonnegative and satisfies  $\frac{1}{2\pi} \int_0^{2\pi} P_r(\theta) d\theta = 1$ ; thus (5) implies that  $2 \operatorname{Re} h_E(z) \in [0, 1]$  for all  $z \in \mathbb{D}$ , and  $h_E$  maps  $\mathbb{D}$  into  $S$ .

As  $r$  increases to 1, the  $P_r$  converge uniformly to the zero function on the complement of any neighborhood of 0 (see, e.g., [2, §X.2]). From (5) we conclude

$$(6) \quad \lim_{r \uparrow 1} 2 \operatorname{Re} h_E(rz) = \chi_E(z)$$

at any  $z \in \mathbb{T}$  at which  $\chi_E$  is continuous. We conclude that for any such  $z$  the limit  $\lim_{r \uparrow 1} (\phi \circ h_E)(rz)$  exists and is in  $\mathbb{T}$ .

We claim that  $\phi \circ h_E$  is a rational function. In the case  $n = 0$  this is clear. Otherwise, from the definition of  $\phi$  it suffices to show that  $\exp(2\pi i h_E)$  is a rational function, and for this it suffices to treat the case  $n = 1$ . In this case there are real numbers  $a < b$  with  $b - a < 2\pi$  satisfying  $E = \{e^{it} : t \in [a, b]\}$ , and  $\widehat{E}(k) = \frac{\exp(-ikb) - \exp(-ika)}{-2\pi ik}$  for all  $k > 0$ . Let  $\log$  denote the analytic logarithm defined on  $\mathbb{C} \setminus \{z \in \mathbb{C} : z \leq 0\}$  that is real on the positive real axis and recall that  $\log(1 - z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}$  for all  $z \in \mathbb{D}$ . A comparison of power series shows

$$h_E(z) = \frac{b - a}{4\pi} + \frac{1}{2\pi i} \left( \log(1 - e^{-ib}z) - \log(1 - e^{-ia}z) \right), \quad z \in \mathbb{D},$$

so  $\exp(2\pi i h_E) = \exp\left(i \frac{b-a}{2}\right) \frac{1 - e^{-ib}z}{1 - e^{-ia}z}$  is rational.

At this point we know that  $b_E = \phi \circ h_E$  is a rational function mapping  $\mathbb{D}$  into itself. From (6) we deduce that  $b_E$  maps  $\mathbb{T}$  into itself, so  $b_E$  is a Blaschke product by Proposition 1. The equality  $U_{b_E} = E$  then follows from (6). The order of  $b_E$  is  $n$  by Corollary 1.  $\square$

If  $E_1$  and  $E_2$  are two unions of arcs related by (1), it is clear from the definition that  $h_{E_1}$  and  $h_{E_2}$  have the same  $n$ th order Taylor polynomial at 0. As  $\phi$  is analytic at 0, the same is true of  $b_{E_1}$  and  $b_{E_2}$ .

**Corollary 2.** *If  $n \geq 0$  and  $E_1$  and  $E_2$  are each unions of at most  $n$  arcs satisfying*

$$(7) \quad \widehat{E}_1(k) = \widehat{E}_2(k), \quad 0 \leq k \leq n,$$

*then there are Blaschke products  $b_1$  and  $b_2$ , each of order at most  $n$ , satisfying  $E_j = U_{b_j}$  for  $j = 1, 2$  and*

$$(8) \quad \widehat{b}_1(k) = \widehat{b}_2(k), \quad 0 \leq k \leq n.$$

### 3. Blaschke products from Toeplitz matrices

Fix a positive integer  $n$  for the remainder of this section. Our goal is to show that Blaschke products  $b_1$  and  $b_2$  having order at most  $n$  and satisfying

(8) must be equal. Let  $L^2$  denote the space of square-integrable functions  $\mathbb{T} \rightarrow \mathbb{C}$ , with inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \overline{g(e^{it})} dt, \quad f, g \in L^2.$$

(We identify two functions if they agree almost everywhere.)

For  $0 \leq k \leq n$  we let  $\zeta^k$  denote the function  $\mathbb{T} \rightarrow \mathbb{C}$  given by  $z \mapsto z^k$ . It is immediate that  $\{\zeta^k : 0 \leq k \leq n\}$  is an orthonormal subset of  $L^2$ . We denote its span, the space of analytic polynomials of degree at most  $n$ , by  $P$ ; we let  $\pi : L^2 \rightarrow P$  denote the orthogonal projection.

**Definition.** If  $f : \mathbb{T} \rightarrow \mathbb{C}$  is bounded,  $T_f : P \rightarrow P$  denotes the linear map given by

$$T_f \xi = \pi(f\xi), \quad \xi \in P.$$

Here  $f\xi$  is the pointwise product of  $f$  and  $\xi$ .

If we let  $\|T_f\|$  denote the norm of  $T_f$  regarded as a linear operator on  $P$  and write  $\|f\|_\infty = \sup_{z \in \mathbb{T}} |f(z)|$ , it is clear that

$$\|T_f\| \leq \|f\|_\infty$$

for any bounded  $f$ . It is also clear that for any such  $f$

$$\langle T_f \zeta^k, \zeta^j \rangle = \widehat{f}(j - k), \quad 0 \leq j, k \leq n,$$

so the matrix of  $T_f$  with respect to the orthonormal basis  $\{\zeta^k : 0 \leq k \leq n\}$  is constant along its diagonals (it is a *Toeplitz matrix*).

If  $f$  is a Blaschke product, then  $f$  is analytic on  $\overline{\mathbb{D}}$ , so the matrix of  $T_f$  is lower triangular with first column  $(\widehat{f}(k))_{k=0}^n$ . Our hypothesis (8) is thus that  $T_{b_1} = T_{b_2}$ , and to deduce that  $b_1 = b_2$  it suffices to show how to recover a Blaschke product  $b$  of order at most  $n$  from the operator  $T_b$  it induces on  $P$ .

**Lemma 1.** *If  $b$  is a Blaschke product of order at most  $n$ , then  $\|T_b\| = 1$ , and for any nonzero  $r \in P$  satisfying  $\|T_b r\| = \|r\|$  one has  $T_b r = br$ .*

**Proof.** There are nonzero polynomials  $p$  and  $q$ , each of degree at most  $n$ , satisfying  $b = p/q$ . Clearly  $T_b q = p$ , and as  $b$  maps  $\mathbb{T}$  to itself, we have  $|p(z)| = |q(z)|$  for all  $z \in \mathbb{T}$ , so  $\|p\| = \|q\|$ . We deduce that  $\|T_b q\| = \|q\|$  and thus  $\|T_b\| \geq 1$ ; since also  $\|T_b\| \leq \|b\|_\infty = 1$ , we conclude  $\|T_b\| = 1$ .

If  $r \in P$  satisfies  $\|T_b r\| = \|r\|$  we have

$$\|r\|^2 = \|T_b r\|^2 = \|\pi(br)\|^2 \leq \|br\|^2 = \int_0^{2\pi} |b(e^{it})|^2 |r(e^{it})|^2 dt = \|r\|^2,$$

from which  $\|\pi(br)\| = \|br\|$  and thus  $\pi(br) = br$  as desired.  $\square$

**Remark 1.** The argument of Lemma 1 can be modified to show that if  $f$  is bounded and analytic on  $\overline{\mathbb{D}}$  and  $\|f\|_\infty = 1$ , then  $\|T_f\| \leq 1$  with equality if and only if  $f$  is a Blaschke product of order at most  $n$ . With more work, one can prove the Caratheodory–Fejer theorem: every lower triangular

$(n + 1) \times (n + 1)$  Toeplitz  $M$  satisfying  $\|M\| = 1$  is of the form  $T_f$  for such an  $f$ .

We are now in a position to give a short proof of Theorem 1.

**Proof of Theorem 1.** By Corollary 2 there are Blaschke products  $b_1$  and  $b_2$  of order at most  $n$  satisfying  $U_{b_j} = E_j$  for  $j = 1, 2$  and  $\widehat{b_1}(k) = \widehat{b_2}(k)$  for  $0 \leq k \leq n$ . This second fact implies that  $T_{b_1} = T_{b_2}$ . By Lemma 1 there is nonzero  $q \in P$  satisfying  $\|T_{b_1}q\| = \|T_{b_2}q\| = \|q\|$  and

$$b_1 = \frac{T_{b_1}q}{q} = \frac{T_{b_2}q}{q} = b_2,$$

so  $E_1 = U_{b_1} = U_{b_2} = E_2$ . □

As the Fourier coefficients of a bounded function are coefficients with respect to an orthonormal basis of the Hilbert space  $L^2$ , one might wonder if Theorem 1 is a special case of a simpler result about arbitrary orthonormal bases of  $L^2$ . This is not the case. There are, for example, orthonormal bases  $B$  for  $L^2$  with the property that for every finite subset  $F \subseteq B$ , there is an arc  $A$  with the property that every element of  $F$  is constant on  $A$ . (The basis  $(e^{2\pi it})_{f \in H}$ , where  $H$  is the Haar basis of  $L^2[0, 1]$  constructed in [3, §III.1], has this property.) In this situation, if  $E \subseteq A$  and  $E' \subseteq A$  are any two unions of arcs with the same total measure, one will have  $\langle \chi_E, f \rangle = \langle \chi_{E'}, f \rangle$  for all  $f \in F$ : any finite collection of coefficients with respect to  $B$  must fail to distinguish infinitely many unions of  $n$  arcs from one another.

### 4. An algorithm

Let  $\mathcal{F}$  denote the map sending a union of at most  $n$  arcs  $E$  to the tuple  $(\widehat{E}(k))_{k=0}^n$  in  $\mathbb{C}^{n+1}$ . Suppose  $c = (c_k)_{k=0}^n$  is given, and we desire to know whether or not  $c$  is in the range of  $\mathcal{F}$ . The arguments of the previous sections give us the following procedure. (We use the orthonormal basis of §3 to identify linear operators on  $P$  with  $(n + 1) \times (n + 1)$  matrices.)

- (1) Calculate the  $n$ th Taylor polynomial at 0 for  $\phi(\frac{c_0}{2} + \sum_{k=1}^n c_k z^k)$ , and make its coefficients the first column of a lower-triangular Toeplitz matrix  $M$ .
- (2) Evaluate  $\|M\|$ .  
If  $\|M\| \neq 1$ , then  $c$  is not in the range of  $\mathcal{F}$ .
- (3) Otherwise  $\|M\| = 1$  and by the Caratheodory–Fejer theorem (see Remark 1) there is a unique Blaschke product  $f$  of order at most  $n$  satisfying  $M = T_f$ . Find  $F = U_f$  (e.g., by solving  $f(z) = \pm 1$  to get the endpoints of the arcs) and calculate the coefficients of the  $n$ th order Taylor polynomial at 0 for  $b_F$ .

If these coefficients are the first column of  $M$  then  $b_F = f$  and  $c = \mathcal{F}(F)$ ; otherwise  $c$  is not in the range of  $\mathcal{F}$ .

**Remark 2.** The third step of the algorithm is necessary as the map  $E \mapsto b_E$  from unions of  $n$  arcs to Blaschke products of order  $n$  is not surjective. One can check, for example, that of the Blaschke products  $b_t(z) = \frac{z^n - t}{1 - tz^n}$  for real  $|t| < 1$ , all of which satisfy  $U_{b_t} = U_{b_0}$ , only  $b_0$  is in the range of  $E \mapsto b_E$ .

If we know in advance that  $c = \mathcal{F}(E)$  is in the range of  $\mathcal{F}$ , this algorithm can recover  $E$  from  $c$  in a somewhat explicit fashion. The matrix  $M$  constructed from  $c$  is  $T_{b_E}$ ; Lemma 1 implies that if we choose a nonzero  $q \in P$  satisfying  $\|Mq\| = \|q\|$ , we will have  $b_E = \frac{Mq}{q}$ . If  $q$  is chosen so as to have minimal degree, the polynomials  $Mq$  and  $q$  will have no nontrivial common factors. In this case the degree of  $q$  is the order of  $b_E$ , and the endpoints of the arcs of  $E$  — the solutions to  $b_E(z) = 1$  and  $b_E(z) = -1$  — are the roots of the polynomials  $Mq - q$  and  $Mq + q$ . A computer has no difficulty carrying out this procedure to find the arcs of  $E$  to any given precision from the tuple  $c = \mathcal{F}(E)$ .

As this algorithm involves solving polynomial equations, we cannot expect symbolic formulas for these endpoints of the arcs of  $E$  in terms of the Fourier coefficients  $\widehat{E}(k)$ . Formulas for the polynomials  $Mq \pm q$ , however, can be obtained with some effort. The entries of  $M$  are polynomials in  $\exp(2\pi i \widehat{E}(0))$ ,  $\widehat{E}(1)$ ,  $\dots$ ,  $\widehat{E}(n)$  with complex coefficients. As  $M$  has norm 1, a vector  $q$  will satisfy  $\|Mq\| = \|q\|$  if and only if  $q$  is an eigenvector for the self-adjoint matrix  $M^*M$  corresponding to the eigenvalue 1; we can find such a  $q$  by using Gaussian elimination, for example. As the entries of  $M^*M$  are polynomials in the entries of  $M$  and their complex conjugates, the coefficients of  $q$  and  $Mq \pm q$  will be rational functions in  $\exp(2\pi i \widehat{E}(0))$ ,  $\widehat{E}(1)$ ,  $\dots$ ,  $\widehat{E}(n)$  and their complex conjugates. Cases may arise in computing  $Mq \pm q$  symbolically: in row reducing the symbolic matrix  $M^*M - I$ , one needs to know whether or not certain functions of the matrix entries are zero — but explicit formulas can be obtained in every case.

We give one example. Suppose that  $E$  is a union of at most two arcs, with  $\widehat{E}(0)$ ,  $\widehat{E}(1)$ , and  $\widehat{E}(2)$  given. Write  $E_0 = \exp(2\pi i \widehat{E}(0))$  and  $E_k = -2\pi i k \widehat{E}(k)$  for  $k = 1, 2$ . Carrying out the above procedure, one finds that if both  $E_1$  and the denominator of

$$a = \frac{E_2 \overline{E_1} + 2E_1 - E_1^2 \overline{E_1} - 2E_1 E_0}{E_1^2 E_0 + E_2 E_0 - E_2 + E_1^2},$$

are nonzero, then the starting points of the arcs of  $E$  are the solutions  $z$  of the equation

$$z^2 - az + \left( \frac{\overline{E_1} + (1 - E_0)a}{E_1 E_0} \right) = 0.$$

The endpoints of the arcs of  $E$  are given by a similar formula.



## References

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This paper is available via <http://nyjm.albany.edu/j/2010/16-11.html>.